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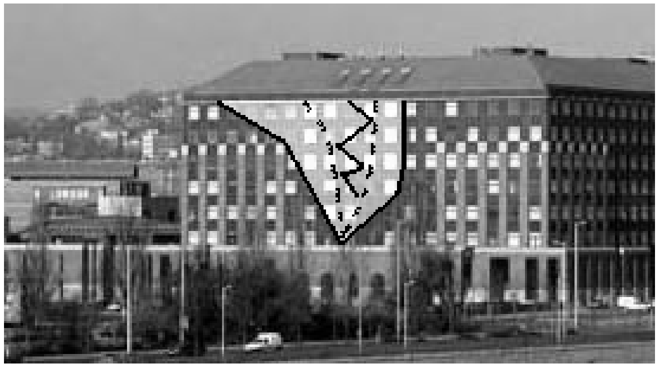

On closedness conditions, strong separation, and convex duality

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Abstract. In the paper, we describe various applications of the closedness and duality theorems of [7] and [8]. First, the strong separability of a polyhedron and a linear image of a convex set is characterized. Then, it is shown how stability conditions (known from the generalized Fenchel-Rockafellar duality theory) can be reformulated as closedness conditions. Finally, we present a generalized Lagrange duality theorem for Lagrange programs described with cone-convex/cone-polyhedral mappings.

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1 Introduction

Closedness conditions require the closedness of convex sets of the form

$$(AC_1) + C_2 := \{Ax + y : x \in C_1,\, y \in C_2\}$$

or

$$C_1 + A^{-1}(C_2) := \{x + v : x \in C_1,\, Av \in C_2\},$$

where A is an m by n real matrix, C_1 and C_2 are convex sets in $\mathcal{R}^n$ and $\mathcal{R}^m$, respectively. These conditions play an important role in the theory of duality in convex programming, see [7] and [8]. In this paper our aim is to describe further applications.

We begin this paper with stating the main results of [7] and [8]. First we fix some notation.

Let us denote by $\operatorname{rec} C$ and $\operatorname{bar} C$ the *recession cone* and the *barrier cone* of a convex set C in $\mathcal{R}^d$, that is let

$$\operatorname{rec} C := \left\{z \in \mathcal{R}^d : x + \lambda z \in C \ (x \in C,\, \lambda \geq 0)\right\},$$
$$\operatorname{bar} C := \left\{a \in \mathcal{R}^d : \inf\{a^T x : x \in C\} > -\infty\right\}.$$

Then $\operatorname{rec} C$ and $\operatorname{bar} C$ are convex cones.

Let us denote by $\operatorname{ri} C$ ($\operatorname{cl} C$) the *relative interior* (*closure*) of the convex set C in $\mathcal{R}^d$. The relative interior of a convex set C is convex, and is nonempty if the convex set C is nonempty. (See [4] for the definition and properties of the relative interior.)

The main result of [7] and [8] is the following closedness theorem. See [8] for an extension of Theorem 1.1 with statements concerning the recession cones. See [3], [7] for further closedness theorems.

THEOREM 1.1. *Let A be an m by n real matrix. Let C_1 be a closed convex set in $\mathcal{R}^n$, and let P_2 be a polyhedron in $\mathcal{R}^m$. Then between the statements*

a) $(A^T \operatorname{bar} P_2) \cap \operatorname{ri}(\operatorname{bar} C_1) \neq \emptyset$,

b) $A^{-1}(-\operatorname{rec} P_2) \cap (\operatorname{rec} C_1) \subseteq -\operatorname{rec} C_1$,

c) $(AC_1) + P_2$ *is closed,*

d) $C_1 + A^{-1}(P_2)$ *is closed,*

hold the following logical relations: a) is equivalent to b); c) is equivalent to d); a) or b) implies c) and d).

In [7] two applications of Theorem 1.1 are mentioned. These duality theorems are stated in Theorems 1.2 and 1.3.

We will use the terminology and notations of [5] here. Let $f : \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ be a convex function, and let $g : \mathcal{R}^m \to \mathcal{R} \cup \{-\infty\}$ be a concave function. Let $A \in \mathcal{R}^{m\times n}$, $b \in \mathcal{R}^m$, $c \in \mathcal{R}^n$. We will consider the following pair of programs from [5]:

$$\begin{array}{ll} (P): & \inf f(x) - g(Ax - b) + c^T x,\ x \in \mathcal{R}^n, \\ (D): & \sup g^c(y) - f^c(A^T y - c) + b^T y,\ y \in \mathcal{R}^m. \end{array}$$

Here f^c and g^c denote the *convex conjugate function* of f and the *concave conjugate function* of g, respectively, i.e. let

$$f^c(a) := \sup\left\{a^T x - f(x) : x \in \mathcal{R}^n\right\},\ g^c(y) := \inf\left\{y^T z - g(z) : z \in \mathcal{R}^m\right\}.$$

Let $[f]$ and $[g]$ denote the *epigraph* of f and the *hypograph* of g, respectively, i.e.

$$[f] := \{(x,\mu) \in \mathcal{R}^{n+1} : f(x) \leq \mu\},\ [g] := \{(z,\nu) \in \mathcal{R}^{m+1} : g(z) \geq \nu\}.$$

Let $F(f)$ and $F(g)$ denote the domain of finiteness of the functions f and g, respectively, i.e. let

$$F(f) := \{x \in \mathcal{R}^n : f(x) < +\infty\},\, F(g) := \{z \in \mathcal{R}^m : g(z) > -\infty\}.$$

The points of the set

$$\mathbf{P} := F(f) \cap \{x : Ax - b \in F(g)\}$$

are called the *feasible solutions* of program (P). We denote by v_P the *optimal value* of program (P), i.e. let

$$v_P := \inf \left\{ f(x) - g(Ax - b) + c^T x : x \in \mathbf{P} \right\}.$$

For the program (D) the set $\mathbf{D}$ and the value v_D can be defined similarly.

With this notation the main duality results of [7] can be stated as follows.

THEOREM 1.2. *Let f be a convex function on $\mathcal{R}^n$, and let $-g$ be a polyhedral convex function on $\mathcal{R}^m$. Then between the statements*

a) the function f is closed, and there exists a strict feasible solution of the program (D), that is a point $y_0 \in \mathcal{R}^m$ such that $y_0 \in F(g^c)$ and $A^T y_0 - c \in \operatorname{ri} F(f^c)$,

b) it holds that $\mathbf{P} \cup \mathbf{D} \neq \emptyset$, and the primal closedness assumption is satisfied, that is the set

$$C_P := \begin{pmatrix} A & 0 \\ c^T & 1 \end{pmatrix} [f] + (-[g])$$

is closed,

c) the optimal values of programs (P) and (D) are equal, and the primal optimal value v_P is attained if it is finite,

hold the following logical relations: a) implies b); b) implies c).

The next theorem is an immediate consequence of Theorem 1.2, as for closed convex functions f and $-g$ the equations $f^{cc} = f$ and $g^{cc} = g$ hold, so Theorem 1.2 can be dualized.

THEOREM 1.3. *Let f be a closed convex function on $\mathcal{R}^n$, and let $-g$ be a polyhedral convex function on $\mathcal{R}^m$. Then between the statements*

a) there exists a strict feasible solution of the program (P), that is a point $x_0 \in \mathcal{R}^n$ such that $x_0 \in \operatorname{ri} F(f)$ and $Ax_0 - b \in F(g)$,

b) it holds that $\mathbf{P} \cup \mathbf{D} \neq \emptyset$, and the dual closedness assumption is satisfied, that is the set

$$C_D := \begin{pmatrix} A^T & 0 \\ b^T & 1 \end{pmatrix} [g^c] + (-[f^c])$$

is closed,

c) the optimal values of programs (P) and (D) are equal, and the dual optimal value v_D is attained if it is finite,

hold the following logical relations: a) implies b); b) implies c).

In the paper, we describe various applications of these closedness and duality theorems: Theorems 1.1, 1.2, and 1.3 will be applied in Sections 2, 3, and 4, respectively. In Section 2 an analogue of Theorem 1.1 is proved, where the property closedness is replaced by strong separability. In Section 3 we reformulate stability conditions (known from the generalized Fenchel-Rockafellar duality theory [5]) as closedness conditions. Generalized Lagrange duality (for programs with cone-convex constraints) is the topic of several papers, see for example [9], [2], and [1]. Our approach is different: in Section 4 we study Lagrange programs described with cone-convex/cone-polyhedral mappings.

2 Strong separation

In this section we will prove an analogue of Theorem 1.1 for strong separation, where the property "closed" is replaced with the property "the origin is not an element of the closure".

Two nonempty convex sets C_1 and C_2 in $\mathcal{R}^n$ are called *strongly separable* if there exists a vector $a \in \mathcal{R}^n$ such that

$$\sup\{a^T x_1 : x_1 \in C_1\} < \inf\{a^T x_2 : x_2 \in C_2\}.$$

It is well-known (see [4], Theorem 11.4) that the sets C_1 and C_2 are strongly separable if and only if $0 \notin \operatorname{cl}(C_2 + (-C_1))$. (Note that the sets C_1 and C_2 are disjoint if and only if $0 \notin C_2 + (-C_1)$.)

The next theorem is an immediate consequence of Theorem 1.1.

THEOREM 2.1. *Let A be an m by n real matrix. Let C_1 be a convex set in $\mathcal{R}^n$, and let P_2 be a polyhedron in $\mathcal{R}^m$. Then between the statements*

a) $0 \notin (AC_1) + P_2$ (that is the sets AC_1 and $-P_2$ are disjoint),

b) $0 \notin C_1 + A^{-1}(P_2)$ (that is the sets $-C_1$ and $A^{-1}(P_2)$ are disjoint),

c) $0 \notin \operatorname{cl}((AC_1) + P_2)$ (that is the sets AC_1 and $-P_2$ are strongly separable),

d) $0 \notin \operatorname{cl}(C_1 + A^{-1}(P_2))$ (that is the sets $-C_1$ and $A^{-1}(P_2)$ are strongly separable),

hold the following logical relations: a) is equivalent to b); a) is equivalent to c) if the set $(AC_1) + P_2$ is closed; b) is equivalent to d) if the set $C_1 + A^{-1}(P_2)$ is closed.

Specially, all the four statements are equivalent if from Theorem 1.1 statement a), b), c) or d) holds. $\square$

The statements c) and d) in Theorem 2.1 are equivalent in the general case as well, as the following theorem shows.

THEOREM 2.2. *Let A be an m by n real matrix. Let C_1 and C_2 be convex sets in $\mathcal{R}^n$ and $\mathcal{R}^m$, respectively. Then,*

a) if $0 \notin \mathrm{cl}\,((AC_1)+C_2)$ then $0 \notin \mathrm{cl}\,(C_1+A^{-1}(C_2))$ (in other words the strong separability of the sets AC_1 and $-C_2$ implies the strong separability of the sets $-C_1$ and $A^{-1}(C_2)$),

b) the statement a) can be reversed if $C_2 \subseteq A(\mathcal{R}^n)$,

c) the statement a) can be reversed if the set C_2 is a polyhedron.

Proof. a) We will show that $0 \in \mathrm{cl}\,(C_1 + A^{-1}(C_2))$ implies $0 \in \mathrm{cl}\,((AC_1) + C_2)$. Let $x_i \in C_1$, $v_i \in A^{-1}(C_2)$ for $i = 1, 2, \ldots$, and suppose that $x_i + v_i \to 0$ $(i \to \infty)$. Then $A(x_i + v_i) \to 0$ $(i \to \infty)$ also holds. As $Av_i \in C_2$ for $i = 1, 2, \ldots$ by definition, we can see that $0 \in \mathrm{cl}\,((AC_1) + C_2)$; the statement a) is proved.

b) Let us suppose now that the set C_2 is a subset of the image space of the matrix A. We will show that then $0 \notin \mathrm{cl}\,(C_1 + A^{-1}(C_2))$ implies $0 \notin \mathrm{cl}\,((AC_1) + C_2)$. By Corollary 11.4.2 in [4], the origin can be strongly separated from the convex set $C_1 + A^{-1}(C_2)$, that is there exists a vector $a \in \mathcal{R}^n$ such that

$$0 < \inf\{a^T x : x \in C_1 + A^{-1}(C_2)\}. \tag{1}$$

As the recession cone of the set $A^{-1}(C_2)$ contains the null space of the matrix A, the inequality (1) implies that the vector a is an element of the image space $A^T(\mathcal{R}^m)$: there exists a vector $z \in \mathcal{R}^m$ such that $a = A^T z$.

Suppose indirectly, that $0 \in \mathrm{cl}\,((AC_1) + C_2)$. Then there exist points $x_i \in C_1$, $y_i \in C_2$ $(i = 1, 2, \ldots)$ such that

$$Ax_i + y_i \to 0 \ (i \to \infty).$$

By assumption, the set C_2 is a subset of the image space of the matrix A, so for some vectors $v_i \in \mathcal{R}^n$ (actually, $v_i \in A^{-1}(C_2)$), the equalities $y_i = Av_i$ $(i = 1, 2, \ldots)$ hold. But then

$$a^T(x_i + v_i) = z^T(Ax_i + y_i) \to 0 \ (i \to \infty),$$

contradicting (1). Hence, $0 \notin \mathrm{cl}\,((AC_1) + C_2)$; statement b) is proved as well.

c) Let us suppose that the set C_2 is a polyhedron. We will show that then the strong separability of the sets $-C_1$ and $A^{-1}(C_2)$ implies the strong separability of the sets AC_1 and $-C_2$. Notice that $A^{-1}(C_2) = A^{-1}(C_2 \cap$

$A(\mathcal{R}^n))$. Here the set $C_2 \cap A(\mathcal{R}^n)$ is a subset of the image space of the matrix A, so by the statement b) the strong separability of the sets $-C_1$ and $A^{-1}(C_2)$ implies the strong separability of the sets AC_1 and $-C_2 \cap A(\mathcal{R}^n)$. Hence, there exist a vector $b \in \mathcal{R}^m$ and a constant $\beta \in \mathcal{R}$ such that the set AC_1 is a subset of the closed halfspace $H^+ := \{y : b^T y \leq \beta\}$, and the polyhedrons $H^+ \cap A(\mathcal{R}^n)$ and $-C_2$ are disjoint. It is well-known that disjoint polyhedrons are strongly separable (see [4], Corollary 19.3.3), so the strong separability of the sets AC_1 and $-C_2$ follows, which finishes the proof of the theorem. □

Finally, we remark that the statement a) in Theorem 2.2 can not be reversed generally, even if the sets C_1 and C_2 are supposed to be closed and convex: there exist closed convex sets C_1 and C_2 such that

$$0 \in \mathrm{cl}\,((AC_1) + C_2),\ 0 \notin \mathrm{cl}\,(C_1 + A^{-1}(C_2))$$

for some linear mapping A.

Really, let

$$A : (\lambda, \mu) \mapsto \lambda \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ (\lambda, \mu \in \mathcal{R});$$

$$C_1 := \mathcal{R} \times \{0\} \subseteq \mathcal{R}^2;\ C_2 := \mathrm{PSD}_2 - \begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix},$$

where PSD_2 denotes the closed convex cone of the 2 by 2 real symmetric positive semidefinite matrices, that is (see [6]),

$$\mathrm{PSD}_2 = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \in \mathcal{R}^{2\times 2} : \alpha, \gamma, \alpha\gamma - \beta^2 \geq 0 \right\}.$$

Then,

$$\begin{pmatrix} 1+i+1/i & 1/2+i \\ 1/2+i & i \end{pmatrix} - i \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix} \ (i \to \infty)$$

shows that

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix} \in \mathrm{cl}\,(\mathrm{PSD}_2 + AC_1).$$

Hence, $0 \in \mathrm{cl}\,((AC_1) + C_2)$.

On the other hand, it can be easily verified that

$$A^{-1}(C_2) = \{(\lambda, \mu) : \lambda \geq -1/2,\ \mu = -1/2\},$$

thus indeed $0 \notin \mathrm{cl}\,(C_1 + A^{-1}(C_2))$; the sets C_1 and C_2 meet the requirements.

3 Stable points

In this section, after describing a geometric and an equivalent algebraic definition of stable points, we reformulate the stability condition as a closedness condition.

The following lemma, concerning the programs (P) and (D), will be needed.

LEMMA 3.1. *Let us suppose that* $\mathbf{D} \neq \emptyset$. *Then the primal closedness assumption is satisfied (that is the set* C_P *is closed) if and only if for every vector* $b \in \mathcal{R}^m$ *the optimal values of programs* (P) *and* (D) *are equal, and the primal optimal value* v_P *is attained if it is finite.*

Proof. As the definition of the set C_P does not depend on the vector b, so the "only if" part of the lemma is a consequence of Theorem 1.2.

On the other hand, with minor modification of the proof of Theorem 4.1 in [7], it can be shown that:

$$(b, \delta) \in C_P \Leftrightarrow \exists x \in \mathcal{R}^n : f(x) - g(Ax - b) + c^T x \leq \delta;$$

and, in case of $\mathbf{P} \cup \mathbf{D} \neq \emptyset$,

$$(b, \delta) \notin \operatorname{cl} C_P \Leftrightarrow \exists y \in \mathcal{R}^m : g^c(y) - f^c(A^T y - c) + b^T y > \delta.$$

Hence, to prove the "if" part of the lemma, it is enough to verify that for every vector $b \in \mathcal{R}^m$ and for every constant $\delta \in \mathcal{R}$,

$$\exists x \in \mathcal{R}^n : f(x) - g(Ax - b) + c^T x \leq \delta \tag{2}$$

or

$$\exists y \in \mathcal{R}^m : g^c(y) - f^c(A^T y - c) + b^T y > \delta \tag{3}$$

holds. For a given vector $b \in \mathcal{R}^m$ two cases are possible:

Case 1: $\mathbf{P} = \emptyset$. Then $v_P = v_D = \infty$, and (3) holds for every $\delta \in \mathcal{R}$.

Case 2: $\mathbf{P} \neq \emptyset$. Then by assumption $v_P = v_D$ with primal attainment, so (2) holds for $\delta \geq v_P$, and (3) holds for $\delta < v_P$.

This way we have proved the "if" part of the theorem as well. $\square$

The following stability conditions appear in the generalized Fenchel-Rockafellar duality theory concerning programs (P) and (D), see [5]. First, we recall the geometric definition of stability.

Let C be a convex set in $\mathcal{R}^d$, and let $e \in \operatorname{rec} C$. A point $x_0 \in C$ is called a *stable point* of the set C if for every affine set N in $\mathcal{R}^d$ satisfying

$$N \cap (\{x_0\} + \mathcal{R}e) \neq \emptyset \text{ and } N \cap (C + \mathcal{R}_{++}e) = \emptyset, \tag{4}$$

there exists a hyperplane H in $\mathcal{R}^d$ such that

$$N \subseteq H \text{ and } H \cap (C + \mathcal{R}_{++}e) = \emptyset. \tag{5}$$

(Here let $\mathcal{R}_{++}e := \{\lambda e : 0 < \lambda \in \mathcal{R}\}$, and let $\mathcal{R}e := \{\mu e : \mu \in \mathcal{R}\}$. It can be easily seen that (4) implies $e \notin \operatorname{rec} N$, and that (5) implies $e \notin \operatorname{rec} H$.)

For a convex function h defined on $\mathcal{R}^n$ the point $u_0 \in F(h)$ is called a *stable point* of the function h, if (u_0, μ_0) is a stable point of the epigraph $[h]$ (with $e_1 := (0, 1) \in \operatorname{rec}[h]$) for some $\mu_0 \in \mathcal{R}$. In this case the function h is called u_0-*stable*. For example, it is proved in [5], that for every $u_0 \in \operatorname{ri} F(h)$, the function h is u_0-stable.

The next lemma, describing an algebraic characterization of u_0-stability, can also be found in [5], see Lemma 5.5.8.

LEMMA 3.2. *Let* $u_0 \in F(h)$. *A convex function* h *on* $\mathcal{R}^n$ *is* u_0*-stable if and only if for every* $n \times m$*-matrix* B *and for every vector* $w \in \mathcal{R}^n$ *with* $u_0 = By_0 - w$ *for some* $y_0 \in \mathcal{R}^m$, *the relation*

$$\hat{h}^c(v) = \min\{h^c(x) + w^T x : B^T x = v\} \tag{6}$$

holds for all $v \in \mathcal{R}^m$. *Here* $\hat{h}(y) := h(By - w)$.

Now we can derive, as an immediate consequence of Lemmas 3.1 and 3.2,

THEOREM 3.1. *Let* $u_0 \in F(h)$. *A closed convex function* h *on* $\mathcal{R}^n$ *is* u_0*-stable if and only if for every* $n \times m$*-matrix* B *and for every vector* $w \in \mathcal{R}^n$ *with* $u_0 = By_0 - w$ *for some* $y_0 \in \mathcal{R}^m$, *the set*

$$\begin{pmatrix} B^T & 0 \\ w^T & 1 \end{pmatrix} [h^c] \tag{7}$$

is closed.

Proof. Apply Lemma 3.1 to the programs

$$\begin{aligned} (P_0) \quad &: \quad \inf f_0(x) - g_0(A_0 x - b_0) + c_0^T x, \ x \in \mathcal{R}^n, \\ (D_0) \quad &: \quad \sup g_0^c(y) - f_0^c(A_0^T y - c_0) + b_0^T y, \ y \in \mathcal{R}^m, \end{aligned}$$

where

$$f_0 := h^c, \ g_0(z) := \begin{cases} 0, & \text{if } z = 0, \\ -\infty & \text{otherwise} \end{cases} \quad (z \in \mathcal{R}^m),$$
$$A_0 := B^T, \ b_0 := v, \ c_0 := w.$$

We obtain that the set in (7) is closed if and only if for all $b_0 \in \mathcal{R}^m$ the optimal values of programs (P_0) and (D_0) are equal, and the primal optimal value v_{P_0} is attained if it is finite. This means that the set in (7) is closed if and only if (6) holds for all $v \in \mathcal{R}^m$. (Note that $\hat{h}^c(v)$ is the optimal value of the dual program (D_0), while the minimum on the right hand side of the equality in (6) is the optimal value of the program (P_0).) Then, Lemma 3.2 gives the statement. $\Box$

Specially, let p be a polyhedral convex function on $\mathcal{R}^n$. Then the conjugate function p^c is also a polyhedral convex function. In other words, the epigraph $[p^c]$, and with it its linear images, are polyhedrons. Hence, by Theorem 3.1, for any vector $u_0 \in F(p)$, the function p is u_0-stable. For another proof of this fact, see [5], Theorem 5.5.9.

4 Lagrange duality

In this section a strong duality theorem concerning generalized Lagrange programs will be derived from a strengthened version of Theorem 1.3.

The following lemma shows that the implication "a)⇒c)" in Theorem 1.3 can also be proved without the assumption that the function f is closed.

LEMMA 4.1. *Let f be a convex function on $\mathcal{R}^n$, and let $-g$ be a polyhedral convex function on $\mathcal{R}^m$. Let us suppose that the program (P) has a strict feasible solution: a point $x_0 \in \mathcal{R}^n$ such that $x_0 \in \operatorname{ri} F(f)$ and $Ax_0 - b \in F(g)$. Then, the optimal values of programs (P) and (D) are equal. Furthermore, the dual optimal value v_D is attained if it is finite.*

Proof. Let us denote by $(\overline{P})$ the program, which we obtain by replacing the functions f and g with their closures $\operatorname{cl} f$ and $\operatorname{cl} g = g$, that is let

$$(\overline{P}): \ \inf (\operatorname{cl} f)(x) - g(Ax - b) + c^T x, \, x \in \mathcal{R}^n.$$

Then the dual of program $(\overline{P})$ is program (D). The point x_0 is also a strict feasible solution of program $(\overline{P})$, so by Theorem 1.3 the optimal values of programs $(\overline{P})$ and (D) are equal, and the optimal value of program (D) is attained if it is finite.

We will show that the optimal values of programs (P) and $(\overline{P})$ are equal. It is obvious, that $v_{\overline{P}} \leq v_P$, as $\operatorname{cl} f \leq f$. On the other hand, for a given $\mu > v_{\overline{P}}$, let x_1 be a feasible solution of program $(\overline{P})$ with corresponding value

$$\mu_1 := (\operatorname{cl} f)(x_1) - g(Ax_1 - b) + c^T x_1 < \mu.$$

Then, for $0 \leq \lambda < 1$ the point $x_\lambda := \lambda x_1 + (1 - \lambda) x_0$ is a strict feasible solution of program (P). Moreover, it can be easily seen from Theorem 7.5 and Corollary 7.5.1 in [4] that

$$f(x_\lambda) \to (\operatorname{cl} f)(x_1), \, g(Ax_\lambda - b) \to g(Ax_1 - b) \ (0 \leq \lambda < 1, \, \lambda \to 1).$$

Consequently, we have for all $\mu > v_{\overline{P}}$,

$$v_{\overline{P}} \leq v_P \leq f(x_\lambda) - g(Ax_\lambda - b) + c^T x_\lambda \to \mu_1 < \mu \ (0 \leq \lambda < 1, \, \lambda \to 1).$$

Thus $v_P = v_{\overline{P}}$, which proves the statement. $\Box$

Now, we describe the definition of the generalized Lagrange programs.

Let $C \subseteq \mathcal{R}^n$ be a convex set, and let $P \subseteq \mathcal{R}^n$ be a polyhedron. Let $K \subseteq \mathcal{R}^m$ be a convex cone, and let $R \subseteq \mathcal{R}^l$ be a polyhedral cone. Let $\tilde{f} : C \to \mathcal{R}$ be a convex function, and let $\tilde{p} : P \to \mathcal{R}$ be a polyhedral convex function. Let $\tilde{g} : C \to \mathcal{R}^m$ be a K-convex mapping, and let $\tilde{h} : P \to \mathcal{R}^l$ be an R-polyhedral mapping. (A mapping $\tilde{g} : C \to \mathcal{R}^m$ is K-*convex*, if the epigraph

$$[\tilde{g}]_K := \{(x, y) \in \mathcal{R}^n \times \mathcal{R}^m : x \in C, \, \tilde{g}(x) \leq_K y\}$$

is convex. A mapping $\tilde{h} : P \to \mathcal{R}^l$ is R-*polyhedral*, if the epigraph $[\tilde{h}]_R$ is a polyhedron. For example, every affine mapping is R-polyhedral. Here $x \leq_K y$ denotes that $y - x \in K$.)

Let us consider the following program pair:

$$\begin{array}{lcl} (LP) & : & \inf \tilde{f}(x) + \tilde{p}(x), \, \tilde{g}(x) \leq_K 0, \, \tilde{h}(x) \leq_R 0, \, x \in C \cap P \\ (LD) & : & \sup \inf \{(\tilde{f} + \tilde{p} + y^T \tilde{g} + z^T \tilde{h})(x) : x \in C \cap P\}, \, y \in K^*, \, z \in R^*, \end{array}$$

where K^* denotes the dual cone of K, that is $K^* := \{y : y^T x \geq 0 \ (x \in K)\}$.

The program (LP) is equivalent to the following program $(\hat{P})$:

$$(\hat{P}) : \qquad \inf \hat{f}(\hat{x}) - \hat{g}(\hat{x}), \, \hat{x} = (x, b_1, b_2, b_3, b_4).$$

Here

$$\hat{f}(\hat{x}) := \begin{cases} \tilde{f}(x), & \text{if } \hat{x} \in \hat{C}_1, \\ \infty & \text{otherwise,} \end{cases}$$

$$\hat{g}(\hat{x}) := \begin{cases} -\tilde{p}(x), & \text{if } \hat{x} \in \hat{C}_2, \\ -\infty & \text{otherwise,} \end{cases}$$

where

$$\begin{array}{lcl} \hat{C}_1 & := & \{\hat{x} : x \in C, \, \tilde{g}(x) + b_1 \leq_K 0, \, b_2 = b_4, \, b_3 \in K\}, \\ \hat{C}_2 & := & \{\hat{x} : x \in P, \, \tilde{h}(x) + b_2 \leq_R 0, \, b_1 = b_3, \, b_4 \in R\}. \end{array}$$

Note that due to our assumptions on the defining functions and mappings, $\hat{f}$ is a convex function, $-\hat{g}$ is a polyhedral convex function, finite on the convex set $\hat{C}_1$ and the polyhedron $\hat{C}_2$, respectively.

The dual of the program $(\hat{P})$ is

$$(\hat{D}): \qquad \sup \hat{g}^c(\hat{y}) - \hat{f}^c(\hat{y}),\ \hat{y} = (a, y_1, y_2, y_3, y_4).$$

It can be easily seen, that

$$\hat{g}^c(\hat{y}) = \begin{cases} \inf\{a^T x + \tilde{p}(x) + y_2^T b_2 : x \in P,\ \tilde{h}(x) + b_2 \leq_R 0\}, \\ \qquad \text{if } y_1 = -y_3,\ y_4 \in R^*, \\ -\infty \qquad \text{otherwise,} \end{cases}$$

and similarly

$$\hat{f}^c(\hat{y}) = \begin{cases} \sup\{a^T x - \tilde{f}(x) + y_1^T b_1 : x \in C,\ \tilde{g}(x) + b_1 \leq_K 0\}, \\ \qquad \text{if } y_2 = -y_4,\ y_3 \in -K^*, \\ \infty \qquad \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} &\hat{g}^c(\hat{y}) - \hat{f}^c(\hat{y}) = \\ &= \begin{cases} \inf\{a^T x + \tilde{p}(x) + y_2^T b_2 : x \in P,\ \tilde{h}(x) + b_2 \leq_R 0\} + \\ \qquad + \inf\{-a^T x + \tilde{f}(x) + y_3^T b_1 : x \in C,\ \tilde{g}(x) + b_1 \leq_K 0\}, \\ \qquad \text{if } -y_3 = y_1 \in K^*,\ -y_2 = y_4 \in R^*, \\ -\infty \qquad \text{otherwise} \end{cases} \\ &= \begin{cases} \inf\{a^T x + \tilde{p}(x) + y_4^T \tilde{h}(x) : x \in P\} + \\ \qquad + \inf\{-a^T x + \tilde{f}(x) + y_1^T \tilde{g}(x) : x \in C\}, \\ \qquad \text{if } -y_3 = y_1 \in K^*,\ -y_2 = y_4 \in R^*, \\ -\infty \qquad \text{otherwise.} \end{cases} \end{aligned}$$

We can see that the program (LD) is a relaxation of the program $(\hat{D})$: if the vector $\hat{y}$ is a feasible solution of the program $(\hat{D})$ then $y := y_1$, $z := y_4$ is a feasible solution of the program (LD), for which between the corresponding values the inequality

$$\hat{g}^c(\hat{y}) - \hat{f}^c(\hat{y}) \leq \inf\{(\tilde{f} + \tilde{p} + y^T \tilde{g} + z^T \tilde{h})(x) : x \in C \cap P\}$$

holds.

From these considerations immediately follows

LEMMA 4.2. *For the optimal values of the programs* (LP), (LD), $(\hat{P})$, *and* $(\hat{D})$ *defined above, the following statements hold:*

a) $v_{\hat{P}} = v_{LP} \geq v_{LD} \geq v_{\hat{D}}$ *(weak duality),*

b) if $v_{\hat{P}} = v_{\hat{D}}$, *then* $v_{LP} = v_{LD}$,

c) if $v_{\hat{P}} = v_{\hat{D}}$ *and the optimal value of the program* $(\hat{D})$ *is attained, then the optimal value of program* (LD) *is attained as well.* □

Now, we can state our strong duality result. The program (LP) is said to satisfy the *weak Slater condition* if there exists a point $x_0 \in \mathcal{R}^n$ such that

$$x_0 \in P \cap \operatorname{ri} C,\ \tilde{g}(x_0) <_K 0,\ \tilde{h}(x_0) \leq_R 0.$$

Then x_0 is called a *weak Slater point.* (Here $x <_K y$ denotes that $y - x \in \operatorname{ri} K$.)

THEOREM 4.1. *Let us suppose that the program* (LP) *satisfies the weak Slater condition. Then the optimal values of programs* (LP) *and* (LD) *are equal. Furthermore, the dual optimal value* v_{LD} *is attained if it is finite.*

Proof. It is proved in [1] (see Theorem 2.3) that

$$\operatorname{ri}\{(x, b) : x \in C,\ \tilde{g}(x) + b \leq_K 0\} = \{(x, b) : x \in \operatorname{ri} C,\ \tilde{g}(x) + b <_K 0\}.$$

Consequently,

$$\operatorname{ri} \hat{C}_1 = \{\hat{x} : x \in \operatorname{ri} C,\ \tilde{g}(x) + b_1 <_K 0,\ b_2 = b_4,\ b_3 \in \operatorname{ri} K\},$$

and we can see that

$$\hat{x}_0 := (x_0, -\tilde{g}(x_0)/2, -\tilde{h}(x_0), -\tilde{g}(x_0)/2, -\tilde{h}(x_0)) \in (\operatorname{ri} \hat{C}_1) \cap \hat{C}_2$$

for any weak Slater point x_0 of the program (LP). Hence, $\hat{x}_0$ is a strict feasible solution of program $(\hat{P})$, and we can apply Lemma 4.1 to the programs $(\hat{P})$ and $(\hat{D})$. We obtain that $v_{\hat{P}} = v_{\hat{D}}$, and that the optimal value of the program $(\hat{D})$ is attained if it is finite. The statement now follows from Lemma 4.2. □

Finally, we remark that an analogue of Corollary 4.1 in [2], for programs (LP) and (LD), can be derived as a consequence of Theorem 4.1: the existence of a weak Slater point and a primal optimal solution implies the existence of a saddle point of the Lagrangian function. (The *Lagrangian function* $L : (C \cap P) \times K^* \times R^* \to \mathcal{R}$ is defined as

$$L(x, y, z) := \tilde{f}(x) + \tilde{p}(x) + y^T \tilde{g}(x) + z^T \tilde{h}(x).$$

A point $(\overline{x}, \overline{y}, \overline{z}) \in (C \cap P) \times K^* \times R^*$ is called a *saddle point* of the Lagrangian function L if

$$L(\overline{x}, y, z) \leq L(\overline{x}, \overline{y}, \overline{z}) \leq L(x, \overline{y}, \overline{z}),$$

for every $x \in C \cap P$, $y \in K^*$, $z \in R^*$.) The proof is an adaptation of the proof of Corollary 4.1 in [2], and is left to the reader.

Conclusion. In this paper, three applications of closedness and duality theorems from previous works of the authors, are presented. First, a separation result was proved, a characterization of the strong separability of a polyhedron and a linear image of a convex set. Secondly, stability conditions were rewritten in the form of closedness conditions. Finally, we studied generalized Lagrange programs, where in addition to the classical cone-convex constraints we supposed the presence of cone-polyhedral constraints also.

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