TWO-STAGE ROBUST UNIT COMMITMENT PROBLEM

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ABSTRACT. As an energy market transforms from a regulated market to a deregulated one, the demands for a power plant are highly uncertain. In this paper, we study a two-stage robust optimization formulation and provide a tractable solution approach for the problem. The computational experiments show the effectiveness of our approach.

KEYWORDS. Unit commitment, mixed integer programming, separation, robust optimization

1. INTRODUCTION

For a thermal plant to generate power for its customers, there are two phases: 1) Unit commitment, i.e., deciding which generator should be on-line to generate power. 2) Economic dispatch, i.e., deciding the output for each on-line generator. The objective is to minimize total cost (or equivalent to maximize total profit) while satisfying customer demands and generator physical constraints. The physical constraints can be summarized as follows (e.g., see [16]).

- Once a thermal unit generator is started up (or shut down), it will stay on (or off) for a minimum amount of time, referred as the minimum-up (or -down) time, before it can be shut down (or started up) again.
- Once a thermal unit is on, the output for the unit should be between a certain range. For instance, it should be between the minimum and the maximum output levels.
- There are start-up and shut-down costs for unit operations. These two costs mainly include labor and maintenance costs.
- The heat rate profile is a non-decreasing convex function. Accordingly, the fuel cost can be written as a non-decreasing quadratic function of the generation level. For instance, let the fuel cost $c(x) = c_0 x^2 + c_1 x + c_2$, where x represents the generation level. Parameters c_0, c_1, and c_2 are non-negative.

Currently, energy markets in US are transforming from regulated markets to deregulated ones. The demands for utility companies are highly uncertain (e.g., see [10]). Under a deregulated market, there are day-ahead and real-time markets. On the day-ahead market, a utility company decides the commitment for each power generation unit for the next day based on the forecasted demand. On the real-time market, the utility company adjusts the output level for each unit, at each time unit (e.g., each operating hour), so as to satisfy the actual demand. If it is necessary, the utility company will sell or buy power through Independent System Operator (ISO) energy markets.

Due to the demand uncertainty, significant research progress has been made on stochastic programming approaches to solve the problem with the objective of minimizing total expected cost. Readers are referred to [15], [6], [7], [8], [12], [14], among others, for detailed two-stage and multi-stage stochastic programming formulations and decomposition methods to solve the problem.

In this research, we study the two-stage robust version of the problem to address the case that the distribution of the demand for the next day is unknown. The main results of robust optimization techniques are described in [2] and [3]. These techniques have been applied to solve different problems such as inventory theory [4, 5], network design [1], and lot-sizing [1]. However, to the best of our knowledge, there is no previous research on robust optimization for unit commitment problems. In this paper, we study the robust unit commitment problem in which the demand at

each time unit is uncertain and within an uncertainty set. The objective is to minimize the total power generation cost under the worst case scenario.

The remaining part of this paper is organized as follows. Section 2 describes the mathematical formulation of the robust unit-commitment problem. Section 3 studies the solution approach to solve the problem. Section 4 provides discussions and extensions on the problem. Finally, Section 5 reports the computational results and Section 6 concludes our study.

2. Mathematical formulation

Nominal model. Let us start with the nominal model of the unit commitment problem, where the demand of each time period is known. In our study, we assume that there are N generators. The planning horizon is T time units and the demand at each time unit t is d_t. In the case that the electricity output does not match the demand, the utility company can buy (or sell) the shortage (or excess) through ISO. The corresponding unit buying and selling prices at time t are τ_t and γ_t respectively.

For each generator i, we assume the minimum-up time is G_i and the minimum-down time is H_i. We also let μ_i and ξ_i be the start-up and shut-down costs for generator i. Once a generator (e.g., generator i) is on, the output of the electricity is within the interval $[\ell_i, u_i]$, $i = 1, \ldots, N$. For the fuel cost, in this section, we start with using one linear piece to approximate the quadratic cost function. We approximate the cost function $c_{it}(x) = a_{it} + \beta_{it}(x - \ell_i)$ of generator i at time t. Without loss of generality, for each generator i, we can assume

$$(1) \qquad\qquad\qquad \gamma_t < \beta_{it} < \tau_t, \ t = 1, ..., T.$$

The nominal unit commitment problem is to determine the on/off times and electricity output of each generator with the objective of minimizing the total power generation cost. We present the nominal model after introducing the following decision variables.

- y_{it}: Binary decision variable to indicate if generator i is on at time t. $y_{it} = 1$ if yes and 0 o.w.
- o_{it}: Binary decision variable to indicate if generator i is started up at time t. $o_{it} = 1$ if yes and 0 o.w.
- v_{it}: Binary decision variable to indicate if generator i is shut down at time t. $v_{it} = 1$ if yes and 0 o.w.
- x_{it}: The amount of electricity generated by generator i at time t.
- z_t: The amount of electricity to sell at time t to ISO.
- w_t: The amount of electricity to buy at time t from ISO.

For the nominal model, demands d_t, $t = 1, ..., T$, are given. The corresponding formulation is

$$\min \quad \sum_{t=1}^{T}\sum_{i=1}^{N}(\mu_i o_{it} + \xi_i v_{it} + \alpha_{it}y_{it} + \beta_{it}x_{it}) + \sum_{t=1}^{T}(\tau_t w_t - \gamma_t z_t)$$

$$(2) \qquad \text{s.t.} \quad \ell_i y_{it} \leq x_{it} \leq u_i y_{it}, \ i = 1, ..., N, t = 1, ..., T$$

$$(3) \qquad\qquad \sum_{i=1}^{N} x_{it} + w_t - z_t = d_t, \ t = 1, ..., T$$

$$(4) \qquad\qquad -y_{i(t-1)} + y_{it} - y_{ik} \leq 0, \ i = 1, \ldots, N, t = 1, \ldots, T, \ \text{and} \ 1 \leq k - (t-1) \leq G_i$$

$$(5) \quad \text{(UC)} \quad y_{i(t-1)} - y_{it} + y_{ik} \leq 1, \ i = 1, \ldots, N, t = 1, \ldots, T, \ \text{and} \ 1 \leq k - (t-1) \leq H_i$$

$$(6) \qquad\qquad -y_{i(t-1)} + y_{it} - o_{it} \leq 0, \ i = 1, \ldots, N, t = 1, \ldots, T$$

$$(7) \qquad\qquad y_{i(t-1)} - y_{it} - v_{it} \leq 0, \ i = 1, \ldots, N, t = 1, \ldots, T$$

$$x_{it}, z_t, w_t \geq 0, y_{it}, o_{it}, v_{it} \in \{0, 1\}, i = 1, ..., N, t = 1, ..., T,$$

where $\alpha_{it} = a_{it} - \beta_{it}\ell_i$. In the above formulation, the objective is to minimize the total cost that includes start-up cost, shut-down cost, fuel cost and the cost for the trading between the utility company and ISO energy markets. Constraints (2) represent the electricity output range of generator i if it is on at time t. Constraints (3) describe the power flow balance at each time unit. Constraints (4) are minimum-up time constraints that describe the minimum-up time required for generator i if it is started up. Accordingly, constraints (5) are minimum-down time constraints that describe the minimum-down time required for generator i if it is shut down. Constraints (6) and (7) indicate the start-up and shut-down operations for each generator i.

Two-stage robust model. When we decide the unit commitment on the day-ahead market and allow demand uncertainty on the real-time market, we need to decide the setup of the generators (y, o, v) on the first stage and (x, z, w) on the second stage after realizing the demand d. Then, in the first stage, the objective becomes minimizing the total cost with the consideration of the worst case scenario due to demand uncertainty. In the robust model, we assume that the demand realization is in a *non-empty* given uncertainty set $\mathcal{D}$. The robust optimization formulation of the first stage can be described as follows.

$$
\min_{y,o,v} \quad \sum_{t=1}^{T}\sum_{i=1}^{N}(\alpha_{it}y_{it} + \mu_i o_{it} + \xi_i v_{it}) + \max_{d\in\mathcal{D}}\min_{(x,z,w)\in\mathcal{M}}\sum_{t=1}^{T}\left(\sum_{i=1}^{N}\beta_{it}x_{it} + \tau_t w_t - \gamma_t z_t\right)
$$

(RUC) s.t. (4), (5), (6), (7),

$$
y_{it}, o_{it}, v_{it} \in \{0,1\}, i = 1, ..., N, \ t = 1, \ldots, T,
$$

where

$$
\mathcal{M} = \{(x, z, w): \quad \ell_i y_{it} \le x_{it} \le u_i y_{it}, \ i = 1, ..., N, \ t = 1, ..., T
$$
$$
\sum_{i=1}^{N} x_{it} + w_t - z_t = d_t, \ t = 1, ..., T
$$
$$
x_{it}, z_t, w_t \ge 0, i = 1, ..., N, \ t = 1, \ldots, T\}.
$$

In our robust model, we consider the uncertain demand for each operating hour t is among a range between a lower bound $\underline{d}_t$ and an upper bound $\bar{d}_t$. And there is a demand budget restriction that $\sum_{t=1}^{T} \pi_t d_t \le \pi_0$. A special case for this is that the total demand is less than certain value during the planning horizon. For instance, $\sum_{t=1}^{T} d_t \le \pi_0$. Thus, the corresponding polyhedron uncertainty set is

(8)
$$
\mathcal{D} = \{d : \sum_{t=1}^{T} \pi_t d_t \le \pi_0, \underline{d}_t \le d_t \le \bar{d}_t, \ t = 1, \ldots, T\}.
$$

For further probability proposition of this uncertainty set, readers are referred to [1].

3. Solution method

In this section, we describe the solution method to solve RUC. First, we analyze the optimal value function $f_t(d_t)$ for the subproblem at the second stage once the first stage decision variable (y, o, v) is fixed. Under this case, the problem RUC is decomposed into T subproblems. Corresponding to each time period t, we have

$$f_t(d_t) \quad = \quad \min \sum_{i=1}^N \beta_{it} x_{it} + \tau_t w_t - \gamma_t z_t$$

(9) $$\text{s.t.} \qquad \ell_i y_{it} \le x_{it} \le u_i y_{it}, \qquad i = 1, ..., N,$$

(10) $$\sum_{i=1}^N x_{it} + w_t - z_t = d_t,$$

$$x_{it}, z_t, w_t \ge 0, \qquad i = 1, \dots, N.$$

In the optimal solution for the subproblem corresponding to time period t, we have $z_t^* = (\sum_{i=1}^N \ell_i y_{it} - d_t)^+$, $w_t^* = (d_t - \sum_{i=1}^N u_i y_{it})^+$, where $x^+ = \max\{x, 0\}$.

Proposition 1. *The value function $f_t(d_t)$ is piecewise linear and convex.*

Proof: Without loss of generality, assume

(11) $$\beta_{1t} \le \beta_{2t} \le \ldots \le \beta_{Nt}$$

From constraints (9) and (10), based on the cost relationship shown in (1), we can first observe that

- If $d_t \le \sum_{i=1}^N \ell_i y_{it}$, then in the optimal solution, each generator generates at its lower bound and sells the over generated power in the amount of $\sum_{i=1}^N \ell_i y_{it} - d_t$ to ISO. Accordingly, the total cost

(12) $$f_t(d_t) = \phi_{t0}(d_t) = \sum_{i=1}^N \beta_{it} \ell_i y_{it} - \gamma_t \Big(\sum_{i=1}^N \ell_i y_{it} - d_t \Big) = \varphi_{t0}(y) + \gamma_t d_t,$$

  where $\varphi_{t0}(y) = \sum_{i=1}^N \beta_{it} \ell_i y_{it} - \gamma_t \sum_{i=1}^N \ell_i y_{it}$.

- If $d_t \ge \sum_{i=1}^N u_i y_{it}$, then in the optimal solution, each generator generates at its upper bound and purchases the shortage part in the amount of $d_t - \sum_{i=1}^N u_i y_{it}$ from ISO. Accordingly, the total cost

(13) $$f_t(d_t) = \phi_{t(N+1)}(d_t) = \sum_{i=1}^N \beta_{it} u_i y_{it} + \tau_t \Big(d_t - \sum_{i=1}^N u_i y_{it} \Big) = \varphi_{t(N+1)}(y) + \tau_t d_t,$$

  where $\varphi_{t(N+1)}(y) = \sum_{i=1}^N \beta_{it} u_i y_{it} - \tau_t \sum_{i=1}^N u_i y_{it}$.

- For the general case, assuming $\big(\sum_{i=\theta_t}^N \ell_i y_{it} + \sum_{i=1}^{\theta_t-1} u_i y_{it} \big) \le d_t \le \big(\sum_{i=\theta_t+1}^N \ell_i y_{it} + \sum_{i=1}^{\theta_t} u_i y_{it} \big)$, $\theta_t = 1, \dots, N$, the total cost

(14)
$$f_t(d_t) = \phi_{t\theta_t}(d_t) = \sum_{i=\theta_t+1}^N \beta_{it} \ell_i y_{it} + \sum_{i=1}^{\theta_t-1} \beta_{it} u_i y_{it} + \beta_{\theta_t t} \Big(d_t - \sum_{i=\theta_t+1}^N \ell_i y_{it} - \sum_{i=1}^{\theta_t-1} u_i y_{it} \Big) = \varphi_{t\theta_t}(y) + \beta_{\theta_t t} d_t,$$

  where $\varphi_{t\theta_t}(y) = \sum_{i=\theta_t+1}^N \beta_{it} \ell_i y_{it} + \sum_{i=1}^{\theta_t-1} \beta_{it} u_i y_{it} - \beta_{\theta_t t} \sum_{i=\theta_t+1}^N \ell_i y_{it} - \beta_{\theta_t t} \sum_{i=1}^{\theta_t-1} u_i y_{it}$.

From (12), (13) and (14), we can observe that $\phi_{t\theta_t}(d_t), \theta_t = 0, \ldots, N+1$, is a linear function of d_t. Based on (1) and (11), we also have that

$$\gamma_t < \beta_{1t} \le \beta_{2t} \le \ldots \le \beta_{Nt} < \tau_t.$$

Together with $\phi_{t\theta_t}(\sum_{i=\theta_t+1}^{N} \ell_i y_{it} + \sum_{i=1}^{\theta_t} u_i y_{it}) = \phi_{t(\theta_t+1)}(\sum_{i=\theta_t+1}^{N} \ell_i y_{it} + \sum_{i=1}^{\theta_t} u_i y_{it}) = \sum_{i=\theta_t+1}^{N} \beta_{it}\ell_i y_{it} + \sum_{i=1}^{\theta_t} \beta_{it}u_i y_{it}$, we have that the value function $f_t(d_t)$ is piecewise linear and convex. Therefore, the conclusion holds. $\qquad \square$

Since the value function $f_t(d_t)$ is piecewise linear and convex, the following corollary holds.

Corollary 1. *The value function* $f_t(d_t) = \max\limits_{\theta_t=0,\ldots,N+1} \phi_{t\theta_t}(d_t).$

Based on the conclusion obtained in Proposition 1 and Corollary 1, RUC can be reformulated as follows:

$$
\begin{aligned}
\min_{y,o,v} \quad & \sum_{t=1}^{T}\sum_{i=1}^{N}(\alpha_{it}y_{it} + \mu_i o_{it} + \xi_i v_{it}) + \max_{d\in\mathcal{D}}\sum_{t=1}^{T} f_t(d_t) \\
= & \sum_{t=1}^{T}\sum_{i=1}^{N}(\alpha_{it}y_{it} + \mu_i o_{it} + \xi_i v_{it}) + \max_{d\in\mathcal{D}}\sum_{t=1}^{T}(\max_{\theta_t=0,\ldots,N+1} \phi_{t\theta_t}(d_t)) \\
\text{s.t.} \quad & \text{Constraints (4), (5), (6), (7),} \\
& y \in \mathbb{B}^{N\times T}, o \in \mathbb{B}^{N\times T}, v \in \mathbb{B}^{N\times T}.
\end{aligned}
$$

Then we can introduce a new continuous decision variable ω for the second stage as follows:

$$
\begin{aligned}
\min_{y,o,v} \quad & \sum_{t=1}^{T}\sum_{i=1}^{N}(\alpha_{it}y_{it} + \mu_i o_{it} + \xi_i v_{it}) + \omega \\
\text{s.t.} \quad & w \ge \sum_{t=1}^{T}\max_{\theta_t=0,\ldots,N+1} \phi_{t\theta_t}(d_t) \text{ for all } d \in \mathcal{D}, \\
& \text{Constraints (4), (5), (6), (7),} \\
& y \in \mathbb{B}^{N\times T}, o \in \mathbb{B}^{N\times T}, v \in \mathbb{B}^{N\times T}, \omega \in \mathbb{R}.
\end{aligned}
$$

In the optimal solution, for a given (y, o, v), since $\phi_{t\theta_t}(d_t)$ and $\phi_{t'\theta_{t'}}(d_{t'})$ are mutually independent, we have

$$(15) \qquad \omega = \max_{\theta_t, 1\le t\le T} \left\{ \max \sum_{t=1}^{T} \phi_{t\theta_t}(d_t) : d \in \mathcal{D} \right\}.$$

For a given θ_t, $1 \le t \le T$, and the corresponding uncertainty set $\mathcal{D}$ as defined in (8), the dual of (15) can be described as follows

$$
\begin{aligned}
\min \quad & \pi_0 \zeta + \sum_{t=1}^{T}(\bar{d}_t\eta_t - \underline{d}_t\rho_t) + \sum_{t=1}^{T}\varphi_{t\theta_t}(y) \\
\text{(Dual)} \quad \text{s.t.} \quad & \pi_t\zeta + \eta_t - \rho_t \ge \psi_{t\theta_t}, 1 \le t \le T, \\
& \zeta, \eta, \rho \ge 0,
\end{aligned}
$$

where $\psi_{t\theta_t}$ is the coefficient of d_t in function $\phi_{t\theta_t}(d_t)$. That is, $\psi_{t\theta_t} = \gamma_t$ if $\theta_t = 0$, $\psi_{t\theta_t} = \tau_t$ if $\theta_t = N+1$, and $\psi_{t\theta_t} = \beta_{\theta_t t}$ if $1 \le \theta_t \le N$. We also notice that $\varphi_{t\theta_t}(y)$ is a constant number here for the dual problem once t, θ_t and y are given.

Therefore, if the vector $\theta = (\theta_1, \ldots, \theta_T)$ is given, we can add the dual constraints to the first stage master problem. Note here the dual constraints are linear. In this case, there are exponential number of combinations of θ_t, $1 \le t \le T$, which corresponds to exponential number of dual constraints. In our approach, we add the constraints gradually by running a separation algorithm. Next, we will describe a separation algorithm to discover the θ value given a solution of the master problem. It is hard to determine the θ value directly, we will use the fact that the value function is piecewise linear.

Separation. The separation problem of (15) can be stated as:

Given a solution (y, o, v, w), does there exist $(\theta_1, ..., \theta_T)$ and $d \in \mathcal{D}$, such that $w < \sum_{t=1}^{T} \phi_{t\theta_t}(d_t)$?

According to Proposition 1, the value function $f_t(d_t)$ is piecewise linear and convex. Therefore, both $f_t(d_t)$ and d_t can be represented as a linear combination of two consecutive breakpoints. For instance, if θ_t is given, then $d_t = \lambda(\sum_{i=\theta_t}^{N} \ell_i y_{it} + \sum_{i=1}^{\theta_t-1} u_i y_{it}) + (1-\lambda)(\sum_{i=\theta_t+1}^{N} \ell_i y_{it} + \sum_{i=1}^{\theta_t} u_i y_{it})$ and $f_t(d_t) = \lambda f_t(\sum_{i=\theta_t}^{N} \ell_i y_{it} + \sum_{i=1}^{\theta_t-1} u_i y_{it}) + (1-\lambda) f_t(\sum_{i=\theta_t+1}^{N} \ell_i y_{it} + \sum_{i=1}^{\theta_t} u_i y_{it})$ for some $\lambda : 0 \le \lambda \le 1$. Based on this, we can formulate the separation problem as a mixed integer program.

Let binary decision variable $\tau_{ti} = 1$ if $\theta_t = i$, and 0, otherwise. Let λ_{ti} be the proportion of the function with respect to function value at $\phi_{t\theta_t}(\sum_{j=\theta_t}^{N} \ell_j y_{jt} + \sum_{j=1}^{\theta_t-1} u_j y_{jt})$.

Then the separation problem (SP) can be formulated as

$$
z_{sp} = \max \quad \sum_{t=1}^{T} \left(\sum_{i=1}^{N+1} \lambda_{ti} \left(\sum_{j=i}^{N} \beta_{jt}\ell_j y_{jt} + \sum_{j=1}^{i-1} \beta_{jt} u_j y_{jt} \right) \right.
$$

$$
+ \lambda_{t0} \left(\sum_{j=1}^{N} \beta_{jt}\ell_j y_{jt} - \gamma_t \left[\sum_{j=1}^{N} \ell_j y_{jt} - \underline{d}_t \right]^+ \right)
$$

$$
\text{(SP)} \qquad \left. + \lambda_{t(N+2)} \left(\sum_{j=1}^{N} \beta_{jt} u_j y_{jt} + \tau_t \left[\overline{d}_t - \sum_{j=1}^{N} u_j y_{jt} \right]^+ \right) \right)
$$

$$
\text{(16)} \qquad \text{s.t.} \quad d_t = \sum_{i=1}^{N+1} \lambda_{ti} \left(\sum_{j=i}^{N} \ell_j y_{jt} + \sum_{j=1}^{i-1} u_j y_{jt} \right)
$$

$$
+ \lambda_{t0} \left(\sum_{j=1}^{N} \ell_j y_{jt} - \left[\sum_{j=1}^{N} \ell_j y_{jt} - \underline{d}_t \right]^+ \right)
$$

$$
+ \lambda_{t(N+2)} \left(\sum_{j=1}^{N} u_j y_{jt} + \left[\overline{d}_t - \sum_{j=1}^{N} u_j y_{jt} \right]^+ \right), \forall 1 \le t \le T,
$$

(17) $$\lambda_{ti} \leq \tau_{t(i-1)} + \tau_{ti}, \quad \forall 1 \leq t \leq T, 1 \leq i \leq N+1,$$

(18) $$\lambda_{t0} \leq \tau_{t0},$$

(19) $$\lambda_{t(N+2)} \leq \tau_{t(N+1)},$$

(20) $$\sum_{i=0}^{N+1} \tau_{ti} = 1, \quad \forall 1 \leq t \leq T,$$

(21) $$\sum_{i=0}^{N+2} \lambda_{ti} = 1, \quad \forall 1 \leq t \leq T,$$

(22) $$d \in \mathcal{D},$$

$$\lambda_{ti} \geq 0, \tau_{ti} \in \{0,1\}.$$

In the above formulation, the objective function indicates that $f_t(d_t)$ can be written as a linear combination of the values at breakpoints. Constraints (16) represent that the demand can be represented as a linear combination of the demand values at breakpoints. Constraints (17) indicate that the value corresponding to i will not be used (i.e., $\lambda_{ti} = 0$) if neither $\theta_t = i - 1$ nor $\theta_t = i$. Constraints (18) and (19) represent the relationship corresponding to the two extreme points. Constraints (20) represent that the objective value can only be located in one linear piece segment. Finally, constraints (21) and (22) represent the linear combination and feasible region for the demand respectively. For any solution (λ, τ) of SP, we can obtain $\theta_t = \{i : \tau_{ti} = 1\}, 1 \leq t \leq T$.

Based on the above analysis, we can summarize the detailed steps of our algorithm in the remaining part of this section. This algorithm is slightly different from Benders' decomposition as shown in [11]. The overall problem to be solved can be described as follows:

$$\min \quad \sum_{t=1}^{T}\sum_{i=1}^{N}(\alpha_{it}y_{it} + \mu_i o_{it} + \xi_i v_{it}) + \omega$$

(RUC') s.t. $$\omega \geq \pi_0 \zeta^\theta + \sum_{t=1}^{T}(\overline{d}_t \eta_t^\theta - \underline{d}_t \rho_t^\theta) + \sum_{t=1}^{T} \varphi_{t\theta_t}(y), \ \forall \theta \in \Theta,$$

$$\pi_t \zeta^\theta + \eta_t^\theta - \rho_t^\theta \geq \psi_{t\theta_t}, \ \forall \theta \in \Theta, 1 \leq t \leq T,$$

Constraints (4), (5), (6), (7),

$$y \in \mathbb{B}^{N \times T}, o \in \mathbb{B}^{N \times T}, v \in \mathbb{B}^{N \times T}, \omega \in \mathbb{R}, \zeta \in \mathbb{R}_+^{|\Theta|}, \eta \in \mathbb{R}_+^{|\Theta| \times T}, \rho \in \mathbb{R}_+^{|\Theta| \times T},$$

where $\Theta = \{(\theta_1, ...\theta_T) : \theta_t \in \{0, 1, ..., N+1\}, t = 1, ..., T\}$ is the set of θ.

The detailed steps of our algorithm are shown as follows.

Initialization: Find set $\Theta^1 \subseteq \Theta$ (Θ^1 may be an empty set).

Let $X_R^1 = \{\omega \in \mathbb{R}, y, o, v \in \mathbb{B}^{N \times T} : \omega \geq \pi_0 \zeta^\theta + \sum_{t=1}^{T}(\overline{d}_t \eta_t^\theta - \underline{d}_t \rho_t^\theta) + \sum_{t=1}^{T} \varphi_{t\theta_t}(y), \pi_t \zeta^\theta + \eta_t^\theta - \rho_t^\theta \geq \psi_{t\theta_t}, \ \forall \theta \in \Theta^1, 1 \leq t \leq T$, constraints (4), (5), (6), (7)$\}$. Set $r = 1$.

Iteration r:

Step 1: Solve the relaxation of RUC':

$$(M^r) \quad z^r = \min\left\{\sum_{t=1}^{T}\sum_{i=1}^{N}(\alpha_{it}y_{it} + \mu_i o_{it} + \xi_i v_{it}) + \omega : (y, o, v, \omega) \in X_R^r\right\}.$$

a. If the master problem M^r is infeasible, stop. The original RUC is infeasible. Note here in our problem setting, M^r is always feasible since RUC always has a feasible solution.

b. If M^r is unbounded, find a feasible solution pair $(\omega^r, y^r, o^r, v^r)$ with $\omega^r < \Delta$ for some small value Δ. In our problem, the total cost is always bounded by 0.

c. Otherwise, record the optimal solution $(\omega^r, y^r, o^r, v^r)$.

Step 2: Solve SP to obtain the optimal θ_t, denoted as θ_t^r, for $1 \leq t \leq T$. According to our problem setting, we know that θ_t^r exists and the problem is bounded. That is, z_{sp}^r is finite.

a. Based on the optimal value τ_{ti} obtained from SP, let $\theta_t^r = i$ if $\tau_{ti} = 1$.

b. *Optimality test.* If $z_{sp}^r \leq \omega^r$, stop. (y^r, o^r, v^r) is an optimal solution of RUC.

c. *Violation.* If $z_{sp}^r > \omega^r$, then the constraint $\omega \geq \pi_0 \zeta^r + \sum_{t=1}^{T}(\bar{d}_t \eta_t^r - \underline{d}_t \rho_t^r) + \sum_{t=1}^{T} \varphi_{t\theta_t^r}(y)$ is violated. Update $\Theta^{r+1} = \Theta^r \cup \{\theta^r\}$ and

(23)
$$X_R^{r+1} = X_R^r \cap \left\{ (y, o, v, \omega) : \omega \geq \pi_0 \zeta^r + \sum_{t=1}^{T}(\bar{d}_t \eta_t^r - \underline{d}_t \rho_t^r) \right.$$
$$\left. + \sum_{t=1}^{T} \varphi_{t\theta_t^r}(y), \quad \pi_t \zeta^r + \eta_t^r - \rho_t^r \geq \psi_{t\theta_t^r} \quad 1 \leq t \leq T \right\}.$$

d. Update $r = r + 1$.

4. DISCUSSIONS AND EXTENSIONS

Turn on/off inequalities. The turn on/off inequalities described in [9] and [13] can be utilized to strengthen the formulation. The turn on inequalities can be described as follows:

$$\sum_{j=\max\{1, t-G_i+1\}}^{t} o_{ij} \leq y_{it}, \quad \forall 1 \leq i \leq N, 1 \leq t \leq T.$$

This inequality lies in the fact that if generator i is off in time period t, it could not have be started up during the last G_i period (including time period t) because of the minimum-up constraints. If generator i is on in time period t, it could have been started up at most once during the last G_i time periods (including time period t), because generator i cannot be started up and shut down, respectively, within G_i time periods.

Similarly, the turn off inequalities can be described as follows:

$$\sum_{j=\max\{1, t-H_i+1\}}^{t} v_{ij} \leq 1 - y_{it}, \quad \forall 1 \leq i \leq N, 1 \leq t \leq T.$$

Multiple linear piece approximation. We can also extend our study to investigate using a multiple piece piecewise linear function to approximate the quadratic non-decreasing fuel cost function. Corresponding to each pair (i, t), assume there are L linear pieces. The slopes for each piece are $\beta_{it} = \beta_{it}^1 \leq \beta_{it}^2 \leq \ldots \leq \beta_{it}^L$ and the breakpoints are ℓ_i^0(i.e., ℓ_i), $\ell_i^1, \ldots, \ell_i^L$(i.e., u_i). Then the value function corresponding to the jth piece is

$$c(x_{it}) = a_{it}^{j-1} + \beta_{it}^j(x_{it} - \ell_i^{j-1}),$$

where $a_{it}^{j-1} = a_{it}^{j-2} + \beta_{it}^{j-1}(\ell_i^{j-1} - \ell_i^{j-2})$ with $a_{it}^0 = a_{it}$.

Then, we can observe that once the first stage decision (y, o, v) is given, for a given d_t, each online generator i will produce $\ell_i^{r(i)}$ where $r(i)$ is an integer and $0 \leq r(i) \leq L$, except one generator (e.g., generator k) generates the amount between $\ell_k^{r(k)}$ and $\ell_k^{r(k)+1}$, $0 \leq r(k) \leq L - 1$. We also notice that $\beta_{it}^{r(i)} \leq \beta_{kt}^{r(k)+1}$ if $r(i) > 0$ and $\beta_{it}^{r(i)+1} \geq \beta_{kt}^{r(k)+1}$ if $r(i) + 1 \leq L$ for each $i \neq k$ and $y_{it} = 1$. Thus, $f_t(d_t)$ is still a piecewise linear convex function. For instance, if $\sum_{i=1}^{N} \ell_i^{r(i)} y_{it} \leq d_t \leq \sum_{i=1}^{k-1} \ell_i^{r(i)} y_{it} + \ell_k^{r(k)+1} y_{kt} + \sum_{i=k+1}^{N} \ell_i^{r(i)} y_{it}$, it can be described as follows:

$$(24) \qquad f_t(d_t) = \sum_{i=1}^{N} (a_{it}^{r(i)} - \alpha_{it}) y_{it} + \beta_{kt}^{r(k)+1} (d_t - \sum_{i=1}^{N} \ell_i^{r(i)}).$$

The number of linear pieces can reach $NL + 2$. Therefore, we can still apply our algorithm to obtain the optimal solution for the corresponding robust unit commitment problem.

5. Computational experiment

In this section, we present the numerical experiments of the proposed algorithm. In the experiments, we assume that there are $N = 30$ generators and $T = 24$ time periods. The upper and lower bounds $\underline{d}$ and $\bar{d}$ of demand in each time period is randomly generated in the following way. First, $\hat{d}$ and $\tilde{d}$ are randomly generated in the interval $[0, 40]$ and $[0, 20]$, respectively. Then $\underline{d} = (\hat{d} - \tilde{d})^+$ and $\bar{d} = \hat{d} + \tilde{d}$. The budget restriction of the uncertainty set $\mathcal{D}$ can be described as $\sum_{t=1}^{T} (d_t - \hat{d}_t)/\tilde{d}_t \leq \pi_0$. We can control the conservatism of the robust optimization approach by controlling π_0. Note π_0 is between $-T$ and T. When $\pi_0 = -T$, the only possible scenario is that all demands are at the lower bounds. When $\pi_0 = T$, the demand of each time period can take any value within the interval of the lower and upper bounds. Costs are randomly generated in the following way. The turn on/off costs μ_i and ξ_i of unit i are in the interval $[0, 30]$. The unit buying and selling prices γ_t and τ_t at time t are in the intervals $[0, 20]$ and $[\gamma_t, \gamma_t + 50]$, respectively. The generation cost parameters β_{it} and α_{it} are in the intervals $[\gamma_t, \tau_t]$ and $[0, 20]$, respectively. All the experiments are performed by ILOG CPLEX 10.2, at Pentium Dual Core 2.40GHZ with 2GB memory. The computational results are summarized in the following table. All the results are the average of 10 random instances. We report the optimal objective value, the number of iterations of our approach, and the average computational time.

π_0	Objective Value	Number of Iterations	CPU Time (seconds)
6	8863	22	27.2
9	9109	25	32.5
12	9337	30	43.9
15	9486	28	28.0
18	9546	29	34.5

From this experiment, we observe first that as π_0 increases, the uncertainty set becomes larger and more scenarios are taking into the consideration. The corresponding objective value increases as the problem becomes more conservative. Second, in these experiments, the size of Θ is 32^{24}. However, in the experiments, the optimal solution can be achieved within 30 iterations. This shows that the algorithm is efficient for this unit commitment problem.

6. Conclusion and future research

In this paper, we proposed a robust optimization approach to address demand uncertainty for the unit-commitment problem under a deregulated energy market. In our approach, we developed a robust integer programming formulation and the corresponding algorithm to solve the problem. Our study shows that the problem is tractable and the computational results verify the effectiveness

of our proposed approach. In the future study, we will further study the problem by incorporating ramping as well as transmission constraints into our model.

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