Most Tensor Problems are NP-Hard

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We show that multilinear (tensor) analogues of many efficiently computable problems in numerical linear algebra are NP-hard. Our list here includes: determining the feasibility of a system of bilinear equations, deciding whether a tensor possesses a given eigenvalue, singular value, or spectral norm; approximating an eigenvalue, eigenvector, singular vector, or spectral norm; determining a best rank-1 approximation to a tensor; and determining the rank of a tensor. Additionally, we prove that some of these problems have no polynomial time approximation schemes, some are undecidable over $\mathbb{Q}$, and at least one enumerative version is #P-complete. We also show that restricting these problems to symmetric tensors does not alleviate their NP-hardness and that the problem of deciding nonnegative definiteness of a symmetric 4-tensor is also NP-hard. Except for this nonnegative definiteness problem, all our results apply to 3-tensors. We shall argue that these 3-tensor problems may be viewed as the boundary separating the computational tractability of linear/convex problems from the intractability of nonlinear/nonconvex ones.

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1. INTRODUCTION

Frequently a problem in science or engineering can be reduced to solving a linear (matrix) system of equations and inequalities. Other times, solutions involve the extraction of certain quantities from matrices such as eigenvectors or singular values. In computer vision, for instance, segmentations of a digital picture along object boundaries can be found by computing the top eigenvectors of a certain matrix produced from the image [Shi and Malik 2000]. Another common problem formulation is to find low-rank matrix approximations that explain a given two-dimensional array of data, accomplished, as is now standard, by zeroing the smallest singular values in a singular value decomposition (SVD) of the array [Golub and Kahan 1965; Golub and Reinsch 1970]. In general, efficient and reliable routines

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computing answers to these and similar problems of linear algebra have been a workhorse for real-world applications of computation.

Recently, there has been a flurry of work on multilinear analogues to these problems. These "tensor methods" have found applications in many fields, including computational biology, computer vision, data analysis, machine learning, neuroimaging, quantum computing, scientific computing, signal processing, spectroscopy, and others (examples include [Allman and Rhodes 2008; Brubaker and Vempala 2009; Cartwright et al. 2009; Comon 2004; 1994; Coppi and Bolasco 1989; De La Vega et al. 2005; Friedman 1991; Friedman and Wigderson 1995; Huggins et al. 2008; Kofidis and Regalia 2001/02; Miyake and Wadati 2002; Shashua and Hazan 2005; Schultz and Seidel 2008; Sidiropoulos et al. 2000; Smilde et al. 2004; Vasilescu 2002; Vasilescu and Terzopoulos 2002; 2004]; see also the 244 references in the survey [Kolda and Bader 2009]). Thus, tensor generalizations to the basic algorithms of linear algebra have the potential to substantially enlarge the arsenal of core tools in numerical computation.

The main results of this paper, however, support the view that tensor problems are almost invariably computationally hard. Indeed, we shall prove that many naturally occurring problems for 3-tensors are NP-hard; that is, unless $P = NP$, there are no polynomial-time algorithms to solve them. A full list of the problems we study can be found in Table II below. Since we deal with mathematical questions over fields (such as the real numbers $\mathbb{R}$), algorithmic complexity is a somewhat subtle notion. Our perspective here will be the Turing model of computation and the Cook-Karp-Levin model of complexity involving NP-hard and NP-complete problems [Cook 1971; Karp 1972; Levin 1973], as opposed to other computational models over fields [Turing 1936; Weihrauch 2000; Valiant 1979a; Blum et al. 1989]. We explain our framework in Subsection 1.3 along with a comparison to other models.

Another interpretation of these findings is that, in the sense of computational tractability, 3-tensor problems form the boundary separating linear/convex problems from nonlinear/nonconvex ones. More specifically, linear algebra is concerned with vector-valued functions that are locally of the form $f(\mathbf{x}) = \mathbf{b} + A\mathbf{x}$; while convex analysis deals with scalar-valued functions that are locally $f(\mathbf{x}) = c + \mathbf{b}^\top \mathbf{x} + \mathbf{x}^\top A\mathbf{x}$ with $A \succeq 0$. These functions involve tensors of order 0, 1, and 2: $c \in \mathbb{R}$, $\mathbf{b} \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n\times n}$. However, as soon as we move on to bilinear vector-valued or trilinear real-valued functions (the next Taylor expansion term for f), we encounter 3-tensors $\mathcal{A} \in \mathbb{R}^{n\times n\times n}$ and NP-hardness.

The primary audience for this article are numerical analysts and computational algebraists, although we hope that it will be of interest to users of tensor methods in various communities. Parts of our exposition contain standard material (e.g., complexity theory to computer scientists, hyperderminants to algebraic geometers, Lagrangians to optimization theorists, etc.), but to appeal to the widest possible audience at the intersection of computer science, linear and multilinear algebra, algebraic geometry, numerical analysis, and optimization, we have tried to keep our discussion as self-contained as possible. A side contribution of this article is a useful framework for incorporating features of computation over $\mathbb{R}$ and $\mathbb{C}$ with classical tools and models of algorithmic complexity involving Turing machines that we think is unlike any existing treatments [Blum et al. 1989; Blum et al. 1998; Hochbaum and Shanthikumar 1990; Vavasis 1991].

1.1. Tensors

Let us first define our basic mathematical objects. Fix a field $\mathbb{F}$, which for us will be either the rationals $\mathbb{Q}$, the reals $\mathbb{R}$, or the complex numbers $\mathbb{C}$. Also, let l, m, and n be positive integers. A 3-*tensor* $\mathcal{A}$ over $\mathbb{F}$ is an $l \times m \times n$ array of elements of $\mathbb{F}$:

$$\mathcal{A} = [\![a_{ijk}]\!]_{i,j,k=1}^{l,m,n} \in \mathbb{F}^{l\times m\times n}. \tag{1}$$

These objects are natural multilinear generalizations of matrices in the following way.

For any positive integer d, let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ denote the standard column basis[1] in the $\mathbb{F}$-vector space $\mathbb{F}^d$. A bilinear function $f : \mathbb{F}^m \times \mathbb{F}^n \to \mathbb{F}$ can be encoded by a matrix $A = [a_{ij}]_{i,j=1}^{m,n} \in \mathbb{F}^{m\times n}$, in which the entry a_{ij} records the value of $f(\mathbf{e}_i, \mathbf{e}_j) \in \mathbb{F}^m \times \mathbb{F}^n$. By linearity in each coordinate, specifying A determines the values of f on all of $\mathbb{F}^m \times \mathbb{F}^n$; in fact, we have $f(\mathbf{u}, \mathbf{v}) = \mathbf{u}^\top A\mathbf{v}$ for any column vectors $\mathbf{u} \in \mathbb{F}^m$ and $\mathbf{v} \in \mathbb{F}^n$. Thus, matrices both encode 2-dimensional arrays of numbers and specify all bilinear functions. Notice also that if $m = n$ and $A = A^\top$ is symmetric, then

$$f(\mathbf{u}, \mathbf{v}) = \mathbf{u}^\top A\mathbf{v} = (\mathbf{u}^\top A\mathbf{v})^\top = \mathbf{v}^\top A^\top \mathbf{u} = \mathbf{v}^\top A\mathbf{u} = f(\mathbf{v}, \mathbf{u}).$$

Thus, symmetric matrices encode bilinear maps invariant under exchanging of coordinates.

These notions generalize: a 3-tensor is a trilinear function $f : \mathbb{F}^l \times \mathbb{F}^m \times \mathbb{F}^n \to \mathbb{F}$ which has a coordinate representation given by a *hypermatrix*[2] $\mathcal{A}$ as in (1). The subscripts and superscripts in (1) will be dropped whenever the range of i, j, k is obvious or unimportant. Also, a 3-tensor $[\![a_{ijk}]\!]_{i,j,k=1}^{n,n,n} \in \mathbb{F}^{n\times n\times n}$ is *symmetric* if

$$a_{ijk} = a_{ikj} = a_{jik} = a_{jki} = a_{kij} = a_{kji}.$$

These objects correspond to coordinate representations of trilinear maps $f : \mathbb{F}^n \times \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ with

$$f(\mathbf{u}, \mathbf{v}, \mathbf{w}) = f(\mathbf{u}, \mathbf{w}, \mathbf{v}) = f(\mathbf{v}, \mathbf{u}, \mathbf{w}) = f(\mathbf{v}, \mathbf{w}, \mathbf{u}) = f(\mathbf{w}, \mathbf{u}, \mathbf{v}) = f(\mathbf{w}, \mathbf{v}, \mathbf{u}).$$

We focus here on 3-tensors mainly for expositional purposes. A notable exception is the problem of deciding positive definiteness of a tensor, a notion nontrivial only in even orders.

1.2. Tensor Eigenvalue

We now explain in detail our findings for the tensor eigenvalue problem since it is the simplest multilinear generalization. We shall also use the problem to illustrate many of the concepts that arise when studying other, more difficult, tensor problems. The basic notions for eigenvalues of tensors were introduced independently in [Lim 2005] and [Qi 2005], with more developments appearing in [Ni et al. 2007; Qi 2007]. Additional theory from the perspective of toric algebraic geometry and intersection theory was provided recently in [Cartwright and Sturmfels 2012]. We will describe the ideas more formally in Section 4, but for now it suffices to say that the usual eigenvalues and eigenvectors of a matrix $A \in \mathbb{R}^{n\times n}$ are the stationary values and points of its Rayleigh quotient, and this point of view generalizes to higher order tensors. This gives, for example, an *eigenvector* of a tensor $\mathcal{A} = [\![a_{ijk}]\!]_{i,j,k=1}^{n,n,n} \in \mathbb{F}^{n\times n\times n}$ as a nonzero column vector $\mathbf{x} = [x_1, \ldots, x_n]^\top \in \mathbb{F}^n$ satisfying:

$$\sum_{i,j=1}^{n} a_{ijk} x_i x_j = \lambda x_k, \quad k = 1, \ldots, n, \tag{2}$$

for some $\lambda \in \mathbb{F}$, which is called an *eigenvalue* of $\mathcal{A}$. Notice that if $(\lambda, \mathbf{x})$ is an eigenpair, then so is $(t\lambda, t\mathbf{x})$ for any $t \neq 0$; thus, eigenpairs are more naturally defined projectively.

As in the matrix case, generic tensors over $\mathbb{F} = \mathbb{C}$ have a finite number of eigenvalues and eigenvectors (up to this scaling equivalence), although their count is exponential in n. Still, it is possible for a tensor to have an infinite number of eigenvalues, but in that case they comprise a cofinite set of complex numbers. Another important fact is that over the reals ($\mathbb{F} = \mathbb{R}$), every 3-tensor has a real eigenpair $(\lambda, \mathbf{x})$. All of these results and more can be found in [Cartwright and Sturmfels 2012]. The following problem is natural for applications.

[1]Formally, $\mathbf{e}_i$ is the column vector in $\mathbb{F}^d$ with a 1 in the ith coordinate and zeroes everywhere else. In this article, vectors in $\mathbb{F}^n$ will always be columns.

[2]We will not use the term hypermatrix but will simply regard a tensor as synonymous with its coordinate representation.

Table I. Decidability of Tensor Problems

Problem	Decidability
Bilinear System over $\mathbb{Q}$	Undecidable (Theorems 2.4, 3.9)
Eigenvalue over $\mathbb{Q}$	Undecidable (Theorem 1.2)
Singular Value over $\mathbb{Q}$	Undecidable (Theorem 5.2)
Rank over $\mathbb{Q}$	Conjecture 11.3

Note: All problems refer to the 3-tensor case.

PROBLEM 1.1. *Given $\mathcal{A} \in \mathbb{F}^{n\times n\times n}$, determine $(\lambda, \mathbf{x}) \in \mathbb{F} \times \mathbb{F}^n$ with $\mathbf{x}$ nonzero such that* (2) *holds.*

We first discuss the computability of this problem. When the entries of the tensor $\mathcal{A}$ are real numbers ($\mathbb{F} = \mathbb{R}$), there is an effective procedure that will output a finite presentation of all real eigenpairs. A good reference for such methods in real algebraic geometry is [Bochnak et al. 1998], and an overview of recent intersections between mathematical logic and algebraic geometry, more generally, can be found in [Haskell et al. 2000]. Over $\mathbb{C}$, these two problems can be tackled directly by computing a Gröbner basis with Buchberger's algorithm since an eigenpair is a solution to a system of polynomial equations over an algebraically closed field (e.g. [Cartwright and Sturmfels 2012, Example 3.5]). Yet another approach is to work with Macaulay matrices of multivariate resultants [Ni et al. 2007]. References for such techniques suitable for numerical analysts are [Cox et al. 2007; 2005].

Although computable over $\mathbb{R}$ and $\mathbb{C}$, Problem 1.1 over the rational numbers is undecidable.

THEOREM 1.2. *Determining whether zero is an eigenvalue with rational eigenvector of a rational tensor is undecidable.*

We give a proof of Theorem 1.2 in Section 2, but it will be a straightforward consequence of the seminal works [Davis et al. 1961; Matijasevič 1970; Jones 1982; Jones and Matijasevič 1984] on Hilbert's 10th Problem. In fact, we will show that three of the four problems in Table I are undecidable.

Even when tensor problems are computable, however, all known methods quickly become impractical as the tensors become larger (i.e., as n grows). In principle, this occurs because simply listing the output to Problem 1.1 is already prohibitive. It is natural, therefore, to ask for faster methods checking whether a given $\lambda \in \mathbb{F}$ is an eigenvalue or approximating a single eigenpair. We first analyze the following easier decision problem.

PROBLEM 1.3 (TENSOR λ-EIGENVALUE). *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and fix $\lambda \in \mathbb{F}$. Decide if λ is an eigenvalue (with corresponding eigenvector in $\mathbb{F}^n$) of a tensor $\mathcal{A} \in \mathbb{F}^{n\times n\times n}$.*

Before explaining our results on Problem 1.3 and other tensor questions, we define the model of computational complexity that we shall utilize to study them.

1.3. Computational Complexity

We hope this article will be useful to casual users of computational complexity — such as numerical analysts and optimization theorists — who nevertheless desire to understand the tractability of their problems in light of modern complexity theory. This section and the next provide a high-level overview for such an audience. In addition, we also carve out a perspective for real computation within the Turing machine framework that we feel is easier to work with than those proposed in [Blum et al. 1989; Blum et al. 1998; Hochbaum and Shanthikumar 1990; Vavasis 1991]. For readers who have no particular interest in tensor problems, the remainder of our article may then be viewed as a series of instructive examples showing how one may deduce the tractability of a numerical computing problem from the rich collection of NP-complete combinatorial problems.

Computational complexity is usually specified on the following three levels.

I. *Model of Computation*: What are inputs and outputs? What is a computation? For us, inputs will be rational tensors and outputs will be rational vectors or FEASIBLE/INFEASIBLE responses. Also, computations are performed on a **Turing Machine**. Alternatives for inputs include Turing computable numbers [Turing 1936; Weihrauch 2000] and alternatives for computation include the Blum-Shub-Smale Machine [Blum et al. 1989; Blum et al. 1998] and the Quantum Turing Machine [Deutsch 1985].

II. *Model of Complexity*: What is the cost of a computation? Here, it is the **time complexity** measured in units of bit operations; i.e., the number of READ, WRITE, MOVE, and other tape-level instructions on bits. This is the same for the ε-accuracy complexity model[3] [Hochbaum and Shanthikumar 1990]. In the Blum-Cucker-Shub-Smale (BCSS) model, it is time complexity measured in units of arithmetic and branching operations on lists of n real or complex numbers. In quantum computing, it is time complexity measured in units of unitary operations on qubits. There are yet other models of complexity that measure other types of computational costs. For example, the Valiant model is based on arithmetic circuit complexity.

III. *Model of Reducibility*: Which problems do we consider equivalent in hardness? For us, it is the **Cook-Karp-Levin** (CKL) sense of reducibility [Cook 1971; Karp 1972; Levin 1973] and its corresponding problem classes: P, NP, NP-complete, NP-hard, etc. Reducibility in the BCSS model is essentially based on CKL. There is also reducibility in the Valiant sense, which applies to the aforementioned Valiant model and gives rise to the complexity classes *VP* and *VNP* [Valiant 1979a; Bürgisser 2000].

Computability is a question to be answered in Level I, whereas hardness is a question to be answered in Levels II and III. In Level II, we have restricted ourselves to time complexity since this is the most basic measure. We do not think it is necessary to consider other measures like space complexity, communication complexity, circuit complexity, etc, when consideration of time complexity alone already reveals that these tensor problems are hard.

Before describing our model more fully, we recall the well-known Blum-Cucker-Shub-Smale framework for computing with real and complex numbers [Blum et al. 1989; Blum et al. 1998]. In this model, an input is a finite list of n real or complex numbers, without regard to how they are represented. In this case, algorithmic computation (essentially) corresponds to arithmetic and branching on equality using a finite number of states, and a measure of computational complexity is the number of these basic operations[4] needed to solve a problem as a function of n. The central message of our paper is that many problems in linear algebra that are efficiently solvable on a Turing machine become NP-hard in multilinear algebra. Under the BCSS model, however, this distinction is not yet possible. For example, while it is well-known that the feasibility of a linear program is in P under the traditional CKL notion of complexity [Khachiyan 1979], the same problem studied within BCSS is among the most daunting open problems in Mathematics (it is the 9th "Smale Problem" [Smale 2000]). The BCSS model has nonetheless produced significant contributions to computational mathematics, especially to the theory of polynomial root approximation (e.g., see [Beltrán and Pardo 2009] and the references therein).

[3]While the ε-accuracy complexity model is more realistic for numerical computations, it is not based on the IEEE floating-point standards [Kahan 1997; Overton 2001]. On the other hand, a model that combines both the flexibility of the ε-accuracy complexity model and the reality of floating-point arithmetic would inevitably be enormously complicated [Vavasis 1991, Section 2.4].

[4]To illustrate the difference between BCSS/CKL, consider the problem of deciding whether two integers r, s multiply to give an integer t. For BCSS, the time complexity is constant since one can compute $u := rs - t$ and check "$u = 0$?" in constant time. Under CKL, however, the problem has best-known time complexity of $N \log(N) 2^{O(\log^* N)}$, where N is the number of bits to specify r, s, and t [Fürer 2007; De et al. 2008].

We now explain our model of computation. First of all, inputs will always be rational numbers[5] and specified by finite length strings of bits. Also, all computations are assumed to be performed on a Turing machine with the standard notion of time complexity involving operations on bits, and outputs will consist of rational numbers or FEASIBLE/INFEASIBLE responses. We note that although quantities such as eigenvalues, singular values, spectral norms, etc., of a tensor will in general not be rational, our reductions have been carefully constructed such that they are rational (or at least finite bit-length) in the cases we study.

The next subsection describes our notion of reducibility for tensor decision problems (such as Problem 1.3 encountered above).

1.4. NP-hardness

The following is the notion of NP-hardness for decision problems that we shall use throughout this paper. As described above, input tensors $\mathcal{A}$ will always consist of rational numbers, and input size is measured in the number of bits required to specify the input. Briefly, we say that a decision problem $\mathscr{D}_1$ is *polynomially reducible* to a decision problem $\mathscr{D}_2$ if the following holds: any input to $\mathscr{D}_1$ can be transformed in polynomially many steps (in the input size) into a set of polynomially larger inputs to $\mathscr{D}_2$ problems such that the corresponding answers can be used to correctly deduce (again, in a polynomial number of steps) the answer to the original $\mathscr{D}_1$ question. Furthermore, we call a decision problem *NP-hard* if one can polynomially reduce any NP-complete decision problem (such as whether a graph is 3-colorable) to it. Of course, by the Cook-Levin Theorem, if one can polynomially reduce any particular NP-complete problem to a decision problem $\mathscr{D}$, then all NP-complete problems are so reducible to $\mathscr{D}$.

Although tensor eigenvalue is computable for $\mathbb{F} = \mathbb{R}$, it is nonetheless NP-hard, as our next theorem explains. For two sample reductions, see Example 1.5 below.

THEOREM 1.4. *Graph* 3*-colorability is polynomially reducible to tensor* 0*-eigenvalue over* $\mathbb{R}$. *Thus, deciding tensor eigenvalue over* $\mathbb{R}$ *is NP-hard.*

Note that NP-hard problems may not be in the class *NP*. A basic open question is whether deciding tensor eigenvalue is also NP-complete. In other words, if a solution to (2) exists, is there a polynomial-time verifiable certificate of this fact? A natural candidate for the certificate is the eigenvector itself, whose coordinates would be, in general, represented as certain zeroes of univariate polynomials with rational coefficients. The relationship between the size of these coefficients and the size of the (rational) input, however, is subtle and beyond our scope. For instance, only recently has the relationship between these quantities been worked out in the case of a set of homogenous linear equations [Freitas et al. 2011].

Example 1.5 (*Real tensor* 0*-eigenvalue solves* 3*-Colorability*). Let $G = (V, E)$ be a simple, undirected graph with vertices $V = \{1, \dots, n\}$ and edges E. Recall that a *proper* (*vertex*) 3*-coloring* of G is an assignment of one of three colors to each of its vertices such that adjacent vertices receive different colors. We also say that G is 3*-colorable* if it has a proper 3-coloring. Figure 1 contains examples. Determining whether a graph G is 3-colorable is a well-known NP-complete decision problem.

As we shall see below in Section 2, the proper 3-colorings of the left-hand side graph G in Figure 1 can be encoded as the nonzero real solutions to the following square set of $n = 35$

[5]The only exception is when we prove NP-hardness of symmetric tensor eigenvalue (Section 8), where we allow input eigenvalues λ to be in the field $\mathbb{F} = \{a + b\sqrt{d} : a, b \in \mathbb{Q}\}$ for a positive integer d. Note that such inputs may also be specified in finite bit-length.

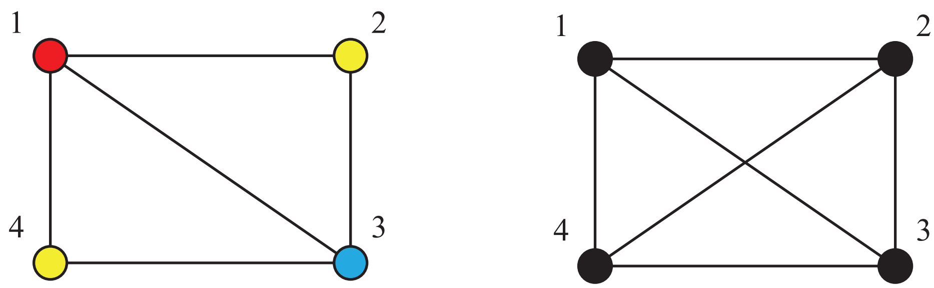

Fig. 1. Simple graphs with six proper 3-colorings (graph at the left) or none (graph at the right).

quadratic polynomials in 35 real unknowns a_i, b_i, c_i, d_i $(i = 1, \dots, 4)$, u, w_i $(i = 1, \dots, 18)$:

$$\begin{aligned}
&a_1c_1 - b_1d_1 - u^2,\ b_1c_1 + a_1d_1,\ c_1u - a_1^2 + b_1^2,\ d_1u - 2a_1b_1,\ a_1u - c_1^2 + d_1^2,\ b_1u - 2d_1c_1,\\
&a_2c_2 - b_2d_2 - u^2,\ b_2c_2 + a_2d_2,\ c_2u - a_2^2 + b_2^2,\ d_2u - 2a_2b_2,\ a_2u - c_2^2 + d_2^2,\ b_2u - 2d_2c_2,\\
&a_3c_3 - b_3d_3 - u^2,\ b_3c_3 + a_3d_3,\ c_3u - a_3^2 + b_3^2,\ d_3u - 2a_3b_3,\ a_3u - c_3^2 + d_3^2,\ b_3u - 2d_3c_3,\\
&a_4c_4 - b_4d_4 - u^2,\ b_4c_4 + a_4d_4,\ c_4u - a_4^2 + b_4^2,\ d_4u - 2a_4b_4,\ a_4u - c_4^2 + d_4^2,\ b_4u - 2d_4c_4,\\
&a_1^2 - b_1^2 + a_1a_3 - b_1b_3 + a_3^2 - b_3^2,\ a_1^2 - b_1^2 + a_1a_4 - b_1b_4 + a_4^2 - b_4^2,\ a_1^2 - b_1^2 + a_1a_2 - b_1b_2 + a_2^2 - b_2^2,\\
&a_2^2 - b_2^2 + a_2a_3 - b_2b_3 + a_3^2 - b_3^2,\ a_3^2 - b_3^2 + a_3a_4 - b_3b_4 + a_4^2 - b_4^2,\ 2a_1b_1 + a_1b_2 + a_2b_1 + 2a_2b_2,\\
&2a_2b_2 + a_2b_3 + a_3b_2 + 2a_3b_3,\ 2a_1b_1 + a_1b_3 + a_2b_1 + 2a_3b_3,\ 2a_1b_1 + a_1b_4 + a_4b_1 + 2a_4b_4,\\
&2a_3b_3 + a_3b_4 + a_4b_3 + 2a_4b_4,\ w_1^2 + w_2^2 + \cdots + w_{17}^2 + w_{18}^2.
\end{aligned} \tag{3}$$

Using symbolic algebra or numerical algebraic geometry software (see the Appendix for a list), one can solve these equations to find six real solutions (without loss of generality, we may take $u = 1$ and all $w_j = 0$), which correspond to the proper 3-colorings of the graph G as follows. Fix one such solution and define $x_k := a_k + \mathrm{i}b_k \in \mathbb{C}$ for $k = 1, \dots, 4$ (we set $\mathrm{i} := \sqrt{-1}$). By construction, these x_k are one of the three cube roots of unity $\{1, \alpha, \alpha^2\}$ where $\alpha = \exp(2\pi\mathrm{i}/3) = -\frac{1}{2} + \frac{\sqrt{3}}{2}$; see Figure 4.

To determine a 3-coloring of G from this solution, one "colors" each vertex i by the root of unity that equals x_i. One can check that no two adjacent vertices share the same color in a coloring; thus, they represent proper 3-colorings of G. For example, one solution is:

$$x_1 = -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}, \quad x_2 = 1, \quad x_3 = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}, \quad x_4 = 1.$$

Polynomials for the right-hand side graph in Figure 1 are the same as (3) except for two additional ones for encoding a new restriction for colorings, the extra edge $\{2, 4\}$:

$$a_2^2 - b_2^2 + a_2a_4 - b_2b_4 + a_4^2 - b_4^2, \quad 2a_2b_2 + a_2b_4 + a_4b_2 + 2a_4b_4.$$

In this case, one can check with the same software (and also certify algebraically) that these extra equations force the system to have no nonzero real solutions, and thus no proper 3-colorings.

Finally, note that since equivalence classes of eigenvectors correspond to proper 3-colorings in this reduction, if we could count real (projective) eigenvectors with eigenvalue $\lambda = 0$, we would solve the #P enumeration problem for proper 3-colorings of graphs (in particular, this proves Corollary 1.17 below). $\square$

Table II. Complexity of Tensor Problems

Problem	Complexity
Bilinear System over $\mathbb{R}$, $\mathbb{C}$	NP-hard (Theorems 1.2, 3.8, 3.7)
Eigenvalue over $\mathbb{R}$	NP-hard (Theorem 1.4)
Symmetric Eigenvalue over $\mathbb{R}$	NP-hard (Theorem 8.3)
Approximating Symmetric Eigenvalue over $\mathbb{R}$	NP-hard (Theorem 8.6)
Approximating Eigenvector over $\mathbb{R}$	NP-hard (Theorem 1.6)
Singular Value over $\mathbb{R}$, $\mathbb{C}$	NP-hard (Theorem 1.8)
Symmetric Singular Value over $\mathbb{R}$	NP-hard (Theorem 9.2)
Approximating Singular Vector over $\mathbb{R}$, $\mathbb{C}$	NP-hard (Theorem 5.4)
Spectral Norm	NP-hard (Theorem 1.11)
Symmetric Spectral Norm	NP-hard (Theorem 9.2)
Approximating Spectral Norm	NP-hard (Theorem 1.12)
Nonnegative Definiteness	NP-hard (Theorem 10.2)
Best Rank-1 Approximation	NP-hard (Theorem 1.14)
Best Symmetric Rank-1 Approximation	NP-hard (Theorem 9.2)
Rank over $\mathbb{R}$ or $\mathbb{C}$	NP-hard (Theorem 7.2)
Enumerating Eigenvectors over $\mathbb{R}$	#P-complete (Corollary 1.17)
Hyperdeterminant	Conjectures 1.10, 11.1
Symmetric Rank	Conjecture 11.2
Bilinear Programming	Conjecture 11.4
Bilinear Least Squares	Conjecture 11.5

Note: Except for positive definiteness, which applies to 4-tensors, all problems refer to the 3-tensor case.

1.5. Approximation Schemes

Although the tensor eigenvalue decision problem is NP-hard and the eigenvector enumeration problem #P-complete, it might be possible to approximate *some* eigenpair $(\lambda, \mathbf{x})$ efficiently. For those unfamiliar with these notions, an *approximation scheme* for a tensor problem (such as finding a tensor eigenvector) is an algorithm producing a (rational) approximate solution to within ε of some solution. An approximation scheme is said to run in *polynomial time* (PTAS) if its running-time is polynomial in the input size for any fixed $\varepsilon > 0$, and *fully polynomial time* (FPTAS) if its running-time is polynomial in both the input size and $1/\varepsilon$. There are other notions of approximation [Hochbaum 1997; Vazirani 2003], but we limit our discussion to these.

Fix $\lambda \in \mathbb{F}$, which we assume is an eigenvalue of a tensor $\mathcal{A}$. Formally, we say it is *NP-hard to approximate an eigenvector* $\mathbf{x} \in \mathbb{F}^n$ of an eigenpair $(\lambda, \mathbf{x})$ to within $\varepsilon > 0$ if (unless $P = NP$) there is no polynomial-time algorithm that always produces an approximate nonzero solution $\hat{\mathbf{x}} = [\hat{x}_1, \dots, \hat{x}_n]^\top \in \mathbb{F}^n$ to system (2) satisfying for all i:

$$|\hat{x}_i/\hat{x}_j - x_i/x_j| < \varepsilon, \quad \text{whenever } x_j \neq 0. \tag{4}$$

This measure of approximation is natural in our context because of the scale invariance of eigenpairs, and it is very related to standard relative error $\|\hat{x} - x\|_\infty / \|x\|_\infty$ in numerical analysis. We shall prove the following inapproximability result in Section 4.

THEOREM 1.6. *It is NP-hard to approximate tensor eigenvector over $\mathbb{R}$ to within $\varepsilon = \frac{3}{4}$.*

COROLLARY 1.7. *Unless $P = NP$, there is no PTAS approximating tensor eigenvector.*

1.6. Tensor Singular Value, Spectral Norm, and Hyperdeterminant

Our next results involve the singular value problem. We postpone definitions until Section 5, but state the main results here.

THEOREM 1.8. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and fix $\sigma \in \mathbb{Q}$. Deciding whether σ is a singular value over $\mathbb{F}$ of a tensor is NP-hard.*

There is also a notion of *hyperdeterminant*[6] for tensors, which we discuss in more depth in Section 3. Similar to the determinant, this is a homogeneous polynomial (with integer coefficients) in the entries of a tensor that vanishes if and only if the tensor has a zero singular value. The following problem is important for multilinear equation solving (e.g., Example 3.2).

PROBLEM 1.9. *Decide if the hyperdeterminant of a tensor is zero.*

We were unable to determine the hardness of Problem 1.9, but conjecture that it is difficult.

CONJECTURE 1.10. *It is NP-hard to decide the vanishing of the hyperdeterminant.*

We are, however, able to evaluate the complexity of deciding spectral norm (see Definition 5.7).

THEOREM 1.11. *Fix any nonzero $\sigma \in \mathbb{Q}$. Deciding whether σ is the spectral norm of a tensor is NP-hard.*

Determining the spectral norm is an optimization (maximization) problem. Thus, while it is NP-hard to decide tensor spectral norm, there might be efficient ways to approximate it. A famous example of approximating solutions to problems whose decision problems are NP-hard is the classical result of [Goemans and Williamson 1995], which gives a polynomial-time algorithm to determine a cut size of a graph that is at least .878 times that of a maximum cut. In fact, it has been shown, assuming the Unique Games Conjecture, that Goemans-Williamson's approximation factor is best possible [Khot et al. 2007]. For some more recent work in the field of approximation algorithms, we refer the reader to [Alon and Naor 2004; Bachoc and Vallentin 2008; Bachoc et al. 2009; Briët et al. 2010a; 2010b; He et al. 2010].

Formally, we say that it is *NP-hard to approximate the spectral norm* of a tensor to within $\varepsilon > 0$ if (unless $P = NP$) there is no polynomial-time algorithm giving a guaranteed lower bound for the spectral norm that is at least a $(1-\varepsilon)$-factor of its true value. Note that ε here might be a function of the input size. A proof of the following can be found in Section 5.

THEOREM 1.12. *It is NP-hard to approximate the spectral norm of a tensor $\mathcal{A}$ to within*

$$\varepsilon = 1 - \left(1 + \frac{1}{N(N-1)}\right)^{-1/2} = \frac{1}{2N(N-1)} + O(N^{-4}),$$

where N is the input size of $\mathcal{A}$.

COROLLARY 1.13. *Unless $P = NP$, there is no FPTAS to approximate spectral norm.*

PROOF. Suppose there is a FPTAS for the tensor spectral norm problem and take $\varepsilon = 1/(4N^2)$ as the approximation error desired for a tensor of input size N. Then, in time polynomial in $1/\varepsilon = 4N^2$ (and thus in N), it would be possible to approximate the spectral norm of a tensor with input size N to within $1 - \left(1 + \frac{1}{N(N-1)}\right)^{-1/2}$ for all large N. From Theorem 1.12, this is only possible if $P = NP$. □

1.7. Tensor Rank

The *outer product* $\mathcal{A} = \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$ of vectors $\mathbf{x} \in \mathbb{F}^l$, $\mathbf{y} \in \mathbb{F}^m$, and $\mathbf{z} \in \mathbb{F}^n$ is the tensor $\mathcal{A} = [\![a_{ijk}]\!]_{i,j,k=1}^{l,m,n}$ given by $a_{ijk} = x_i y_j z_k$. A nonzero tensor that can be expressed as an outer product of vectors is called *rank*-1. More generally, the *rank* of a tensor $\mathcal{A} = [\![a_{ijk}]\!] \in$

[6]We remark that Barvinok has another definition of the hyperdeterminant whose corresponding decision problem can be shown NP-hard [Barvinok 1995]. However, this notion is quite different; for instance, it is zero for odd-ordered tensors.

$\mathbb{F}^{l\times m\times n}$, denoted rank($\mathcal{A}$), is the minimum r for which $\mathcal{A}$ may be expressed as a sum of r rank-1 tensors [Hitchcock 1927a; 1927b] with $\lambda_i \in \mathbb{F}$, $\mathbf{x}_i \in \mathbb{F}^l$, $\mathbf{y}_i \in \mathbb{F}^m$, and $\mathbf{z}_i \in \mathbb{F}^n$:

$$\operatorname{rank}(\mathcal{A}) := \min\Big\{r : \mathcal{A} = \sum\nolimits_{i=1}^{r} \lambda_i\, \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i\Big\}. \tag{5}$$

For a symmetric tensor $\mathcal{S} \in \mathbb{F}^{n\times n\times n}$, we shall require that the vectors in the outer product be the same:

$$\operatorname{rank}(\mathcal{S}) := \min\Big\{r : \mathcal{S} = \sum\nolimits_{i=1}^{r} \lambda_i\, \mathbf{x}_i \otimes \mathbf{x}_i \otimes \mathbf{x}_i\Big\}. \tag{6}$$

The number in (6) is called the *symmetric rank* of $\mathcal{S}$. It is still not known whether a symmetric tensor's symmetric rank is always its rank (this is the Comon conjecture [Landsberg 2012]), although the best symmetric rank-1 approximation and the best rank-1 approximation coincide (see Section 9).

Note that these definitions of rank agree with matrix rank when applied to a 2-tensor, i.e. a matrix. Our next result says that approximating a tensor with a single rank-1 element is already hard.

THEOREM 1.14. *Rank-1 tensor approximation is NP-hard.*

As will become clear, tensor rank as defined in (5) implicitly depends on the choice of field. Suppose that $\mathbb{F} \subseteq \mathbb{E}$ is a subfield of a field $\mathbb{E}$. If $\mathcal{A} \in \mathbb{F}^{l\times m\times n}$ is as in (5), but we allow $\lambda_i \in \mathbb{E}$, $\mathbf{x}_i \in \mathbb{E}^l$, $\mathbf{y}_i \in \mathbb{E}^m$, and $\mathbf{z}_i \in \mathbb{E}^n$, then the number computed in (5) is called the *rank of $\mathcal{A}$ over* $\mathbb{E}$. We will write $\operatorname{rank}_{\mathbb{E}}(\mathcal{A})$ (a notation that we will use whenever the choice of field is important) for the rank of $\mathcal{A}$ over $\mathbb{E}$. In general, it is possible that

$$\operatorname{rank}_{\mathbb{E}}(\mathcal{A}) < \operatorname{rank}_{\mathbb{F}}(\mathcal{A}).$$

We discuss this in detail in Section 7, where we give a new result about the rank of tensors over changing fields (in contrast, the rank of a matrix does not change when the ground field is enlarged). The proof uses symbolic and computational algebra in a fundamental way[7].

THEOREM 1.15. *There is a rational tensor $\mathcal{A} \in \mathbb{Q}^{2\times 2\times 2}$ with $\operatorname{rank}_{\mathbb{R}}(\mathcal{A}) < \operatorname{rank}_{\mathbb{Q}}(\mathcal{A})$.*

Håstad has famously shown that tensor rank over $\mathbb{Q}$ is NP-hard [Håstad 1990]. Since tensor rank over $\mathbb{Q}$ differs in general from tensor rank over $\mathbb{R}$, it is natural to ask if tensor rank might still be NP-hard over $\mathbb{R}$ and $\mathbb{C}$. In Section 7, we shall explain how the argument in [Håstad 1990] also proves the following.

THEOREM 1.16 (HÅSTAD). *Tensor rank is NP-hard over $\mathbb{R}$ and $\mathbb{C}$.*

1.8. Symmetric Tensors

One may wonder if NP-hard problems for general nonsymmetric tensors might perhaps become tractable for symmetric ones. We show that restricting these problems to the class of symmetric tensors does not remove NP-hardness. As with their nonsymmetric counterparts, eigenvalue, singular value, spectral norm, and best rank-1 approximation problems for symmetric tensors all remain NP-hard. In particular, the NP-hardness of symmetric spectral norm in Theorem 9.2 answers an open problem in [Brubaker and Vempala 2009].

1.9. #P-completeness

As is evident from the title of our article and the list in Table II, we have used NP-hardness as our primary measure of computational intractability. To give an indication that Valiant's notion of #P-completeness [Valiant 1979b] is nonetheless relevant to tensor problems, we prove the following result about tensor eigenvalue over $\mathbb{R}$ (see Example 1.5).

[7]For code used in this paper, go to: `http://www.msri.org/people/members/chillar/code.html`.

COROLLARY 1.17. *It is #P-complete to count tensor eigenvectors over* $\mathbb{R}$.

Because of space constraints we will not elaborate on the notion of #P-completeness except to say that it applies to enumeration problems associated with NP-complete decision problems. For example, deciding whether a graph is 3-colorable is an NP-complete decision problem, but counting the number of proper 3-colorings of a graph is a #P-complete enumeration problem. Evidently, #P-complete problems are at least as hard as their corresponding NP-complete counterparts.

1.10. Quantum Computers

Another question sometimes posed to the authors is whether quantum computers might help with these problems. This is believed unlikely because of the seminal works [Bernstein and Vazirani 1997] and [Fortnow and Rogers 1999] (see also the survey [Fortnow 2009]). These authors have demonstrated that the complexity class of bounded error quantum polynomial time (BQP) problems is not expected to overlap with the complexity class of NP-hard problems. Since *BQP* encompasses the decision problems solvable by a quantum computer in polynomial time, the NP-hardness results in this article show that quantum computers are unlikely to be effective for tensor problems.

1.11. Finite Fields

We have restricted our discussion in this article to field extensions of $\mathbb{Q}$ as these are most relevant for the numerical computation arising in science and engineering. Corresponding results over finite fields are nonetheless also of interest in computer science; for instance, quadratic feasibility arises in cryptography [Courtois et al. 2002] and tensor rank arises in boolean satisfiability problems [Håstad 1990].

2. QUADRATIC FEASIBILITY IS NP-HARD

Since it will be a basic tool for us later, we examine the complexity of quadratic feasibility. First, we consider general quadratic equations. The following problem formulation will be useful in proving results about tensor eigenvalue (e.g., Theorems 1.2 and 1.4).

PROBLEM 2.1. *Let* $\mathbb{F} = \mathbb{Q}$, $\mathbb{R}$, *or* $\mathbb{C}$. *For* $i = 1, \ldots, m$, *let* $A_i \in \mathbb{F}^{n \times n}$, $\mathbf{b}_i \in \mathbb{F}^n$, *and* $c_i \in \mathbb{F}$. *Also, let* $\mathbf{x} = [x_1, \ldots, x_n]^\top$ *be a column vector of unknowns, and set* $G_i(\mathbf{x}) = \mathbf{x}^\top A_i \mathbf{x} + \mathbf{b}_i^\top \mathbf{x} + c_i$. *Decide if the system* $\{G_i(\mathbf{x}) = 0\}_{i=1}^m$ *has a solution* $\mathbf{x} \in \mathbb{F}^n$.

Another quadratic problem that is more natural in our setting is the following.

PROBLEM 2.2 (QUADRATIC FEASIBILITY). *Let* $\mathbb{F} = \mathbb{Q}$, $\mathbb{R}$, *or* $\mathbb{C}$. *For* $i = 1, \ldots, m$, *let* $A_i \in \mathbb{F}^{n \times n}$ *and set* $G_i(\mathbf{x}) = \mathbf{x}^\top A_i \mathbf{x}$. *Decide if the system of equations* $\{G_i(\mathbf{x}) = 0\}_{i=1}^m$ *has a nonzero solution* $\mathbf{x} \in \mathbb{F}^n$.

Remark 2.3. It is elementary that the (polynomial) complexity of Problem 2.1 is the same as that of Problem 2.2 when $\mathbb{F} = \mathbb{Q}$ or $\mathbb{R}$. To see this, homogenize each equation $G_i = 0$ in Problem 2.1 by introducing a new unknown z: $A_i \mathbf{x} + \mathbf{b}_i^\top \mathbf{x} z + c_i z^2 = 0$. Next, introduce the quadratic equation $x_1^2 + \cdots + x_n^2 - z^2 = 0$. Now, this new set of equations is easily seen to have a nonzero solution if and only if the original system has any solution at all. The main trick used here is that over the reals, we have: $\sum x_i^2 = 0 \;\Rightarrow\; x_i = 0$ for all i.

Problem 2.2 for $\mathbb{F} = \mathbb{R}$ was studied in [Barvinok 1993]. There, it is shown that for fixed n, one can decide the real feasibility of $m \gg n$ such quadratic equations $\{G_i(\mathbf{x}) = 0\}_{i=1}^m$ in n unknowns in a number of arithmetic operations that is polynomial in m. In contrast, we shall show that quadratic feasibility over the rationals is undecidable and NP-hard over both $\mathbb{R}$ and $\mathbb{C}$.

To prepare for our proof of the former result, we first explain, in very simplified form, the connection of quadratic feasibility to the Halting Problem established in the works

[Davis et al. 1961; Matijasevič 1970; Jones 1982; Jones and Matijasevič 1984]. For a detailed exposition of the ideas involved, see the introductory book [Matijasevič 1993]. Collectively, these papers resolve (in the negative) Hilbert's 10th Problem: whether there is a finite procedure to decide the solvability of general polynomial equations over the integers (the *Diophantine Problem* over $\mathbb{Z}$).

The following fact in theoretical computer science is a basic consequence of these papers. Fix a universal Turing machine. There is a listing of all Turing machines $\mathscr{T}_x$ $(x = 1, 2, \ldots)$ and a finite set S of integral polynomial equations in the parameter x and other unknowns with the following property: For each particular positive integer $x = 1, 2, \ldots$, the system $S(x)$ of integral polynomial equations has a solution in positive integers if and only if the Turing machine $\mathscr{T}_x$ halts with no input. In particular, polynomial equation solving over the positive integers is undecidable. With this background in place, we prove the following.

THEOREM 2.4. *Problem 2.2, quadratic feasibility over $\mathbb{Q}$ is undecidable.*

PROOF. A refinement of the above discussion on Hilbert's 10th Problem (see [Jones 1982, pp. 552]) shows that whether the Turing machine $\mathscr{T}_x$ halts can be encoded as whether a finite set $S_0(x)$ of integral quadratic equations in 58 unknowns $y_1, \ldots, y_{58}$, in the parameter x has a solution in positive integers. For each of these unknowns y_i in $S_0(x)$, we add an equation with four new variables a_i, b_i, c_i, d_i encoding (by Lagrange's Four-Square Theorem) that y_i should be a positive integer:

$$y_i = a_i^2 + b_i^2 + c_i^2 + d_i^2 + 1.$$

Let $S_1(x)$ denote this new system; note that $S_1(x)$ has a solution in integers if and only if $S_0(x)$ has a solution in positive integers. By Remark 2.3, there is a family of homogeneous integral quadratic equations $S_2(x)$ in the positive integer parameter x with the following property: $S_2(x)$ has a nonzero integral solution if and only if the Turing machine $\mathscr{T}_x$ halts. Since a set of homogeneous equations have a nonzero rational solution if and only if they have a nonzero integral solution, equations $S_2(x)$ have a nonzero rational solution if and only if $\mathscr{T}_x$ halts. Thus, if we could decide quadratic feasibility over $\mathbb{Q}$ with a Turing machine, we could also solve the Halting problem with one as well. □

Remark 2.5. It is interesting to note how similar Theorem 2.4 is to the statement that the Diophantine Problem over $\mathbb{Q}$ is undecidable, a problem that is still wide open.

We next study quadratic feasibility when $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$ and show that it is NP-hard. Variations of this basic result can be found in [Bayer 1982], [Lovász 1994], and [Grenet et al. 2010].

THEOREM 2.6. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Graph 3-colorability is polynomially reducible to quadratic feasibility over $\mathbb{F}$. Thus, Problem 2.2 over $\mathbb{F}$ is NP-hard.*

The idea of turning colorability problems into questions about polynomials appears to originate with Bayer's thesis although it has arisen in several other places, including [Lovász 1994; De Loera 1995; De Loera et al. 2008]. For a recent application of polynomial algebra to deciding unique 3-colorability of graphs, see [Hillar and Windfeldt 2008].

To prove Theorem 2.6, we shall reduce graph 3-colorability to quadratic feasibility over $\mathbb{C}$. The result for $\mathbb{F} = \mathbb{R}$ then follows from the following fact.

LEMMA 2.7. *Let $\mathbb{F} = \mathbb{R}$ and A_i, G_i be as in Problem 2.2. Consider a new system $H_j(\mathbf{x}) = \mathbf{x}^\top B_j \mathbf{x}$ of $2m$ equations in $2n$ unknowns given by:*

$$B_i = \begin{bmatrix} A_i & 0 \\ 0 & -A_i \end{bmatrix}, \quad B_{m+i} = \begin{bmatrix} 0 & A_i \\ A_i & 0 \end{bmatrix}, \quad i = 1, \ldots, m.$$

The equations $\{H_j(\mathbf{x}) = 0\}_{j=1}^{2m}$ *have a nonzero real solution* $\mathbf{x} \in \mathbb{R}^{2n}$ *if and only if the equations* $\{G_i(\mathbf{z}) = 0\}_{i=1}^{m}$ *have a nonzero complex solution* $\mathbf{z} \in \mathbb{C}^n$.

PROOF. By construction, a nonzero complex solution $\mathbf{z} = \mathbf{u} + i\mathbf{v} \in \mathbb{C}^n$ with $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ to the equations $\{G_i(\mathbf{z}) = 0\}_{i=1}^{m}$ corresponds to a nonzero real solution $\mathbf{x} = (\mathbf{u}^\top, \mathbf{v}^\top)^\top$ for $\{H_j(\mathbf{x}) = 0\}_{j=1}^{2m}$. □

The trivial observation below also gives flexibility in specifying quadratic feasibility problems over $\mathbb{R}$. This will be useful since the set of equations defining a tensor eigenpair is a square system.

LEMMA 2.8. *Let* $G_i(\mathbf{x}) = \mathbf{x}^\top A_i \mathbf{x}$ *for* $i = 1, \ldots, m$ *with each* $A_i \in \mathbb{R}^{n\times n}$. *Consider a new system* $H_j(\mathbf{x}) = \mathbf{x}^\top B_j \mathbf{x}$ *of* $r \geq m+1$ *equations in* $s \geq n$ *unknowns given by* $s \times s$ *matrices:*

$$B_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \ i = 1, \ldots, m; \quad B_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ j = m+1, \ldots, r-1; \quad B_r = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix};$$

in which I *is the* $(s-n)\times(s-n)$ *identity matrix. Equations* $\{H_j(\mathbf{x}) = 0\}_{j=1}^{r}$ *have a nonzero solution* $\mathbf{x} \in \mathbb{R}^s$ *if and only if* $\{G_i(\mathbf{x}) = 0\}_{i=1}^{m}$ *have a nonzero solution* $\mathbf{x} \in \mathbb{R}^n$.

The following set of polynomials C_G allows us to relate feasibility of a polynomial system to 3-colorability of a graph G. An instance of this encoding (after applying Lemmas 2.7 and 2.8) is Example 1.5.

Definition 2.9. The *color encoding* of a graph $G = (V, E)$ with n vertices is the set of $4n$ polynomials in $2n + 1$ unknowns:

$$C_G := \begin{cases} x_i y_i - z^2, \quad y_i z - x_i^2, \quad x_i z - y_i^2, & i = 1, \ldots, n, \\ \sum_{\{i,j\}\in E} (x_i^2 + x_i x_j + x_j^2), & i = 1, \ldots, n. \end{cases} \tag{7}$$

LEMMA 2.10. C_G *has a common nonzero complex solution if and only if the graph* G *is* 3*-colorable.*

PROOF. Suppose that G is 3-colorable and let $[x_1, \ldots, x_n]^\top \in \mathbb{C}^n$ be a proper 3-coloring of G, encoded using cube roots of unity as in Example 1.5. Set $z = 1$ and $y_i = 1/x_i$ for $i = 1, \ldots, n$; we claim that these numbers are a common zero of C_G. It is clear that the first $3n$ polynomials in (7) evaluate to zero. Next consider any expression of the form $p_i = \sum_{\{i,j\}\in E} x_i^2 + x_i x_j + x_j^2$ in which $\{i, j\} \in E$. Since we have a 3-coloring, it follows that $x_i \neq x_j$; thus,

$$0 = x_i^3 - x_j^3 = \frac{x_i^3 - x_j^3}{x_i - x_j} = x_i^2 + x_i x_j + x_j^2.$$

In particular, each p_i evaluates to zero as desired.

Conversely, suppose that the set of polynomials C_G has a common nontrivial solution,

$$\mathbf{0} \neq [x_1, \ldots, x_n, y_1, \ldots, y_n, z]^\top \in \mathbb{C}^{2n+1}.$$

If $z = 0$, then all of the x_i and y_i must be zero as well. Thus $z \neq 0$, and since the equations are homogenous, we may assume that our solution has $z = 1$. It follows that $x_i^3 = 1$ for all i so that $[x_1, \ldots, x_n]^\top$ is a 3-coloring of G. We are left with verifying that it is proper. If $\{i, j\} \in E$ and $x_i = x_j$, then $x_i^2 + x_i x_j + x_j^2 = 3x_i^3$; otherwise, if $x_i \neq x_j$, then $x_i^2 + x_i x_j + x_j^2 = 0$. Thus, $p_i = 3r x_i^2$, where r is the number of vertices j adjacent to i that have $x_i = x_j$. It follows that $r = 0$ so that $x_i \neq x_j$ for all $\{i, j\} \in E$, and thus G has a proper 3-coloring. □

We close with a proof that quadratic feasibility over $\mathbb{C}$ (and therefore over $\mathbb{R}$) is NP-hard.

PROOF OF THEOREM 2.6. Given a graph G, construct the color encoding C_G. From Lemma 2.10, the homogeneous quadratic polynomials C_G have a nonzero complex solution if and only if G is 3-colorable. Thus, solving Problem 2.2 over $\mathbb{C}$ in polynomial time would allow us to do the same for graph 3-colorability. □

3. BILINEAR SYSTEM IS NP-HARD

We consider some natural bilinear extensions to the quadratic feasibility problems encountered earlier, generalizing Example 3.2 below. The main result is Theorem 3.7, which shows that the following bilinear feasibility problem over $\mathbb{R}$ or $\mathbb{C}$ is NP-hard. In Section 5, we use this result to show that certain singular value problems for tensors are also NP-hard.

PROBLEM 3.1 (TENSOR BILINEAR FEASIBILITY). *Let* $\mathbb{F} = \mathbb{Q}, \mathbb{R},$ *or* $\mathbb{C}$. *Let* $\mathcal{A} = [\![a_{ijk}]\!] \in \mathbb{F}^{l\times m\times n}$ *and set* $A_i(j,k) = a_{ijk}$, $B_j(i,k) = a_{ijk}$, *and* $C_k(i,j) = a_{ijk}$ *to be all the slices of* $\mathcal{A}$. *Decide if the following set of equations:*

$$\begin{cases} \mathbf{v}^\top A_i \mathbf{w} = 0, & i = 1,\dots,l; \\ \mathbf{u}^\top B_j \mathbf{w} = 0, & j = 1,\dots,m; \\ \mathbf{u}^\top C_k \mathbf{v} = 0, & k = 1,\dots,n; \end{cases} \tag{8}$$

has a solution $\mathbf{u} \in \mathbb{F}^l$, $\mathbf{v} \in \mathbb{F}^m$, $\mathbf{w} \in \mathbb{F}^n$, *with all* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *nonzero.*

Multilinear systems of equations have been studied since the early 19th century. For instance, the following result was known more than 150 years ago [Cayley 1845].

Example 3.2 ($2 \times 2 \times 2$ *hyperdeterminant*). For $\mathcal{A} = [\![a_{ijk}]\!] \in \mathbb{C}^{2\times2\times2}$, define

$$\mathrm{Det}_{2,2,2}(\mathcal{A}) := \frac{1}{4}\left[\det\left(\begin{bmatrix} a_{000} & a_{010} \\ a_{001} & a_{011}\end{bmatrix} + \begin{bmatrix} a_{100} & a_{110} \\ a_{101} & a_{111}\end{bmatrix}\right) - \det\left(\begin{bmatrix} a_{000} & a_{010} \\ a_{001} & a_{011}\end{bmatrix} - \begin{bmatrix} a_{100} & a_{110} \\ a_{101} & a_{111}\end{bmatrix}\right)\right]^2 - 4\det\begin{bmatrix} a_{000} & a_{010} \\ a_{001} & a_{011}\end{bmatrix}\det\begin{bmatrix} a_{100} & a_{110} \\ a_{101} & a_{111}\end{bmatrix}.$$

Given a matrix $A \in \mathbb{C}^{n\times n}$, the equation $\mathbf{x}^\top A\mathbf{y} = 0$ has a nontrivial solution ($\mathbf{x}, \mathbf{y}$ both nonzero) if and only if $\det(A) = 0$. Cayley proved a multilinear version that parallels the matrix case. The following system of bilinear equations:

$$\begin{aligned} &a_{000}x_0y_0 + a_{010}x_0y_1 + a_{100}x_1y_0 + a_{110}x_1y_1 = 0, \quad a_{001}x_0y_0 + a_{011}x_0y_1 + a_{101}x_1y_0 + a_{111}x_1y_1 = 0, \\ &a_{000}x_0z_0 + a_{001}x_0z_1 + a_{100}x_1z_0 + a_{101}x_1z_1 = 0, \quad a_{010}x_0z_0 + a_{011}x_0z_1 + a_{110}x_1z_0 + a_{111}x_1z_1 = 0, \\ &a_{000}y_0z_0 + a_{001}y_0z_1 + a_{010}y_1z_0 + a_{011}y_1z_1 = 0, \quad a_{100}y_0z_0 + a_{101}y_0z_1 + a_{110}y_1z_0 + a_{111}y_1z_1 = 0, \end{aligned}$$

has a nontrivial solution ($\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^2$ all nonzero) if and only if $\mathrm{Det}_{2,2,2}(\mathcal{A}) = 0$. □

A remarkable result established in [Gelfand et al. 1994; 1992] is that hyperdeterminants generalize to tensors of arbitrary orders, provided that certain dimension restrictions (9) are satisfied. We do not formally define the hyperdeterminant[8] here; however, it suffices to know that $\mathrm{Det}_{l,m,n}$ is a homogeneous polynomial with integer coefficients in the variables x_{ijk} where $i = 1,\dots,l$, $j = 1,\dots,m$, $k = 1,\dots,n$. Such a polynomial defines a function $\mathrm{Det}_{l,m,n} : \mathbb{C}^{l\times m\times n} \to \mathbb{C}$ by evaluation at $\mathcal{A} = [\![a_{ijk}]\!] \in \mathbb{C}^{l\times m\times n}$, i.e. setting $x_{ijk} = a_{ijk}$. The following generalizes Example 3.2.

[8]Roughly speaking, the hyperdeterminant is a polynomial that defines the set of all tangent hyperplanes to the set of rank-1 tensors in $\mathbb{C}^{l\times m\times n}$. Gelfand, Kapranov, and Zelevinsky showed that this set is a hypersurface (i.e., defined by the vanishing of a single polynomial) if and only if the condition (9) is satisfied. Also, to be mathematically precise, these sets lie in projective space.

THEOREM 3.3 (GELFAND, KAPRANOV, ZELEVINSKY). *Given a tensor $\mathcal{A} \in \mathbb{C}^{l\times m\times n}$, the hyperdeterminant* $\mathrm{Det}_{l,m,n}$ *is defined if and only if l, m, n satisfy:*

$$l \le m + n - 1, \quad m \le l + n - 1, \quad n \le l + m - 1. \tag{9}$$

In particular, hyperdeterminants exist when $l = m = n$. Given any $\mathcal{A} = [\![a_{ijk}]\!] \in \mathbb{C}^{l\times m\times n}$ with (9) *satisfied, the system*

$$\begin{aligned} \sum\nolimits_{j,k=1}^{m,n} a_{ijk} v_j w_k = 0 \quad & i = 1, \dots, l; \\ \sum\nolimits_{i,k=1}^{l,n} a_{ijk} u_i w_k = 0, \quad & j = 1, \dots, m; \\ \sum\nolimits_{i,j=1}^{l,m} a_{ijk} u_i v_j = 0, \quad & k = 1, \dots, n; \end{aligned} \tag{10}$$

has a nontrivial complex solution if and only if $\mathrm{Det}_{l,m,n}(\mathcal{A}) = 0$.

Remark 3.4. Condition (9) is the 3-tensor equivalent of '$m \le n$ and $n \le m$' for the existence of determinants of 2-tensors (i.e., matrices).

We shall examine the following closely related problem. Such systems of bilinear equations have also appeared in other contexts [Cohen and Tomasi 1997].

PROBLEM 3.5 (TRIPLE BILINEAR FEASIBILITY). *Let $\mathbb{F} = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. Let $A_k, B_k, C_k \in \mathbb{F}^{n\times n}$ for $k = 1, \dots, n$. Decide if the following set of equations:*

$$\begin{cases} \mathbf{v}^\top A_i \mathbf{w} = 0, & i = 1, \dots, n; \\ \mathbf{u}^\top B_j \mathbf{w} = 0, & j = 1, \dots, n; \\ \mathbf{u}^\top C_k \mathbf{v} = 0, & k = 1, \dots, n; \end{cases} \tag{11}$$

has a solution $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{F}^n$, with all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ nonzero.

The difference between Problem 3.5 and Problem 3.1 is that the coefficient matrices A_i, B_j, C_k in (11) are allowed to be arbitrary rather than slices of a tensor $\mathcal{A}$ as in (8). Furthermore, we always require $l = m = n$ in Problem 3.5 whereas Problem 3.1 has no such requirement.

If one could show that Problem 3.5 is NP-hard for A_i, B_j, C_k coming from a tensor $\mathcal{A} \in \mathbb{C}^{n\times n\times n}$ or that Problem 3.1 is NP-hard on the subset of problems where $\mathcal{A} \in \mathbb{C}^{l\times m\times n}$ with l, m, n satisfying (9), then this would imply that deciding whether the bilinear system (10) has a non-trivial solution is NP-hard. It then would follow that deciding whether the hyperdeterminant of a tensor is zero is also NP-hard. Unfortunately, our proofs below do not achieve either of these. The NP-hardness of the hyperdeterminant is therefore still open; see Conjecture 1.10.

Before proving Theorem 3.7, we first verify that, as in the case of quadratic feasibility, it is enough to show NP-hardness of the problem over $\mathbb{C}$.

LEMMA 3.6. *Let $\mathcal{A} \in \mathbb{R}^{l\times m\times n}$. There is a tensor $\mathcal{B} \in \mathbb{R}^{2l\times 2m\times 2n}$ such that tensor bilinear feasibility over $\mathbb{R}$ for $\mathcal{B}$ is the same as tensor bilinear feasibility over $\mathbb{C}$ for $\mathcal{A}$*

PROOF. Let $A_i = [a_{ijk}]_{j,k=1}^{m,n}$ for $i = 1, \dots, l$. Consider the tensor $\mathcal{B} = [\![b_{ijk}]\!] \in \mathbb{R}^{2l\times 2m\times 2n}$ given by setting its slices $B_i(j, k) = b_{ijk}$ as follows:

$$B_i = \begin{bmatrix} A_i & 0 \\ 0 & -A_i \end{bmatrix}, \; B_{l+i} = \begin{bmatrix} 0 & A_i \\ A_i & 0 \end{bmatrix}, \quad i = 1, \dots, l.$$

It is straightforward to check that nonzero real solutions to (8) for the tensor $\mathcal{B}$ correspond in a one-to-one manner with nonzero complex solutions to (8) for the tensor $\mathcal{A}$. □

We now come to the proof of the main theorem of this section. For the argument, we shall need the following elementary fact of linear algebra (easily proved by induction on the number of unknowns): a system of m linear equations in n unknowns with $m > n$ has at least two solutions (in particular, at least one nonzero solution).

THEOREM 3.7. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Graph 3-colorability is polynomially reducible to Problem 3.1 (tensor bilinear feasibility). Thus, Problem 3.1 is NP-hard.*

PROOF OF THEOREM 3.7. Given a graph $G = (V, E)$ with $v = |V|$, we shall form a tensor $\mathcal{A} = \mathcal{A}(G) \in \mathbb{Z}^{l\times m\times n}$ with $l = v(2v + 5)$ and $m = n = (2v + 1)$ having the property that system (8) has a nonzero complex solution if and only if G has a proper 3-coloring.

Consider column vectors $\mathbf{v} = [x_1, \ldots, x_v, y_1, \ldots, y_v, t]^\top$ and $\mathbf{w} = [\hat{x}_1, \ldots, \hat{x}_v, \hat{y}_1, \ldots, \hat{y}_v, \hat{t}]^\top$ of unknowns. The 2×2 minors of the matrix formed by placing $\mathbf{v}$ and $\mathbf{w}$ side-by-side are $v(2v + 1)$ quadratics, $\mathbf{v}^\top A_i\mathbf{w}$, $i = 1, \ldots, v(2v + 1)$, for matrices $A_i \in \mathbb{Z}^{(2v+1)\times(2v+1)}$ with entries in $\{-1, 0, 1\}$. By construction, these equations are satisfied for some $\mathbf{v}, \mathbf{w}$ both nonzero if and only if there is a nonzero $c \in \mathbb{C}$ such that $\mathbf{v} = c\mathbf{w}$. Next, we write down the $3v$ polynomials $\mathbf{v}^\top A_i\mathbf{w}$ for $i = v(2v + 1) + 1, \ldots, v(2v + 1) + 3v$ whose vanishing (along with the equations above) implies that the x_i are (projective) cubic roots of unity; see (7). We also encode v equations $\mathbf{v}^\top A_i\mathbf{w}$ for $i = v(2v+4)+1, \ldots, v(2v+4)+v$ whose vanishing implies that x_i and x_j are different if $\{i, j\} \in E$. Finally, the tensor $\mathcal{A}(G) = [\![a_{ijk}]\!] \in \mathbb{Z}^{l\times m\times n}$ is defined by $a_{ijk} = A_i(j, k)$.

We verify that $\mathcal{A}$ has the claimed property. Suppose that there are three nonzero complex vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ which satisfy tensor bilinear feasibility over $\mathbb{C}$. Then from construction, $\mathbf{v} = c\mathbf{w}$ for some $c \neq 0$ and also $\mathbf{v}$ encodes a proper 3-coloring of the graph G by Lemma 2.10. Conversely, suppose that G is 3-colorable with a coloring represented using a vector $[x_1, \ldots, x_v]^\top \in \mathbb{C}^v$ of cube roots of unity. Then, the vectors $\mathbf{v} = \mathbf{w} = [x_1, \ldots, x_v, x_1^{-1}, \ldots, x_v^{-1}, 1]^\top$ satisfy the first set of equations in (8). The other sets of equations define a linear system for the vector $\mathbf{u}$ consisting of $4v + 2$ equations in $l = v(2v + 5) > 4v + 2$ unknowns. In particular, there is always a nonzero $\mathbf{u}$ solving them, proving that complex tensor bilinear feasibility is true for $\mathcal{A}$. □

Note that our choice of l, m, n in this construction does not satisfy (9); thus, NP-hardness of the hyperdeterminant does not follow from Theorem 3.7. We now prove the following.

THEOREM 3.8. *Problem 3.5, triple bilinear feasibility, is NP-hard over $\mathbb{R}$.*

PROOF. Since the encoding in Theorem 2.6 has more equations than unknowns, we may use Lemma 2.8 to further transform this system into an equivalent one that is square (see Example 1.5). Thus, if we could solve square quadratic feasibility ($m = n$ in Problem 2.1) over $\mathbb{R}$ in polynomial time, then we could do the same for graph 3-colorability. Using this observation, it is enough to prove that a given square, quadratic feasibility problem can be polynomially reduced to Problem 3.5.

Therefore, suppose that A_i are given $n \times n$ real matrices for which we would like to determine if $\mathbf{x}^\top A_i\mathbf{x} = 0$ $(i = 1, \ldots, n)$ has a solution $0 \neq \mathbf{x} \in \mathbb{R}^n$. Let E_{ij} denote the matrix with a 1 in the (i, j) entry and 0's elsewhere. Consider a system S as in (11) in which we also define

$$B_1 = C_1 = E_{11} \quad \text{and} \quad B_i = C_i = E_{1i} - E_{i1}, \quad \text{for } i = 2, \ldots, n.$$

We shall construct a decision tree based on the answer to feasibility questions involving systems having the form of S. This will give us an algorithm to determine whether the original quadratic problem is feasible.

Consider changing system S by replacing B_1 and C_1 with matrices consisting of all zeroes, and call this system S'. We first make two claims about solutions to S and S'.

Claim 1: If S has a solution, then $(u_1, v_1) = (0, 0)$. Moreover, in this case, $w_1 = 0$.

First note that $u_1 v_1 = 0$ since S has a solution. Suppose that $u_1 = 0$ and $v_1 \neq 0$. Then the form of the matrices C_i forces $u_2 = \cdots = u_n = 0$ as well. But then $\mathbf{u} = 0$, which contradicts S having a solution. A similar examination with $u_1 \neq 0$ and $v_1 = 0$ proves the claim. It is also easy to see that $w_1 = 0$ in this case.

Claim 2: Suppose that S has no solution. If S' has a solution, then $\mathbf{v} = c\mathbf{u}$ and $\mathbf{w} = d\mathbf{u}$ for some $0 \neq c, d \in \mathbb{R}$. Moreover, if S' has no solution, then the original quadratic problem has no solution.

To verify Claim 2, suppose first that S' has a solution $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ but S does not. In that case we must have $u_1 \neq 0$. Also, $v_1 \neq 0$ since otherwise the third set of equations $\{u_1 v_i - u_i v_1 = 0\}_{i=2}^{n}$ would force $\mathbf{v} = 0$. But then $\mathbf{v} = c\mathbf{u}$ for $c = \frac{v_1}{u_1}$ and $\mathbf{w} = d\mathbf{u}$ for $d = \frac{w_1}{u_1}$ as desired. On the other hand, suppose that both S and S' have no solution. We claim that $\mathbf{x}^\top A_i \mathbf{x} = 0$ $(i = 1, \ldots, n)$ has no solution $\mathbf{x} \neq 0$ either. Indeed, if it did, then setting $\mathbf{u} = \mathbf{v} = \mathbf{w} = \mathbf{x}$, we would get a solution to S' (as one can easily check), a contradiction.

We are now prepared to give our method for solving quadratic feasibility using at most $n+2$ queries to the restricted version ($B_i = C_i$ for all i) of Problem 3.5.

First check if S has a solution. If it does not, then ask if S' has a solution. If it does not, then output "INFEASIBLE". This answer is correct by Claim 2. If S has no solution but S' does, then there is a solution with $\mathbf{v} = c\mathbf{u}$ and $\mathbf{w} = d\mathbf{u}$, both c and d nonzero. But then $\mathbf{x}^\top A_i \mathbf{x} = 0$ for $\mathbf{x} = \mathbf{u}$ and each i. Thus, we output "FEASIBLE".

If instead, S has a solution, then the solution necessarily has $(u_1, v_1, w_1) = (0, 0, 0)$. Consider now the $n-1$-dimensional system T in which A_i becomes the lower-right $(n-1) \times (n-1)$ block of A_i, and C_i and D_i are again of the same form as the previous ones. This is a smaller system with 1 less unknown. We now repeat the above examination inductively with start system T replacing S.

If we make it to the final stage of this process without outputting an answer, then the original system S has a solution with

$$u_1 = \cdots = u_{n-1} = v_1 = \cdots = v_{n-1} = w_1 = \cdots = w_{n-1} = 0 \text{ and } u_n, v_n, w_n \text{ are all nonzero.}$$

It follows that the (n, n) entry of each A_i $(i = 1 \ldots, n)$ is zero. Thus, it is clear that there is a solution $\mathbf{x}$ to the the original quadratic feasibility problem, and so we output "FEASIBLE".

We have therefore verified the algorithm terminates with the correct answer and it does so in polynomial time with an oracle that can solve Problem 3.5. □

Although in the above proof, we have $l = m = n$ and thus (9) is satisfied, our choice of the coefficient matrices A_k, B_k, C_k do not arise from slices of a single tensor $\mathcal{A}$. So again, NP-hardness of deciding the vanishing of the hyperdeterminant does not follow from Theorem 3.8.

With the same arguments, we can show the following undecidability result over $\mathbb{Q}$.

THEOREM 3.9. *Problem 3.1, tensor bilinear feasibility, is undecidable over* $\mathbb{Q}$.

PROOF. As in the proof of Theorem 3.7, reduce quadratic feasibility over $\mathbb{Q}$ to Problem 3.1 over $\mathbb{Q}$; then appeal to Theorem 2.4. □

4. TENSOR EIGENVALUE IS NP-HARD

The eigenvalues and eigenvectors of a symmetric matrix $A \in \mathbb{R}^{n \times n}$ are the stationary values and points of its Rayleigh quotient $\mathbf{x}^\top A\mathbf{x}/\mathbf{x}^\top\mathbf{x}$, equivalently defined by the stationarity conditions for maximizing the quadratic form $\mathbf{x}^\top A\mathbf{x}$ constrained to the unit ℓ^2-sphere:

$$\|\mathbf{x}\|_2^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = 1. \tag{12}$$

The stationarity conditions of the Lagrangian, $L(\mathbf{x}, \lambda) = \mathbf{x}^\top A\mathbf{x} - \lambda(\|\mathbf{x}\|_2^2 - 1)$, at a stationary point $(\lambda, \mathbf{x})$ give the familiar eigenvalue equation $A\mathbf{x} = \lambda\mathbf{x}$, which is then used to define eigenvalue/eigenvector pairs for any square matrices.

The above discussion extends to give a notion of eigenvalues and eigenvectors for 3-tensors. They are suitably constrained stationary values and points of the cubic form:

$$\mathcal{A}(\mathbf{x},\mathbf{x},\mathbf{x}) := \sum_{i,j,k=1}^{n} a_{ijk}x_ix_jx_k, \tag{13}$$

associated with a tensor $\mathcal{A} \in \mathbb{R}^{n\times n\times n}$. However, one now has several natural generalizations of the constraint. One may retain (12). Alternatively, one may choose

$$\|\mathbf{x}\|_3^3 = |x_1|^3 + |x_2|^3 + \cdots + |x_n|^3 = 1 \tag{14}$$

or a unit sum-of-cubes,

$$x_1^3 + x_2^3 + \cdots + x_n^3 = 1. \tag{15}$$

Each of these choices has its advantage: condition (14) defines a compact set while condition (15) defines an algebraic set, and both result in eigenvectors that are scale-invariant. Condition (12) defines a set that is both compact and algebraic, but makes eigenvectors that are not scale-invariant. These were proposed independently in [Lim 2005; Qi 2005].

By considering the stationarity conditions of the Lagrangian, $L(\mathbf{x},\lambda) = \mathcal{A}(\mathbf{x},\mathbf{x},\mathbf{x}) - \lambda c(\mathbf{x})$, for $c(\mathbf{x})$ defined by one of the conditions in (12), (14), or (15), we obtain the following.

Definition 4.1. Fix a field $\mathbb{F} = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. The number $\lambda \in \mathbb{F}$ is called an **ℓ^2-eigenvalue** of the tensor $\mathcal{A} \in \mathbb{F}^{n\times n\times n}$ and $\mathbf{0} \neq \mathbf{x} \in \mathbb{F}^n$ its corresponding **ℓ^2-eigenvector** if (2) holds. Similarly, $\lambda \in \mathbb{F}$ is called an **ℓ^3-eigenvalue** and $\mathbf{0} \neq \mathbf{x} \in \mathbb{F}^n$ its corresponding **ℓ^3-eigenvector** if

$$\sum_{i,j=1}^{n} a_{ijk}x_ix_j = \lambda x_k^2, \quad k = 1,\ldots,n. \tag{16}$$

Using all of the tools we have developed thus far, we now prove that tensor eigenvalue over $\mathbb{Q}$ is undecidable and that real tensor eigenvalue is NP-hard.

Proof of Theorem 1.2 and Theorem 1.4. The case $\lambda = 0$ of tensor λ-eigenvalue becomes square quadratic feasibility ($m = n$ in Problem 2.2) as discussed in the proof of Theorem 3.8. A similar situation holds when $\lambda \in \mathbb{Q}$ and we use (16) to define ℓ^3-eigenpairs. Thus, by Lemma 2.8, deciding if $\lambda = 0$ is an eigenvalue of a tensor over $\mathbb{Q}$ is undecidable by Theorem 2.4 and NP-hard over $\mathbb{R}$ by Theorem 1.4. □

We will see in Section 8 that the eigenvalue problem for *symmetric* 3-tensors is also NP-hard. We close this section with a proof that it is even NP-hard to approximate an eigenvector of a tensor.

Proof of Theorem 1.6. Suppose that one could approximate in polynomial time a tensor eigenvector with eigenvalue $\lambda = 0$ to within $\varepsilon = \frac{3}{4}$ as in (4). Then by the discussion in Section 2, it would be possible to pick out proper 3-colorings from solutions to a square set of polynomial equations (see Example 1.5), since they represent cube roots of unity separated by a distance of at least $\frac{3}{2}$ in each real or imaginary part (see Figure 4). Thus, finding an approximate real eigenvector to within $\varepsilon = \frac{3}{4}$ of a true one in polynomial time would allow one to also decide graph 3-colorability in polynomial time. □

5. TENSOR SINGULAR VALUE AND SPECTRAL NORM ARE NP-HARD

It is easy to verify that the singular values and singular vectors of a matrix $A \in \mathbb{R}^{m\times n}$ are the stationary values and stationary points of the quotient $\mathbf{x}^\top A\mathbf{y}/\|\mathbf{x}\|_2\|\mathbf{y}\|_2$. Indeed, at a stationary point $(\mathbf{x},\mathbf{y},\sigma)$, the first order condition of the associated Lagrangian,

$$L(\mathbf{x},\mathbf{y},\sigma) = \mathbf{x}^\top A\mathbf{y} - \sigma(\|\mathbf{x}\|_2\|\mathbf{y}\|_2 - 1), \tag{17}$$

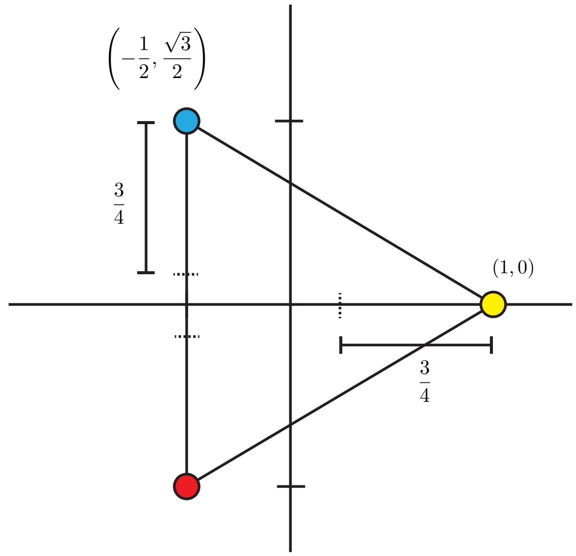

Fig. 2. **It is NP-hard to approximate a real eigenvector.** Each dot (i.e. cube root of unity) above in the complex plane represents a pair of real numbers which are coordinates of a real eigenvector of a rational tensor (see Example 1.5). Therefore, if one could approximate this eigenvector to within $\varepsilon = \frac{3}{4}$ in each real coordinate, then one would be able to decide which cube roots of unity (i.e. colors) to label vertices of a 3-colorable graph G such that the coloring is proper.

produces the familiar singular value equations:

$$A\mathbf{v} = \sigma\mathbf{u}, \quad A^\top\mathbf{u} = \sigma\mathbf{v},$$

where $\mathbf{u} = \mathbf{x}/\|\mathbf{x}\|_2$ and $\mathbf{v} = \mathbf{y}/\|\mathbf{y}\|_2$.

This derivation has been extended to define a notion of singular values and singular vectors for higher-order tensors [Lim 2005]. For $\mathcal{A} \in \mathbb{R}^{l\times m\times n}$, we have the trilinear form[9]

$$\mathcal{A}(\mathbf{x},\mathbf{y},\mathbf{z}) := \sum_{i,j,k=1}^{n} a_{ijk} y_i x_j z_k, \tag{18}$$

and consideration of its stationary values on a product of unit ℓ^p-spheres leads to the Lagrangian,

$$L(\mathbf{x},\mathbf{y},\mathbf{z},\sigma) = \mathcal{A}(\mathbf{x},\mathbf{y},\mathbf{z}) - \sigma(\|\mathbf{x}\|_p\|\mathbf{y}\|_p\|\mathbf{z}\|_p - 1).$$

The only ambiguity is choice of p. As for eigenvalues, natural choices are $p = 2$ or 3.

Definition 5.1. Fix a field $\mathbb{F} = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. Let $\sigma \in \mathbb{F}$, and suppose that $\mathbf{u} \in \mathbb{F}^l$, $\mathbf{v} \in \mathbb{F}^m$, and $\mathbf{w} \in \mathbb{F}^n$ are all nonzero. The number $\sigma \in \mathbb{F}$ is called an ℓ^2-**singular value** and the nonzero $\mathbf{u},\mathbf{v},\mathbf{w}$ are called ℓ^2-**singular vectors** of $\mathcal{A}$ if

$$\begin{aligned} \textstyle\sum_{j,k=1}^{m,n} a_{ijk} v_j w_k &= \sigma u_i, \quad i = 1,\dots,l; \\ \textstyle\sum_{i,k=1}^{l,n} a_{ijk} u_i w_k &= \sigma v_j, \quad j = 1,\dots,m; \\ \textstyle\sum_{i,j=1}^{l,m} a_{ijk} u_i v_j &= \sigma w_k, \quad k = 1,\dots,n. \end{aligned} \tag{19}$$

[9]When $l = m = n$ and $\mathbf{x} = \mathbf{y} = \mathbf{z}$, the trilinear form in (18) becomes the cubic form in (13).

Similarly, σ is called an ℓ^3**-singular value** and nonzero $\mathbf{u}, \mathbf{v}, \mathbf{w}$ ℓ^3**-singular vectors** if

$$\begin{aligned} \sum\nolimits_{j,k=1}^{m,n} a_{ijk} v_j w_k &= \sigma u_i^2, \quad i = 1, \dots, l; \\ \sum\nolimits_{i,k=1}^{l,n} a_{ijk} u_i w_k &= \sigma v_j^2, \quad j = 1, \dots, m; \\ \sum\nolimits_{i,j=1}^{l,m} a_{ijk} u_i v_j &= \sigma w_k^2, \quad k = 1, \dots, n. \end{aligned} \tag{20}$$

When $\sigma = 0$, definitions (19) and (20) agree and reduce to tensor bilinear feasibility (Problem 3.1). In particular, if condition (9) holds, then $\operatorname{Det}_{l,m,n}(\mathcal{A}) = 0$ iff 0 is an ℓ^2-singular value of $\mathcal{A}$ iff 0 is an ℓ^3-singular value of $\mathcal{A}$ (an observation first made in [Lim 2005]).

Our first result about tensor singular value follows directly from the undecidability of tensor bilinear feasibility over the rationals (Theorem 3.9).

THEOREM 5.2. *Tensor singular value is undecidable over the rationals.*

The following is immediate from Theorem 3.7, which was proved by a reduction from 3-colorability.

THEOREM 5.3. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Deciding whether $\sigma = 0$ is an (ℓ^2 or ℓ^3)-singular value over $\mathbb{F}$ of a tensor is NP-hard.*

The tools we developed in Section 3 also directly apply to give an analogue of Theorem 1.6 for approximating singular vectors of a tensor corresponding to singular value $\sigma = 0$.

THEOREM 5.4. *It is NP-hard to approximate a triple of tensor singular vectors over $\mathbb{R}$ to within $\varepsilon = \frac{3}{4}$ and over $\mathbb{C}$ to within $\varepsilon = \frac{\sqrt{3}}{2}$.*

COROLLARY 5.5. *Unless P = NP, there is no PTAS for approximating tensor singular vectors.*

Note that verifying whether $0 \neq \sigma \in \mathbb{Q}$ is a singular value of a tensor $\mathcal{A}$ is the same as checking whether 1 is a singular value of $\mathcal{A}' = \mathcal{A}/\sigma$. In this section, we shall reduce computing the max-clique number of a graph to the singular value decision problem for $\sigma = 1$, extending some ideas from [Nesterov 2003] and [He et al. 2010]. In particular, we shall prove the following.

THEOREM 5.6. *Fix $0 \neq \sigma \in \mathbb{Q}$. Deciding whether σ is an ℓ^2-singular value over $\mathbb{R}$ of a tensor is NP-hard.*

We next define the spectral norm of a tensor, a concept closely related to singular value.

Definition 5.7. The **spectral norm** of a tensor $\mathcal{A}$ is

$$\|\mathcal{A}\|_{2,2,2} := \sup_{\mathbf{x},\mathbf{y},\mathbf{z} \neq \mathbf{0}} \frac{|\mathcal{A}(\mathbf{x}, \mathbf{y}, \mathbf{z})|}{\|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \|\mathbf{z}\|_2}.$$

The spectral norm is either the maximum or minimum value of $\mathcal{A}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ constrained to the set $\{(\mathbf{x}, \mathbf{y}, \mathbf{z}) : \|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = \|\mathbf{z}\|_2 = 1\}$, and thus is an ℓ^2-singular value of $\mathcal{A}$. At the end of this section, we will show that the corresponding spectral norm questions are NP-hard (Theorems 1.11 and 1.12).

We now explain our setup for the proof of Theorem 5.6. Let $G = (V, E)$ be a simple graph on vertices $V = \{1, \dots, n\}$ with e edges E, and let $\omega = \omega(G)$ be the *clique number* of G (that is, the number of vertices in a largest clique). An important result linking an optimization problem to ω is the following classical theorem [Motzkin and Straus 1965]. It can be used to give an elegant proof of Turán's Graph Theorem, which bounds the number of edges in a graph in terms of its clique number (e.g., see [Aigner 1995]).

THEOREM 5.8 (MOTZKIN-STRAUS). *Let* $\Delta_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n x_i = 1\}$ *and let* $G = (V, E)$ *be a graph on* n *vertices with clique number* $\omega(G)$. *Then,*

$$1 - \frac{1}{\omega(G)} = 2 \cdot \max_{\mathbf{x} \in \Delta_n} \sum\nolimits_{\{i,j\} \in E} x_i x_j.$$

Let A_G be the adjacency matrix of the graph G. For each positive integer l, define $Q_l := A_G + \frac{1}{l} J$, in which J is the all-ones matrix. Also, let

$$M_l := \max_{\mathbf{x} \in \Delta_n} \mathbf{x}^\top Q_l \mathbf{x} = 1 + \frac{\omega - l}{l\omega}.$$

We have $M_\omega = 1$ and also

$$M_l < 1 \ \text{ if } \ l > \omega; \quad M_l > 1 \ \text{ if } \ l < \omega. \tag{21}$$

For $k = 1, \ldots, e$, let $E_k = \frac{1}{2} E_{i_k j_k} + \frac{1}{2} E_{j_k i_k}$ in which $\{i_k, j_k\}$ is the kth edge of G. Here, the $n \times n$ matrix E_{ij} has a 1 in the (i, j)-th spot and zeroes elsewhere. For each positive integer l, consider the following optimization problem (having rational input):

$$N_l := \max_{\|\mathbf{u}\|_2 = 1} \left\{ \sum_{i=1}^{l} \left(\mathbf{u}^\top \frac{1}{l} I \mathbf{u} \right)^2 + \sum_{k=1}^{e} (\mathbf{u}^\top E_k \mathbf{u})^2 + \sum_{k=1}^{e} (\mathbf{u}^\top E_k \mathbf{u})^2 \right\}.$$

LEMMA 5.9. *For any graph* G, *we have* $M_l = N_l$.

PROOF. By construction, $N_l = \frac{1}{l} + 2 \cdot \max_{\|\mathbf{u}\|_2 = 1} \sum_{\{i,j\} \in E} u_i^2 u_j^2$, which is easily seen to equal M_l. □

Using an idea that we learned from [He et al. 2010], we obtain the following.

LEMMA 5.10.

$$M_l = \max_{\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1} \left\{ \sum_{i=1}^{l} \left(\mathbf{u}^\top \frac{1}{l} I \mathbf{v} \right)^2 + \sum_{k=1}^{e} (\mathbf{u}^\top E_k \mathbf{v})^2 + \sum_{k=1}^{e} (\mathbf{u}^\top E_k \mathbf{v})^2 \right\}.$$

PROOF. The proof can be found in [He et al. 2010]. The argument uses the Karush-Kuhn-Tucker conditions to find a maximizer with $\mathbf{v} = \pm\mathbf{u}$ for any optimization problem involving sums of squares of bilinear forms in $\mathbf{u}, \mathbf{v}$. □

Next, let

$$T_l := \max_{\|\mathbf{u}\| = \|\mathbf{v}\|_2 = \|\mathbf{w}\|_2 = 1} \left\{ \sum_{i=1}^{l} \left(\mathbf{u}^\top \frac{1}{l} I \mathbf{v} \right) w_i + \sum_{k=1}^{e} (\mathbf{u}^\top E_k \mathbf{v}) w_{l+k} + \sum_{k=1}^{e} (\mathbf{u}^\top E_k \mathbf{v}) w_{m+l+k} \right\}. \tag{22}$$

LEMMA 5.11. *The optimization problem* (22) *has* $T_l = M_l^{1/2}$. *Thus,*

$$T_l = 1 \textit{ iff } l = \omega; \quad T_l > 1 \textit{ iff } l < \omega; \quad \textit{and } \ T_l < 1 \textit{ iff } l > \omega.$$

PROOF. Fixing $\mathbf{a} = [a_1, \ldots, a_s]^\top \in \mathbb{R}^s$, the Cauchy-Schwarz inequality implies that a sum $\sum_{i=1}^s a_i w_i$ with $\|\mathbf{w}\|_2 = 1$ achieves a maximum value of $\|\mathbf{a}\|_2$ (with $w_i = a_i / \|\mathbf{a}\|_2$ if

$\|\mathbf{a}\|_2 \neq 0$). Thus,

$$\begin{aligned}
T_l &= \max_{\|\mathbf{u}\|_2=\|\mathbf{v}\|_2=\|\mathbf{w}\|_2=1} \sum_{i=1}^{l}\left(\mathbf{u}^\top \frac{1}{l} I\mathbf{v}\right) w_i + \sum_{k=1}^{e}(\mathbf{u}^\top E_k\mathbf{v})w_{l+k} + \sum_{k=1}^{e}(\mathbf{u}^\top E_k\mathbf{v})w_{e+l+k}\\
&= \max_{\|\mathbf{u}\|_2=\|\mathbf{v}\|_2=1} \sqrt{\sum_{i=1}^{l}\left(\mathbf{u}^\top \frac{1}{l} I\mathbf{v}\right)^2 + \sum_{k=1}^{e}(\mathbf{u}^\top E_k\mathbf{v})^2 + \sum_{k=1}^{e}(\mathbf{u}^\top E_k\mathbf{v})^2}\\
&= M_l^{1/2}.
\end{aligned}$$

□

We are now prepared to prove Theorem 5.6, and Theorems 1.11 and 1.12 from the introduction.

Proof of Theorem 5.6. We cast (22) in the form of a tensor singular value problem. Set $\mathcal{A}_l$ to be the three dimensional tensor with a_{ijk} equal to the coefficient of the term $u_i v_j w_k$ in the multilinear form (22). Then, T_l is just the maximum ℓ^2-singular value of $\mathcal{A}_l$. We now show that if we could decide whether $\sigma = 1$ is an ℓ^2-singular value of $\mathcal{A}_l$, then we would solve the max-clique problem.

Given a graph G, construct the tensor $\mathcal{A}_l$ for each integer $l \in \{1, \dots, n\}$. The largest value of l for which 1 is a singular value of $\mathcal{A}_l$ is $\omega(G)$. To see this, notice that if l is larger than $\omega(G)$, the maximum singular value of $\mathcal{A}_l$ is smaller than 1 by Lemma 5.11. Therefore, $\sigma = 1$ cannot be a singular value of $\mathcal{A}_l$ in these cases. However, $\sigma = 1$ is a singular value of the tensor $\mathcal{A}_\omega$. □

Proof of Theorem 1.11. In the reduction above used to prove Theorem 5.6, it suffices to decide whether the tensors $\mathcal{A}_l$ have spectral norm equal to 1. □

Proof of Theorem 1.12. Suppose that we could approximate spectral norm to within a factor of $1-\varepsilon = (1+1/N(N-1))^{-1/2}$, where N is tensor input size. Consider the tensors $\mathcal{A}_l$ as in the proof of Theorems 5.6 and 1.11 above, which have input size N at least the number of vertices n of the graph G. For each l, we are guaranteed an approximation for the spectral norm of $\mathcal{A}_l$ of at least

$$(1-\varepsilon)\cdot M_l^{1/2} > \left(1 + \frac{1}{n(n-1)}\right)^{-1/2}\left(1 + \frac{\omega - l}{l\omega}\right)^{1/2}. \tag{23}$$

It is easy to verify that (23) implies that any spectral norm approximation of $\mathcal{A}_l$ is greater than 1 whenever $l \leq \omega - 1$. In particular, as $\mathcal{A}_\omega$ has spectral norm exactly 1, we can determine $\omega(G)$ by finding the largest $l = 1, \dots, n$ for which a spectral norm approximation of $\mathcal{A}_l$ is 1 or less. □

6. BEST RANK-1 TENSOR APPROXIMATION IS NP-HARD

For $r > 1$, the best rank-r approximation problem for tensors (in Frobenius norm $\|\cdot\|_F$),

$$\min_{\mathbf{x}_i,\mathbf{y}_i,\mathbf{z}_i} \|\mathcal{A} - \lambda_1\mathbf{x}_1\otimes\mathbf{y}_1\otimes\mathbf{z}_1 - \cdots - \lambda_r\mathbf{x}_r\otimes\mathbf{y}_r\otimes\mathbf{z}_r\|_F,$$

does not necessarily have a solution because the set $\{\mathcal{A} \in \mathbb{F}^{l\times m\times n} : \operatorname{rank}_{\mathbb{F}}(\mathcal{A}) \leq r\}$ is in general not closed when $r > 1$. The following simple example is based on an exercise in [Knuth 1998].

Example 6.1. Let $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{F}^m$, $i = 1, 2, 3$. Let

$$\mathcal{A} = \mathbf{x}_1\otimes\mathbf{x}_2\otimes\mathbf{y}_3 + \mathbf{x}_1\otimes\mathbf{y}_2\otimes\mathbf{x}_3 + \mathbf{y}_1\otimes\mathbf{x}_2\otimes\mathbf{x}_3,$$

and for $n \in \mathbb{N}$, let

$$\mathcal{A}_n = \mathbf{x}_1 \otimes \mathbf{x}_2 \otimes (\mathbf{y}_3 - n\mathbf{x}_3) + \left(\mathbf{x}_1 + \frac{1}{n}\mathbf{y}_1\right) \otimes \left(\mathbf{x}_2 + \frac{1}{n}\mathbf{y}_2\right) \otimes n\mathbf{x}_3.$$

One may show that $\operatorname{rank}_{\mathbb{F}}(\mathcal{A}) = 3$ if and only if $\mathbf{x}_i, \mathbf{y}_i$ are linearly independent, $i = 1, 2, 3$. Since $\operatorname{rank}_{\mathbb{F}}(\mathcal{A}_n) \leq 2$ and

$$\lim_{n\to\infty} \mathcal{A}_n = \mathcal{A},$$

the rank-3 tensor $\mathcal{A}$ has no best rank-2 approximation. $\square$

The phenomenon of a tensor failing to have a best rank-r approximation is quite widespread, occurring over a wide range of dimensions, orders, and ranks, regardless of the choice of norm (or Brègman divergence) used. These counterexamples occur with positive probability and in some cases with certainty (in $\mathbb{F}^{2\times2\times2}$, no tensor of rank-3 has a best rank-2 approximation). We refer the reader to [De Silva and Lim 2008] for further details.

On the other hand, the set of rank-1 tensors (together with zero) is closed. In fact, it is the Segre variety in classical algebraic geometry [Landsberg 2012]. Consider the problem of finding the best rank-1 approximation to a tensor $\mathcal{A}$:

$$\min_{\mathbf{x},\mathbf{y},\mathbf{z}} \|\mathcal{A} - \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}\|_F. \tag{24}$$

By introducing an additional parameter $\sigma \geq 0$, we may rewrite the rank-1 term in the form $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} = \sigma\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}$ where $\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = \|\mathbf{w}\|_2 = 1$. Then,

$$\begin{aligned}\|\mathcal{A} - \sigma\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}\|_F^2 &= \|\mathcal{A}\|_F^2 - 2\sigma\langle\mathcal{A}, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}\rangle + \sigma^2\|\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}\|_F^2 \\ &= \|\mathcal{A}\|_F^2 - 2\sigma\langle\mathcal{A}, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}\rangle + \sigma^2.\end{aligned}$$

This expression is minimized when

$$\sigma = \max_{\|\mathbf{u}\|_2=\|\mathbf{v}\|_2=\|\mathbf{w}\|_2=1} \langle\mathcal{A}, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}\rangle = \|\mathcal{A}\|_{2,2,2}$$

since

$$\langle\mathcal{A}, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}\rangle = \mathcal{A}(\mathbf{u}, \mathbf{v}, \mathbf{w}).$$

If $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a solution to the optimization problem (24), then σ may be computed as

$$\sigma = \|\sigma\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}\|_F = \|\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}\|_F = \|\mathbf{x}\|_2\|\mathbf{y}\|_2\|\mathbf{z}\|_2.$$

We conclude that the best rank-1 approximation is also NP-hard, which is Theorem 1.14 from the introduction. We will see in Section 9 that restricting to symmetric 3-tensors does not make the best rank-1 approximation problem any more tractable.

7. TENSOR RANK IS NP-HARD

It was shown in [Håstad 1990] that any 3SAT boolean formula[10] can be encoded as a 3-tensor $\mathcal{A}$ over a finite field or $\mathbb{Q}$ and that the satisfiability of the formula is equivalent to checking if $\operatorname{rank}(A) \leq r$ for some r depending on the number of variables and clauses (the tensor rank being taken over the respective field). In particular, tensor rank is NP-hard over $\mathbb{Q}$ and NP-complete over finite fields.

Since the majority of the recent applications of tensor methods are over $\mathbb{R}$ and $\mathbb{C}$, a natural question is whether tensor rank is also NP-hard over these fields. In other words, is it NP-hard to decide whether $\operatorname{rank}_{\mathbb{R}}(\mathcal{A}) \leq r$ or if $\operatorname{rank}_{\mathbb{C}}(\mathcal{A}) \leq r$ for a given tensor $\mathcal{A}$ with rational entries and a given $r \in \mathbb{N}$?

[10]Recall that this a boolean formula in n variables and m clauses where each clause contains exactly three variables, e.g. $(x_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (x_1 \vee x_2 \vee x_4)$.

One difficulty with the notion of tensor rank is that it depends on the base field. For instance, there are real tensors with rank over $\mathbb{C}$ strictly less than their rank over $\mathbb{R}$ [De Silva and Lim 2008]. We will show here that the same can happen for tensors with rational entries. In particular, Håstad's result for tensor rank over $\mathbb{Q}$ does not directly apply to $\mathbb{R}$ and $\mathbb{C}$. Nevertheless, Håstad's proof shows, as we explain below in Theorem 7.2, that tensor rank remains NP-hard over both $\mathbb{R}$ and $\mathbb{C}$.

Proof of Theorem 1.15. We construct a rational tensor $\mathcal{A}$ with $\operatorname{rank}_{\mathbb{R}}(\mathcal{A}) < \operatorname{rank}_{\mathbb{Q}}(\mathcal{A})$ explicitly. Let $\mathbf{x} = [1, 0]^\top$ and $\mathbf{y} = [0, 1]^\top$. First observe that

$$\overline{\mathbf{z}} \otimes \mathbf{z} \otimes \overline{\mathbf{z}} + \mathbf{z} \otimes \overline{\mathbf{z}} \otimes \mathbf{z} = 2\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x} - 4\mathbf{y} \otimes \mathbf{y} \otimes \mathbf{x} + 4\mathbf{y} \otimes \mathbf{x} \otimes \mathbf{y} - 4\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{y} \in \mathbb{Q}^{2\times 2\times 2},$$

where $\mathbf{z} = \mathbf{x} + \sqrt{2}\mathbf{y}$ and $\overline{\mathbf{z}} = \mathbf{x} - \sqrt{2}\mathbf{y}$. Let $\mathcal{A}$ be this tensor; thus, $\operatorname{rank}_{\mathbb{R}}(\mathcal{A}) \le 2$. We claim that $\operatorname{rank}_{\mathbb{Q}}(\mathcal{A}) > 2$. Suppose not and that there exist $\mathbf{u}_i = [a_i, b_i]^\top$, $\mathbf{v}_i = [c_i, d_i]^\top \in \mathbb{Q}^2$, $i = 1, 2, 3$, with

$$\mathcal{A} = \mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \mathbf{u}_3 + \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3. \tag{25}$$

Identity (25) gives eight equations found in (26). Thus, by Lemma 7.1, $\operatorname{rank}_{\mathbb{Q}}(\mathcal{A}) > 2$. □

Lemma 7.1. *The system of 8 equations in 12 unknowns:*

$$\begin{aligned} &a_1a_2a_3 + c_1c_2c_3 = 2,\ a_1a_3b_2 + c_1c_3d_2 = 0,\ a_2a_3b_1 + c_2c_3d_1 = 0,\\ &a_3b_1b_2 + c_3d_1d_2 = -4,\ a_1a_2b_3 + c_1c_2d_3 = 0,\ a_1b_2b_3 + c_1d_2d_3 = -4,\\ &a_2b_1b_3 + c_2d_3d_1 = 4,\ b_1b_2b_3 + d_1d_2d_3 = 0 \end{aligned} \tag{26}$$

has no solution in rational numbers a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, and d_1, d_2, d_3.

Proof. One may verify in exact symbolic arithmetic (see the Appendix) that the following two equations are polynomial consequences of (26):

$$2c_2^2 - d_2^2 = 0 \quad \text{and} \quad c_1d_2d_3 - 2 = 0.$$

Since no rational number when squared equals 2, the first equation implies that any rational solution to (26) must have $c_2 = d_2 = 0$, an impossibility by the second. Thus, no rational solutions exist. □

We now provide an addendum to Håstad's result.

Theorem 7.2. *Tensor rank is NP-hard over fields $\mathbb{F} \supseteq \mathbb{Q}$; in particular, over $\mathbb{R}$ and $\mathbb{C}$.*

Proof. [Håstad 1990] contains a recipe for encoding any given 3SAT Boolean formula in n variables and m clauses as a tensor $\mathcal{A} \in \mathbb{F}^{(n+2m+2)\times 3n\times(3n+m)}$ with the property that the 3SAT formula is satisfiable if and only if $\operatorname{rank}_{\mathbb{F}}(\mathcal{A}) \le 4n + 2m$. The recipe defines $(n + 2m + 2) \times 3n$ matrices for $i = 1, \dots, n$, $j = 1, \dots, m$:

— V_i: 1 in $(1, 2i - 1)$ and $(2, 2i)$, 0 elsewhere;
— S_i: 1 in $(1, 2n + i)$, 0 elsewhere;
— M_i: 1 in $(1, 2i - 1)$, $(2 + i, 2i)$ and $(2 + i, 2n + i)$, 0 elsewhere;
— C_j: depends on jth clause (more involved), and has entries $0, \pm 1$;

and the 3-tensor

$$\mathcal{A} = [V_1, \dots, V_n, S_1, \dots, S_n, M_1, \dots, M_n, C_1, \dots, C_m] \in \mathbb{F}^{(n+2m+2)\times 3n\times(3n+m)}.$$

Observe that the matrices V_i, S_i, M_i, C_i are defined with only $-1, 0, 1$ and that the argument in [Håstad 1990] uses only the axioms for fields. In particular, it holds for any $\mathbb{F} \supseteq \mathbb{Q}$. □

7.1. Small Tensor Rank is in RP

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ in this section and rank be over either choice of fields. We have explained that it is NP-hard to check whether the rank of a specific $\mathcal{A} \in \mathbb{F}^{(n+2m+2)\times 3n\times(3n+m)}$ is not more than $4n + 2m$. In this case,

$$\min\{n + 2m + 2, 3n, 3n + m\} < 4n + 2m.$$

It turns out that if we know *a priori* that $\mathcal{A} \in \mathbb{F}^{l\times m\times n}$ has rank $r < \min\{l, m, n\}$, then r can be determined in polynomial time.

PROBLEM 7.3 (SMALL TENSOR RANK). *Given a tensor $\mathcal{A} \in \mathbb{F}^{l\times m\times n}$ and an integer $1 \leq r \leq \min\{l, m, n\} - 1$, determine if $\operatorname{rank}(\mathcal{A}) \leq r$.*

We first state a well-known result [Bürgisser et al. 1996, Theorem 14.45] giving two other equivalent characterizations of tensor rank.

THEOREM 7.4. *Let $\mathcal{A} = [\![a_{ijk}]\!]_{i,j,k=1}^{l,m,n} = [A_1, \dots, A_n] \in \mathbb{F}^{l\times m\times n}$ where $A_1, \dots, A_n \in \mathbb{F}^{l\times m}$ are given by $A_k = [a_{ijk}]_{i,j=1}^{l,m}$. The following are equivalent statements for $\operatorname{rank}(\mathcal{A}) \leq r$:*

(i) *there exist $\mathbf{x}_1, \dots, \mathbf{x}_r \in \mathbb{F}^l$, $\mathbf{y}_1, \dots, \mathbf{y}_r \in \mathbb{F}^m$, and $\mathbf{z}_1, \dots, \mathbf{z}_r \in \mathbb{F}^m$ such that*

$$\mathcal{A} = \mathbf{x}_1 \otimes \mathbf{y}_1 \otimes \mathbf{z}_1 + \cdots + \mathbf{x}_r \otimes \mathbf{y}_r \otimes \mathbf{z}_r;$$

(ii) *there exist $\mathbf{x}_1, \dots, \mathbf{x}_r \in \mathbb{F}^l$ and $\mathbf{y}_1, \dots, \mathbf{y}_r \in \mathbb{F}^m$ with*

$$\operatorname{span}\{A_1, \dots, A_n\} \subseteq \operatorname{span}\{\mathbf{x}_1\mathbf{y}_1^\top, \dots, \mathbf{x}_r\mathbf{y}_r^\top\};$$

(iii) *there exist diagonal matrices $D_1, \dots, D_n \in \mathbb{F}^{r\times r}$ and $X \in \mathbb{F}^{l\times r}$, $Y \in \mathbb{F}^{m\times r}$ such that*

$$A_k = XD_kY^\top, \quad k = 1, \dots, n.$$

Furthermore if we demand that r be the smallest integer with these properties, then they are each equivalent to $\operatorname{rank}(\mathcal{A}) = r$.

Our result follows from either the second or third characterization (we use the third).

THEOREM 7.5. *The small tensor rank problem is solvable in randomized polynomial time.*

PROOF. Let $r \in \{1, \dots, \min\{l, m, n\} - 1\}$. By Theorem 7.4, if $\operatorname{rank}(\mathcal{A}) \leq r$, then for any $\lambda_1, \dots, \lambda_n \in \mathbb{F}$:

$$\operatorname{rank}(\lambda_1 A_1 + \cdots + \lambda_n A_n) = \operatorname{rank}(X(\lambda_1 D_1 + \cdots + \lambda_n D_n)Y^\top) \leq r,$$

as $D = \lambda_1 D_1 + \cdots + \lambda_n D_n \in \mathbb{F}^{r\times r}$. Since one may determine matrix rank in polynomial time, the inequality may be checked in polynomial time. □

We remark that the small tensor rank problem is related to a special case of Edmonds' problem in matroid theory [Edmonds 1967].

PROBLEM 7.6 (EDMONDS). *Given a linear subspace $V \subseteq \mathbb{F}^{n\times n}$, decide if there exists a nonsingular matrix $A \in V$.*

In the special case when $V = \operatorname{span}\{A_1, \dots, A_n\}$ with the generators $A_1, \dots, A_n \in \mathbb{F}^{n\times n}$ given, Edmonds' problem is solvable in randomized polynomial time since it is equivalent to checking whether

$$\det\left(\sum_{i=1}^{n} x_i A_i\right)$$

is identically the zero polynomial [Gurvits 2003]. This is Problem 7.3 when $r = l = m$.

8. SYMMETRIC TENSOR EIGENVALUE IS NP-HARD

It is natural to ask if the eigenvalue problem remains NP-hard if the general tensor in (2) or (16) is replaced by a symmetric one.

PROBLEM 8.1 (SYMMETRIC TENSOR EIGENVALUE). *Given a symmetric 3-tensor $\mathcal{S} \in \mathbb{R}^{n\times n\times n}$, decide if $\lambda \in \mathbb{R}$ is an eigenvalue satisfying (2) or (16) for some nonzero $\mathbf{x} \in \mathbb{R}^n$.*

Let $G = (V, E)$ be a simple graph with vertices $V = \{1, \ldots, n\}$ and edges E. A subset of vertices $S \subseteq V$ is said to be *stable* (or *independent*) if $\{i, j\} \notin E$ for all $i, j \in S$, and the *stability number* $\alpha(G)$ is defined to be the size of a largest stable set. Nesterov has deduced from the Motzkin-Straus Theorem an analogue for the stability number [Nesterov 2003; De Klerk 2008].

THEOREM 8.2 (NESTEROV). *Let $G = (V, E)$ be a graph with n vertices and stability number $\alpha(G)$. Let $s = n+n(n-1)/2$ and $\mathbb{S}^{s-1} = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n\times\mathbb{R}^{n(n-1)/2} : \|\mathbf{x}\|_2^2+\|\mathbf{y}\|_2^2 = 1\}$. Then,*

$$\sqrt{1-\frac{1}{\alpha(G)}} = 3\sqrt{3}\cdot \max_{(\mathbf{x},\mathbf{y})\in\mathbb{S}^{s-1}} \sum\nolimits_{i<j,\,\{i,j\}\notin E} x_i x_j y_{ij}. \tag{27}$$

We will deduce the NP-hardness of symmetric tensor eigenvalue from the observation that every homogeneous cubic polynomial corresponds to a symmetric 3-tensor whose maximum eigenvalue is the maximum on the right-hand side of (27).

For any $1 \le i < j < k \le n$, let

$$s_{ijk} = \begin{cases} 1 & 1 \le i < j \le n,\ k = n + \varphi(i, j),\ \{i, j\} \notin E, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varphi(i, j) = (i-1)n-i(i-1)/2+j-i$ is a lexicographical enumeration of the $n(n-1)/2$ pairs $i < j$. For the other cases $i < k < j, \ldots, k < j < i$, we set

$$s_{ijk} = s_{ikj} = s_{jik} = s_{jki} = s_{kij} = s_{kji}.$$

Also, whenever two or more indices are equal, we put $s_{ijk} = 0$. This defines a symmetric tensor $\mathcal{S} = [\![s_{ijk}]\!] \in \mathbb{R}^{s\times s\times s}$ with the property that

$$\mathcal{S}(\mathbf{z}, \mathbf{z}, \mathbf{z}) = 6\sum\nolimits_{i<j,\,\{i,j\}\notin E} x_i x_j y_{ij},$$

where $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^{n(n-1)/2} = \mathbb{R}^s$.

Since $\lambda = \max_{\|\mathbf{z}\|_2=1} \mathcal{S}(\mathbf{z}, \mathbf{z}, \mathbf{z})$ is necessarily a stationary value of $\mathcal{S}(\mathbf{z}, \mathbf{z}, \mathbf{z})$ constrained to $\|\mathbf{z}\|_2 = 1$ and therefore an ℓ^2-eigenvalue of $\mathcal{S}$, Nesterov's Theorem implies that

$$\lambda = \sqrt{\frac{4}{3}\left(1-\frac{1}{\alpha(G)}\right)}$$

is an ℓ^2-eigenvalue of $\mathcal{S}$. It is well-known [Garey and Johnson 1979] that deciding whether $\alpha(G)$ is a particular number is NP-hard; therefore, we have the following.

THEOREM 8.3. *Symmetric tensor eigenvalue over $\mathbb{R}$ is NP-hard.*

PROOF. For $l = n, \ldots, 1$, we check if $\lambda_l = \sqrt{\frac{4}{3}\left(1-\frac{1}{l}\right)}$ is an ℓ^2-eigenvalue of $\mathcal{S}$. Since $\alpha(G) \in \{1, \ldots, n\}$, at most n answers to Problem 8.1 with inputs $\lambda_n, \ldots, \lambda_1$ (taken in decreasing order of magnitude so that the first eigenvalue identified would be the maximum) would reveal its value. Hence, Problem 8.1 is NP-hard. □

Remark 8.4. Here we have implicitly used the assumption that inputs to the symmetric tensor eigenvalue decision problem are allowed to be quadratic irrationalities of the form $\sqrt{\frac{4}{3}\left(1-\frac{1}{l}\right)}$ for each integer $l \in \{1, \ldots, n\}$.

It is also known that $\alpha(G)$ is NP-hard to approximate[11] [Håstad 1999; Zuckerman 2006] (see also the survey [De Klerk 2008]).

THEOREM 8.5 (HÅSTAD, ZUCKERMAN). *Unless $P = NP$, one cannot approximate $\alpha(G)$ of a graph G in polynomial time to within a factor of $n^{1-\varepsilon}$ for any $\varepsilon > 0$.*

Theorem 8.5 implies the following inapproximability result for symmetric tensors.

COROLLARY 8.6. *Unless $P = NP$, there is no FPTAS for approximating the largest ℓ^2-eigenvalue of a real symmetric tensor.*

9. SYMMETRIC SINGULAR VALUE, SPECTRAL NORM, AND RANK-1 APPROXIMATION ARE NP-HARD

We will deduce from Theorem 8.3 and Corollary 8.6 a series of hardness results for symmetric tensors parallel to earlier ones for the nonsymmetric case.

First, we state a recent technical result in [Friedland 2011] saying that the best rank-1 approximation of a symmetric tensor (by definition, a tensor $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$) and its best symmetric-rank-1 approximation (by definition, a symmetric tensor $\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x}$) may be chosen to be the same. Although we restrict to symmetric 3-tensors, Friedland's result holds for arbitrary order tensors.

THEOREM 9.1 (FRIEDLAND). *Let $\mathcal{S} \in \mathbb{R}^{n \times n \times n}$ be a symmetric 3-tensor. Then,*

$$\min_{\sigma \geq 0,\ \|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = \|\mathbf{w}\|_2 = 1} \|\mathcal{S} - \sigma \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}\|_F = \min_{\lambda \geq 0,\ \|\mathbf{v}\|_2 = 1} \|\mathcal{S} - \lambda \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v}\|_F. \tag{28}$$

Furthermore, the optimal σ and λ may be chosen to be equal.

Our discussion relating spectral norm, largest singular value, and best rank-1 approximation (for nonsymmetric tensors) in Section 6 carries over to symmetric tensors to give

$$\lambda = \max_{\|\mathbf{v}\|_2 = 1} \langle \mathcal{S}, \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v} \rangle = \|\mathcal{S}\|_{2,2,2},$$

where λ is the optimal solution for the right-hand side of (28). Theorem 9.1 implies that the spectral norm of a symmetric 3-tensor $\mathcal{S}$ may be expressed symmetrically:

$$\|\mathcal{S}\|_{2,2,2} = \sup_{\mathbf{x},\mathbf{y},\mathbf{z} \neq \mathbf{0}} \frac{|\mathcal{S}(\mathbf{x},\mathbf{y},\mathbf{z})|}{\|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \|\mathbf{z}\|_2} = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{|\mathcal{S}(\mathbf{x},\mathbf{x},\mathbf{x})|}{\|\mathbf{x}\|_2^3}.$$

Combining Theorem 8.3 and Corollary 8.6, we obtain the following.

THEOREM 9.2. *The following problems are all NP-hard over $\mathbb{F} = \mathbb{R}$:*

(i) *Deciding the largest ℓ^2-singular value or eigenvalue of a symmetric 3-tensor.*
(ii) *Deciding the spectral norm of a symmetric 3-tensor.*
(iii) *Determining the best symmetric rank-1 approximation of a symmetric 3-tensor.*

Furthermore, unless $P = NP$, there are no FPTAS for these problems.

PROOF. Let $\mathcal{S} \in \mathbb{R}^{n \times n \times n}$ be a symmetric 3-tensor. By Theorem 9.1, the optimal σ in the best rank-1 approximation of $\mathcal{S}$ (left-hand side of (28)) equals the optimal λ in the

[11] Håstad's original result [Håstad 1999] required $NP \neq ZPP$, but Zuckerman was able to weaken this assumption to $P \neq NP$.

best symmetric rank-1 approximation of $\mathcal{S}$ (right-hand side of (28)). Since the optimal σ is also the largest ℓ^2-singular value of $\mathcal{S}$ and the optimal λ is the largest ℓ^2-eigenvalue of $\mathcal{S}$, these coincide. Note that the optimal σ is also equal to the spectral norm $\|\mathcal{S}\|_{2,2,2}$. The NP-hardness and non-existence of FPTAS of problems (i)–(iii) now follow from Theorem 8.3 and Corollary 8.6. □

We note that Theorem 9.2 answers a question in [Brubaker and Vempala 2009] about the computational complexity of spectral norm for symmetric tensors.

10. TENSOR NONNEGATIVE DEFINITENESS IS NP-HARD

We say that $\mathcal{S}$ is *nonnegative definite* if $\mathcal{S}(\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x})$ is a nonnegative polynomial; i.e.,

$$\mathcal{S}(\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x}) = \sum\nolimits_{i,j,k,l=1}^{n} s_{ijkl}x_i x_j x_k x_l \geq 0, \quad \text{for all } \mathbf{x} \in \mathbb{R}^n. \tag{29}$$

We consider tensors of order 4 instead of 3 as in other sections because any symmetric 3-tensor $\mathcal{S} \in \mathbb{R}^{n\times n\times n}$ is *indefinite* since $\mathcal{S}(\mathbf{x},\mathbf{x},\mathbf{x})$ can take both positive and negative values (as $\mathcal{S}(-\mathbf{x},-\mathbf{x},-\mathbf{x}) = -\mathcal{S}(\mathbf{x},\mathbf{x},\mathbf{x})$). In general, odd-ordered symmetric tensors are indefinite.

We will deduce the NP-hardness of nonnegative definiteness from a well-known result in [Murty and Kabadi 1987]. Recall that a matrix $A \in \mathbb{R}^{n\times n}$ is said to be *copositive* if $A(\mathbf{y},\mathbf{y}) = \mathbf{y}^\top A\mathbf{y} \geq 0$ for all $\mathbf{y} \geq \mathbf{0}$.

THEOREM 10.1 (MURTY-KABADI). *Deciding copositiviy is NP-hard.*

Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be given. Consider the symmetric 4-tensor $\mathcal{S} = [\![s_{ijkl}]\!] \in \mathbb{R}^{n\times n\times n\times n}$:

$$s_{ijkl} = \begin{cases} a_{ij} + a_{ji} & \text{if } i = k \text{ and } j = l, \\ 0 & \text{otherwise.} \end{cases}$$

The tensor $\mathcal{S}$ is nonnegative definite if and only if

$$\mathcal{S}(\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x}) = \sum_{i,j,k,l=1}^{n} s_{ijkl}x_i x_j x_k x_l = 2\sum_{i,j=1}^{n} a_{ij}x_i^2 x_j^2 \geq 0,$$

for all $\mathbf{x} \in \mathbb{R}^n$ if and only if $\mathbf{y}^\top A\mathbf{y} \geq 0$ for all $\mathbf{y} \geq \mathbf{0}$ $(y_i = x_i^2)$ if and only if A is copositive. Hence we have deduced the following.

THEOREM 10.2. *Deciding nonnegative definitness of symmetric 4-tensors is NP-hard.*

The following characterization of nonnegative definitness derives from the fact that ℓ^2- and ℓ^4-eigenvalues are Lagrange multipliers [Qi 2005, Theorem 5].

THEOREM 10.3 (QI). *The following are equivalent for a symmetric 4-tensor $\mathcal{S} \in \mathbb{R}^{n\times n\times n\times n}$:*

(i) *$\mathcal{S}$ is nonnegative definite.*
(ii) *All ℓ^2-eigenvalues of $\mathcal{S}$ are nonnegative.*
(iii) *All ℓ^4-eigenvalues of $\mathcal{S}$ are nonnegative.*

With this observation, the following is an immediate corollary from Theorem 10.2.

COROLLARY 10.4. *Determining the sign of the smallest ℓ^2-eigenvalue or ℓ^4-eigenvalue of symmetric 4-tensors is NP-hard.*

11. OPEN PROBLEMS

We have tried to be thorough in our list of tensor problems, but there are several that we have not studied. We state a few of them here as open problems. The first involve the hyperdeterminant. Let $\mathbb{Q}[\mathrm{i}] := \{a + b\mathrm{i} \in \mathbb{C} : a, b \in \mathbb{Q}\}$ be the field of *Gaussian rationals.*

CONJECTURE 11.1. *Let $l, m, n \in \mathbb{N}$ satisfy the GKZ condition* (9)*:*

$$l \leq m + n - 1, \quad m \leq l + n - 1, \quad n \leq l + m - 1,$$

and let $\operatorname{Det}_{l,m,n}$ *be the* $l \times m \times n$ *hyperdeterminant.*

(i) *Deciding* $\operatorname{Det}_{l,m,n}(\mathcal{A}) = 0$ *is an NP-hard decision problem for inputs* $\mathcal{A} \in \mathbb{Q}[\mathrm{i}]^{l \times m \times n}$.
(ii) *Is NP-hard to decide or approximate a value for inputs* $\mathcal{A} \in \mathbb{Q}[\mathrm{i}]^{l \times m \times n}$ *to:*

$$\min_{\operatorname{Det}_{l,m,n}(\mathcal{X})=0} \|\mathcal{A} - \mathcal{X}\|_F. \tag{30}$$

(iii) $\operatorname{Det}_{l,m,n}(\mathcal{A})$ *is #P-complete to compute for inputs* $\mathcal{A} \in \{0,1\}^{l \times m \times n}$.
(iv) *All statements above remain true in the special case* $l = m = n$.

The optimization problem (30) defines a notion of *condition number* for 3-tensors. Note that for a matrix $A \in \mathbb{C}^{n \times n}$, the corresponding problem has solution given by the Moore-Penrose inverse $X = A^\dagger$:

$$\min_{\det(X)=0} \|A - X\|_F = \|A^\dagger\|_F^{-1}.$$

The optimum value normalized by the norm of the input gives the reciprocal of the condition number:

$$\frac{\|A^\dagger\|_F^{-1}}{\|A\|_F} = \kappa_F(A)^{-1}.$$

One reason for our belief in the intractability of problems involving the hyperdeterminant is that checking whether the general multivariate resultant vanishes for a system of n polynomials in n variables is known to be NP-hard over any field [Grenet et al. 2010]. Theorem 3.7 strengthens this result by saying that these polynomials may be chosen to be bilinear forms. Conjecture 11.1(i) further specializes by stating that these bilinear forms (10).can be associated with a 3-tensor satisfying GKZ condition (9).

Another reason for our conjectures is that the hyperdeterminant is a complex object; for instance, the $2\times2\times2\times2$-hyperdeterminant has almost 2.9 million monomial terms [Huggins et al. 2008]. Of course, this does not necessarily force the problems above to be intractable. For instance, both the determinant and permanent of an $n \times n$ matrix have $n!$ terms, but one is efficiently computable while the other is #P-complete [Valiant 1979b].

In Section 7, we explained that tensor rank is NP-hard over any field extension of $\mathbb{Q}$, but we also showed that 'small tensor rank' is in RP. We did not investigate the corresponding questions for the symmetric rank of a symmetric tensor (Definition 6). We conjecture the following.

CONJECTURE 11.2. *Let* $\mathcal{S} \in \mathbb{F}^{n \times n \times n}$ *and* $r \in \mathbb{N}$. *Deciding if* $\operatorname{rank}_{\mathbb{F}}(\mathcal{S}) \leq r$ *is NP-hard.*

While tensor rank is NP-hard over $\mathbb{Q}$, we suspect from Table I that it is also undecidable.

CONJECTURE 11.3. *Tensor rank and symmetric tensor rank over* $\mathbb{Q}$ *are undecidable.*

We have shown that deciding the existence of an exact solution to a system of bilinear equations (11) is NP-hard. There are two closely related problems: (i) when the equalities in (11) are replaced by inequalities and (ii) when we seek an approximate least-squares solution to (11). These lead to multilinear variants of linear programming and linear least squares. We state them formally here.

CONJECTURE 11.4 (BILINEAR PROGRAMMING FEASIBILITY). *Let* $A_k, B_k, C_k \in \mathbb{R}^{n \times n}$ *and* $\alpha_k, \beta_k, \gamma_k \in \mathbb{R}$ *for each* $k = 1, \ldots, n$. *It is NP-hard to decide if the following set of* *inequalities:*

$$\begin{cases} \mathbf{y}^\top A_i \mathbf{z} \le \alpha_k, & i = 1, \dots, n; \\ \mathbf{x}^\top B_j \mathbf{z} \le \beta_k, & j = 1, \dots, n; \\ \mathbf{x}^\top C_k \mathbf{y} \le \gamma_k, & k = 1, \dots, n; \end{cases} \tag{31}$$

defines a nonempty subset of $\mathbb{F}^n$.

CONJECTURE 11.5 (BILINEAR LEAST SQUARES). *Given* $3n$ *coefficient matrices* $A_k, B_k, C_k \in \mathbb{R}^{n \times n}$ *and* $\alpha_k, \beta_k, \gamma_k \in \mathbb{R}$, $k = 1, \dots, n$, *the bilinear least squares problem:*

$$\min_{\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{F}^n} \sum\nolimits_{k=1}^{n} (\mathbf{x}^\top A_k \mathbf{y} - \alpha_k)^2 + (\mathbf{y}^\top B_k \mathbf{z} - \beta_k)^2 + (\mathbf{z}^\top C_k \mathbf{x} - \gamma_k)^2 \tag{32}$$

is NP-hard to approximate.

Unlike the situation of (11), where $\mathbf{x} = \mathbf{y} = \mathbf{z} = \mathbf{0}$ is considered a trivial solution, we can no longer disregard an all-zero solution in (31) or (32). Consequently, the problem of deciding whether a homogeneous system of bilinear equations (11) has a *nonzero* solution is not a special case of (31) or (32).

12. CONCLUSION

Although this paper argues that most tensor problems are NP-hard, we should not be discouraged in our search for solutions to them. For instance, while computations with Gröbner bases are doubly exponential in the worst case [Yap 2000, pp. 400], they nonetheless proved useful in the specific context of Theorem 1.15. It is also important to note that NP-hardness is an asymptotic property; e.g., it applies to scenarios where tensor size n goes to infinity. Nonetheless, in many applications, n is usually fixed and often very small; e.g., $n = 2 : |0\rangle, |1\rangle$ (qubits, [Miyake and Wadati 2002]), $n = 3 : x, y, z$ (spatial coordinates, [Schultz and Seidel 2008]), $n = 4 : A, C, G, T$ (DNA nucleobases, [Allman and Rhodes 2008]), etc. For example, while Theorem 10.2 gives an NP-hardness result for general n, the case $n = 3$ may be formulated as a tractable convex program that is useful in neuroimaging [Lim and Schultz 2012].

Bernd Sturmfels once made the remark to us that "All interesting problems are NP-hard." In light of this, we would like to view our article as evidence that most tensor problems are interesting.

APPENDIX

We give here the complete details for the proof of Lemma 7.1, which was used to prove Theorem 1.15. We used the symbolic computing software[12] SINGULAR, and in particular the function **lift** to find the polynomials $H_1, \dots, H_8$ and $G_1, \dots, G_8$ below. Define three sets

[12]One can use commercially available Maple, `http://www.maplesoft.com/products/maple`, or Mathematica, `http://www.wolfram.com/mathematica`; free SINGULAR, `http://www.singular.uni-kl.de`, Macaulay 2, `http://www.math.uiuc.edu/Macaulay2`, or Sage, `http://www.sagemath.org`; or numerical packages such as Bertini, `http://www.nd.edu/~sommese/bertini` and PHCpack, `http://homepages.math.uic.edu/~jan/download.html`.

of polynomials:

$$F_1 := a_1a_2a_3 + c_1c_2c_3 - 2,\ F_2 := a_1a_3b_2 + c_1c_3d_2,\ F_3 := a_2a_3b_1 + c_2c_3d_1,$$
$$F_4 := a_3b_1b_2 + c_3d_1d_2 + 4,\ F_5 := a_1a_2b_3 + c_1c_2d_3,\ F_6 := a_1b_2b_3 + c_1d_2d_3 + 4,$$
$$F_7 := a_2b_1b_3 + c_2d_1d_3 - 4,\ F_8 := b_1b_2b_3 + d_1d_2d_3.$$

$$G_1 := -\tfrac{1}{8}b_1b_2b_3c_2d_2 + \tfrac{1}{8}a_2b_1b_3d_2^2,\ G_2 := \tfrac{1}{8}b_1b_2b_3c_2^2 - \tfrac{1}{8}a_2b_1b_3c_2d_2,\ G_3 := -\frac{1}{2}c_2d_2,\ G_4 := \frac{1}{2}c_2^2,$$
$$G_5 := \frac{1}{8}a_3b_1b_2c_2d_2 - \frac{1}{8}a_2a_3b_1d_2^2,\ G_6 := -\frac{1}{8}a_3b_1b_2c_2^2 + \frac{1}{8}a_2a_3b_1c_2d_2,\ G_7 := -\tfrac{1}{2}c_1^2,$$
$$G_8 := \tfrac{1}{8}a_2a_3b_1c_1^2 - \tfrac{1}{8}a_1a_2a_3c_1d_1.$$
$$H_1 := 0,\ H_2 := -\tfrac{1}{32}b_1b_2b_3c_2d_1d_3 + \tfrac{1}{32}a_2b_1b_3d_1d_2d_3, H_3 := \tfrac{1}{32}b_1b_2b_3c_1d_2d_3 - \tfrac{1}{32}a_1b_2b_3d_1d_2d_3,$$
$$H_4 := \tfrac{1}{32}a_1b_2b_3c_2d_1d_3 - \tfrac{1}{32}a_2b_1b_3c_1d_2d_3,\ H_5 := -\tfrac{1}{8}b_1b_2b_3,\ H_6 := \tfrac{1}{2}, H_7 := \tfrac{1}{8}a_1b_2b_3 - \tfrac{1}{8}c_1d_2d_3,$$
$$H_8 := \tfrac{1}{8}c_1c_2d_3.$$

The polynomials $g = 2c_2^2 - d_2^2$ and $h = c_1d_2d_3 - 2$ are polynomial linear combinations of $F_1, \dots, F_8$:

$$g = \sum_{k=1}^{8} F_kG_k \quad \text{and} \quad h = \sum_{k=1}^{8} F_kH_k. \tag{33}$$

Thus, if a rational point makes $F_1, \dots, F_8$ all zero, then both g and h must also vanish on it. We remark that expressions such as (33) are far from unique.

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REFERENCES

M. Aigner, "Turán's graph theorem," *Amer. Math. Monthly*, **102** (1995), no. 9, pp. 808–816.

E.S. Allman and J.A. Rhodes, "Phylogenetic ideals and varieties for the general Markov model," *Adv. in Appl. Math.*, **40** (2008), no. 2, pp. 127–148.

N. Alon and A. Naor, "Approximating the cut-norm via Grothendieck's inequality," *SIAM J. Comput.*, **35** (2006), no. 4, pp. 787–803.

C. Bachoc, G. Nebe, F.M. de Oliveira Filho, and F. Vallentin, "Lower bounds for measurable chromatic numbers," *Geom. Funct. Anal.*, **19** (2009), no. 3, pp. 645–661.

C. Bachoc and F. Vallentin, "New upper bounds for kissing numbers from semidefinite programming," *J. Amer. Math. Soc.*, **21** (2008), no. 3, pp. 909–924.

A.I. Barvinok, "Feasibility testing for systems of real quadratic equations," *Discrete Comput. Geom.*, **10** (1993), no. 1, pp. 1–13.

A.I. Barvinok, "New algorithms for linear k-matroid intersection and matroid k-parity problems," *Math. Programming*, **69** (1995), no. 3, pp. 449–470.

D. Bayer, *The Division Algorithm and the Hilbert Scheme*, Ph.D. Thesis, Department of Mathematics, Harvard University, 1982.

C. Beltrán and L. M. Pardo, "Efficient polynomial system-solving by numerical methods," *J. Fixed Point Theory Appl.*, **6** (2009), no. 1, pp. 63–85.

E. Bernstein and U. Vazirani, "Quantum complexity theory," *SIAM J. Comput.*, **26** (1997), no. 5, pp. 1411–1473.

L. Blum, F. Cucker, M. Shub, and S. Smale, *Complexity and Real Computation*, Springer-Verlag, New York, NY, 1998.

L. Blum, M. Shub, and S. Smale, "On a theory of computation and complexity over the real numbers," *Bull. Amer. Math. Soc.*, **21** (1989), no. 1, pp. 1–46.

J. Bochnak, M. Coste, and M.F. Roy, *Real Algebraic Geometry*, 2nd Ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, **36**, Springer-Verlag, Berlin, 1998.

J. Briët, F.M. de Oliveira Filho, and F. Vallentin, "The positive semidefinite Grothendieck problem with rank constraint," pp. 31–42, in S. Abramsky et al. (Eds), *Automata, Languages and Programming*, Lecture Notes in Computer Science, **6198**, Springer-Verlag, Berlin, 2010.

J. Briët, F.M. de Oliveira Filho, and F. Vallentin, "Grothendieck inequalities for semidefinite programs with rank constraint," *preprint*, (2010), `http://arxiv.org/abs/1011.1754`.

S.C. Brubaker and S.S. Vempala , "Random tensors and planted cliques," pp. 409–419, in Dinur et al. (Eds.), *APPROX and RANDOM 2009*, Lecture Notes in Computer Science, **5687**, Springer-Verlag, Berlin, 2009.

P. Bürgisser, *Completeness and Reduction in Algebraic Complexity Theory*, Algorithms and Computations in Mathematics, **7**, Springer-Verlag, Berlin, 2000.

P. Bürgisser, M. Clausen, and M.A. Shokrollahi, *Algebraic Complexity Theory*, Grundlehren der mathematischen Wissenschaften, **315**, Springer-Verlag, Berlin, 1996.

D.A. Cartwright, S.M. Brady, D.A. Orlando, B. Sturmfels, and P.N. Benfey, "Reconstructing spatiotemporal gene expression data from partial observations," *Bioinform.*, **25** (2009) no. 19, pp. 2581–2587.

D.A. Cartwright and B. Sturmfels, "The number of eigenvalues of a tensor," *Linear Algebra Appl.*, (2012), to appear, `http://dx.doi.org/10.1016/j.laa.2011.05.040`.

A. Cayley, "On the theory of linear transformation," *Cambridge Math. J.*, **4** (1845), pp. 193–209.

S. Cohen and C. Tomasi. "Systems of bilinear equations," *Technical Report*, **97-1588**, Department of Computer Science, Stanford University, 1997.

P. Comon, "Blind identification and source separation in 2×3 under-determined mixtures," *IEEE Trans. Signal Process.*, **52** (2004), no. 1, pp. 11–22.

P. Comon, "Independent component analysis: a new concept?," *Signal Process.*, **36** (1994), no. 3, pp. 287–314.

P. Comon, G. Golub, L.-H. Lim, and B. Mourrain, "Symmetric tensors and symmetric tensor rank," *SIAM J. Matrix Anal. Appl.*, **30** (2008), no. 3, pp. 1254–1279.

S. Cook, "The complexity of theorem proving procedures," *Proc. ACM Symp. Theory Comput.* (STOC), **3** (1971), pp. 151–158.

R. Coppi and S. Bolasco (Eds.), *Multiway Data Analysis*, Elsevier Science, Amsterdam, Netherlands, 1989.

N. Courtois, L. Goubin, W. Meier, and J.-D. Tacier, "Solving underdefined systems of multivariate quadratic equations," pp. 211–227, in D. Naccache and P. Paillier (Eds.), *Public Key Cryptography* (PKC 2002), Lecture Notes in Computer Science, **2274**, Springer-Verlag, Berlin, 2002.

D.A. Cox, J.B. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms*, 3rd Ed., Springer-Verlag, New York, NY, 2007.

D.A. Cox, J.B. Little, and D. O'Shea, *Using Algebraic Geometry*, 2nd Ed., Springer-Verlag, New York, NY, 2005.

M. Davis, H. Putnam, and J. Robinson, "The decision problem for exponential diophantine equations," *Ann. of Math.*, **74** (1961), no. 3, pp. 425–436.

A. De, P.P. Kurur, C. Saha, and R. Saptharishi, "Fast integer multiplication using modular arithmetic," *Proc. ACM Symp. Theory Comput.* (STOC), **40** (2008), pp. 499–506.

E. De Klerk, "The complexity of optimizing over a simplex, hypercube or sphere: a short survey," *Cent. Eur. J. Oper. Res.*, **16** (2008), no. 2, pp. 111–125.

W.F. De La Vega, R. Kannan, M. Karpinski, and S. Vempala, "Tensor decomposition and approximation schemes for constraint satisfaction problems," *Proc. ACM Symp. Theory Comput.* (STOC), **37** (2005), pp. 747–754.

J.A. De Loera, "Gröbner bases and graph colorings," *Beiträge Algebra Geom.*, **36** (1995), no. 1, pp. 89–96.

J.A. De Loera, J. Lee, P.N. Malkin, and S. Margulies, "Hilbert's nullstellensatz and an algorithm for proving combinatorial infeasibility," *Proc. Int. Symposium Symb. Algebr. Comput.* (ISSAC), **21** (2008), pp. 197–206.

V. De Silva and L.-H. Lim, "Tensor rank and the ill-posedness of the best low-rank approximation problem," *SIAM J. Matrix Anal. Appl.*, **30** (2008), no. 3, pp. 1084–1127.

D. Deutsch, "Quantum theory, the Church-Turing principle and the universal quantum computer," *Proc. Roy. Soc. London Ser. A*, **400** (1985), no. 1818, pp. 97–117.

J. Edmonds, "Systems of distinct representatives and linear algebra," *J. Res. Nat. Bur. Standards*, **71B** (1967), no. 4, pp. 241–245.

L. Fortnow, "The status of the P versus NP problem," *Comm. ACM*, **52** (2009), no. 9, pp. 78–86.

L. Fortnow and J. Rogers, "Complexity limitations on quantum computation," *J. Comput. System Sci.*, **59** (1999), no. 2, pp. 240–252.

P.J. Freitas, S. Friedland, and G. Porta, "Upper bounds on the magnitude of solutions of certain linear systems with integer coefficients," *preprint*, (2011), `http://arxiv.org/abs/1108.4078`.

S. Friedland, "Best rank one approximation of real symmetric tensors can be chosen symmetric," *preprint*, (2011), `http://arxiv.org/abs/1110.5689`.

J. Friedman, "The spectra of infinite hypertrees," *SIAM J. Comput.*, **20** (1991), no. 5, pp. 951–961.

J. Friedman and A. Wigderson, "On the second eigenvalue of hypergraphs," *Combinatorica*, **15** (1995), no. 1, pp. 43–65.

M. Fürer, "Faster integer multiplication," *Proc. ACM Symp. Theory Comput.* (STOC), **39** (2007), pp. 57–66.

M.R. Garey and D.S. Johnson, *Computers and Intractability*, W.H. Freeman, San Francisco, CA, 1979.

I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser Publishing, Boston, MA, 1994.

I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, "Hyperdeterminants," *Adv. Math.*, **96** (1992), no. 2, pp. 226–263.

M.X. Goemans and D.P. Williamson, "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming," *J. Assoc. Comput. Mach.*, **42** (1995), no. 6, pp. 1115–1145.

G.H. Golub and W. Kahan, "Calculating the singular values and pseudo-inverse of a matrix," *J. Soc. Indust. Appl. Math. Ser. B*, **2** (1965), pp. 205–224.

G.H. Golub and C. Reinsch, "Singular value decomposition and least squares solutions," *Numer. Math.*, **14** (1970), no. 5, pp. 403–420.

B. Grenet, P. Koiran, and N. Portier, "The multivariate resultant is NP-hard in any characteristic," pp. 477–488 in P. Hliněný and A. Kučera (Eds), *Mathematical Foundations of Computer Science* (MFCS), Lecture Notes in Computer Science, **6281**, Springer-Verlag, Berlin, 2010.

M. Gu, "Finding well-conditioned similarities to block-diagonalize nonsymmetric matrices is NP-hard," *J. Complexity*, **11** (1995), no. 3, pp. 377–391.

F. Grunewald and D. Segal, "On the integer solutions of quadratic equations," *J. Reine Angew. Math.*, **569** (2004), pp. 13–45.

L. Gurvits, "Classical deterministic complexity of Edmond's problem and quantum entanglement," *Proc. ACM Symp. Theory Comput.* (STOC), **35** (2003), pp. 10–19.

D. Haskell, A. Pillay, and C. Steinhorn (Eds.), *Model Theory, Algebra, and Geometry*, Mathematical Sciences Research Institute Publications, **39**, Cambridge University Press, Cambridge, 2000.

J. Håstad, "Clique is hard to approximate within $n^{1-\varepsilon}$," *Acta Math.*, **182** (1999), no. 1, pp. 105–142.

J. Håstad, "Tensor rank is NP-complete," *J. Algorithms*, **11** (1990), no. 4, pp. 644–654.

S. He, Z. Li, and S. Zhang, "Approximation algorithms for homogeneous polynomial optimization with quadratic constraints," *Math. Programming Ser. B*, **125** (2010), no. 2, pp. 353–383.

C. Hillar and T. Windfeldt, "Algebraic characterization of uniquely vertex colorable graphs," *J. Combin. Theory Ser. B*, **98** (2008), no. 2, pp. 400–414.

F.L. Hitchcock, "The expression of a tensor or a polyadic as a sum of products," *J. Math. Phys.*, **6** (1927), no. 1, pp. 164–189.

F.L. Hitchcock, "Multiple invariants and generalized rank of a p-way matrix or tensor," *J. Math. Phys.*, **7** (1927), no. 1, pp. 39–79.

D. Hochbaum (Ed.), *Approximation Algorithms for NP-Hard Problems*, PWS Publishing, Boston, MA, 1997.

D.S. Hochbaum and J.G. Shanthikumar, "Convex separable optimization is not much harder than linear optimization," *J. Assoc. Comput. Mach.*, **37** (1990), no. 4, pp. 843–862.

P. Huggins, B. Sturmfels, J. Yu, and D.S. Yuster, "The hyperdeterminant and triangulations of the 4-cube," *Math. Comp.*, **77** (2008), no. 263, pp. 1653–1679.

J.P. Jones, "Universal Diophantine equation," *J. Symbolic Logic*, **47** (1982), no. 3, pp. 549–571.

J.P. Jones and Y.V. Matijasevič, "Register machine proof of the theorem on exponential diophantine representation of enumerable sets," *J. Symbolic Logic*, **49** (1984), no. 3, pp. 818–829.

W. Kahan, *IEEE Standard 754 for Binary Floating-Point Arithmetic*, `http://www.cs.berkeley.edu/~wkahan/ieee754status/ieee754.ps`, Berkeley, CA, 1997.

R.M. Karp, "Reducibility among combinatorial problems," pp. 85–103, in R.E. Miller and J.W. Thatcher (Eds), *Complexity of Computer Computations*, Plenum, New York, NY, 1972.

L.G. Khachiyan, "A polynomial algorithm in linear programming," *Dokl. Akad. Nauk SSSR*, **244** (1979), no. 5, pp. 1093–1096, English translation in *Soviet Math. Dokl.*, **20** (1979), no. 1, pp. 191–194.

S. Khot, G. Kindler, E. Mossel, and R. O'Donnell, "Optimal inapproximability results for MAX-CUT and other 2-variable CSPs?," *SIAM J. Comput.*, **37** (2007), no. 1, pp. 319–357.

D.E. Knuth, *The Art of Computer Programming*, **2**: seminumerical algorithms, 3rd Ed., Addision Wesley, Reading, MA, 1998.

E. Kofidis and P.A. Regalia, "On the best rank-1 approximation of higher-order supersymmetric tensors," *SIAM J. Matrix Anal. Appl.*, **23** (2001/02), no. 3, pp. 863–884.

T. Kolda and B. Bader, "Tensor decompositions and applications," *SIAM Rev.*, **51** (2009), no. 3, pp. 455–500.

B. Lambov, "RealLib: An efficient implementation of exact real arithmetic," *Mathematical. Structures in Comp. Sci.*, **17** (2007), no. 1, pp. 81–98.

J.M. Landsberg, *Tensors: Geometry and Applications*, AMS, Providence, RI, 2012.

L.A. Levin, "Universal sequential search problems," *Probl. Inf. Transm.*, **9** (1973) no. 3, pp. 265–266.

L.-H. Lim, "Singular values and eigenvalues of tensors: a variational approach," *Proc. IEEE Int. Workshop on Computational Advances in Multi-Sensor Adaptive Process.* (CAMSAP), **1** (2005), pp. 129–132.

L.-H. Lim and T. Schultz, "Harmonic analysis and algebraic geometry in diffusion magnetic resonance imaging," *preprint*, (2012), `http://www.stat.uchicago.edu/~lekheng/work/dmri.pdf`.

L. Lovász, "Stable sets and polynomials," *Discrete Math.*, **124** (1994), no. 1–3, pp. 137–153.

Y.V. Matijasevič, *Hilbert's Tenth Problem*, MIT Press, Cambridge, MA, 1993.

Y.V. Matijasevič, "The Diophantineness of enumerable sets," *Dokl. Akad. Nauk SSSR*, **191** (1970), no. 2, pp. 279–282, English translation in *Soviet Math. Dokl.*, **11** (1970), no. 2, pp. 354–358.

A. Miyake and M. Wadati, "Multipartite entanglement and hyperdeterminants," *Quantum Inf. Comput.*, **2** (2002), suppl., pp. 540–555.

T. Motzkin and E.G. Straus, "Maxima for graphs and a new proof of a theorem of Turán," *Canad. J. Math.*, **17** (1965), pp. 533–540.

K.G. Murty and S.N. Kabadi, "Some NP-complete problems in quadratic and nonlinear programming," *Math. Programming*, **39** (1987), no. 2, pp. 117–129.

Yu. Nesterov, "Random walk in a simplex and quadratic optimization over convex polytopes," *preprint*, (2003), `http://edoc.bib.ucl.ac.be:83/archive/00000238/01/dp2003-71.pdf`.

G. Ni, L. Qi, F. Wang, and Y. Wang, "The degree of the E-characteristic polynomial of an even order tensor," *J. Math. Anal. Appl.*, **329** (2007), no. 2, pp. 1218–1229.

M.L. Overton, *Numerical Computing with IEEE Floating Point Arithmetic*, SIAM, Philadelphia, PA, 2001.

V.Y. Pan, "Solving a polynomial equation: some history and recent progress," *SIAM Rev.*, **39** (1997), no. 2, pp. 187-220.

L. Qi, "Eigenvalues and invariants of tensors," *J. Math. Anal. Appl.* **325** (2007), no. 2, pp. 1363–1377.

L. Qi, "Eigenvalues of a real supersymmetric tensor," *J. Symbolic Comput.*, **40** (2005), no. 6, pp. 1302–1324.

B. Reznick, "On the length of binary forms," *preprint*, (2010), `http://arxiv.org/abs/1007.5485`.

B. Reznick, "Sums of even powers of real linear forms," *Mem. Amer. Math. Soc.*, **96** (1992), no. 463.

T. Schultz and H.-P. Seidel, "Estimating crossing fibers: a tensor decomposition approach," *IEEE Trans. Vis. Comput. Graphics*, **14** (2008), no. 6, pp. 1635–1642.

A. Shashua and T. Hazan, "Non-negative tensor factorization with applications to statistics and computer vision," *Proc. Int. Conf. Mach. Learn.* (ICML), **22** (2005), pp. 792–799.

J. Shi and J. Malik, "Normalized cuts and image segmentation," *IEEE Trans. Pattern Anal. Mach. Intell.*, **22** (2000), no. 8, pp. 888–905.

N.D. Sidiropoulos, R. Bro, and G.B. Giannakis, "Parallel factor analysis in sensor array processing," *IEEE Trans. Signal Process.*, **48** (2000), no. 8, pp. 2377–2388.

C.L. Siegel, "Zur Theorie der quadratischen Formen," *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, (1972), pp. 21–46.

S. Smale, "Mathematical problems for the next century," pp. 271–294 in: V. Arnold, M. Atiyah, P. Lax, and B. Mazur (Eds.), *Mathematics: Frontiers and Perspectives*, AMS, Providence, RI, 2000.

A. Smilde, R. Bro, and P. Geladi, *Multi-way Analysis: Applications in the Chemical Sciences*, John Wiley, West Sussex, 2004.

V. Strassen, "Gaussian elimination is not optimal," *Numer. Math.*, **13** (1969), no. 4, pp. 354–356.

A.M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proc. Lond. Math. Soc.*, 2nd Series, **42** (1937), no. 1, pp. 230–265, see also correction in **43** (1937), no. 6, pp. 544–546.

L.G. Valiant, "Completeness classes in algebra," *Proc. ACM Symp. Theory Comput.* (STOC), **11** (1979), pp. 249–261.

L.G. Valiant, “The complexity of computing the permanent,” *Theoret. Comput. Sci.*, **8** (1979), no. 2, pp. 189–201.

V. Vazirani, *Approximation Algorithms*, Springer-Verlag, Berlin, 2003.

M.A.O. Vasilescu, “Human motion signatures: analysis, synthesis, recognition,” *Proc. Int. Conf. Pattern Recognition* (ICPR), **3** (2002), pp. 456–460.

M.A.O. Vasilescu and D. Terzopoulos, “Multilinear image analysis for facial recognition,” *Proc. Int. Conf. Pattern Recognition* (ICPR), **2** (2002), pp. 511–514.

M.A.O. Vasilescu and D. Terzopoulos, “TensorTextures: multilinear image-based rendering,” *Proc. ACM SIGGRAPH*, **31** (2004), pp. 336–342.

S.A. Vavasis, *Nonlinear Optimization: Complexity Issues*, International Series of Monographs on Computer Science, **8**, Oxford University Press, New York, NY, 1991.

K. Weihrauch, *Computable Analysis: An Introduction*, Springer-Verlag, Berlin, 2000.

C.K. Yap, *Fundamental Problems of Algorithmic Algebra*, Oxford University Press, New York, NY, 2000.

T. Zhang and G.H. Golub, “Rank-one approximation to high order tensors,” *SIAM J. Matrix Anal. Appl.*, **23** (2001), no. 2, pp. 534–550.

D. Zuckerman, “Linear degree extractors and the inapproximability of max clique and chromatic number,” *Proc. ACM Symp. Theory Comput.* (STOC), **36** (2006), pp. 681–690.