ON THE NONEXISTENCE OF SUM OF SQUARES CERTIFICATES FOR THE BMV CONJECTURE

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ABSTRACT. The algebraic reformulation of the BMV conjecture is equivalent to a family of dimensionfree tracial inequalities involving positive semidefinite matrices. Sufficient conditions for these to hold in the form of algebraic identities involving polynomials in noncommuting variables have been given by Markus Schweighofer and the second author. Later the existence of these certificates has been settled for all but one case, which is resolved in this short note.

1. INTRODUCTION

In an attempt to simplify the calculation of partition functions of quantum mechanical systems Bessis, Moussa and Villani [BMV75] conjectured in 1975 that for any two symmetric matrices A, B, where B is positive semidefinite, the function $t \mapsto \operatorname{tr}(e^{A-tB})$ is the Laplace transform of a positive Borel measure with real support. This would permit the calculation of explicit upper and lower bounds of energy levels in multiple particle systems. For an overview of mostly analytical approaches before 1998 we refer the reader to Moussa's survey [Mou00].

In 2004, Lieb and Seiringer [LS04] restated the conjecture in the following purely algebraic form: all the coefficients of the polynomial

$$p_m = \operatorname{tr}((A + tB)^m) \in \mathbb{R}[t]$$

are nonnegative whenever $m \in \mathbb{N}$ and A and B are positive semidefinite matrices of the same size. The coefficient of t^k in p_m is the trace of $S_{m,k}(A, B) :=$ the sum of all words of length m in A and B in which B appears exactly k times (and therefore A exactly $m - k$ times).

In 2008, Schweighofer and the second author [KS08b] established a new approach to the BMV conjecture using sums of hermitian squares of polynomials in noncommuting variables combined with Hillar's descent theorem [Hil07] and proved the conjecture for $m \leq 13$. To describe the method in detail we introduce some notation.

1.1. **Notation.** The main feature of this method is to model the matrices as *noncommuting variables*. Systematizing this approach leads us to the ring of polynomials in two noncommuting variables. Let $\langle X, Y \rangle$ be the monoid freely generated by $\{X, Y\}$, i.e., $\langle X, Y \rangle$ consists of **words** in the two noncommuting letters X, Y (including the empty word denoted by

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1). We consider the free algebra $\mathbb{R}\langle X,Y\rangle$ on $\langle X,Y\rangle$, i.e., the ring of polynomials in the noncommuting variables X,Y with coefficients from $\mathbb{R}$. The elements of $\mathbb{R}\langle X,Y\rangle$ are linear combinations of words from $\langle X,Y\rangle$ and are called **NC polynomials**. The length of the longest word in an NC polynomial $f\in\mathbb{R}\langle X,Y\rangle$ is the **degree** of f and is denoted by $\deg f$. Likewise we consider the X-degree $\deg_X f$ and the Y-degree $\deg_Y f$. An element of the form aw where $0\neq a\in\mathbb{R}$ and $w\in\langle X,Y\rangle$ is called a **monomial** and a its **coefficient**. Hence words are monomials whose coefficient is 1.

Definition 1.1. Two polynomials $f,g\in\mathbb{R}\langle X,Y\rangle$ are called **cyclically equivalent** ($f\overset{\text{cyc}}{\sim} g$) if $f-g$ is a sum of commutators in $\mathbb{R}\langle X,Y\rangle$, i.e., there are $p_i,q_i\in\mathbb{R}\langle X,Y\rangle$ with $f-g=\sum(p_iq_i-q_ip_i)=\sum[p_i,q_i]$.

This definition reflects the fact that $\operatorname{tr}(AB)=\operatorname{tr}(BA)$ for square matrices A and B of the same size. The following proposition shows that cyclic equivalence can easily be checked and will be used tacitly in the sequel.

Proposition 1.2.

(a) *For $v,w\in\langle X,Y\rangle$, we have $v\overset{\text{cyc}}{\sim} w$ if and only if there are $v_1,v_2\in\langle X,Y\rangle$ such that $v=v_1v_2$ and $w=v_2v_1$.*
(b) *Two polynomials $f=\sum_{w\in\langle X,Y\rangle}a_ww$ and $g=\sum_{w\in\langle X,Y\rangle}b_ww$ ($a_w,b_w\in\mathbb{R}$) are cyclically equivalent if and only if for each $v\in\langle X,Y\rangle$,*

$$\sum_{\substack{w\in\langle X,Y\rangle\\ w\overset{\text{cyc}}{\sim} v}}a_w=\sum_{\substack{w\in\langle X,Y\rangle\\ w\overset{\text{cyc}}{\sim} v}}b_w.$$

We equip $\mathbb{R}\langle X,Y\rangle$ with the **involution** $*$ that fixes $\mathbb{R}\cup\{X,Y\}$ pointwise and thus reverses words, e.g. $(X^2-XY^3)^*=X^2-Y^3X$. So $\mathbb{R}\langle X,Y\rangle$ is the $*$-algebra freely generated by two symmetric letters. Let $\operatorname{Sym}\mathbb{R}\langle X,Y\rangle$ denote the set of all **symmetric elements**, that is, $\operatorname{Sym}\mathbb{R}\langle X,Y\rangle=\{f\in\mathbb{R}\langle X,Y\rangle\mid f^*=f\}$. The involution $*$ extends naturally to matrices (in particular, to vectors) over $\mathbb{R}\langle X,Y\rangle$. For instance, if $V=(v_i)$ is a (column) vector of NC polynomials $v_i\in\mathbb{R}\langle X,Y\rangle$, then V^* is the row vector with components v_i^*. We shall also use V^t to denote the row vector with components v_i.

The following lemma is a special case of [KS08a, Theorem 2.1] and was the main motivation for the definition of cyclic equivalence. For a dimension dependent extension we refer the reader to [BK09].

Lemma 1.3. *Suppose $f\in\mathbb{R}\langle X,Y\rangle$ and $f^*=f$. Then $f\overset{\text{cyc}}{\sim}0$ if and only if $\operatorname{tr}(f(A,B))=0$ for all real symmetric matrices A and B of the same size.*

We now turn to notions related to positivity.

Definition 1.4. We denote by

$$\Sigma^2=\{\sum_{i=1}^m g_i^*g_i\mid m\in\mathbb{N},\ g_i\in\mathbb{R}\langle X,Y\rangle\}\subseteq\operatorname{Sym}\mathbb{R}\langle X,Y\rangle \tag{1}$$

the convex cone of all **sums of hermitian squares** and by

$$\begin{aligned} \Theta^2 &= \{f \in \mathbb{R}\langle X, Y\rangle \mid \exists g \in \Sigma^2 : f \overset{\text{cyc}}{\sim} g\} \\ &= \Sigma^2 + \{\sum_{i=1}^{m} [g_i, h_i] \mid m \in \mathbb{N},\ g_i, h_i \in \mathbb{R}\langle X, Y\rangle\} \subseteq \mathbb{R}\langle X, Y\rangle \end{aligned} \tag{2}$$

the convex cone of all polynomials that are **cyclically equivalent to a sum of hermitian squares**.

The importance of these sets for us is given by the following elementary observations:

Proposition 1.5. *Let $f \in \mathbb{R}\langle X, Y\rangle$.*

(1) *If $f \in \Sigma^2$, then $f(A, B) \succeq 0$ for all symmetric matrices A and B of the same size.*
(2) *If $f \in \Theta^2$, then $\operatorname{tr}(f(A, B)) \geq 0$ for all symmetric matrices A and B of the same size.*

By Helton's theorem [Hel02], the converse of (1) holds: if for $f \in \mathbb{R}\langle X, Y\rangle$, $f(A, B)$ is positive semidefinite for all symmetric matrices A and B of the same size, then $f \in \Sigma^2$. On the other hand, the converse of (2) fails in general, that is, there are examples of NC polynomials f satisfying the trace positivity condition of part (2) of Proposition 1.5 yet $f \notin \Theta^2$, cf. [KS08a, Example 4.4] or [KS08b, Example 3.5]. Nevertheless, this part of Proposition 1.5 yields a useful sufficient condition for tracial positivity and was exploited by Schweighofer and the second author [KS08b] to prove the BMV conjecture for $m \leq 13$.

Observation 1.6. To model *positive semidefiniteness* with the aid of *symmetric* noncommuting variables, we consider polynomials in X^2, Y^2: if $S_{m,k}(X^2, Y^2) \in \Theta^2$ for some m, k, then the t^k coefficient of p_m is nonnegative for all positive semidefinite matrices A, B of all sizes.

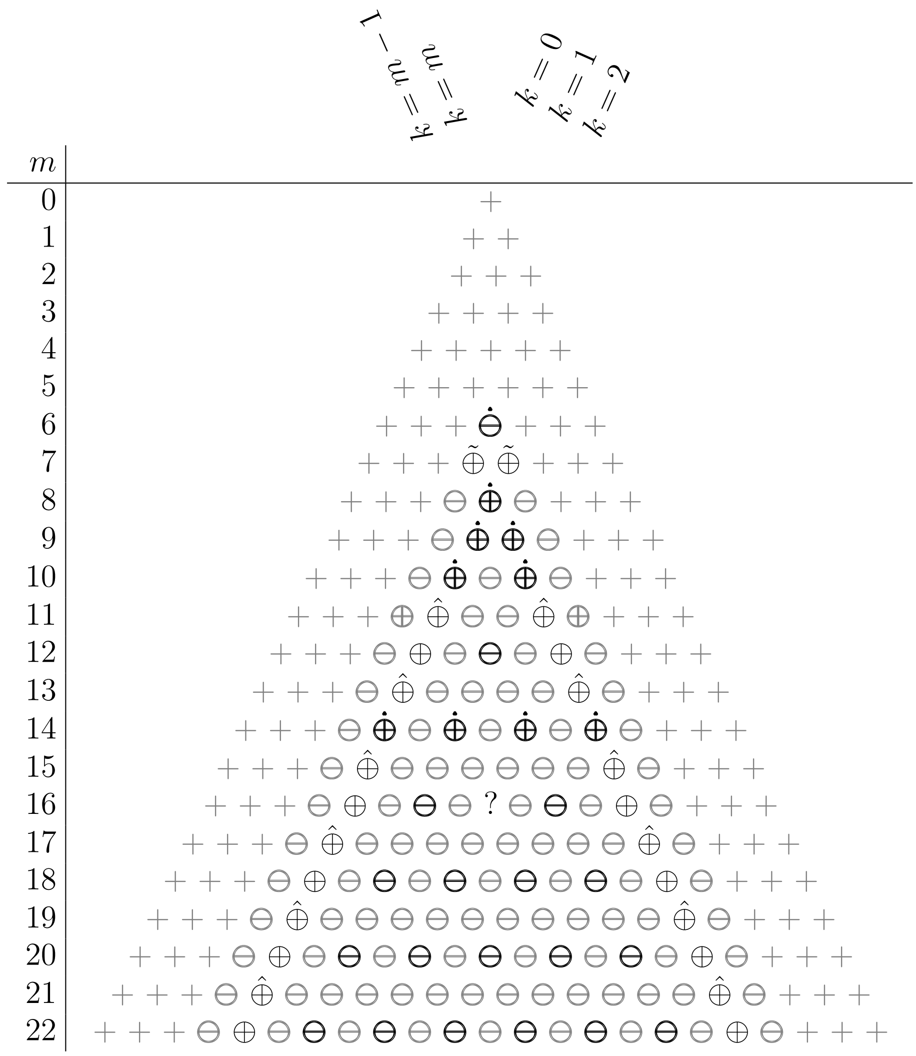
1.2. **A xmas tree.** Much work has been done in determining whether $S_{m,k}(X^2, Y^2) \in \Theta^2$ for a given pair (m, k). It is easy to see $S_{m,k}(X^2, Y^2) \in \Theta^2$ for $k \leq 2$ or $m - k \leq 2$. In our terminology, the first (nontrivial) certificate can be extracted from Hägele [Häg07] to show $S_{7,3}(X^2, Y^2) \in \Theta^2$. This was followed upon in [KS08b] where among the main results were $S_{6,3}(X^2, Y^2) \notin \Theta^2$, $S_{14,4}(X^2, Y^2) \in \Theta^2$ and $S_{14,6}(X^2, Y^2) \in \Theta^2$. The latter results combined with Hillar's descent theorem [Hil07] imply that the BMV conjecture holds for $m \leq 13$.

Here is a brief overview of the latest developments. Landweber and Speer [LS09] proved for example that $S_{m,4}(X^2, Y^2) \in \Theta^2$ for odd m and that $S_{11,3}(X^2, Y^2) \in \Theta^2$. They also give results on the negative side, implying that $S_{m,k}(X^2, Y^2) \notin \Theta^2$ in the following cases:

(1) m is odd and $5 \leq k \leq m - 5$;
(2) $m \geq 13$ is odd and $k = 3$;
(3) m is even, k is odd and $3 \leq k \leq m - 3$;
(4) $(m, k) = (9, 3)$.

Independently of the work of Landweber and Speer, Burgdorf [Bur] found a combinatorial proof of $S_{m,4}(X^2, Y^2) \in \Theta^2$ for all m. The last contribution of negative results is given by Collins, Dykema and Torres-Ayala [CDTA]: $S_{12,6}(X^2, Y^2) \notin \Theta^2$ and for even m, k with $6 \leq k \leq m - 10$, $S_{m,k}(X^2, Y^2) \notin \Theta^2$.

We present the state-of-the-art knowledge conveniently in the form of a table:



Is $S_{m,k}(X^2, Y^2) \in \Theta^2$?

(The tree continues following the pattern in rows 20, 21 and 22.)

authors	color
Hägele [Häg07]	$\tilde{\oplus}$
Klep and Schweighofer [KS08b]	$\dot{\oplus}$ $\dot{\ominus}$
Burgdorf [Bur], Landweber and Speer [LS09]	$\hat{\oplus}$
Burgdorf [Bur]	$\oplus$
Landweber and Speer [LS09]	$\oplus$ $\ominus$
Collins, Dykema and Torres-Ayala [CDTA]	$\ominus$

symbol	meaning
$+$	$S_{m,k}$ is in Θ^2 for trivial reasons
$\oplus$	$S_{m,k}$ is in Θ^2 (with proof)
$\ominus$	$S_{m,k}$ is not in Θ^2 (with proof)

Legend

The aim of this article is to settle the remaining case, i.e., we prove (what was conjectured in [KS08b, pg. 754] based on numerical evidence) $S_{16,8}(X^2,Y^2) \notin \Theta^2$.

2. Gram matrix method and semidefinite programming

In this section we explain how a desired nonmembership certificate can be obtained. The main idea is to construct a linear map $L : \mathbb{R}\langle X,Y\rangle \to \mathbb{R}$ satisfying

$$L(\Theta^2) \subseteq [0,\infty), \quad L(S_{16,8}(X^2,Y^2)) < 0. \tag{3}$$

2.1. **Gram matrix method.** Checking whether a polynomial in noncommuting variables is an element of Σ^2 and Θ^2, respectively, is most efficiently done via the so-called Gram matrix method [KS08b, KP], well-known in the commutative setting [CLR95, PS03]. We explain this for the case needed in the sequel.

Theorem 2.1 (Klep, Schweighofer [KS08b, Proposition 3.3])**.** *Suppose m,k are even and set*

$$\begin{aligned} V_1 &:= \left\{v \in \{X^2,Y^2\}^{\frac{m}{2}} \mid \deg_X v = m-k,\ \deg_Y v = k\right\}, \\ V_2 &:= \left\{v \in X\{X^2,Y^2\}^{\frac{m}{2}-1}X \mid \deg_X v = m-k,\ \deg_Y v = k\right\}, \\ V_3 &:= \left\{v \in Y\{X^2,Y^2\}^{\frac{m}{2}-1}Y \mid \deg_X v = m-k,\ \deg_Y v = k\right\}. \end{aligned} \tag{4}$$

Let $\bar v_i$ denote the vector $[v]_{v\in V_i}$. Then $S_{m,k}(X^2,Y^2) \in \Theta^2$ if and only if there exist positive semidefinite matrices $G_i \in \operatorname{Sym}\mathbb{R}^{V_i\times V_i}$ such that

$$S_{m,k}(X^2,Y^2) \overset{\text{cyc}}{\sim} \sum_i \bar v_i^* G_i \bar v_i. \tag{5}$$

If $G_i = C_i^* C_i$ and $C_i \in \mathbb{R}^{J_i\times V_i}$ (J_i some index set), then with $[p_{i,j}]_{j\in J_i} := C_i \bar v_i$ we have

$$S_{m,k}(X^2,Y^2) \overset{\text{cyc}}{\sim} \sum_{i,j} p_{i,j}^* p_{i,j}. \tag{6}$$

Any symmetric block matrix $G = \begin{bmatrix} G_1 & & \\ & G_2 & \\ & & G_3 \end{bmatrix}$ satisfying $f \overset{\text{cyc}}{\sim} \sum_i \bar v_i^* G_i \bar v_i$ for some $f \in \mathbb{R}\langle X,Y\rangle$, is called a **Gram matrix** for f. If $f = \sum_i \bar v_i^* G_i \bar v_i$, then we call G an **exact Gram matrix**. (We emphasize this is not the standard definition.)

In the case we are interested in, $k = 8$ is even and $m = 2k = 16$. This can be used to obtain a small refinement of the above theorem by way of symmetries under group actions (cf. [GP04] for a study of invariant sum of squares polynomials in *commuting* variables). Let $\sigma : \mathbb{R}\langle X,Y\rangle \to \mathbb{R}\langle X,Y\rangle$ be the automorphism $X \mapsto Y$, $Y \mapsto X$.

Corollary 2.2. *Suppose k is even and set*

$$\begin{aligned} U_1 &:= \left\{u \in \{X^2,Y^2\}^{k} \mid \deg_X u = k,\ \deg_Y u = k\right\}, \\ U_2 &:= \left\{u \in X\{X^2,Y^2\}^{k-1}X \mid \deg_X u = k,\ \deg_Y u = k\right\}, \\ U_3 &:= \left\{\sigma(u) \mid u \in U_2\right\}. \end{aligned} \tag{7}$$

Let $\bar u_i$ denote the vector $[u]_{u\in U_i}$. Then $S_{2k,k}(X^2,Y^2) \in \Theta^2$ if and only if there exist positive semidefinite matrices $G_i \in \operatorname{Sym}\mathbb{R}^{U_i\times U_i}$, $i = 1,2$, such that

$$S_{2k,k}(X^2,Y^2) \overset{\text{cyc}}{\sim} \bar u_1^* G_1 \bar u_1 + \bar u_2^* G_2 \bar u_2 + \bar u_3^* G_2 \bar u_3. \tag{8}$$

2.2. **The certificate.** Let us now return to the question whether $S_{16,8}(X^2,Y^2) \in \Theta^2$. Following Theorem 2.1 we must determine whether there exists a block diagonal positive semidefinite matrix G such that $S_{16,8}(X^2,Y^2) \overset{\text{cyc}}{\sim} W^*GW$, where W is the vector $\begin{bmatrix} \bar{v}_1^t & \bar{v}_2^t & \bar{v}_3^t \end{bmatrix}^t$ obtained above. This is a semidefinite feasibility problem in the block diagonal matrix variable G, where the constraints

$$\langle A_w, G\rangle = a_w \tag{9}$$

express that for each product of monomials $w \in \{p^*q \mid p,q \in W\}$ we have

$$\sum_{\substack{p,q\in W\\ p^*q \overset{\text{cyc}}{\sim} w}} G_{p,q} = \sum_{u \overset{\text{cyc}}{\sim} w} a_u, \tag{10}$$

where a_u is the coefficient of u in $S_{16,8}(X^2,Y^2)$. We may restrict ourselves to monomials $w = p^*q$ with $p,q \in V_i$ for some $i = 1,2,3$. Moreover, since $S_{16,8}(X^2,Y^2)$ is symmetric we can merge for each pair of equivalence classes $[w]$, $[w^*]$, $w \in \{p^*q \mid p,q \in W\}$, the constraints (10) into a single constraint

$$\sum_{\substack{p,q\in W\\ p^*q \overset{\text{cyc}}{\sim} w \ \vee \ p^*q \overset{\text{cyc}}{\sim} w^*}} G_{p,q} = \sum_{u \overset{\text{cyc}}{\sim} w \ \vee \ u \overset{\text{cyc}}{\sim} w^*} a_u. \tag{11}$$

After further restricting to equivalence classes $[w]$, $[w^*]$ with $w \in \{p^*q \mid p,q \in V_i \text{ for some } i = 1,2,3\}$ we finally obtain 440 linearly independent linear constraints of this type (there are 4485 equivalence classes $[w]$, $w \in \{p^*q \mid p,q \in W\}$, yielding 4485 constraints (10). Hence we reduced the number of constraints for about 90 %).

To prove this problem is *infeasible*, we find a separating hyperplane with the help of semidefinite programming. Fix $m = 16$, $k = 8$ and let $\mathcal{V}$ denote the vector space of all block diagonal symmetric matrices as in Theorem 2.1. To each $0 \neq G \in \mathcal{V}$ we can associate the NC polynomial $W^*GW \in \mathbb{R}\langle X,Y\rangle$ of degree 32. Let $\mathcal{P}$ denote the vector space of all such polynomials. Each $f \in \mathcal{P}$ has an exact Gram matrix. It is even *unique* since f is *homogeneous* [KP, Proposition 2.3]. Let $\Theta^2\mathcal{P}$ denote the set of NC polynomials in $\mathcal{P}$ with a positive semidefinite Gram matrix.

Lemma 2.3. *$\Theta^2\mathcal{P} = \Theta^2 \cap \mathcal{P}$.*

Proof. This is a straightforward extension of the proof of [KS08b, Proposition 3.3]. ∎

Every linear map $L : \mathcal{P} \to \mathbb{R}$ can be presented as

$$f \mapsto \langle B_1, G_1\rangle + \langle B_2, G_2\rangle + \langle B_3, G_3\rangle = \operatorname{tr}(B_1G_1) + \operatorname{tr}(B_2G_2) + \operatorname{tr}(B_3G_3) \tag{12}$$

for some (symmetric) block matrix $B_L = \begin{bmatrix} B_1 & & \\ & B_2 & \\ & & B_3 \end{bmatrix}$, where $\begin{bmatrix} G_1 & & \\ & G_2 & \\ & & G_3 \end{bmatrix}$ is an exact Gram matrix for f. Conversely, equation (12) can be used to define $L : \mathcal{P} \to \mathbb{R}$ due to the uniqueness of the exact Gram matrix for polynomials in $\mathcal{P}$.

Now suppose in addition that $L(\Theta^2\mathcal{P}) \subseteq [0,\infty)$. By the self-duality of the cone of positive semidefinite matrices, this implies $B_L \succeq 0$. Also, $L(f) = 0$ for all $f \in \mathcal{P}$ with $f \overset{\text{cyc}}{\sim} 0$ by Lemma 2.3. Hence

$$\langle H, B_L\rangle = 0 \tag{13}$$

for all $H \in \mathcal{V}$ satisfying $W^*HW \overset{\text{cyc}}{\sim} 0$. The later condition can be rephrased as follows: $\langle A_w, H\rangle = 0$ for all A_w (the constraint matrices from our original feasibility SDP (11)). Let $\{C_j \mid j \in J\}$ denote a basis of $\{A_{p^*q} \mid i \in \{1,2,3\}, (p,q) \in V_i \times V_i\}^\perp \subseteq \mathcal{V}$. Then such an H is in the span of the C_j. So (13) can be equivalently written as

$$\langle C_j, B_L\rangle = 0 \quad \text{for all } j \in J. \tag{14}$$

We are now in a position to present the desired SDP constructing a separating hyperplane. Let G_0 denote any Gram matrix for $S_{16,8}(X^2, Y^2)$. Consider the semidefinite feasibility problem

$$\begin{aligned} B = \begin{bmatrix} B_1 & & \\ & B_2 & \\ & & B_3 \end{bmatrix} &\succeq 0, \\ \langle B, G_0\rangle &= -100, \\ \langle B, C_i\rangle &= 0 \quad \text{for all } C_i. \end{aligned} \tag{15}$$

Theorem 2.4. (15) *is feasible.*

2.3. **Proof of Theorem 2.4.** We explain how this was verified using a computer.

It is easy to run a general SDP solver (such as SDPT3 [TTT09] or SeDuMi [SeD09]; see also Mittelman's website [Mit09] for a benchmark of state-of-the-art solvers) to produce a *floating point* feasible solution for (15). However, looking for a *symbolic* (*e.g. rational*) feasible point, additional work is required. We proceed as follows: run (15) as an SDP with trivial objective function, since under a strict feasibility assumption the interior point methods yield solutions in the relative interior of the optimal face, which is in our case the whole feasibility set. If strict complementarity is additionally provided the interior point methods lead to the analytic center of the feasibility set [GS98, HdKR02, HdKR05].

In our example this leads to a matrix B' with smallest eigenvalue approximately $\varepsilon = 0.41$ and distance to the affine subspace generated by the linear constraints of (15) being approximately $\delta = 1.7 \cdot 10^{-7}$. Taking a very close rational approximation B'' of B' and then projecting onto the affine subspace yields a *rational* matrix B feasible for (15), cf. [PP08, Proposition 8 and Fig. 1 on pg. 276].

We also *explicitly* computed a rational (even integer) feasible point for a small modification of (15). All the data needed to verify the correctness is available from our NCSOStools [CKP09] website

`http://ncsostools.fis.unm.si/`

See also the Appendix for a fuller explanation. ■

Corollary 2.5. $S_{16,8}(X^2, Y^2) \notin \Theta^2$.

Proof. Any feasible point B for (15) gives rise to a linear map $L : \mathcal{P} \to \mathbb{R}$ as in (12). This linear form satisfies $L(\Theta^2\mathcal{P}) \subseteq [0, \infty)$ and $L(S_{16,8}(X^2, Y^2)) = -100 < 0$. Thus $S_{16,8}(X^2, Y^2) \notin \Theta^2$ by Theorem 2.1. ■

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Appendix: The Matlab verification

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The data package

bmv168_data_27_10_09.mat,

available from  http://ncsostools.fis.unm.si/download contains the following data:

- S168  ... the BMV polynomial S_{16,8}(X^2,Y^2)

- V ... the vector of all monomials of order 16, which can appear in an
SOHS polynomial cyclically equivalent to S168.
Note that V=[V1;V2:V3], where Vi is a vector as in Theorem 2.1

- G0 =  a (block diagonal) Gram matrix for S168 - the one we used in (15).

- A ... a matrix of order 19600 x 440 ... each column of A corresponds
to an equation in <Aw,X>=aw, as in (9) or (10)

- C ... a matrix of order 19600 x 3305 ... columns of C are pairwise
orthogonal and also orthogonal to columns of A ... matrix reformulations
of columns of C are exactly matrices Ci from (15).
Note: we kept in A and C only columns which corresponds to the diagonal
blocks, as described in Lemma 2.3, hence we have in A and C altogether
70*71/2 + 2*35*36/2 = 3745 columns.

- B ...  a solution of (15). Note that B = blockDiag(B1,B2,B3) - a PsD matrix
of order 140x140.


Instructions (some of this requires NCSOStools):

1. To reproduce the polynomial S_{16,8}(X^2,Y^2), run

         S168=BMVq(16,8);

2. To check that G0 is a Gram matrix for S168 call

         Snew=V'*G0*V; NCisCycEq(S168,Snew)

(Caution: the last command must give answer 1 and takes quite some time
to evaluate.)

3. To check that A contains the equations (11) compute

         trace(reshape(A(:,i),140,140)*G0)
```

```
which must be the number of all monomials in S168, which are cyclically
equivalent to the monomials w or w^*, underlying the i-th equation

4. To check that B is feasible for the linear constraints in (15) run

         norm(C'*B(:))==0, trace(B*G0)<0

5. B is an integer matrix. To see that is it PsD, compute

        min(eig(B))>0

Alternatively, for a symbolic verification, please use our Mathematica
notebook

bmv_16_8-ldlt.nb

available from http://ncsostools.fis.unm.si/download where the LDU factorization
is given.
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