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A Facial Reduction Algorithm for Finding Sparse SOS Representations

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Abstract

Facial reduction algorithm reduces the size of the positive semidefinite cone in SDP. The elimination method for a sparse SOS polynomial ([3]) removes unnecessary monomials for an SOS representation. In this paper, we establish a relationship between a facial reduction algorithm and the elimination method for a sparse SOS polynomial.

Key words: Facial Reduction Algorithm, Semidefinite Programming, Polynomial Optimization.

1. Introduction

Since Lasserre [4] and Parrilo [7] proposed semidefinite programming (SDP) relaxation for polynomial optimization problems (POPs), various powerful algorithms for solving POPs by using SDP and sums of square (SOS) polynomials have been proposed. These results are summarized in an excellent survey [5].

In general, if a POP is large-scale, *e.g.* it has hundreds of variables, then the resulting SDP becomes too huge to compute the optimal value. It is necessary to exploit a structure of a given POP, *e.g.* sparsity and/or symmetry, for reducing the size of the SDP. For this, Kojima *et al.* [3] proposed a method to reduce the size of the SDP obtained from a sparse SOS polynomial. This method was also discussed in [6]. In this paper, we call the method the *elimination method for a sparse SOS polynomial* (EMSSOSP). EMSSOSP removes unnecessary monomials for an SOS representation of a sparse SOS polynomial.

A facial reduction algorithm (FRA) was proposed by Borwein and Wolokowicz [1, 2]. Ramana *et al.* [10] showed that FRA for SDP can generate an SDP which has an interior feasible solution.

The purpose of this paper is to establish a relationship between FRA and EMSSOSP.

As a by-procut of this result, we prove that a computationally heavy part of EMSSOSP proposed in [3] is redundant for finding a set of unnecessary monomials for an SOS representation of a sparse SOS polynomial. This part enumerates all integer vectors in the convex hull of a set, and the authors in [3] reported that the part has much more computational cost than the other part.

In this paper, let $\mathbb{R}$ and $\mathbb{N}$ be the sets of real and natural numbers, respectively. We define for $n, r \in \mathbb{N}$, $\mathbb{N}^n_r = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \mid \sum_{i=1}^n \alpha_i \le r\}$. For $n \in \mathbb{N}$, $\mathbb{S}^n$ and $\mathbb{S}^n_+$ denote the set of $n \times n$ symmetric matrices and positive semidefinite matrices, respectively. For $A \subseteq \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we define $\alpha A := \{\alpha a \mid a \in A\}$.

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2. The elimination method for a sparse SOS representation

EMSSOSP removes unnecessary monomials for an SOS representation of a given sparse polynomial f. The resulting SDP is equivalent to the original SDP constructed by Parrilo [7] and the size of the SDP becomes smaller than the original SDP. In [12], EMSSOSP was demonstrated that computational efficiency of SDP relaxation was improved.

Let f be a polynomial with degree $2r$ and we write $f(x) = \sum_{\alpha\in F} f_\alpha x^\alpha$, where $x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, f_α denotes the coefficient corresponding with the monomial x^α and F is the set of $\alpha \in \mathbb{N}^n$ such that f_α is nonzero. Then F, a finite subset of $\mathbb{N}^n_{2r}$, it is called the *support* of f. If the number of elements in F is small, then we call f *sparse*.

We assume that f is an SOS polynomial, $f(x) = \sum_{j=1}^k g_j(x)^2$, where k and the coefficients of polynomials g_j are unknown. Now because g_j is a polynomial, we write g_j using a finite set $G_j \subseteq \mathbb{N}^n_r$ as follows: $g_j(x) = \sum_{\alpha\in G_j}(g_j)_\alpha x^\alpha$, where $(g_j)_\alpha$ is the coefficient corresponding with the monomial x^α. Let $G = \cup_{j=1}^k G_j$. Then we can rewrite polynomials g_j by G instead of G_j. Indeed, if $G \setminus G_j \neq \emptyset$, set the coefficient $(g_j)_\alpha$ to be zero for $\alpha \in G\setminus G_j$. In this case, we say that f has *an SOS representation with* G. Also if the number of the set G is small, then we say that f has *a sparse SOS representation with* G.

Once G is found, we can construct an SDP by using the following lemma. We remark that the lemma is equivalent to Theorem 1 in [9] if $G = \mathbb{N}^n_r$.

Lemma 2.1. *(Lemma 2.1 in* [3]*) Let G be a finite subset of $\mathbb{N}^n$ and $u_G(x) = (x^\alpha : \alpha \in G)$. Then, f has an SOS representation with G if and only if there exists a positive semidefinite matrix $V \in \mathbb{S}_+^{\#(G)}$ such that $f(x) = u_G(x)^T V u_G(x)$ for all $x \in \mathbb{R}^n$.*

From Lemma 2.1, to find an SOS representation with G of f , we consider the following problem:

$$\begin{cases} \text{Find} & V \in \mathbb{S}_+^{\#(G)} \\ \text{subj. to} & f(x) = u_G(x)^T V u_G(x) \ (\forall x \in \mathbb{R}^n). \end{cases} \tag{1}$$

We regard the constraint of (1) as an identity on x. By comparing coefficients of all monomials in the both sides of the identity, we obtain an SDP. If $G = \mathbb{N}^n_r$, then the resulting SDP is identical to Parrilo's SDP relaxation.

If f is a sparse SOS polynomial, we can expect that an SOS representation of f are sparse, *i.e.*, the number of elements in G is small. To find such a small set G, EMSSOSP was proposed in [3].

Later, we describe the detail of EMSSOSP. To this end, we give the following theorem and lemma, which play an essential role on EMSSOSP.

Theorem 2.2. *(Theorem 1 and Lemma in Section 3 of* [11]*) Let f and F be a polynomial and its support, respectively. We define $F^e := F \cap (2\mathbb{N}^n)$. Assume that f is a nonnegative polynomial, i.e., $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. Then F is included in conv(F^e), where conv(A) is the convex hull of $A \subseteq \mathbb{R}^n$. Moreover, if f has an SOS representation with G, then $(g_j)_\alpha = 0$ for all $j = 1,\dots,k$ and $\alpha \in G\setminus\bar{G}$, where $\bar{G} := \frac{1}{2}$conv$(F^e)\cap\mathbb{N}^n$.*

Lemma 2.3. *(Lemma 3.1 in* [3]*) Assume that f has an SOS representation with G and that there exist nonempty sets $B, H \subseteq G$ such that the triplet (B, H, G) satisfies*

$$\begin{cases} H \subseteq G, B \subseteq G, G = H \cup B, \\ (B + B) \cap F = \emptyset \text{ and } (B + B) \cap (G + H) = \emptyset. \end{cases} \tag{2}$$

Then f has an SOS representation with H.

We remark that it is easy to prove $B \cap H = \emptyset$ from (2).

For a given $G \subseteq \mathbb{N}^n_r$, we give an algorithm of EMSSOSP(G):

Algorithm 2.4. *(EMSSOSP(G))*

- **Input::** $G \subseteq \mathbb{N}^n_r$.
- **Step 1::** *Set $G^0 = G$ and $i = 0$.*
- **Step 2::** *If there do not exist $B^i, H^i \subseteq G^i$ such that the triplet $(B, H, G) = (B^i, H^i, G^i)$ satisfies (2), then stop and return G^i.*
- **Step 3::** *Otherwise set $G^{i+1} = H^i$ and $i = i + 1$, and go back to Step 2.*

We remark that EMSSOSP($\bar{G}$) is EMSSOSP proposed in [3]. Step 1 and Step 2 of EMSSOSP(G) are based on Theorem 2.2 and Lemma 2.3, respectively.

Let G^* be the finite set returned by EMSSOSP($\bar{G}$). Clearly, we have $(g_j)_\alpha = 0$ for all $j = 1, \ldots, k$ and $\alpha \in \mathbb{N}^n_r \setminus G^*$. Moreover, it follows that $\frac{1}{2}F^e \subseteq G^*$. EMSSOSP($G$) removes unnecessary monomials which are not needed to construct an SOS representation of f.

In [3], the authors reported that before executing EMSSOSP($\bar{G}$), we need to enumerate all integer points of $\bar{G}$ and that we need much computational cost for this part. The following theorem guarantees that we can obtain the same set G^* of monomials as EMSSOSP($\bar{G}$) even if we start EMSSOSP(G) from an arbitrary set G including $\bar{G}$. This proposition is one of our contribution of this paper.

Theorem 2.5. *Assume that f is an SOS polynomial. If $G \supseteq \bar{G}$, then EMSSOSP(G) returns G^*.*

We postpone the proof till Appendix A.

3. An FRA and a relationship between FRA and EMSSOSP

3.1. **A facial reduction algorithm.** We consider the following SDP:

$$\sup\ b^T y \ \text{ subj. to } \ C - \sum_{i=1}^m A_i y_i \in \mathbb{S}^n_+, \tag{3}$$

where $b \in \mathbb{R}^m, C, A_1, \ldots, A_m \in \mathbb{S}^n$. For SDP (3) which does not have any interior feasible solutions, FRA reduces the closed convex cone $\mathbb{S}^n_+$ to a smaller closed convex subcone. If we generate a smaller SDP by replacing $\mathbb{S}^n_+$ by the smaller subcone, then (i) the resulting SDP is equivalent to (3), and (ii) it has an interior feasible solution. Because of (ii), we can expect that the numerical stability of the primal-dual interior-point methods is improved for the resulting SDP.

FRA was first proposed by Borwein and Wolkowicz [1, 2], and later simplified by Pataki [8]. Although the FRA works for conic programming (CP) with nonempty feasible region, FRA for CP without assuming the feasibility was proposed in [13].

We give the detail of FRA. A closed subcone $\mathcal{F}$ of $\mathbb{S}^n_+$ is a face of $\mathbb{S}^n_+$ if for any $x, y \in \mathbb{S}^n_+$, $x + y \in \mathcal{F}$ implies that $x, y \in \mathcal{F}$. Let $\mathcal{A} = \{C - \sum_{i=1}^m A_i y_i \mid y \in \mathbb{R}^m\}$. FRA reduces $\mathbb{S}^n_+$ to the smallest face including $\mathcal{A} \cap \mathbb{S}^n_+$, which is called the *minimal cone* for (3).

We give an algorithm of FRA for SDP (3):

Algorithm 3.1. *(Facial Reduction Algorithm)*

- **Step 1::** *Set $i = 0$ and $\mathcal{F}_0 = \mathbb{S}^n_+$.*
- **Step 2::** *If $\ker A \cap H_c^- \cap \mathcal{F}_i^* \subseteq span(W_1, \ldots, W_i)$, then stop and return $\mathcal{F}_i$.*
- **Step 3::** *Find $W_{i+1} \in (\ker A \cap H_c^- \cap \mathcal{F}_i^*) \setminus span(W_1, \ldots, W_i)$.*
- **Step 4::** *If $C \bullet W_{i+1} < 0$, then stop. SDP (3) is infeasible.*
- **Step 5::** *Set $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{W_{i+1}\}^\perp$ and $i = i + 1$, and go back to Step 2.*

In this algorithm, H_c^- denotes the half-space $\{W \in \mathbb{S}^n \mid C \bullet W \leq 0\}$ and we define $\ker A := \{W \in \mathbb{S}^n \mid A_i \bullet W = 0\ (i = 1, \ldots, m)\}$. Moreover, we set $\text{span}(\emptyset) = \{0\}$. In [13], it was showed that Algorithm 3.1 can find the minimal cone for SDP (3) or detect the infeasibility of SDP (3) in a finite number of iterations. In addition, if we know in advance that SDP (3) has a feasible solution, then we can replace H_c^- by $\ker c^T := \{W \in \mathbb{S}^n \mid C \bullet W = 0\}$ in FRA.

3.2. **A relationship between FRA and EMSSOSP.** In this subsection, we reveal a relationship between FRA and EMSSOSP(G). Specially, we show that EMSSOSP(G) can be interpreted as FRA

Let f be a polynomial with degree $2r$. We assume that f has a sparse SOS representation with G satisfying $\bar{G} \subseteq G \subseteq \mathbb{N}^n_r$. Applying EMSSOSP($G$) into f, it generates G^s and the sequence $\{(B^i, H^i, G^i)\}_{i=0}^{s-1}$ satisfying (2) for each $i = 0, 1, \ldots, s-1$, where $G^0 = G$ and $G^s = G^*$. Then we can construct an SDP by G^*.

From (1), we obtain the following SDP:

$$\sup\ 0 \ \text{ subj. to } \ f_\alpha = E_\alpha \bullet V \ (\alpha \in G + G),\ V \in \mathbb{S}^{\#(G)}_+, \tag{4}$$

where we define E_α by

$$(E_\alpha)_{\beta,\gamma} = \begin{cases} 1 & \alpha = \beta + \gamma, \\ 0 & \text{o.w.} \end{cases} \quad \text{for all } \beta, \gamma \in G.$$

SDP (4) has a feasible solution because we have assumed that f has a sparse SOS representation with G. Therefore we can replace H_c^- by $\ker c^T$ in FRA. It is not difficult to verify that the set corresponding to the set $\ker A \cap \ker c^T$ is

$$\left\{ W \;\middle|\; \begin{array}{l} W = \sum_{\alpha \in G+G} y_\alpha E_\alpha = (y_{\alpha+\beta})_{\alpha,\beta\in G} \\ \sum_{\alpha \in F} f_\alpha y_\alpha = 0 \end{array} \text{ for some } y_\alpha \right\}. \tag{5}$$

The following lemma shows that one can construct $W \in \ker A \cap \ker c^T \cap \mathcal{F}_i^*$ from (B^i, H^i, G^i) satisfying (2).

Lemma 3.2. *Let $B \subseteq \mathbb{N}^n$ be a nonempty finite set. We define y_α for all $\alpha \in B+B$ as follows:*

$$y_\alpha = \int_S x^\alpha dx,$$

where $S \subseteq \mathbb{R}^n$ is a compact set with nonempty interior. Then $W = (y_{\alpha+\beta})_{\alpha,\beta\in B}$ is positive definite.

Proof : Clearly, W is positive semidefinite. We prove that $z^T W z = 0$ implies $z = 0$. From the definition of W, $z^T W z = 0$ implies that the polynomial $z(x)$ is zero on S. Because S has nonempty interior, $z(x)$ is the zero polynomial, and thus $z = 0$. □

We give our main theorem. This theorem implies that EMSSOSP(G) can be interpreted as FRA.

Theorem 3.3. *Let $G \subseteq \mathbb{N}_r^n$. We assume that f has a sparse SOS representation with G and that EMSSOSP(G) generates $G^* = G^s$ and the triplets (B^i, H^i, G^i) for all $i = 0, \ldots, s-1$ which satisfy (2). Then FRA for SDP (4) can generate the following faces:*

$$\mathcal{F}_{i+1} = \left\{ V \in \mathbb{S}^{\#(G)} \;\middle|\; V = \begin{pmatrix} \tilde{V} & O & O \\ O & O & O \\ O & O & O \end{pmatrix} \text{ and } \tilde{V} \in \mathbb{S}_+^{\#(H^i)} \right\}. \tag{6}$$

In particular, the face $\mathcal{F}_s$ is

$$\mathcal{F}_s = \left\{ V \in \mathbb{S}^{\#(G)} \;\middle|\; V = \begin{pmatrix} \tilde{V} & O \\ O & O \end{pmatrix} \text{ and } \tilde{V} \in \mathbb{S}_+^{\#(G^*)} \right\}. \tag{7}$$

Proof : The triplets (B^i, H^i, G^i) for all $i = 0, \ldots, s-1$ satisfy (2). We construct $W_{i+1} = (y^{i+1}_{\alpha+\beta})_{\alpha,\beta\in G}$ from B^i as follows:

$$y^{i+1}_\alpha = \begin{cases} \int_S x^\alpha \, dx & \text{for all } \alpha \in B^i + B^i, \\ 0 & \text{for all } \alpha \in (G+G) \setminus (B^i + B^i), \end{cases}$$

where S is a compact set with nonempty interior.

Claim 1. *$W_{i+1} \notin$ span$(W_1, \ldots, W_i)$.*

Proof of Claim 1 : Because $(B^i + B^i) \cap (G^i + H^i) = \emptyset$, we have

$$W_{i+1} = \begin{array}{c} \\ H^i \\ B^i \\ G \setminus G^i \end{array} \begin{array}{c} \begin{array}{ccc} H^i & B^i & G \setminus G^i \end{array} \\ \begin{pmatrix} O & O & S_1^i \\ O & S_2^i & S_3^i \\ (S_1^i)^T & (S_3^i)^T & S_4^i \end{pmatrix} \end{array}, \tag{8}$$

where S_2^i is positive definite for all $i = 0, 1, \ldots, s-1$ by Lemma 3.2. On the other hand, because $B^i \subseteq G^i$ and the submatrices $(W_j)_{G^i,G^i} = O$ for all $j = 1, \ldots, i$, therefore, $W_{i+1} \notin \text{span}(W_1, \ldots, W_i)$. □

We prove this theorem by induction on i. We consider the case of $i = 0$. From $G \setminus G^0 = \emptyset$ and the form of W_{i+1} in (8), we have

$$W_1 = \begin{array}{c} \\ H^0 \\ B^0 \end{array} \begin{array}{c} \begin{array}{cc} H^0 & B^0 \end{array} \\ \begin{pmatrix} O & O \\ O & S_2^0 \end{pmatrix} \end{array},$$

where S_2^0 is positive semidefinite. Therefore, we have

$$\mathbb{S}_+^{\#(G)} \cap \{W_1\}^\perp = \left\{ V \;\middle|\; V = \begin{pmatrix} \tilde{V} & O \\ O & O \end{pmatrix}, \tilde{V} \in \mathbb{S}_+^{\#(H^0)} \right\},$$

and this coincides with the face $\mathcal{F}_1$.

We assume that the i-th face $\mathcal{F}_i$ is as follows:

$$\mathcal{F}_i = \left\{ V \in \mathbb{S}^{\#(G)} \middle| V = \begin{pmatrix} \tilde{V} & O & O \\ O & O & O \\ O & O & O \end{pmatrix} \text{ and } \tilde{V} \in \mathbb{S}_+^{\#(H^{i-1})} \right\}.$$

Because $H^{i-1} = G^i$, the dual $\mathcal{F}_i^*$ is

$$\mathcal{F}_i^* = \left\{ W \in \mathbb{S}^{\#(G)} \middle| \begin{array}{l} W = \begin{pmatrix} \tilde{W} & S_1 \\ S_1^T & S_2 \end{pmatrix} \text{ and } \tilde{W} \in \mathbb{S}_+^{\#(G^i)}, \\ S_1 \in \mathbb{R}^{\#(G^i)\times\#(G\setminus G^i)}, S_2 \in \mathbb{S}^{\#(G\setminus G^i)} \end{array} \right\}.$$

Claim 2. $W_{i+1} \in (\ker A \cap \ker c^T \cap \mathcal{F}_i^*) \setminus \mathit{span}(W_1, \ldots, W_i)$

Proof : From Claim 1, $W_{i+1} \notin \mathrm{span}(W_1, \ldots, W_i)$. It follows from Lemma 3.2 that S_2^i in (8) is positive definite. From this fact and $G^i = H^i \cup B^i$, $W_{i+1} \in \mathcal{F}_i^*$. In addition, from the definition of W_{i+1} and $F \subseteq H^i + H^i$, W_{i+1} belongs to the set of (5). Consequently, we obtain the desired result. □

Now, the face $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{W_{i+1}\}^\perp$ is

$$\mathcal{F}_{i+1} = \left\{ V \in \mathbb{S}^{\#(G)} \middle| V = \begin{pmatrix} \tilde{V} & O & O \\ O & O & O \\ O & O & O \end{pmatrix} \text{ and } \tilde{V} \in \mathbb{S}_+^{\#(H^{i})} \right\}.$$

Therefore, (6) is proved by induction. Specially, it follows from $B^{s-1} = G^{s-1} \setminus H^{s-1}$ and $H^{s-1} = G^*$ that $B^{s-1} \cup (G \setminus G^{s-1}) = G \setminus G^*$. Therefore, we obtain the s-th face (7) written by G^*. □

We show that SDP obtained by EMSSOSP(G) is equivalent to an SDP obtained by replacing $\mathbb{S}_+^{\#(G)}$ by $\mathcal{F}_s$.

From (1), the SDP obtained by EMSSOSP(G) is

$$\sup\ 0\ \text{ subj. to }\ f_\alpha = \tilde{E}_\alpha \bullet \tilde{V}\ (\alpha \in G^* + G^*),\ \tilde{V} \in \mathbb{S}_+^{\#(G^*)}, \tag{9}$$

where $\tilde{E}_\alpha := (E_\alpha)_{G^*,G^*}$.

On the other hand, we generate an SDP by replacing $\mathbb{S}_+^{\#(G)}$ by $\mathcal{F}_s$ for SDP (4), we obtain the following SDP:

$$\sup\ 0\ \text{ subj. to }\ f_\alpha = E_\alpha \bullet V\ (\alpha \in G + G),\ V \in \mathcal{F}_s. \tag{10}$$

Then, the feasible region of SDP (10) is equivalent to that of SDP (4). From the form of $V \in \mathcal{F}_s$, we obtain the following SDP:

$$\sup\ 0\ \text{ subj. to }\ f_\alpha = E_\alpha \bullet \begin{pmatrix} \tilde{V} & O \\ O & O \end{pmatrix}\ (\alpha \in G + G),\ \tilde{V} \in \mathbb{S}_+^{\#(G^*)}. \tag{11}$$

For $\alpha \in (G+G) \setminus (G^* + G^*)$, from the definition of E_α, $(E_\alpha)_{\beta,\gamma} = 0$ for all $\beta, \gamma \in G^*$, and thus the linear equalities $f_\alpha = E_\alpha \bullet V$ for all $\alpha \in (G+G) \setminus (G^* + G^*)$ are equal to $f_\alpha = 0$. On the other hand, because $F \subseteq G^* + G^*$, $f_\alpha = 0$ for all $\alpha \in (G+G) \setminus (G^* + G^*)$, and thus these equalities are trivial in SDP (11). It follows from this discussion that SDP (11) is equivalent with SDP (9). Consequently, we conclude that the SDP obtained by EMSSOSP(G) is equivalent to the SDP obtained by replacing $\mathbb{S}_+^{\#(G)}$ by $\mathcal{F}_s$.

Note that FRA may generate a smaller SDP than EMSSOSP(G). We give such an example.

Example 3.4. We consider $f = x_1^4 - 2x_1^2x_2^2 + x_2^4$. Applying EMSSOSP($\mathbb{N}_2^2$) into f, we obtain $G^* = \{(2,0),(0,2),(1,1)\}$and the following SDP from SDP (9):

$$\begin{cases} \sup & 0 \\ \text{subj. to} & 1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet V, 0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bullet V, \\ & -2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bullet V, 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \bullet V, \\ & 1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet V, V \in \mathbb{S}_+^{\#(G^*)}. \end{cases} \tag{12}$$

The set (5) is

$$\left\{ W \in \mathbb{S}^{\#(G^*)} \;\middle|\; \begin{array}{c} W = (y_{\alpha+\beta})_{\alpha,\beta \in G^*}, \\ y_{(4,0)} - 2y_{(2,2)} + y_{(0,4)} = 0 \end{array} \quad \text{for some } y_\alpha \right\}.$$

For W in the set (5), we define y_α as follows:

$$y_{(4,0)} = y_{(0,4)} = y_{(2,2)} = 1 \text{ and } y_{(3,1)} = y_{(1,3)} = 0$$

Then we have

$$W = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \ker A \cap \ker c^T \cap \mathbb{S}_+^{\#(G^*)}.$$

The face $\mathcal{F}$ generated by W is

$$\mathcal{F} \quad = \quad \left\{ V \in \mathbb{S}_+^{\#(G^*)} \;\middle|\; V \bullet \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0 \right\},$$

which is smaller than $\mathbb{S}_+^{\#(G^*)}$. By replacing $\mathbb{S}_+^{\#(G^*)}$ by $\mathcal{F}$ for SDP (12) and removing trivial equalities, we obtain the following SDP:

$$\begin{cases} \sup & 0 \\ \text{subj. to} & \tilde{V}_{(2,0),(2,0)} = 1, \tilde{V}_{(2,0),(0,2)} = -1, \tilde{V}_{(0,2),(0,2)} = 1 \\ & \tilde{V}_{(2,0),(2,0)} + 2\tilde{V}_{(2,0),(0,2)} + \tilde{V}_{(0,2),(0,2)} = 0 \\ & \tilde{V} = \begin{pmatrix} \tilde{V}_{(2,0),(2,0)} & \tilde{V}_{(2,0),(0,2)} \\ \tilde{V}_{(2,0),(0,2)} & \tilde{V}_{(0,2),(0,2)} \end{pmatrix} \in \mathbb{S}_+^2. \end{cases} \tag{13}$$

From SDP (13), we can find an SOS representation of f which is $f = (x_1^2 - x_2^2)^2$. From this example, we see that FRA can find a smaller set $\tilde{G} = \{(2,0),(0,2)\}$ than $G^* = \{(2,0),(0,2),(1,1)\}$. The reason is that EMSSOSP(G) uses only the information of the support F of f, while FRA exploits the information of the coefficients of f.

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Appendix A. A proof of Theorem 2.5

We define a set family Γ as follows:

(i) $\bar{G} := \frac{1}{2}\mathrm{conv}(F^e) \cap \mathbb{N}^n \in \Gamma$.
(ii) if $G \in \Gamma$ and there exist nonempty sets $B, H \subseteq G$ such that the triplet (B, H, G) satisfies (2), $H \in \Gamma$.

In [3], the authors proved that if, $G, G' \in \Gamma$, then $G \cap G' \in \Gamma$. This implies the existence of the smallest finite set $\check{G}$ in Γ in the sense that $\check{G} \subseteq G$ for any $G \in \Gamma$. From the definition of Γ, we can construct the triplets (B^i, H^i, G^i) for all $i = 0, \ldots, s-1$ which satisfy (2) and we have $G^0 = \bar{G}$ and $G^s = \check{G}$. Therefore, EMSSOSP($\bar{G}$) can return $\check{G}$. It is not difficult to verify by using Lemma 3.3 in [3] that the set G^* returned by EMSSOSP($\bar{G}$) is always $\check{G}$.

To prove Theorem 2.5, we fix $G^0 \supseteq \bar{G}$ and replace $\bar{G}$ in the definition of Γ by G^0. We define Γ' as follows:

(i) $G^0 \in \Gamma'$.
(ii) if $G \in \Gamma'$ and there exist nonempty sets $B, H \subseteq G$ such that the triplet (B, H, G) satisfies (2), $H \in \Gamma'$.

By applying a similar argument in the proofs of Lemma 3.3 and Theorem 3.1 in [3], we can prove that the existence of the smallest finite set $\hat{G} \in \Gamma'$. Furthermore, using the same argument as the case of EMSSOSP($\bar{G}$), we can see that EMSSOSP(G^0) returns $\hat{G}$. This implies that for Theorem 2.5, it is sufficient to prove $G^* = \hat{G}$.

To prove $G^* \subseteq \hat{G}$, we use the following lemma:

Lemma A.1. *Assume that $\emptyset \neq B \subseteq P, G \subseteq P$, $B \cap G \neq \emptyset$ and that the triplet (B, H, P) satisfies*

$$\begin{cases} H \subseteq P, B \subseteq P, P = H \cup B, \\ (B + B) \cap F = \emptyset \textit{ and } (B + B) \cap (P + H) = \emptyset. \end{cases} \tag{14}$$

Then the triplet $(B \cap G, H \cap G, G)$ satisfies (2).

Proof: It is sufficient to prove $G = (H \cap G) \cup (B \cap G)$, $((B \cap G) + (B \cap G)) \cap F = \emptyset$ and $((B \cap G) + (B \cap G)) \cap (G + (H \cap G)) = \emptyset$. We omit the proofs because it is easy to check these equalities. □

For the triplet (B, H, G^0) satisfying (14), if $B \cap \bar{G} \neq \emptyset$, we can remove at least $B \cap \bar{G}$ from $\bar{G}$ and thus $\bar{G} \setminus (B \cap \bar{G}) \subseteq G^0 \setminus B$. Otherwise, we have $\bar{G} \subseteq G^0 \setminus B$. These imply that the resulting set obtained by the first iteration of EMSSOSP($\bar{G}$) is included in the resulting set obtained by the first iteration of EMSSOSP(G^0) because $\bar{G} \subseteq G^0$. By applying Lemma A.1 into these sets repeatedly, we have $G^* \subseteq \hat{G}$.

On the other hand, to prove $G^* \supseteq \hat{G}$, it is sufficient to show that $\Gamma \subseteq \Gamma'$. From the definition of Γ', if $\bar{G} \in \Gamma'$, then $\Gamma \subseteq \Gamma'$. To prove $\bar{G} \in \Gamma'$, we use Algorithm A.3 based on the following lemma. Because we have assumed that f is an SOS polynomial, it follows from Theorem 2.2 that $F \subseteq \mathrm{conv}(F^e)$.

Lemma A.2. *Assume that there exist nonempty sets $B, H \subseteq G$ such that the triplet (B, H, G) satisfies*

$$\begin{cases} H \subseteq G, B \subseteq G, G = H \cup B, \\ (B + B) \cap \textit{conv}(F^e) = \emptyset \textit{ and } (B + B) \cap (G + H) = \emptyset. \end{cases} \tag{15}$$

Then the triplet (B, H, G) also satisfies (2).

Proof : Because of $F \subseteq \mathrm{conv}(F^e)$, the triplet (B, H, G) satisfies (2). □

We give an algorithm to find $\bar{G}$ from G^0.

Algorithm A.3. *(The restricted version of EMSSOSP(G^0))*

- **Step 1::** *Set $i = 0$.*
- **Step 2::** *If there do not exist $B^i, H^i \subseteq G^i$ such that the triplet (B^i, H^i, G^i) satisfies (15), then stop and return G^i.*
- **Step 3::** *Otherwise set $G^{i+1} = H^i$ and $i = i + 1$, and go back to Step 2.*

We assume that Algorithm A.3 requires s iterations. Then it generates triplets (B^i, H^i, G^i) for $i = 0, 1, \ldots, s-1$ which satisfy (15) and we obtain a finite set G^s. From Lemma A.2 and the definition of Γ', it follows that $G^i \in \Gamma'$ for all $i = 0, 1, \ldots, s$. The following proposition ensures that $\bar{G} \in \Gamma'$.

Proposition A.4. *We have $G^s = \bar{G}$.*

Proof : It follows from assumption (15) in Lemma A.2 that $\frac{1}{2}\mathrm{conv}(F^e) \cap \mathbb{N}^n \subseteq G^i$ for all $i = 0, \ldots, s$. Thus $\bar{G} \subseteq G^i$ for all $i = 0, 1, \ldots, s$.

We prove $G^s \subseteq \bar{G}$. Let $\alpha \in G^s$ be a vertex of $\mathrm{conv}(G^s)$. Then $\alpha \in \mathbb{N}^n$ and $2\alpha \in \mathrm{conv}(F^e)$. Indeed, because $(B, H, G) = (\{\alpha\}, G^s \setminus \{\alpha\}, G^s)$ does not satisfy (15), we obtain the condition $2\alpha \in \mathrm{conv}(F^e)$ or $2\alpha \in G^s + G^s \setminus \{\alpha\}$. The latter does not hold because α is a vertex of $\mathrm{conv}(G^s)$. This shows $2\alpha \in \mathrm{conv}(F^e)$. This implies $2\mathrm{conv}(G^s) \subseteq \mathrm{conv}(F^e)$, and thus $2G^s \subseteq \mathrm{conv}(F^e)$. Therefore, we obtain $G^s \subseteq \frac{1}{2}\mathrm{conv}(F^e) \cap \mathbb{N}^n = \bar{G}$. □

From Proposition A.4, we have $\bar{G} \in \Gamma'$, and thus $\bar{G} \subseteq G^*$. This completes to prove Theorem 2.5.