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Enclosing Ellipsoids and Elliptic Cylinders of Semialgebraic Sets and Their Application to Error Bounds in Polynomial Optimization

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B-459 Enclosing Ellipsoids and Elliptic Cylinders of Semialgebraic Sets and Their Application to Error Bounds in Polynomial Optimization

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Abstract.
This paper is concerned with a class of ellipsoidal sets (ellipsoids and elliptic cylinders in $\mathbb{R}^m$) which are determined by a freely chosen $m \times m$ positive semidefinite matrix. All ellipsoidal sets in this class are similar to each other through a parallel transformation and a scaling around their centers by a constant factor. Based on the basic idea of lifting, we first present a conceptual min-max problem to determine an ellipsoidal set with the smallest size in this class which encloses a given subset of $\mathbb{R}^m$. Then we derive a numerically tractable enclosing ellipsoidal set of a given semialgebraic subset of $\mathbb{R}^m$ as a convex relaxation of the min-max problem in the lifting space. A main feature of the proposed method is that it is designed to incorporate into existing SDP relaxations with exploiting sparsity for various optimization problems to compute error bounds of their optimal solutions. We discuss how we adapt the method to a sparse variant of Lasserre's hierarchy SDP relaxation for polynomial optimization problems and to a standard SDP relaxation for quadratic optimization problems. Some numerical results on polynomial optimization problems and the sensor network localization problem are also presented.

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1 Introduction

We can describe an ellipsoid in the m-dimensional Euclidean space $\mathbb{R}^m$ by using two parameters, $\boldsymbol{z} \in \mathbb{R}^m$ and an $m \times m$ positive definite matrix $\boldsymbol{Q}$, such that

$$\mathcal{E}(\boldsymbol{z}, \boldsymbol{Q}) = \left\{ \boldsymbol{v} \in \mathbb{R}^m : (\boldsymbol{v} - \boldsymbol{z})^T \boldsymbol{Q} (\boldsymbol{v} - \boldsymbol{z}) \leq 1 \right\}.$$

Here $\boldsymbol{z} \in \mathbb{R}^m$ denotes the *center* of the ellipsoid and $\boldsymbol{Q}$ determines the shape and size of the ellipsoid. If a set $F \subset \mathbb{R}^m$ is bounded, there always exists an enclosing ellipsoid of F. But such an ellipsoid is not unique. Among enclosing ellipsoids, the minimum volume enclosing ellipsoid of F, which we denote by MVEE(F), is the most important one in theory and also in practice. If C is a compact convex subset of $\mathbb{R}^m$, MVEE(C) exists and it satisfies a nice property [13] $\frac{1}{m}$MVEE(C) $\subset C \subset$ MVEE(C), where the ellipsoid on the left-hand side is obtained by scaling MVEE(C) around its center $\boldsymbol{z}$ by a factor $1/m$. When F is a finite set, MVEE(F) has a lot of applications and several numerical methods has been proposed. See [14, 23, 34, etc.] and the references therein. It is well-known that the volume of an ellipsoid $\mathcal{E}(\boldsymbol{z}, \boldsymbol{Q})$ is proportional to $\sqrt{\det \boldsymbol{Q}^{-1}}$. Therefore we can write the problem of computing MVEE(F) as

$$\text{minimize} \ \ -\log\det \boldsymbol{Q} \ \text{subject to} \ \ F \subset \mathcal{E}(\boldsymbol{z}, \boldsymbol{Q}).$$

We note that the objective function $-\log\det \boldsymbol{Q}$ to be minimized is a smooth convex function in the $m \times m$ positive definite matrix variable $\boldsymbol{Q}$. What makes the problem difficult to solve is the description of the feasible region. If we can replace it by a numerically tractable description for a convex feasible region, we can solve the transformed problem. When F consists of a finite number of ellipsoids such a replacement is possible by linear matrix inequalities [4], and we can apply the interior-point point method [35, 37] to the transformed problem. Also Yildirim [40] proposed a numerical method for this case as a modification and extension of Khachiyan's algorithm [14] for the case where F is finite.

This paper is concerned with a more general case where $F \subset \mathbb{R}^m$ forms a semialgebraic set described by a finite number of polynomial inequalities. Recently, Nie and Demmel [29] proposed a numerical method for approximating a "minimum" ellipsoid for this case based on sum of squares relaxation. They employed Trace $\boldsymbol{Q}^{-1}$ to measure the size of $\mathcal{E}(\boldsymbol{z}, \boldsymbol{Q})$, and formulated the problem of finding an enclosing ellipsoid $\mathcal{E}(\boldsymbol{z}, \boldsymbol{Q})$ of F with the smallest Trace $\boldsymbol{Q}^{-1}$ as

$$\text{minimize} \ \ \text{Trace}\, \boldsymbol{P} \quad \text{subject to} \quad 1 - (\boldsymbol{v} - \boldsymbol{z})^T \boldsymbol{P}^{-1} (\boldsymbol{v} - \boldsymbol{z}) \leq 0 \text{ for every } \boldsymbol{v} \in F.$$

Applying Putinar's Lemma [31] on sum of squares of polynomials to the constraint, they derived a sum of squares relaxation of the problem.

The Nie–Demmel sum of square relaxation method is very powerful in theory. In fact, they showed under a moderate assumption that the optimal value of the resulting relaxed problem, which can be solved as an SDP, converges to the optimal value of the original problem as the degree of the polynomials used in the sum of squares relaxation tends to infinity. They also presented some small-size numerical examples. In practice, however, the direct use of their method is expected too expensive to solve a larger-size instance because the resulting SDP to solve becomes larger exponentially as the underlying dimension m is larger, the description of F involves more polynomials and/or the degrees of some polynomials there are higher. This drawback might be inevitable in some extent as long as we use sum of squares (or SDP) relaxation technique since it is known expensive itself.

The purpose of the paper is to propose a less expensive and more practical ellipsoidal enclosing method for a semialgebraic set $F \subset \mathbb{R}^m$. For this purpose, we abandon MVEE(F). Instead, we restrict ourselves to minimization of the size of an enclosing ellipsoidal set (ellipsoid or elliptic cylinder) of F with a given fixed shape. More specifically, we consider a class of ellipsoidal sets of the form

$$E(\boldsymbol{z}, \rho) = \{\boldsymbol{v} \in \mathbb{R}^m : (\boldsymbol{v} - \boldsymbol{z})^T \boldsymbol{M} (\boldsymbol{v} - \boldsymbol{z}) \le \rho\}.$$

Here $\boldsymbol{M}$ denotes an $m \times m$ positive semidefinite matrix chosen freely in advance. Note that we allow the case where $\det \boldsymbol{M} = 0$ so that the resulting $E(\boldsymbol{z}, \rho)$ forms an elliptic cylinder. We want to find an $E(\boldsymbol{z}, \rho)$ with the smallest ρ that contains a semialgebraic subset F of $\mathbb{R}^m$. This restriction not only reduces the number of parameters $m(m+3)/2$ in MVEE(F) to $1 + m$ but also makes it possible for us to design a method which we can incorporate smoothly into SDP relaxation methods (with or without exploiting sparsity) developed for various problems including nonconvex quadratic optimization problems [5, 30, 32, etc.], polynomial optimization problems [24, 38, etc.], polynomial SDPs [12, 19, 25], polynomial optimization problems over symmetric cones [21, 22] and the sensor network localization problem [1, 15].

Our ellipsoidal enclosing method is based on lifting. Its basic idea is to embed a nonconvex optimization problem in a convex optimization problem, which serves a convex relaxation of the original problem, in a higher dimensional space. It has been playing an essential role explicitly or implicitly in many SDP relaxation methods referred above. Using the idea of lifting, we first present a conceptual min-max problem for computing an $E(\boldsymbol{z}, \rho)$ with the smallest ρ that encloses F, where F can be any subset of $\mathbb{R}^m$, not necessarily restricted to be semialgebraic. We can reduce this problem to a maximization of a concave quadratic objective function over a convex feasible region in the lifted space. But the feasible region may not have any tractable representation, so we are not able to solve this problem in general. When F is a semialgebraic subset of $\mathbb{R}^m$, we can utilize existing SDP relaxation methods to obtain a tractable convex relaxation of the feasible region. This case is of particular interest because the resulting maximization problem to determine $\boldsymbol{z}$ and ρ becomes an SDP; hence we can compute them by the primal-dual interior-point method [3, 6, 33, 36].

A major motivation behind this work is to develop a numerical method for estimating error bounds in polynomial optimization. Consider an optimization problem

$$\text{minimize } f_0(\boldsymbol{x}) \text{ subject to } \boldsymbol{x} \in F_0. \tag{1}$$

Here $f_0(\boldsymbol{x})$ denotes a real valued polynomial in $\boldsymbol{x} \in \mathbb{R}^m$ and F_0 a semialgebraic set in $\mathbb{R}^m$. Suppose that we have computed a lower bound of the optimal value and a rough approximate optimal solution $\boldsymbol{x}^0$ of the problem by applying an SDP relaxation to this problem. In general, $\boldsymbol{x}^0$ is not a feasible solution of the problem, but we may apply a local optimization method with taking $\boldsymbol{x}^0$ as an initial point to the problem and compute a more accurate global optimal solution $\tilde{\boldsymbol{x}}$ with a tighter upper bound $\tilde{f}_0 = f_0(\tilde{\boldsymbol{x}})$ for the global minimum objective value of the problem (1). We assume that $\tilde{\boldsymbol{x}} \in F_0$. Then, if we take $F = \{\boldsymbol{x} \in F_0 : f_0(\boldsymbol{x}) \leq \tilde{f}_0\}$ and $\boldsymbol{M}$ to be an $m \times m$ positive semidefinite matrix (for example, take $\boldsymbol{M}$ to be the $m \times m$ identity matrix), the enclosing ellipsoidal set $E(\boldsymbol{z}, \rho)$ of F provides an error bound for the approximate optimal solution $\tilde{\boldsymbol{x}} \in F_0$. Note that (1) covers various combinatorial and nonconvex optimization problems such as 0-1 integer linear programs, nonconvex quadratic programs, polynomial optimization problems and even polynomial SDPs [12, 19, 25]. We can incorporate our method in a wide class of SDP relaxation methods for such problems to compute error bounds for their approximate optimal solutions.

In Section 2, we formulate the problem of finding an enclosing ellipsoidal set with the minimum size, and derive a tractable convex relaxation of the problem using the basic idea of lifting. In Section 3, we incorporate the convex relaxation method described in Section 2 into a sparse variant [38] of Lasserre's hierarchy SDP relaxation [24] for general polynomial optimization problems. We place the main emphasis on exploiting sparsity in our method to compute error bounds for approximate optimal solutions of polynomial optimization problems. In Section 4, we apply our method to general quadratic optimization problems, and then specialize the discussion there to the Biswas-Ye SDP relaxation [1] of the sensor network localization problem. This enables us to compute error bounds for locations of sensors obtained by the SDP relaxation. Some numerical results on our method combined with a sparse variant [16] of the Biswas-Ye SDP relaxation are also presented.

2 Main results

2.1 Notation and symbols

Let $\mathbb{R}^{m \times \ell}$ denote the space of $m \times \ell$ real matrices, $\mathbb{S}^m$ the space of $m \times m$ real symmetric matrices and $\mathbb{S}^m_+$ the cone of positive semidefinite matrices. We write $\boldsymbol{Y} \succeq \boldsymbol{O}$ if $\boldsymbol{Y} \in \mathbb{S}^m_+$ for some m.

2.2 A conceptual min-max formulation for the smallest enclosing ellipsoidal set with a given shape

We deal with ellipsoidal sets in the space of $m \times \ell$ real matrices to adapt our ellipsoidal enclosing method to SDP relaxation in matrix variables. The Biswas-Ye SDP relaxation for the sensor network localization problem, which we will discuss in Section 4.2, is such an example. Let F be a subset of $\mathbb{R}^{m \times \ell}$, and take a matrix $\boldsymbol{M} \in \mathbb{S}^m_+$. We consider the class of ellipsoidal sets of the form $E(\boldsymbol{Z}, \gamma) = \left\{\boldsymbol{V} \in \mathbb{R}^{m \times \ell} : \varphi(\boldsymbol{V}, \boldsymbol{Z}) \leq \gamma\right\}$ that contains F, where

$$\begin{aligned}
\varphi(\boldsymbol{V}, \boldsymbol{Z}) &= \text{Trace}\left((\boldsymbol{V} - \boldsymbol{Z})^T \boldsymbol{M} (\boldsymbol{V} - \boldsymbol{Z})\right) \\
&= \text{Trace}(\boldsymbol{M}\boldsymbol{V}\boldsymbol{V}^T) - 2\text{Trace}((\boldsymbol{M}\boldsymbol{V})^T \boldsymbol{Z}) + \text{Trace}((\boldsymbol{M}\boldsymbol{Z})^T \boldsymbol{Z}) \\
&\quad \text{for every } (\boldsymbol{V}, \boldsymbol{Z}) \in \mathbb{R}^{m \times \ell} \times \mathbb{R}^{m \times \ell}.
\end{aligned}$$

We formulate the problem of finding an ellipsoidal set $E(\boldsymbol{Z},\gamma)$ with the smallest $\gamma=\gamma^*$ as

$$\gamma^* = \min_{\boldsymbol{Z}\in\mathbb{R}^{m\times\ell}} \max_{\boldsymbol{V}\in F} \varphi(\boldsymbol{V},\boldsymbol{Z}) \tag{2}$$

Now we transform the inner maximization problem to a maximization of a linear function over a convex set in a higher dimensional lifting space. Define

$$\begin{aligned}
C^* &= \text{the convex hull of } \left\{(\boldsymbol{V},\boldsymbol{V}\boldsymbol{V}^T)\in\mathbb{R}^{m\times\ell}\times\mathbb{S}^m : \boldsymbol{V}\in F\right\},\\
\hat{\varphi}(\boldsymbol{V},\boldsymbol{W},\boldsymbol{Z}) &= \text{Trace}(\boldsymbol{M}\boldsymbol{W}) - 2\text{Trace}((\boldsymbol{M}\boldsymbol{V})^T\boldsymbol{Z}) + \text{Trace}((\boldsymbol{M}\boldsymbol{Z})^T\boldsymbol{Z})\\
&\quad \text{for every } (\boldsymbol{V},\boldsymbol{W},\boldsymbol{Z})\in\mathbb{R}^{m\times\ell}\times\mathbb{S}^m\times\mathbb{R}^{m\times\ell}.
\end{aligned}$$

Lemma 2.1.

(i) Let $\boldsymbol{Z}\in\mathbb{R}^{m\times\ell}$ *be fixed. Then* $\displaystyle\max_{\boldsymbol{V}\in F}\varphi(\boldsymbol{V},\boldsymbol{Z}) = \max_{(\boldsymbol{V},\boldsymbol{W})\in C^*}\hat{\varphi}(\boldsymbol{V},\boldsymbol{W},\boldsymbol{Z})$.

(ii) $\displaystyle\gamma^* = \min_{\boldsymbol{Z}\in\mathbb{R}^{m\times\ell}} \max_{(\boldsymbol{V},\boldsymbol{W})\in C^*}\hat{\varphi}(\boldsymbol{V},\boldsymbol{W},\boldsymbol{Z})$.

Proof: It suffices to show (i) since (ii) follows from the definition (2) of γ^* and (i). By construction, we know that $\displaystyle\max_{\boldsymbol{V}\in F}\varphi(\boldsymbol{V},\boldsymbol{Z}) \leq \max_{(\boldsymbol{V},\boldsymbol{W})\in C^*}\hat{\varphi}(\boldsymbol{V},\boldsymbol{W},\boldsymbol{Z})$. Hence we only have to show the converse inequality. Let $(\overline{\boldsymbol{V}},\overline{\boldsymbol{W}})\in C^*$. By the definition of C^*, we can take $\boldsymbol{V}^j\in F$ and $\lambda_j\geq 0$ $(j=1,2,\ldots,k)$ such that

$$(\overline{\boldsymbol{V}},\overline{\boldsymbol{W}}) = \sum_{j=1}^m \lambda_j(\boldsymbol{V}^j,\boldsymbol{V}^j(\boldsymbol{V}^j)^T),\ 1=\sum_{j=1}^m\lambda_j.$$

Since $\hat{\varphi}(\boldsymbol{V},\boldsymbol{W},\boldsymbol{Z})$ is linear with respect to $(\boldsymbol{V},\boldsymbol{W})\in\mathbb{R}^{m\times\ell}\times\mathbb{S}^m$, we see that

$$\hat{\varphi}(\overline{\boldsymbol{V}},\overline{\boldsymbol{W}},\boldsymbol{Z}) \ = \ \sum_{j=1}^m\lambda_j\hat{\varphi}(\boldsymbol{V}^j,\boldsymbol{V}^j(\boldsymbol{V}^j)^T,\boldsymbol{Z}) = \sum_{j=1}^m\lambda_j\varphi(\boldsymbol{V}^j,\boldsymbol{Z}),$$

which implies that $\hat{\varphi}(\overline{\boldsymbol{V}},\overline{\boldsymbol{W}},\boldsymbol{Z})$ is an weighted average of $\varphi(\boldsymbol{V}^j,\boldsymbol{Z})$ $(j=1,2,\ldots,k)$. Hence there is at least one j for which $\varphi(\boldsymbol{V}^j,\boldsymbol{Z})\geq\hat{\varphi}(\overline{\boldsymbol{V}},\overline{\boldsymbol{W}},\boldsymbol{Z})$ and $\lambda_j>0$ hold. Hence $\displaystyle\max_{\boldsymbol{V}\in F}\varphi(\boldsymbol{V},\boldsymbol{Z})\geq\max_{(\boldsymbol{V},\boldsymbol{W})\in C^*}\hat{\varphi}(\boldsymbol{V},\boldsymbol{W},\boldsymbol{Z})$ follows. ∎

2.3 Convex relaxation in a lifting space

The transformed inner maximization of the linear objective function $\hat{\varphi}(\boldsymbol{V},\boldsymbol{W},\boldsymbol{Z})$ over the convex feasible region C^* in (i) of Lemma 2.1 is still difficult to solve in general, because C^* may not have any numerically tractable representation. To solve the min-max problem stated in (ii) of Lemma 2.1 approximately, we consider a class of convex super sets $\widehat{C}$ of C^* satisfying a certain additional property. By construction, C^* is a convex subset of

$$\mathbb{L} \;=\; \left\{ (\boldsymbol{V}, \boldsymbol{W}) \in \mathbb{R}^{m\times\ell} \times \mathbb{S}^m : \begin{pmatrix} \boldsymbol{I}_\ell & \boldsymbol{V}^T \\ \boldsymbol{V} & \boldsymbol{W} \end{pmatrix} \in \mathbb{S}^{\ell+m}_+ \right\}$$

We see that $(\boldsymbol{V}, \boldsymbol{V}\boldsymbol{V}^T) \in \mathbb{L}$ for every $\boldsymbol{V} \subset \mathbb{R}^{m\times\ell}$, so that $\mathbb{L}$ may be regarded as a lifting of $\mathbb{R}^{m\times\ell}$. In the reminder of this section, we investigate the problem

$$\widehat{\gamma} \;=\; \min_{\boldsymbol{Z} \in \mathbb{R}^{m\times\ell}} \; \max_{(\boldsymbol{V}, \boldsymbol{W}) \in \widehat{C}} \; \hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{Z}).$$

under the assumption that $C^* \subset \widehat{C} \subset \mathbb{L}$.

Now the function $\hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{Z})$ is linear in $(\boldsymbol{V}, \boldsymbol{W})$ over the convex set $\widehat{C}$ and convex quadratic in $\boldsymbol{Z}$ over the linear space $\mathbb{R}^{m\times\ell}$. So we can expect the following min-max and max-min equivalence under an additional assumption, which we will describe later.

$$\begin{aligned} &\min_{\boldsymbol{Z} \in \mathbb{R}^{m\times\ell}} \; \max_{(\boldsymbol{V}, \boldsymbol{W}) \in \widehat{C}} \; \hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{Z}) \\ &\qquad = \max_{(\boldsymbol{V}, \boldsymbol{W}) \in \widehat{C}} \; \min_{\boldsymbol{Z} \in \mathbb{R}^{m\times\ell}} \; \hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{Z}) \end{aligned} \tag{3}$$

Assuming that this identity holds, we will focus our attention to the right hand max-min problem. For each fixed $(\boldsymbol{V}, \boldsymbol{W}) \in \widehat{C}$, its inner minimization problem is written as

$$\text{minimize} \quad \hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{Z}) \quad \text{subject to} \quad \boldsymbol{Z} \in \mathbb{R}^{m\times\ell}.$$

Since the objective function $\hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{Z})$ is convex and quadratic in the variable matrix $\boldsymbol{Z}$, the global minimum value is attained when the gradient $-2\boldsymbol{M}\boldsymbol{V} + 2\boldsymbol{M}\boldsymbol{Z}$ of $\hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{Z})$ with respect to $\boldsymbol{Z}$ vanishes. Specifically we may choose a minimum solution $\boldsymbol{Z} = \boldsymbol{V}$. Then the global minimum value coincides with $\text{Trace}\big(\boldsymbol{M}(\boldsymbol{W} - \boldsymbol{V}\boldsymbol{V}^T)\big)$. Therefore the max-min problem in (3) has been reduced to

$$\text{maximize} \quad \zeta(\boldsymbol{V}, \boldsymbol{W}) \quad \text{subject to} \quad (\boldsymbol{V}, \boldsymbol{W}) \in \widehat{C}, \tag{4}$$

where

$$\begin{aligned} \zeta(\boldsymbol{V}, \boldsymbol{W}) &= \hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{V}) \\ &= \text{Trace}\big(\boldsymbol{M}(\boldsymbol{W} - \boldsymbol{V}\boldsymbol{V}^T)\big) \text{ for every } (\boldsymbol{V}, \boldsymbol{W}) \in \mathbb{R}^{m\times\ell} \times \mathbb{S}^m. \end{aligned}$$

We also observe that

$$\begin{aligned} \zeta(\boldsymbol{V}, \boldsymbol{W}) &\geq 0 \text{ for every } (\boldsymbol{V}, \boldsymbol{W}) \in \mathbb{L}, &(5)\\ \hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{Z}) &= \varphi(\boldsymbol{V}, \boldsymbol{Z}) + \zeta(\boldsymbol{V}, \boldsymbol{W}) \\ &\geq \varphi(\boldsymbol{V}, \boldsymbol{Z}) \text{ for every } (\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{Z}) \in \mathbb{L} \times \mathbb{R}^{m\times\ell}. &(6) \end{aligned}$$

Here (5) follows from $\boldsymbol{M} \in \mathbb{S}^m_+$ and $\boldsymbol{W} - \boldsymbol{V}\boldsymbol{V}^T \in \mathbb{S}^m_+$.

The objective function of the problem (4) to be maximized is concave quadratic function and the feasible region $\widehat{C}$ is convex, so we can solve this problem as long as the convex feasible region $\widehat{C}$ is numerically tractable . In the next subsection, we will show how we solve this problem when $\widehat{C}$ is represented in terms of linear inequalities over symmetric cones including normal linear inequalities, linear matrix inequalities and second order cone inequalities.

The lemma below plays an essential role to prove our main theorem, Theorem 2.3.

Lemma 2.2. *Assume that the maximization problem (4) has a maximum solution* $(\widehat{\boldsymbol{Z}}, \widehat{\boldsymbol{W}})$ *with the objective value* $\hat{\gamma} = \zeta(\widehat{\boldsymbol{Z}}, \widehat{\boldsymbol{W}})$*. Then* $\hat{\gamma} \geq \hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \widehat{\boldsymbol{Z}})$ *for every* $(\boldsymbol{V}, \boldsymbol{W}) \in \widehat{C}$*.*

Proof: Choose an arbitrary $(\overline{\boldsymbol{V}}, \overline{\boldsymbol{W}}) \in \widehat{C}$. Define

$$\left.\begin{array}{rcl} \boldsymbol{V}(\lambda) & = & (1-\lambda)\overline{\boldsymbol{V}} + \lambda\widehat{\boldsymbol{Z}}, \\ \boldsymbol{W}(\lambda) & = & (1-\lambda)\overline{\boldsymbol{W}} + \lambda\widehat{\boldsymbol{W}} \end{array}\right\} \text{ for every } \lambda \in [0,1].$$

Since $\widehat{C}$ is convex, we know the entire line segment $\{(\boldsymbol{V}(\lambda), \boldsymbol{W}(\lambda)) : \lambda \in [0,1]\}$ lies in the feasible region $\widehat{C}$ of the problem (4). We will investigate the change of the objective value $\zeta(\boldsymbol{V}(\lambda), \boldsymbol{W}(\lambda))$ along this line segment. We first see that

$$\zeta(\boldsymbol{V}(\lambda), \boldsymbol{W}(\lambda)) \qquad \text{attains the maximum } \hat{\gamma} \geq 0 \text{ at } \lambda = 1 \text{ over } [0,1]. \tag{7}$$

For every $\lambda \in [0,1]$, we can easily verify that

$$\begin{array}{rcl} \boldsymbol{V}(\lambda)\boldsymbol{V}(\lambda)^T & = & \left((1-\lambda)\overline{\boldsymbol{V}} + \lambda\widehat{\boldsymbol{Z}}\right)\left((1-\lambda)\overline{\boldsymbol{V}} + \lambda\widehat{\boldsymbol{Z}}\right)^T \\ & = & \lambda^2(\overline{\boldsymbol{V}} - \widehat{\boldsymbol{Z}})(\overline{\boldsymbol{V}} - \widehat{\boldsymbol{Z}})^T \\ & & -\lambda\left((\overline{\boldsymbol{V}} - \widehat{\boldsymbol{Z}})(\overline{\boldsymbol{V}} - \widehat{\boldsymbol{Z}})^T - \widehat{\boldsymbol{Z}}\widehat{\boldsymbol{Z}}^T + \overline{\boldsymbol{V}}\overline{\boldsymbol{V}}^T\right) + \overline{\boldsymbol{V}}\overline{\boldsymbol{V}}^T. \end{array}$$

Hence

$$\begin{array}{rl} & \zeta(\boldsymbol{V}(\lambda), \boldsymbol{W}(\lambda)) \\ = & \text{Trace } \boldsymbol{M}\left(\boldsymbol{W}(\lambda) - \boldsymbol{V}(\lambda)\boldsymbol{V}(\lambda)^T\right) \\ = & \text{Trace } \boldsymbol{M}\Big(-\lambda^2(\overline{\boldsymbol{V}} - \widehat{\boldsymbol{Z}})(\overline{\boldsymbol{V}} - \widehat{\boldsymbol{Z}})^T \\ & +\lambda\left((\overline{\boldsymbol{V}} - \widehat{\boldsymbol{Z}})(\overline{\boldsymbol{V}} - \widehat{\boldsymbol{Z}})^T + (\widehat{\boldsymbol{W}} - \widehat{\boldsymbol{Z}}\widehat{\boldsymbol{Z}}^T) - (\overline{\boldsymbol{W}} - \overline{\boldsymbol{V}}\overline{\boldsymbol{V}}^T)\right) + \overline{\boldsymbol{W}} - \overline{\boldsymbol{V}}\overline{\boldsymbol{V}}^T\Big) \\ = & \lambda(1-\lambda)\varphi(\overline{\boldsymbol{V}}, \widehat{\boldsymbol{Z}}) + \lambda\left(\zeta(\widehat{\boldsymbol{Z}}, \widehat{\boldsymbol{W}}) - \zeta(\overline{\boldsymbol{V}}, \overline{\boldsymbol{W}})\right) + \zeta(\overline{\boldsymbol{V}}, \overline{\boldsymbol{W}}). \end{array}$$

By (7), we consequently obtain that

$$\begin{array}{rcl} 0 & \leq & \left.\dfrac{\partial\zeta(\boldsymbol{V}(\lambda), \boldsymbol{W}(\lambda)}{\partial\lambda}\right|_{\lambda=1} = -\varphi(\overline{\boldsymbol{V}}, \widehat{\boldsymbol{Z}}) + \zeta(\widehat{\boldsymbol{Z}}, \widehat{\boldsymbol{W}}) - \zeta(\overline{\boldsymbol{V}}, \overline{\boldsymbol{W}} \\ & = & = -\hat{\varphi}(\overline{\boldsymbol{V}}, \overline{\boldsymbol{W}}, \widehat{\boldsymbol{Z}}) + \hat{\gamma}. \end{array}$$

∎

We are now ready to state the main theoretical results.

Theorem 2.3. *Assume that the problem (4) has a maximum solution* $(\widehat{\boldsymbol{Z}}, \widehat{\boldsymbol{W}})$ *with the objective value* $\hat{\gamma} = \zeta(\widehat{\boldsymbol{Z}}, \widehat{\boldsymbol{W}})$.

(i) $\hat{\gamma} = \min_{\boldsymbol{Z} \in \mathbb{R}^{m\times\ell}} \max_{(\boldsymbol{V},\boldsymbol{W}) \in \widehat{C}} \hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{Z}) = \max_{(\boldsymbol{V},\boldsymbol{W}) \in \widehat{C}} \min_{\boldsymbol{Z} \in \mathbb{R}^{m\times\ell}} \hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{Z})$.

(ii) $\boldsymbol{V} \in E(\widehat{\boldsymbol{Z}}, \hat{\gamma})$ *if* $(\boldsymbol{V}, \boldsymbol{W}) \in \widehat{C}$.

Proof: (i) Recall that we have reduced the max-min problem in (3) to the maximization problem (4). Hence

$$\hat{\gamma} = \zeta(\widehat{\boldsymbol{Z}}, \widehat{\boldsymbol{W}}) = \hat{\varphi}(\widehat{\boldsymbol{Z}}, \widehat{\boldsymbol{W}}, \widehat{\boldsymbol{Z}}) = \max_{(\boldsymbol{V},\boldsymbol{W}) \in \widehat{C}} \min_{\boldsymbol{Z} \in \mathbb{R}^{m\times\ell}} \hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{Z}).$$

In general, we also know that

$$\min_{\boldsymbol{Z} \in \mathbb{R}^{m\times\ell}} \max_{(\boldsymbol{V},\boldsymbol{W}) \in \widehat{C}} \hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{Z}) \geq \max_{(\boldsymbol{V},\boldsymbol{W}) \in \widehat{C}} \min_{\boldsymbol{Z} \in \mathbb{R}^{m\times\ell}} \hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{Z}).$$

On the other hand, we see by Lemma 2.2 that

$$\hat{\gamma} \geq \max_{(\boldsymbol{V},\boldsymbol{W}) \in \widehat{C}} \hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \widehat{\boldsymbol{Z}}) \geq \min_{\boldsymbol{Z} \in \mathbb{R}^{m\times\ell}} \max_{(\boldsymbol{V},\boldsymbol{W}) \in \widehat{C}} \hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{Z}).$$

Therefore (i) holds. (ii) Assume that $(\boldsymbol{V}, \boldsymbol{W}) \in \widehat{C}$. By Lemma 2.2 and (6), we see that $\hat{\gamma} \geq \hat{\varphi}(\boldsymbol{V}, \boldsymbol{W}, \widehat{\boldsymbol{Z}}) \geq \varphi(\boldsymbol{V}, \widehat{\boldsymbol{Z}})$. This implies $\boldsymbol{V} \in E(\widehat{\boldsymbol{Z}}, \hat{\gamma})$. ∎

2.4 On computation of enclosing ellipsoidal sets

Since $\boldsymbol{M}$ is positive semidefinite, we factorize $\boldsymbol{M}$ such that $\boldsymbol{M} = \boldsymbol{B}\boldsymbol{B}^T$ for some $\boldsymbol{B} \in \mathbb{R}^{m\times k}$, where k denotes the rank of $\boldsymbol{M}$. Then the problem (4) is equivalent to

$$\text{maximize} \quad \text{Trace}\, \boldsymbol{M}\boldsymbol{W} - t \quad \text{subject to} \quad (\boldsymbol{V}, \boldsymbol{W}) \in \widehat{C},\ t \geq \left\|\text{vec}(\boldsymbol{B}^T\boldsymbol{V})\right\|^2, \tag{8}$$

where $\text{vec}(\boldsymbol{B}^T\boldsymbol{V})$ denotes the $k \times m$-dimensional column vector sequencing all the columns of $\boldsymbol{B}^T\boldsymbol{V}$. We can further transform the inequality constraint $t \geq \left\|\text{vec}(\boldsymbol{B}^T\boldsymbol{V})\right\|^2$ to a second order cone inequality $1 + t \geq \left((1-t)^2 + \text{vec}(2\boldsymbol{B}^T\boldsymbol{V})^T\text{vec}(2\boldsymbol{B}^T\boldsymbol{V})\right)^{1/2}$ or a linear matrix inequality $\begin{pmatrix} t & \text{vec}(\boldsymbol{B}^T\boldsymbol{V})^T \\ \text{vec}(\boldsymbol{B}^T\boldsymbol{V}) & \boldsymbol{I} \end{pmatrix} \in \mathbb{S}^{1+m\times k}_+$. When the feasible region $\widehat{C}$ of the problem (4) is represented in terms of linear matrix inequalities and/or second order cone inequalities in $(\boldsymbol{V}, \boldsymbol{W}) \in \mathbb{R}^{m\times\ell} \times \mathbb{S}^m$, the transformed problem is an SDP (with second order cone inequalities), which we can solve by the primal-dual interior-point method [3, 6, 33, 36]. It should be noted that the second order cone inequality or the linear matrix inequality to be added is sparse in the sense they does not involve the $m \times m$ symmetric matrix variable $\boldsymbol{W}$. Therefore, if the original description of $\widehat{C}$ in terms of linear matrix inequalities is sparse (or structured sparse [17, 18, 20, 38]) and if we choose an $\boldsymbol{M} \in \mathbb{S}^m_+$ which shares the same sparsity, we can expect to maintain the sparsity (or the structured sparsity) in the resulting SDP to compute $\widehat{\boldsymbol{Z}} \in \mathbb{R}^{m\times\ell}$ and $\hat{\gamma}$. More detailed discussion on how we exploit sparsity will be presented in the next section.

3 Exploiting sparsity in polynomial optimization problems

We consider a polynomial optimization problem (abbreviated by POP)

$$\text{minimize} \;\; f_0(\boldsymbol{x}) \text{ subject to } \;\boldsymbol{x} \in F_0, \tag{9}$$

where $F_0 = \{\boldsymbol{x} \in \mathbb{R}^n : f_k(\boldsymbol{x}) \geq 0 \; (k = 1, 2, \ldots, p)\}$, and each $f_k(\boldsymbol{x})$ denotes a polynomial function in $\boldsymbol{x} \in \mathbb{R}^n$. In Section 3.2, we describe the sparse SDP relaxation proposed by Waki et al. in the paper [38] as a sparse variant of Lasserre's hierarchy SDP relaxation [24]. We present how we incorporate Theorem 2.3 in the sparse SDP relaxation to compute error bounds in the POP (9) in Section 3.3, and some numerical results in Section 3.4.

3.1 Notation and symbols

Let $\mathbb{Z}^n_+$ denote the set of n-dimensional nonnegative integer vectors. For every $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n_+$, we use the notation $\boldsymbol{x}^{\boldsymbol{\alpha}}$ for the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. Specifically we assume that $\boldsymbol{x}^{\boldsymbol{0}} = 1$. Each polynomial $f_k(\boldsymbol{x})$ can be represented as

$$f_k(\boldsymbol{x}) \;\; = \sum_{\boldsymbol{\alpha} \in \mathcal{F}_k} h^k_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \;\; \text{for every } \boldsymbol{x} \in \mathbb{R}^n$$

for some $h^k_{\boldsymbol{\alpha}} \in \mathbb{R}$ and some finite subset $\mathcal{F}_k$ of $\mathbb{Z}^n_+$. The degree of $f_k(\boldsymbol{x})$ is defined by

$$\deg(f_k(\boldsymbol{x})) \;\; = \;\; \max\left\{ \sum_{i=1}^n \alpha_i : \boldsymbol{\alpha} \in \mathcal{F}_k, \; h^k_{\boldsymbol{\alpha}} \neq 0 \right\}.$$

Let $\mathcal{F}$ be a finite subset of $\mathbb{Z}^n_+$, $(x^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{F})$ a column vector of monomials $\boldsymbol{x}^{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} \in \mathcal{F})$, and $(\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{F}, \boldsymbol{\beta} \in \mathcal{F}) = (x^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{F})(x^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{F})^T$ a symmetric matrix of monomials $\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ $(\boldsymbol{\alpha} \in \mathcal{F}, \boldsymbol{\beta} \in \mathcal{F})$. The order of the element monomials in the column vector $(\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{F})$ is arbitrary but has to be fixed. For example, we may assume that they are sequenced according to the graded reverse lexicographic order $\succ_{\text{grevlex}}$; if $\boldsymbol{\beta} \in \mathcal{F}$, $\boldsymbol{\gamma} \in \mathcal{F}$ and $\boldsymbol{x}^{\boldsymbol{\gamma}} \succ_{\text{grevlex}} \boldsymbol{x}^{\boldsymbol{\beta}}$ (or equivalently $\boldsymbol{x}^{\boldsymbol{\beta}} \prec_{\text{grevlex}} \boldsymbol{x}^{\boldsymbol{\gamma}}$), then $\boldsymbol{x}^{\boldsymbol{\beta}}$ is placed above $\boldsymbol{x}^{\boldsymbol{\gamma}}$ in the column vector $(x^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{F})$. If $\mathcal{F} = \emptyset$, we assume that $(\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{F})$ is the empty vector, and $(\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{F}, \boldsymbol{\beta} \in \mathcal{F})$ the empty matrix. We also use the notation $(y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{F})$ for a column vector of real variables $y_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} \in \mathcal{F})$, which is obtained from $(\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{F})$ by replacing each monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$ with a real variable $y_{\boldsymbol{\alpha}}$. In addition, we use $(W_{\boldsymbol{\alpha}\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{F}, \boldsymbol{\beta} \in \mathcal{F})$ a symmetric matrix of real variables $W_{\boldsymbol{\alpha}\boldsymbol{\beta}}$ $(\boldsymbol{\alpha} \in \mathcal{F}, \boldsymbol{\beta} \in \mathcal{F})$. If we replace each monomial $\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ in the symmetric matrix $(\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{F}, \boldsymbol{\beta} \in \mathcal{F})$ with a real variable $y_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$, we obtain $(y_{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{F}, \boldsymbol{\beta} \in \mathcal{F})$, which is often called a moment matrix [24].

Let $\mathcal{N}_0 = \{1, 2, \ldots, n\}$. For every nonnegative integer η and every nonempty $C \subset \mathcal{N}_0$, we define

$$\mathcal{F}(\eta, C) = \begin{cases} \emptyset & \text{if } \eta = 0, \\ \left\{ \boldsymbol{\alpha} \in \mathbb{Z}^n_+ : 1 \leq \sum_{i \in C} \alpha_i \leq \eta, \ \alpha_i = 0 \text{ if } \ i \notin C \right\} & \text{if } \ \eta \geq 1. \end{cases} \tag{10}$$

3.2 A sparse variant of Lasserre's hierarchy SDP relaxation for polynomial optimization problems

To illustrate the construction of the sparse SDP relaxation [38], we use the following example with a variable vector $\boldsymbol{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$.

$$\left.\begin{array}{ll} \text{minimize} & f_0(\boldsymbol{x}) \equiv -x_1 + x_1 x_2 + 2x_3^3 \\ \text{subject to} & \boldsymbol{x} \in F_0 \equiv \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \begin{array}{l} f_1(\boldsymbol{x}) \equiv 1 - x_1^2 \geq 0, \\ f_2(\boldsymbol{x}) \equiv 1 - x_2^2 - x_3^2 \geq 0 \end{array} \right\} \end{array}\right\} \tag{11}$$

For each $k = 1, 2, \ldots, p$, let $I_k = \{i : \alpha_i > 0 \ \text{ and } \ h^k_{\boldsymbol{\alpha}} \neq 0 \ \text{ for some } \ \boldsymbol{\alpha} \in \mathcal{F}_k\}$ denote the set indices i such that x_i is involved in the polynomial $f_k(\boldsymbol{x})$. Define an $n \times n$ symmetric symbolic matrix $\boldsymbol{R}^0$ by

$$R^0_{ij} = \begin{cases} \star & \text{if } i = j, \\ \star & \text{if distinct variables } x_i \text{ and } x_j \text{ are in a monomial of } f_0(\boldsymbol{x}), \\ \star & \text{if } i \in I_k \text{ and } j \in I_k \text{ for some } k = 1, 2, \ldots, p, \\ 0 & \text{otherwise.} \end{cases}$$

We call $\boldsymbol{R}^0$ the *correlative sparsity pattern matrix*. We then introduce the *correlative sparsity pattern graph* $G(\mathcal{N}_0, \mathcal{E}_0)$ with $\mathcal{E}_0 = \{(i, j) \in \mathcal{N}_0 \times \mathcal{N}_0 : i < j, \ R^0_{ij} = \star\}$. Let $G(\mathcal{N}_0, \overline{\mathcal{E}}_0)$ be a chordal extension of $G(\mathcal{N}_0, \mathcal{E}_0)$ or equivalently a chordal graph with the node set $\mathcal{N}_0$ and an edge set $\overline{\mathcal{E}}_0 \supset \mathcal{E}_0$. Here we say a graph is chordal if every cycle with more than three edges has a chord. See [2] for basic properties of chordal graphs. Let Γ denote the set of maximal cliques of $G(\mathcal{N}_0, \overline{\mathcal{E}}_0)$, where $C \subset \mathcal{N}_0$ is a maximal clique of $G(\mathcal{N}_0, \overline{\mathcal{E}}_0)$ if it is the node set of a maximal complete subgraph of $G(\mathcal{N}_0, \overline{\mathcal{E}}_0)$. By construction, each I_k is covered by some $C \in \Gamma$. Let $I_k \subset C_k \in \Gamma$ $(k = 1, 2, \ldots, p)$. In the remainder of Section 3, we implicitly assume that the sizes of all maximal cliques $C \in \Gamma$ are small compared to the dimension n of the vector variable $\boldsymbol{x}$ of the POP (9), although the discussions are valid without this assumption.

In the case of example (11), we see that

$$I_1 = \{1\}, \ I_2 = \{2, 3\}, \ \boldsymbol{R}^0 = \begin{pmatrix} \star & \star & 0 \\ \star & \star & \star \\ 0 & \star & \star \end{pmatrix}.$$

Here $\star$ is assigned at the $(1, 2)$th and $(2, 1)$th elements since x_1 and x_2 are involved in the monomial $x_1 x_2$ of the objective polynomial function $f_0(\boldsymbol{x})$, while $\star$ is assigned at the $(2, 3)$th and $(3, 2)$th elements since $2 \in\in I_2$ and $3 \in I_2$ (or x_2 and x_3 are involved in the polynomial

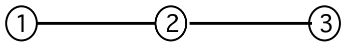

Figure 1: The correlative sparsity pattern graph of example (11).

function $f_2(\boldsymbol{x})$). Figure 1 shows the correlative sparsity pattern graph $G(\mathcal{N}_0, \mathcal{E}_0)$ of the POP (11). This graph is chordal since it has no cycle. Hence $G(\mathcal{N}_0, \overline{\mathcal{E}}_0) = G(\mathcal{N}_0, \mathcal{E}_0)$. The set of maximal cliques are $\Gamma = \{\{1,2\},\ \{2,3\}\}$, and we can take $C_1,\ C_2 \in \Gamma$ such that $I_1 = \{1\} \subset C_1 = \{1,2\} \in \Gamma$ and $I_2 \subset C_2 = \{2,3\} \in \Gamma$.

For each $k = 0, 1, \ldots, p$, let $\omega_k = \lceil \deg(f_k(\boldsymbol{x}))/2 \rceil$ and $\omega_{\max} = \max\{\omega_k : k = 0, 1, \ldots, p\}$. We choose positive integer $\omega \geq \omega_{\max}$, which we call the *relaxation order*. Define

$$\begin{aligned}
\mathcal{G}_k &= \mathcal{F}(\omega - \omega_k, C_k) \ (k = 1, 2, \ldots, p), \\
\mathcal{G}(C) &= \mathcal{F}(\omega, C) \ (C \in \Gamma), \\
\boldsymbol{G}_k(\boldsymbol{x}) &= \begin{pmatrix} 1 & (\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}_k)^T \\ (\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}_k) & (\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{G}_k, \boldsymbol{\beta} \in \mathcal{G}_k) \end{pmatrix} f_k(\boldsymbol{x}) \\
& \qquad \text{for every } \boldsymbol{x} \in \mathbb{R}^n \ (k = 1, 2, \ldots, p).
\end{aligned}$$

Here $\mathcal{F}(\eta, C)$ with $C \in \Gamma$ and $\eta \in \mathbb{Z}_+$ is defined by (10). Note that if $\omega - \omega_k = 0$ then $\mathcal{G}_k = \emptyset$ and $\boldsymbol{G}_k(\boldsymbol{x}) \equiv f_k(\boldsymbol{x})$. Then we see that

$$\begin{aligned}
& f_k(\boldsymbol{x}) \geq 0 \text{ iff } \boldsymbol{G}_k(\boldsymbol{x}) \succeq \boldsymbol{O} \ (k = 1, 2, \ldots, p), \\
& \begin{pmatrix} 1 & (\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C))^T \\ (\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C)) & (\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{G}(C), \boldsymbol{\beta} \in \mathcal{G}(C)) \end{pmatrix} \succeq \boldsymbol{O} \\
& \qquad \text{for every } \boldsymbol{x} \in \mathbb{R}^n \ (C \in \Gamma).
\end{aligned}$$

Hence the POP (9) under consideration is equivalent to the following polynomial SDP.

$$\left.\begin{array}{ll}
\text{minimize} & f_0(\boldsymbol{x}) \\
\text{subject to} & \boldsymbol{G}_k(\boldsymbol{x}) \succeq \boldsymbol{O} \ (k = 1, 2, \ldots, p), \\
& \begin{pmatrix} 1 & (\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C))^T \\ (\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C)) & (\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{G}(C), \boldsymbol{\beta} \in \mathcal{G}(C)) \end{pmatrix} \succeq \boldsymbol{O} \\
& \qquad (C \in \Gamma).
\end{array}\right\} \quad (12)$$

Let $\mathcal{H}_1 = \bigcup_{C \in \Gamma} \mathcal{G}(C)$, $\mathcal{H}_2 = \bigcup_{C \in \Gamma} (\mathcal{G}(C) + \mathcal{G}(C))$, and $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$, where $\mathcal{G}(C) + \mathcal{G}(C) = \{\boldsymbol{\alpha} + \boldsymbol{\beta} : \boldsymbol{\alpha} \in \mathcal{G}(C),\ \boldsymbol{\beta} \in \mathcal{G}(C)\}$. Then all monomials involved in the polynomial function $f_0(\boldsymbol{x})$ and mappings $\boldsymbol{G}_k(\boldsymbol{x})$ $(k = 1, 2, \ldots, p)$ are contained in $\{\boldsymbol{0}\} \cup \{\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}\}$, so we can represent them as

$$\begin{aligned}
f_0(\boldsymbol{x}) &= h^0_{\boldsymbol{0}} + \sum_{\boldsymbol{\alpha} \in \mathcal{H}} h^0_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \ \text{ for every } \boldsymbol{x} \in \mathbb{R}^n, \\
\boldsymbol{G}^k(\boldsymbol{x}) &= \boldsymbol{H}^k_{\boldsymbol{0}} + \sum_{\boldsymbol{\alpha} \in \mathcal{H}} \boldsymbol{H}^k_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \ \text{ for every } \boldsymbol{x} \in \mathbb{R}^n \ (k = 1, 2, \ldots, p).
\end{aligned}$$

for some $h^0_{\boldsymbol{\alpha}} \in \mathbb{R}$ and symmetric matrices $\boldsymbol{H}^k_{\boldsymbol{\alpha}}$ ($\boldsymbol{\alpha} \in \{\boldsymbol{0}\} \cup \mathcal{H},\ k = 1, 2, \ldots, p$). Note that some of $h^0_{\boldsymbol{\alpha}} \in \mathbb{R}$ and symmetric matrices $\boldsymbol{H}^k_{\boldsymbol{\alpha}}$ ($\boldsymbol{\alpha} \in \{\boldsymbol{0}\} \cup \mathcal{H},\ k = 1, 2, \ldots, p$) can vanish. Thus we can rewrite the polynomial SDP (12) as

$$\left.\begin{array}{ll} \text{minimize} & h^0_{\boldsymbol{0}} + \displaystyle\sum_{\boldsymbol{\alpha}\in\mathcal{H}} h^0_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \\ \text{subject to} & \boldsymbol{H}^k_{\boldsymbol{0}} + \displaystyle\sum_{\boldsymbol{\alpha}\in\mathcal{H}} \boldsymbol{H}^k_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \succeq \boldsymbol{O}\ (k = 1, 2, \ldots, p), \\ & \begin{pmatrix} 1 & (\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C))^T \\ (\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C)) & (\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{G}(C), \boldsymbol{\beta} \in \mathcal{G}(C)) \end{pmatrix} \succeq \boldsymbol{O} \\ & \hfill (C \in \Gamma). \end{array}\right\} \tag{13}$$

To derive an SDP relaxation of the POP (9), we apply a linearization to the polynomial SDP (13) by replacing each monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$ by a single variable $y_{\boldsymbol{\alpha}}$.

$$\left.\begin{array}{ll} \text{minimize} & h^0_{\boldsymbol{0}} + \displaystyle\sum_{\boldsymbol{\alpha}\in\mathcal{H}} h^0_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}} \\ \text{subject to} & \boldsymbol{H}^k_{\boldsymbol{0}} + \displaystyle\sum_{\boldsymbol{\alpha}\in\mathcal{H}} \boldsymbol{H}^k_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}} \succeq \boldsymbol{O}\ (k = 1, 2, \ldots, p), \\ & \begin{pmatrix} 1 & (y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C))^T \\ (y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C)) & (y_{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{G}(C), \boldsymbol{\beta} \in \mathcal{G}(C)) \end{pmatrix} \succeq \boldsymbol{O} \\ & \hfill (C \in \Gamma). \end{array}\right\} \tag{14}$$

We illustrate the discussion above using example (11). We see that

$$\begin{aligned} \omega_0 &= \lceil \deg(f_0(\boldsymbol{x}))/2 \rceil = 2,\ \omega_1 = \lceil \deg(f_1(\boldsymbol{x}))/2 \rceil = 1, \\ \omega_2 &= \lceil \deg(f_2(\boldsymbol{x}))/2 \rceil = 1,\ \omega_{\max} = 2. \end{aligned}$$

Let's take $\omega = 2 \geq \omega_{\max}$. Then

$$\begin{aligned} & \mathcal{G}_1 = \mathcal{F}(\omega - \omega_1, C_1) = \mathcal{F}(1, \{1, 2\}) = ((1,0,0),\ (0,1,0))\,, \\ & \mathcal{G}_2 = \mathcal{F}(\omega - \omega_2, C_2) = \mathcal{F}(1, \{2, 3\}) = ((0,1,0),\ (0,0,1))\,, \\ & \mathcal{G}(C_1) = \mathcal{F}(\omega, C_1) = \mathcal{F}(2, \{1, 2\}) \\ & \qquad = \{(1,0,0),\ (0,1,0),\ (2,0,0),\ (1,1,0),\ (0,2,0)\}\,, \\ & \mathcal{G}(C_2) = \mathcal{F}(\omega, C_2) = \mathcal{F}(2, \{2, 3\}) \\ & \qquad = \{(0,1,0),\ (0,0,1),\ (0,2,0),\ (0,1,1),\ (0,0,2)\}\,, \\ & \boldsymbol{G}_1(\boldsymbol{x}) = \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1 x_2 \\ x_2 & x_1 x_2 & x_2^2 \end{pmatrix} (1 - x_1^2), \\ & \qquad = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{x}^{\boldsymbol{0}} + \cdots + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} x_1^2 x_2^2, \\ & \boldsymbol{G}_2(\boldsymbol{x}) = \begin{pmatrix} 1 & x_2 & x_3 \\ x_2 & x_2^2 & x_2 x_3 \\ x_3 & x_2 x_3 & x_3^2 \end{pmatrix} (1 - x_2^2 - x_3^2) \end{aligned}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{x}^{\boldsymbol{0}} + \cdots + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} x_3^4,$$

$$\begin{pmatrix} 1 & (\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C_1))^T \\ (\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C_1)) & (\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{G}(C_1), \boldsymbol{\beta} \in \mathcal{G}(C_1)) \end{pmatrix} = \begin{pmatrix} 1 & x_1 & \dots & x_2^2 \\ x_1 & x_1^2 & \dots & x_1 x_2^2 \\ \vdots & \vdots & & \vdots \\ x_2^2 & x_1 x_2^2 & \dots & x_2^4 \end{pmatrix},$$

$$\begin{pmatrix} 1 & (\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C_1))^T \\ (\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C_2)) & (\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{G}(C_2), \boldsymbol{\beta} \in \mathcal{G}(C_2)) \end{pmatrix} = \begin{pmatrix} 1 & x_2 & \dots & x_3^2 \\ x_2 & x_2^2 & \dots & x_2 x_3^2 \\ \vdots & \vdots & & \vdots \\ x_3^2 & x_2 x_3^2 & \dots & x_3^4 \end{pmatrix}.$$

Thus we have an SDP relaxation of the POP (11).

$$\left. \begin{array}{ll} \text{minimize} & -y_{100} + y_{110} + 2y_{003} \\ \text{subject to} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} y_{220} \succeq \boldsymbol{O}, \\ & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} y_{004} \succeq \boldsymbol{O}, \\ & \begin{pmatrix} 1 & y_{100} & \cdots & y_{020} \\ y_{100} & y_{200} & \cdots & y_{120} \\ \vdots & \vdots & & \vdots \\ y_{020} & y_{120} & \cdots & y_{040} \end{pmatrix} \succeq \boldsymbol{O}, \\ & \begin{pmatrix} 1 & y_{010} & \cdots & y_{002} \\ y_{010} & y_{020} & \cdots & y_{012} \\ \vdots & \vdots & & \vdots \\ y_{002} & y_{012} & \cdots & y_{004} \end{pmatrix} \succeq \boldsymbol{O}. \end{array} \right\} \tag{15}$$

The construction of the sparse SDP relaxation (14) depends not only on the POP (9) but also on the relaxation order ω. Let κ^ω denote the optimal value of the sparse SDP relaxation (14) with the relaxation order ω, and κ^* the optimal value of the POP (9). Then $\kappa^\omega \leq \kappa^{\omega+1} \leq \kappa^*$ for every positive integer $\omega \geq \omega_{\max}$. Lasserre [25] showed under a certain condition which requires that the feasible region of the POP (9) is compact, κ^ω converges κ^* as ω tends to ∞. In practice, the relaxation order ω not greater than $\omega_{\max} + 2$ is often large enough for κ^ω to attain κ^* accurately. When the maximal cliques $C \in \Gamma$ of the chordal extension $G(\mathcal{N}_0, \overline{\mathcal{E}}_0)$ derived from the correlative sparsity pattern graph $G(\mathcal{N}_0, \mathcal{E}_0)$ of the POP (9) are small, the sparse SDP relaxation is much more efficient than the original Lasserre's SDP relaxation. See the paper [38] for more details.

3.3 Computation of error bounds

In this subsection, we show how we effectively incorporate Theorem 2.3 in the sparse SDP relaxation (14) to compute error bounds in the POP (9). Assume that an approximate global optimal solution $\tilde{\boldsymbol{x}} \in F_0$ with the objective function value $\tilde{f}_0 = f_0(\tilde{\boldsymbol{x}})$ is available. In practice, the SDP relaxation (14) may not provide such an approximate global optimal and feasible solution of the POP (9). We can apply a local optimization method to improve the quality of the SDP solution to compute a feasible and better quality approximate global optimal solution $\tilde{\boldsymbol{x}}$. This technique was implemented in SparsePOP [39]. Let

$$\mathcal{J} \;=\; \bigcup_{C \in \Gamma} \left(\mathcal{G}(C) \times \mathcal{G}(C)\right) \subset \mathcal{H}_1 \times \mathcal{H}_1.$$

(Recall that $\mathcal{H}_1 = \bigcup_{C \in \Gamma} \mathcal{G}(C)$). To maintain the sparsity in (14), we restrict the positive semidefinite matrix $\boldsymbol{M}$, which we will choose to define the function φ in the description of an ellipsoidal set and also to define the function ζ in the maximization problem (4), to the class of sparse matrices $\mathcal{M} = \left\{ \left(M_{\boldsymbol{\alpha}\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{H}_1,\ \boldsymbol{\beta} \in \mathcal{H}_1 \right) : M_{\boldsymbol{\alpha}\boldsymbol{\beta}} = 0 \text{ if } (\boldsymbol{\alpha}, \boldsymbol{\beta}) \not\in \mathcal{J} \right\}$. Note that we can take any diagonal matrix with nonnegative entries for $\boldsymbol{M} \in \mathcal{M}$. In the case of example (11), we see that

$$\mathcal{H}_1 \;=\; \mathcal{G}(C_1) \cup \mathcal{G}(C_2) = \left\{ (i,j,k) \in \mathbb{Z}^3_+ :\ 1 \leq i+j+k \leq 2,\ (i,j,k) \neq (1,0,1) \right\},$$

and that the sparsity pattern of matrices in $\mathcal{M}$ is described as

$$\begin{pmatrix} & 100 & 010 & 001 & 200 & 110 & 020 & 011 & 002 \\ 100 & \star & \star & 0 & \star & \star & \star & 0 & 0 \\ 010 & \star & \star & \star & \star & \star & \star & \star & \star \\ 001 & 0 & \star & \star & 0 & 0 & \star & \star & \star \\ 200 & \star & \star & 0 & \star & \star & \star & 0 & 0 \\ 110 & \star & \star & 0 & \star & \star & \star & 0 & 0 \\ 020 & \star & \star & \star & \star & \star & \star & \star & \star \\ 011 & 0 & \star & \star & 0 & 0 & \star & \star & \star \\ 002 & 0 & \star & \star & 0 & 0 & \star & \star & \star \end{pmatrix}, \qquad (16)$$

where $\star$ denotes any real number.

Choose a positive semidefinite $\boldsymbol{M} \in \mathcal{M}$, and consider the problem

$$\left.\begin{array}{ll} \text{maximize} & \displaystyle\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{J}} M_{\boldsymbol{\alpha}\boldsymbol{\beta}} y_{\boldsymbol{\alpha}+\boldsymbol{\beta}} - \sum_{\boldsymbol{\alpha} \in \mathcal{H}_1} \sum_{\boldsymbol{\beta} \in \mathcal{H}_1} M_{\boldsymbol{\alpha}\boldsymbol{\beta}} y_{\boldsymbol{\alpha}} y_{\boldsymbol{\beta}} \\ \text{subject to} & \displaystyle \boldsymbol{h}^0_{\boldsymbol{0}} + \sum_{\boldsymbol{\alpha} \in \mathcal{H}} h^0_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}} \leq \tilde{f}_0,\ \boldsymbol{H}^k_{\boldsymbol{0}} + \sum_{\boldsymbol{\alpha} \in \mathcal{H}} \boldsymbol{H}^k_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}} \succeq \boldsymbol{O}\ (k = 1, 2, \ldots, p), \\ & \begin{pmatrix} 1 & (y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C))^T \\ (y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C)) & (y_{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{G}(C), \boldsymbol{\beta} \in \mathcal{G}(C)) \end{pmatrix} \succeq \boldsymbol{O} \\ & \hfill (C \in \Gamma). \end{array}\right\} \qquad (17)$$

Theorem 3.1. *Let $(\hat{y}_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H})$ and $\hat{\gamma}$ be an optimal solution and the optimal objective value of the problem (17). Then the inequality*

$$\left((x^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}_1) - (\hat{y}_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}_1)\right)^T \boldsymbol{M} \left((x^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}_1) - (\hat{y}_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}_1)\right) \leq \hat{\gamma} \qquad (18)$$

holds for every feasible solution $\boldsymbol{x}$ of the POP (9) with the objective function value $f_0(\boldsymbol{x}) \leq \tilde{f}_0$. (A proof of the theorem will be given in Section 3.5.)

Now we do some discussion on how we solve (17). First we show that $\boldsymbol{M}$ allows a sparse Cholesky factorization $\boldsymbol{M} = \boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^T(\boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^T)^T$ with some permutation matrix $\boldsymbol{P}$ such that $\boldsymbol{B}$ is a lower triangular matrix and $\boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^T \in \mathcal{M}$. To see this, we consider the graph $G(\mathcal{N}_1, \mathcal{E}_1)$ with the the node set $\mathcal{N}_1 = \mathcal{H}_1$ and the edge set $\mathcal{E}_1 = \{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{J} : \boldsymbol{\beta} \succ_{\text{grevlex}} \boldsymbol{\alpha}\}$, which represents the sparsity pattern of matrices in the class $\mathcal{M}$.

Lemma 3.2. *$G(\mathcal{N}_1, \mathcal{E}_1)$ is a chordal graph and and its maximal cliques are $\mathcal{G}(C)$ $(C \in \Gamma)$.*

Proof: We use the fact that a graph is chordal if and only if its maximal cliques satisfies the running intersection property (see, for example, [2]). Since the graph $G(\mathcal{N}_0, \overline{\mathcal{E}}_0)$ is chordal, the family Γ of its maximal cliques satisfies the running intersection property: we can index the maximal cliques in Γ such that

$$\forall r = 1, 2, \dots, q-1,\ \exists s(r) \geq r+1;\ C_r \cap (C_{r+1} \cup C_{r+1} \cup \cdots \cup C_q) \subsetneqq C_{s(r)}. \tag{19}$$

Here q denotes the number of cliques in Γ. By construction, we can verify that the maximal cliques of $G(\mathcal{N}_1, \mathcal{E}_1)$ are $\mathcal{G}(C)$ $(C \in \Gamma)$. We will show the following running intersection property

$$\mathcal{G}(C_r) \cap (\mathcal{G}(C_{r+1}) \cup \mathcal{G}(C_{r+1}) \cup \cdots \cup \mathcal{G}(C_q)) \quad \subsetneqq \quad \mathcal{G}(C_{s(r)})\ (r = 1, 2, \dots, q-1) \tag{20}$$

holds for this family. To show this relation, assume that

$$\boldsymbol{\alpha} \in \mathcal{G}(C_r) \cap (\mathcal{G}(C_{r+1}) \cup \mathcal{G}(C_{r+1}) \cup \cdots \cup \mathcal{G}(C_q)).$$

Let $J = \{j : \alpha_j > 0\}$. Then it follows from the relation above and the construction of $\mathcal{G}(C_r)$ $(r = 1, 2, \dots, q)$ that

$$J \subset C_r \text{ and } \ (J \subset C_{r+1} \text{ or } \ J \subset C_{r+2} \ \text{ or } \ \dots \text{ or } \ J \subset C_{r+2}).$$

Hence we obtain that $J \subset C_r \cap (C_{r+1} \cup C_{r+1} \cup \cdots \cup C_q)$, and $J \subset C_{s(r)}$ by (19). This implies that $\boldsymbol{\alpha} \in \mathcal{G}(C_{s(r)})$. Thus we have shown that the inclusion relation in (20) except that it is proper. It follows from (19) that there is a $j \in C_{s(r)}$ such that

$$j \notin C_r \cap (C_{r+1} \cup C_{r+1} \cup \cdots \cup C_q).$$

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$ such that $\alpha_j = 1$ and $\alpha_i = 0$ $(i \neq j)$. Then we see that

$$\boldsymbol{\alpha} \notin \mathcal{G}(C_r) \cap (\mathcal{G}(C_{r+1}) \cup \mathcal{G}(C_{r+1}) \cup \cdots \cup \mathcal{G}(C_q)) \quad \text{and} \quad \boldsymbol{\alpha} \in \mathcal{G}(C_{s(r)}).$$

Therefore we have shown that the proper inclusion relation in (20). ∎

It is known that a graph is chordal if and only if it has a perfect elimination ordering of its node set [10]. Hence, if $\boldsymbol{P}$ is a matrix that performs a perfect elimination ordering of the node set $\mathcal{N}_1$, we have a Cholesky factorization $\boldsymbol{P}^T\boldsymbol{M}\boldsymbol{P} = \boldsymbol{B}\boldsymbol{B}^T$ such that $\boldsymbol{B}$ is a lower triangular matrix and $\boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^T \in \mathcal{M}$.

Let $\overline{\boldsymbol{B}} = \boldsymbol{P}\boldsymbol{B}$. Then $\boldsymbol{M} = \overline{\boldsymbol{B}}\overline{\boldsymbol{B}}^T$. By applying the same technique as the one that we used to derive the SDP problem (8) with an SOCP constraint from the problem (4), we rewrite the problem (17) as

$$\left.\begin{array}{ll}\text{maximize} & \displaystyle\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{J}} M_{\boldsymbol{\alpha}\boldsymbol{\beta}} y_{\boldsymbol{\alpha}+\boldsymbol{\beta}} - t \\ \text{subject to} & t \geq \|\overline{\boldsymbol{B}}^T(y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}_1)\|^2, \ \boldsymbol{h}^0_{\boldsymbol{0}} + \displaystyle\sum_{\boldsymbol{\alpha}\in\mathcal{H}} h^0_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}} \leq \tilde{f}_0, \\ & \boldsymbol{H}^k_{\boldsymbol{0}} + \displaystyle\sum_{\boldsymbol{\alpha}\in\mathcal{H}} \boldsymbol{H}^k_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}} \succeq \boldsymbol{O} \ (k = 1, 2, \ldots, p), \\ & \begin{pmatrix} 1 & (y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C))^T \\ (y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C)) & (y_{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{G}(C), \boldsymbol{\beta} \in \mathcal{G}(C)) \end{pmatrix} \succeq \boldsymbol{O} \\ & \hfill (C \in \Gamma). \end{array}\right\} \quad (21)$$

This problem forms an SDP with an SOCP constraint, but it is different from the normal SDP to which we can apply the primal-dual interior-point method [3, 6, 33, 36]. In particular, the positive semidefinite constraints imposed on the variable matrices

$$\begin{pmatrix} 1 & (y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C))^T \\ (y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{G}(C)) & (y_{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{G}(C), \boldsymbol{\beta} \in \mathcal{G}(C)) \end{pmatrix} \ (C \in \Gamma) \quad (22)$$

are not independent because some $y_{\boldsymbol{\alpha}}$ may be shared by different variable matrices. Basically there are two types of standard form SDPs which existing SDP solvers [3, 6, 33, 36] accept as their input. The one is the equality standard form and the other is the linear matrix inequality standard form. See [17] for more details on conversion of the SDP (21) to those forms, and [7] for software for the conversion.

Now we apply the discussion above to the POP (11). Table 1 summarizes numerical results. Suppose that $\tilde{\boldsymbol{x}} \in F_0$ is an approximate optimal solution of the POP (11). Let $\tilde{f}_0 = f_0(\tilde{\boldsymbol{x}})$. Recall that we have derived an SDP (15) as its sparse SDP relaxation [38] with the relaxation order $\omega = 2$, and that the sparsity pattern of matrices in the class $\mathcal{M}$ is described as in (16). Choose a diagonal matrix $(M_{\boldsymbol{\alpha}\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{H}_1, \boldsymbol{\beta} \in \mathcal{H}_1) \in \mathcal{M}$ with $M_{100,100} = M_{010,010} = M_{001,001} = 1$ and $M_{\boldsymbol{\alpha}\boldsymbol{\beta}} = 0$ elsewhere. Then the problem (17) turns out to be

$$\left.\begin{array}{ll}\text{maximize} & y_{200} + y_{002} + y_{002} - y_{100}^2 - y_{010}^2 - y_{001}^2 \\ \text{subject to} & -y_{100} + y_{110} + 2y_{003} \leq \tilde{f}_0, \ \text{the constraints of (15).}\end{array}\right\} \quad (23)$$

Let $(\hat{y}_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{J})$ be an optimal solution of (23) with the objective value $\hat{\gamma}$, and let $\hat{\boldsymbol{z}} = (\hat{z}_1, \hat{z}_2, \hat{z}_3)^T$ and $\hat{z}_1 = \hat{y}_{100}$, $\hat{z}_2 = \hat{y}_{010}$, $\hat{z}_3 = \hat{y}_{001}$. Then we have a sphere $E(\hat{\boldsymbol{z}}, \gamma) = \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x} - \hat{\boldsymbol{z}}\| \leq \sqrt{\hat{\gamma}}\}$, which contains all feasible solutions of the POP (11) with their objective values not grater than $\tilde{f}_0$.

If we are interested in an error bound only for a particular variable, say x_3, we choose a diagonal matrix $(M_{\boldsymbol{\alpha}\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{H}_1, \boldsymbol{\beta} \in \mathcal{H}_1) \in \mathcal{M}$ with $M_{001,001} = 1$ and $M_{\boldsymbol{\alpha}\boldsymbol{\beta}} = 0$ elsewhere. In this case, the problem (17) turns out to be

$$\left.\begin{array}{ll}\text{maximize} & y_{002} - y_{001}^2 \\ \text{subject to} & -y_{100} + y_{110} + 2y_{003} \leq \tilde{f}_0, \ \text{the constraints of (15).}\end{array}\right\} \quad (24)$$

Then we obtain $\hat{z}_3 = \hat{y}_{001}$ and $\hat{\gamma}$ such that $\hat{z}_3 - \sqrt{\hat{\gamma}} \leq x_3 \leq \hat{z}_3 + \sqrt{\hat{\gamma}}$ for every feasible solution $\boldsymbol{x}$ of the POP (11) with its objective value not grater than $\tilde{f}_0$. Here $(\hat{y}_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{J})$ denotes an optimal solution of (24) with the objective value $\hat{\gamma}$ and $\hat{z}_3 = \hat{y}_{001}$.

Choice of a diagonal matrix	$\tilde{f}_0$	$\hat{\boldsymbol{z}}$	$\sqrt{\hat{\gamma}}$
(23)	-3.083932	(1.00000,-0.169103,-0.985598)	7.3e-5
(24)	-3.083932	(1.00000,-0.169104,-0.985598)	5.2e-5

Table 1: Numerical results on problems (23) and (24).

3.4 Numerical results

Table 2 shows numerical results on the computation of error bounds described in the previous subsection. The first three test problems are unconstrained POPs. They are minimizations of the Broyden banded function [26], the Broyden tridigonal function [26] and the generalized Rosenbrock function [28], respectively. The rest of the test problems are constrained POPs from [11]. We modified and/or added lower and upper bounds of variables in the constrained POPs so that their SDP relaxation worked effectively. The numerical experiment was done on 2×2.8GHz Quad-Core Intel Xeon with 4GB memory. In each test POP, we first computed a lower bound $\hat{f}_0$ for its optimal value and a rough approximate solution $\hat{\boldsymbol{x}}$ by applying SparsePOP [39] with SeDuMi [33] to the POP. Then we refined the solution $\hat{\boldsymbol{x}}$ which is not necessarily a feasible solution of the POP to get a more accurate and feasible solution $\tilde{\boldsymbol{x}}$ of the POP with the objective value $\tilde{f}_0 = f_0(\tilde{\boldsymbol{x}})$ by applying the MATLAB function fminunc.m (in unconstrained cases) or fmincon.m (in constrained cases) of the MATLAB Optimization Toolbox. Then we chose a diagonal $\boldsymbol{M} \in \mathcal{M}$ which induced an enclosing sphere $E(\hat{\boldsymbol{z}}, \hat{\gamma}) = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \hat{\boldsymbol{z}}\|^2 \leq \hat{\gamma}\}$ of the set of feasible solutions with objective values not greater than $\tilde{f}_0$ as shown for the POP (11). See the problem (23). In Table 2, n denotes the number of variables of each test problem, ω the relaxation order used for the SDP relaxation, RelObjErr $= |\tilde{f}_0 - \hat{f}_0| / \max\{1, |\tilde{f}_0|\}$, and E.time the elapsed time for computing $\hat{\boldsymbol{z}}$ and $\hat{\gamma}$ by SeDuMi. In each case, we observe that the sphere $E(\hat{\boldsymbol{z}}, \hat{\gamma}) = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \hat{\boldsymbol{z}}\| \leq \sqrt{\hat{\gamma}}\}$ generated contains the refined global approximate solution $\tilde{\boldsymbol{x}}$.

The generalized Rosenbrock function has two distinct global minimum solutions; the one is $\tilde{\boldsymbol{x}}^+ = (1, 1, 1, \ldots, 1)$ and the other is $\tilde{\boldsymbol{x}}^- = (-1, 1, 1, \ldots, 1)$. To contain both solutions, the sphere with the center $\hat{\boldsymbol{z}} \approx (0, 1, 1, \ldots, 1)$ and $\sqrt{\hat{\gamma}} \approx 1.0$ was generated. When we added the inequality constraint $x_1 \geq 0$ to exclude the one solution $\tilde{\boldsymbol{x}}^-$, a much smaller sphere with radius $\sqrt{\hat{\gamma}} \approx$ 2.6e-2 enclosing the other solution $\tilde{\boldsymbol{x}}^+$ was generated.

SparsePOP [39] implemented both of the dense SDP relaxation [24] and the sparse SDP relaxation [38] for POPs. The sparse SDP relaxation was not effective for some of the constrained POPs from [11], so that we applied the dense SDP relaxation to such POPs.

The POPs ex9_1_1, ex9_1_2, ..., ex9_2_6 involve complementarity conditions such that $x_i \geq 0,\ x_j \geq 0,\ x_i x_j = 0$ for some variables x_i and x_j. In cases of ex9_1_2, ex9_1_4, ex9_1_5, ex9_1_8 and ex9_2_1, the POP has multiple global optimal solutions, so the radius $\sqrt{\hat{\gamma}}$ of the sphere $E(\hat{\boldsymbol{z}}, \hat{\gamma}) = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \hat{\boldsymbol{z}}\|^2 \leq \hat{\gamma}\}$ containing them is larger than the other

Problem	n	ω	RelObjErr	$\frac{\|\tilde{\boldsymbol{x}}-\hat{\boldsymbol{z}}\|}{\|\tilde{\boldsymbol{x}}\|}$	Error Bound (SeDuMi) $\frac{\sqrt{\hat{\gamma}}}{\|\tilde{\boldsymbol{x}}\|}$	$\sqrt{\hat{\gamma}}$	Elapsed time
BroydenBand	10	3	3.4e-08	1.1e-05	3.1e-04	5.5e-04	129.3
BroydenTri	1000	2	1.1e-05	5.1e-04	3.1e-02	7.0e-01	57.7
Rosenbrock$^{(2)}$	1000	2	7.4e-04	3.2e-02	3.2e-02	1.0e+00	11.6
Rosenbrock$^{(3)}$	1000	2	8.5e-05	1.9e-05	8.4e-04	2.6e-02	10.2
ex2_1_3	13	2	2.6e-09	4.0e-09	9.9e-05	6.0e-04	0.2
ex2_1_5	10	2	3.5e-10	1.5e-08	1.0e-04	2.7e-04	0.9
ex2_1_8	24	2	4.9e-09	4.0e-07	5.7e-04	2.4e-02	10.9
ex5_2_2_case1	9	2	1.1e-01	3.9e-01	8.2e-01	2.1e+02	1.0
ex5_2_2_case1$^{(1)}$	9	2	3.8e-08	4.0e-05	6.4e-03	1.7e+00	1.1
ex5_2_2_case2	9	2	3.0e-01	6.5e-01	9.5e-01	5.2e+02	0.8
ex5_2_2_case2$^{(1)}$	9	2	2.9e-07	8.4e-06	2.8e-03	2.2e+00	1.1
ex5_2_2_case3	9	2	2.2e-02	2.5e-01	4.6e-01	1.3e+02	0.7
ex5_2_2_case3$^{(1)}$	9	2	1.4e-09	6.7e-07	9.8e-04	3.2e-01	0.8
ex9_1_1	13	2	7.5e-09	1.1e-02	3.3e-02	5.3e-01	0.3
ex9_1_2$^{(2)}$	10	3	2.9e-05	7.5e-02	2.7e-01	2.2e+00	4.4
ex9_1_4$^{(2)}$	10	2	2.1e-05	9.5e-01	9.5e-01	1.2e+02	0.4
ex9_1_5$^{(2)}$	13	3	7.1e-06	6.7e-01	1.0e+00	4.7e+00	1.1
ex9_1_8$^{(2)}$	12	2	4.0e-10	3.2e-03	3.8e-01	3.5e+00	0.2
ex9_2_1$^{(2)}$	10	2	2.9e-01	8.6e-01	1.2e+00	1.9e+01	0.3
ex9_2_1$^{(1)(2)}$	10	2	1.2e-09	8.3e-02	7.3e-01	1.2e+01	0.6
ex9_2_2	8	2	2.4e-05	9.7e-05	3.6e-03	7.3e-02	0.5
ex9_2_3	16	2	6.9e-08	2.3e-08	1.6e-04	8.5e-03	0.3
ex9_2_4	8	2	1.5e-07	8.3e-05	7.3e-02	3.3e-01	0.3
ex9_2_5	8	2	1.1e-08	5.1e-06	4.4e-04	4.9e-03	0.2
ex9_2_6	12	2	2.9e-07	4.0e-04	2.7e-02	4.7e-02	0.3
alkyl	14	3	1.8e-09	2.2e-05	1.8e-04	2.2e-03	1.7
st_bpaf1a	10	2	1.0e-09	3.4e-08	1.3e-04	2.8e-03	0.5
st_bpaf1b	10	2	9.5e-10	6.9e-09	9.6e-05	2.0e-03	0.3
st_e07	10	2	2.9e-10	1.7e-08	8.1e-05	2.8e-02	0.3
st_jcbpaf2	10	2	1.2e-08	6.9e-08	2.7e-04	3.9e-02	1.0

Table 2: Polynomial optimization problems. (1) : the dense SDP relaxation was applied because the sparse SDP relaxation was not effective. (2) : the POP has multiple global optimal solutions. (3) : the constraint $x_1 \geq 0$ was added to compute a solution with $x_1 \geq 0$.

cases. Among these POPs, we investigate global optimal solutions of ex9_1_2 in details. The problem is of the form

$$\left.\begin{array}{ll}\text{minimize} & -x_1 - 3x_2 \\ \text{subject to} & -x_1 + x_2 + x_3 = 3,\ x_1 + 2x_2 + x_4 = 12, \\ & 4x_1 - x_2 + x_5 = 12, -x_2 + x_6 = 0, \\ & x_7 + 2x_8 - x_9 - x_{10} = -1, \\ & x_7x_3 = 0,\ x_8x_4 = 0,\ x_9x_5 = 0,\ x_{10}x_6 = 0, \\ & 0 \leq x_j \leq 5\ (j = 1, 2, \dots, 10). \end{array}\right\} \quad (25)$$

Table 3 shows error bounds to each coordinate of the approximate solution $\tilde{\boldsymbol{x}}$ of ex9_1_2 which are obtained by taking an $\boldsymbol{M} \in \mathcal{M}$ for an enclosing ellipsoidal set $E(\hat{\boldsymbol{z}}^i, \hat{\gamma}_i) = \{\boldsymbol{x} \in \mathbb{R}^n : |x_i - \hat{z}^i_i| \leq \sqrt{\hat{\gamma}_i}\}$. From this table, we see that $x_3 > 0,\ x_6 > 0$ and $x_9 > 0$ at every global

i	$\tilde{x}^i_i$	$\hat{z}^i_i$	$\sqrt{\hat{\gamma}_i}$
1	+4.0000	+3.9997	+0.0055
2	+4.0000	+4.0002	+0.0145
3	+3.0000	+2.9995	+0.0089
4	+0.0000	+0.0002	+0.0279
5	+0.0000	+0.0009	+0.0148
6	+4.0000	+4.0002	+0.0123
7	−0.0000	+0.0000	+0.0030
8	+0.7170	+1.0000	+1.0001
9	+2.4340	+3.0000	+2.0004
10	+0.0000	+0.0000	+0.0041

Table 3: Numerical results on ex9_1_2.

optimal solution $\boldsymbol{x}$ of the problem. Hence $x_5 = x_7 = x_{10} = 0$ at every optimal solution $\boldsymbol{x}$ by the complementarity conditions. Eliminating these variables $x_5,\ x_7$ and x_{10}, we obtain

$$\left.\begin{array}{ll}\text{minimize} & -x_1 - 3x_2 \\ \text{subject to} & -x_1 + x_2 + x_3 = 3,\ x_1 + 2x_2 + x_4 = 12, \\ & 4x_1 - x_2 = 12, -x_2 + x_6 = 0, \\ & 2x_8 - x_9 = -1,\ x_8x_4 = 0. \\ & 0 \leq x_j \leq 5\ (j = 1, 2, 3, 4, 6, 8, 9), \end{array}\right\} \quad (26)$$

which is equivalent to (25). The optimal solutions of the reduced problems are given by

$$x_1 = 4,\ x_2 = 4,\ x_3 = 3,\ x_4 = 0,\ x_6 = 4,\ 0 \leq x_8 = (x_9 - 1)/2 \leq 2,\ 1 \leq x_9 \leq 5.$$

In Table 4, we show error bounds to each coordinate of the approximate optimal solution $\tilde{\boldsymbol{x}}$ of the reduced problem of (26). We observe that the coordinates $x_1,\ x_2,\ x_3,\ x_4$ and x_6 of the approximate optimal solution $\tilde{\boldsymbol{x}}$ are accurate, and that the bounds $|x_i - \hat{z}_i| \leq \sqrt{\hat{\gamma}_i}$ $(i = 8, 9)$ for all optimal solutions $\boldsymbol{x}$ of (26) are tight.

i	$\tilde{x}_i$	$\hat{z}_i$	$\sqrt{\hat{\gamma}_i}$
1	+4.0000	+4.0000	+0.0002
2	+4.0000	+4.0000	+0.0002
3	+3.0000	+3.0000	+0.0006
4	+0.0000	+0.0000	+0.0006
6	+4.0000	+4.0000	+0.0004
8	+0.9604	+1.0000	+1.0000
9	+2.9208	+3.0000	+2.0000

Table 4: Numerical results on ex9_1_2.

3.5 Proof of Theorem 3.1

In order to prove the theorem, we utilize some fundamental results about positive semidefinite matrix completion. Let $\overline{\mathcal{H}}_1 = \{\mathbf{0}\} \cup \mathcal{H}_1$. We introduce a symmetric variable matrix of the form $\boldsymbol{Y} = \left(Y_{\boldsymbol{\alpha\beta}} : \boldsymbol{\alpha} \in \overline{\mathcal{H}}_1,\ \boldsymbol{\beta} \in \overline{\mathcal{H}}_1\right)$. Then we can embed all matrix variables listed in (22) involved in the problem (17) into $\boldsymbol{Y}$ with the additional constraints

$$\left.\begin{array}{rcl} Y_{\mathbf{00}} &=& 1,\ Y_{\boldsymbol{\alpha}\mathbf{0}} = y_{\boldsymbol{\alpha}} \ \text{ if } \boldsymbol{\alpha} \in \mathcal{H}_1, \\ Y_{\mathbf{0}\boldsymbol{\beta}} &=& y_{\boldsymbol{\beta}} \ \text{ if } \boldsymbol{\beta} \in \mathcal{H}_1,\ Y_{\boldsymbol{\alpha\beta}} = y_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \ \text{ if } (\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{J}. \end{array}\right\} \tag{27}$$

Thus, instead of the problem (17), we consider the following problem

$$\left.\begin{array}{ll} \text{maximize} & \displaystyle\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{J}} M_{\boldsymbol{\alpha\beta}} y_{\boldsymbol{\alpha}+\boldsymbol{\beta}} - \sum_{\boldsymbol{\alpha}\in\mathcal{H}_1}\sum_{\boldsymbol{\beta}\in\mathcal{H}_1} M_{\boldsymbol{\alpha\beta}} y_{\boldsymbol{\alpha}} y_{\boldsymbol{\beta}} \\ \text{subject to} & \boldsymbol{h}^0_{\mathbf{0}} + \displaystyle\sum_{\boldsymbol{\alpha}\in\mathcal{H}} h^0_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}} \le \tilde{f}_0, \\ & \boldsymbol{H}^k_{\mathbf{0}} + \displaystyle\sum_{\boldsymbol{\alpha}\in\mathcal{H}} \boldsymbol{H}^k_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}} \succeq \boldsymbol{O} \ (k = 1,2,\ldots,p), \\ & \boldsymbol{Y} \succeq \boldsymbol{O} \text{ with the condition (27).} \end{array}\right\} \tag{28}$$

Lemma 3.3. *The problem (17) is equivalent to the problem (28). More precisely,* $(y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H})$ *is a feasible solution of the problem (17) if and only if* $((y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}), (Y_{\boldsymbol{\alpha\beta}} : \boldsymbol{\alpha} \in \overline{\mathcal{H}}_1, \boldsymbol{\beta} \in \overline{\mathcal{H}}_1))$ *is a feasible solution of the problem (28) for some* $(Y_{\boldsymbol{\alpha\beta}} : \boldsymbol{\alpha} \in \overline{\mathcal{H}}_1, \boldsymbol{\beta} \in \overline{\mathcal{H}}_1))$.

Proof: If $\boldsymbol{Y}$ is positive semidefinite then so is every principal submatrix. This implies that if $((y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}), (Y_{\boldsymbol{\alpha\beta}} : \boldsymbol{\alpha} \in \overline{\mathcal{H}}_1, \boldsymbol{\beta} \in \overline{\mathcal{H}}_1))$ is a feasible solution of the problem (28), then $(y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H})$ is a feasible solution of the problem (17). To prove the converse, assume that $(y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H})$ is a feasible solution of the problem (17). Fix the elements

$$Y_{\mathbf{00}},\ Y_{\boldsymbol{\alpha}\mathbf{0}}\ (\boldsymbol{\alpha} \in \mathcal{H}_1),\ Y_{\mathbf{0}\boldsymbol{\beta}}\ (\boldsymbol{\beta} \in \mathcal{H}_1),\ Y_{\boldsymbol{\alpha\beta}}\ ((\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{J})$$

by the condition (27). Then all constraints of the problem (28) except $\boldsymbol{Y} \succeq \boldsymbol{O}$ are satisfied. Hence it is sufficient to prove

(a) we can specify the unfixed elements in $\boldsymbol{Y}$ so that $\boldsymbol{Y}$ becomes positive semidefinite.

(This is exactly a positive semidefinite matrix completion problem). To prove this assertion, we consider a graph $G(\mathcal{N}_2, \mathcal{E}_2)$ with $\mathcal{N}_2 = \overline{\mathcal{H}}_1 = \{\mathbf{0}\} \cup \mathcal{H}_1$ and

$$\begin{aligned} \mathcal{E}_2 &= \{(\mathbf{0}, \boldsymbol{\beta}) \in \mathcal{N}_2 \times \mathcal{N}_2 : \boldsymbol{\beta} \in \mathcal{H}_1\} \\ &\quad \cup \left\{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{N}_2 \times \mathcal{N}_2 : (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{J},\ \boldsymbol{\alpha} \preceq_{\text{grevlex}} \boldsymbol{\beta}\right\}. \end{aligned}$$

We will show that

(b) $G(\mathcal{N}_2, \mathcal{E}_2)$ forms a chordal graph with the maximal cliques $\{\mathbf{0}\} \cup \mathcal{G}(C)$ $(C \in \Gamma)$.

Note that each clique $\{\mathbf{0}\} \cup \mathcal{G}(C)$ is corresponding to the index set of a matrix variable in (22) or a principal submatrix of $\boldsymbol{Y}$ whose values are fixed by (27). Therefore, (a) follows from the fundamental result about positive semidefinite matrix completion (Gron, Johnson, Sá and Wolkowicz [9, Theorem 7]). The assertion (b) follows from the fact that the graph $G(\mathcal{N}_2, \mathcal{E}_2)$ is obtained by adding a node $\{\mathbf{0}\}$ and edges $(\mathbf{0}, \boldsymbol{\alpha})$ $(\boldsymbol{\alpha} \in \mathcal{H}_1)$ to the chordal graph $G(\mathcal{N}_1, \mathcal{E}_1)$. In fact, the running intersection property

$$\begin{aligned} &(\{\mathbf{0}\} \cup \mathcal{G}(C_r)) \cap ((\{\mathbf{0}\} \cup \mathcal{G}(C_{r+1})) \cup (\{\mathbf{0}\} \cup \mathcal{G}(C_{r+1})) \cup \cdots \cup (\{\mathbf{0}\} \cup \mathcal{G}(C_q))) \\ &\qquad \subsetneq \left(\{\mathbf{0}\} \cup \mathcal{G}(C_{s(r)})\right) \ (r = 1, 2, \ldots, q-1) \end{aligned}$$

on the maximal cliques $\{\mathbf{0}\} \cup \mathcal{G}(C)$ $(C \in \Gamma)$ of $G(\mathcal{N}_2, \mathcal{E}_2)$ follows from the running intersection property (20) on the the maximal cliques $\mathcal{G}(C)$ $(C \in \Gamma)$ of $G(\mathcal{N}_1, \mathcal{E}_1)$. ∎

We represent the symmetric matrix variable $\boldsymbol{Y}$ as

$$\boldsymbol{Y} = \begin{pmatrix} 1 & (y_{\boldsymbol{\beta}} : \boldsymbol{\beta} \in \mathcal{H}_1)^T \\ (y_{\boldsymbol{\beta}} : \boldsymbol{\beta} \in \mathcal{H}_1) & (W_{\boldsymbol{\alpha}\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{H}_1, \boldsymbol{\beta} \in \mathcal{H}_1) \end{pmatrix}. \tag{29}$$

Then the condition (27) turns out to be

$$W_{\boldsymbol{\alpha}\boldsymbol{\beta}} = y_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \ \text{ if } (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{J}. \tag{30}$$

We also observe that the objective function of (28) is identical to

$$\text{Trace}\ \boldsymbol{M}\left((W_{\boldsymbol{\alpha}\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{H}_1, \boldsymbol{\beta} \in \mathcal{H}_1) - (y_{\boldsymbol{\beta}} : \boldsymbol{\beta} \in \mathcal{H}_1)(y_{\boldsymbol{\beta}} : \boldsymbol{\beta} \in \mathcal{H}_1)^T\right)$$

under the condition (30). Therefore we obtain the problem

$$\left.\begin{array}{ll} \text{maximize} & \text{Trace}\ \boldsymbol{M}\Big((W_{\boldsymbol{\alpha}\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{H}_1, \boldsymbol{\beta} \in \mathcal{H}_1) \\ & \quad -(y_{\boldsymbol{\beta}} : \boldsymbol{\beta} \in \mathcal{H}_1)(y_{\boldsymbol{\beta}} : \boldsymbol{\beta} \in \mathcal{H}_1)^T\Big) \\ \text{subject to} & \boldsymbol{h}^0_{\mathbf{0}} + \displaystyle\sum_{\boldsymbol{\alpha} \in \mathcal{H}} h^0_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}} \leq \tilde{f}_0, \\ & \boldsymbol{H}^k_{\mathbf{0}} + \displaystyle\sum_{\boldsymbol{\alpha} \in \mathcal{H}} \boldsymbol{H}^k_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}} \succeq \boldsymbol{O} \ (k = 1, 2, \ldots, p), \\ & \begin{pmatrix} 1 & (y_{\boldsymbol{\beta}} : \boldsymbol{\beta} \in \mathcal{H}_1)^T \\ (y_{\boldsymbol{\beta}} : \boldsymbol{\beta} \in \mathcal{H}_1) & (W_{\boldsymbol{\alpha}\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{H}_1, \boldsymbol{\beta} \in \mathcal{H}_1) \end{pmatrix} \succeq \boldsymbol{O} \\ & \text{the condition (30),} \end{array}\right\} \tag{31}$$

which is equivalent to the problem (28).

Now we are ready to prove Theorem 3.1. Let $(\hat{y}_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H})$ and $\hat{\gamma}$ be an optimal solution and the optimal objective value of the problem (17). Then there is a $(\widehat{W}_{\boldsymbol{\alpha}\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{H}_1, \boldsymbol{\beta} \in \mathcal{H}_1)$ such that $\left((\hat{y}_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}), (\widehat{W}_{\boldsymbol{\alpha}\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{H}_1, \boldsymbol{\beta} \in \mathcal{H}_1)\right)$ is an optimal solution of the problem (31) with the same objective value $\hat{\gamma}$. By (ii) of Theorem 2.3, we see that

$$((y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}_1) - (\hat{y}_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}_1))^T \boldsymbol{M} \left((y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}_1) - (\hat{y}_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}_1)\right) \leq \hat{\gamma}$$

holds for every feasible solution $\left((y_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}), (W_{\boldsymbol{\alpha}\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{H}_1, \boldsymbol{\beta} \in \mathcal{H}_1)\right)$ of the problem (31). If $\boldsymbol{x} \in \mathbb{R}^n$ is a feasible solution of the POP (9) with the objective value $f_0(\boldsymbol{x}) \leq \tilde{f}_0$, then

$$\begin{aligned} &\left((\boldsymbol{y}_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}), (W_{\boldsymbol{\alpha}\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{H}_1, \boldsymbol{\beta} \in \mathcal{H}_1)\right) \\ &= \left((\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}), (\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{H}_1, \boldsymbol{\beta} \in \mathcal{H}_1)\right) \end{aligned} \quad (32)$$

is a feasible solution of the problem (31). Therefore the inequality (18) follows from the inequality above. This completes the proof of Theorem 3.1.

Remark 3.4. The objective function

$$\text{Trace}\, \boldsymbol{M} \left((W_{\boldsymbol{\alpha}\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{H}_1, \boldsymbol{\beta} \in \mathcal{H}_1) - (y_{\boldsymbol{\beta}} : \boldsymbol{\beta} \in \mathcal{H}_1)(y_{\boldsymbol{\beta}} : \boldsymbol{\beta} \in \mathcal{H}_1)^T\right)$$

of the problem (31) is nonnegative over the feasible region. (Recall the inequality (5)). It attains the minimum value 0 when (32) holds for some feasible solution $\boldsymbol{x}$ of the POP (9). Therefore the problem (31) (hence the problem (17)) may be regarded as a problem of finding a point with the worst feasibility. On the other hand, if $\boldsymbol{M}$ is positive definite and $\hat{\gamma} = 0$, then $(\widehat{W}_{\boldsymbol{\alpha}\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{H}_1, \boldsymbol{\beta} \in \mathcal{H}_1) = (\hat{y}_{\boldsymbol{\beta}} : \boldsymbol{\beta} \in \mathcal{H}_1)(y_{\boldsymbol{\beta}} : \boldsymbol{\beta} \in \mathcal{H}_1)^T$, which implies that the variable matrix $\boldsymbol{Y}$ given in (29) is rank 1 at the solution of the problem (31). It also follows from $\hat{\gamma} = 0$ that the feasible region whose objective value is less than $\hat{f}_0$ consists of a unique global optimal solution $\boldsymbol{x}$ given by $(\boldsymbol{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H}) = (\hat{y}_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{H})$.

4 Quadratic optimization problems

A quadratic optimization problem (abbreviated by QOP) may be regarded as a special case of the polynomial optimization problem (9) where all polynomials are quadratic or $\omega_k = \lceil \deg(f_k(\boldsymbol{x}))/2 \rceil \leq 1$ $(k = 0, 1, 2, \ldots, p)$. We can apply Lasserre's hierarchy SDP relaxation [24] and its sparse variant [38] to QOPs, and all the results on exploiting sparsity presented in Section 3 remain valid with the relaxation order $\omega \geq \omega_{\max} = 1$. If we take $\omega = 1$, Lasserre's SDP relaxation is essentially equivalent to a classical and standard SDP relaxation originally proposed by Shor [32] for QOPs. See also [5, 30, etc.]. In this case, there is another way of exploiting sparsity, which we present in this section. If the QOP satisfies a structured sparsity characterized by a chordal graph, the resulting SDP inherits the sparsity. We can utilize some techniques [8, 17, 27] developed for exploiting sparsity in SDPs, so we don't need to take care of sparsity of the QOP when we construct its SDP relaxation.

4.1 An SDP relaxation of a general quadratic optimization problem

Let f_k $(k = 0, 1, \ldots, p)$ be a quadratic function in $\boldsymbol{X} \in \mathbb{R}^{m\times\ell}$ of the form

$$f_k(\boldsymbol{X}) \;=\; \mathrm{Trace}\begin{pmatrix} \boldsymbol{G}_k & \boldsymbol{A}_k^T \\ \boldsymbol{A}_k & \boldsymbol{Q}_k \end{pmatrix}\begin{pmatrix} \boldsymbol{I}_\ell & \boldsymbol{X}^T \\ \boldsymbol{X} & \boldsymbol{X}\boldsymbol{X}^T \end{pmatrix}, \tag{33}$$

where $\boldsymbol{G}_k \in \mathbb{S}^\ell$, $\boldsymbol{A}_k \in \mathbb{R}^{m\times\ell}$ and $\boldsymbol{Q}_k \in \mathbb{S}^m$. Let $\boldsymbol{f} : \mathbb{R}^{m\times\ell} \to \mathbb{R}^p$ be a vector function such that $\boldsymbol{f}(\boldsymbol{X}) = (f_1(\boldsymbol{X}), f_2(\boldsymbol{X}), \ldots, f_p(\boldsymbol{X}))^T$ for every $\boldsymbol{X} \in \mathbb{R}^{m\times\ell}$, and J be a symmetric cone in $\mathbb{R}^p$. We consider a QOP

$$\text{minimize} \quad f_0(\boldsymbol{X}) \quad \text{subject to} \quad \boldsymbol{X} \in F_0. \tag{34}$$

Here $F_0 = \left\{\boldsymbol{X} \in \mathbb{R}^{m\times\ell} : \boldsymbol{f}(\boldsymbol{X}) \in J\right\}$. This QOP is quite general. If we take $\ell = 1$ (hence $\boldsymbol{X} \in \mathbb{R}^m$) and the nonnegative orthant $\mathbb{R}^m_+$ of the m-dimensional vector space for J, the problem (34) stands for a usual QOP in a variable vector $\boldsymbol{X} \in \mathbb{R}^m$. The problem (34) covers bilinear or quadratic SDPs, if we take the cone $\mathbb{S}^r_+$ of positive semidefinite matrices for J. In this case, we identify the space $\mathbb{S}^r$ of $r \times r$ symmetric matrices with an $r(r+1)/2$ dimensional subspace of the $p = r \times r$ dimensional column vector space. Also the sensor network localization problem which we will present in the next subsection is regarded as a special case of the QOP (34). We describe a (dense) SDP relaxation which unifies many existing SDP relaxations [1, 5, 30, 32] for those problems.

Define the lifting set $\mathbb{L}$, $\hat{f}_k : \mathbb{L} \to \mathbb{R}$ $(k = 0, 1, \ldots, p)$ and $\hat{\boldsymbol{f}} : \mathbb{L} \to \mathbb{R}^p$ by

$$\begin{aligned}
\mathbb{L} &= \left\{(\boldsymbol{V}, \boldsymbol{W}) \in \mathbb{R}^{m\times\ell} \times \mathbb{S}^m : \begin{pmatrix} \boldsymbol{I}_\ell & \boldsymbol{V}^T \\ \boldsymbol{V} & \boldsymbol{W} \end{pmatrix} \in \mathbb{S}^{\ell+m}_+ \right\}, \\
\hat{f}_k(\boldsymbol{V}, \boldsymbol{W}) &= \mathrm{Trace}\begin{pmatrix} \boldsymbol{G}_k & \boldsymbol{A}_k^T \\ \boldsymbol{A}_k & \boldsymbol{Q}_k \end{pmatrix}\begin{pmatrix} \boldsymbol{I}_\ell & \boldsymbol{V}^T \\ \boldsymbol{V} & \boldsymbol{W} \end{pmatrix} \\
&\qquad\qquad \text{for every } (\boldsymbol{V}, \boldsymbol{W}) \in \mathbb{R}^{m\times\ell} \times \mathbb{S}^m, \\
\hat{\boldsymbol{f}}(\boldsymbol{V}, \boldsymbol{W}) &= (\hat{f}_1(\boldsymbol{V}, \boldsymbol{W}), \hat{f}_2(\boldsymbol{V}, \boldsymbol{W}), \ldots, \hat{f}_p(\boldsymbol{V}, \boldsymbol{W}))^T \\
&\qquad\qquad \text{for every } (\boldsymbol{V}, \boldsymbol{W}) \in \mathbb{R}^{m\times\ell} \times \mathbb{S}^m.
\end{aligned}$$

Then the problem below serves as an SDP relaxation of the QOP (34).

$$\text{minimize} \quad \hat{f}_0(\boldsymbol{V}, \boldsymbol{W}) \quad \text{subject to } (\boldsymbol{V}, \boldsymbol{W}) \in \widehat{F}_0, \tag{35}$$

where $\widehat{F}_0 = \left\{(\boldsymbol{V}, \boldsymbol{W}) \in \mathbb{L} : \hat{\boldsymbol{f}}(\boldsymbol{V}, \boldsymbol{W}) \in J\right\}$. In fact, we note that $\hat{f}_k(\boldsymbol{X}, \boldsymbol{X}\boldsymbol{X}^T) = f_k(\boldsymbol{X})$ $(k = 0, 1, \ldots, p)$ and $\hat{\boldsymbol{f}}(\boldsymbol{X}, \boldsymbol{X}\boldsymbol{X}^T) = \boldsymbol{f}(\boldsymbol{X})$ for every $\boldsymbol{X} \in \mathbb{R}^{m\times\ell}$. Hence, if $\boldsymbol{X} \in \mathbb{R}^{m\times\ell}$ is a feasible solution of the QOP (34), then $(\boldsymbol{V}, \boldsymbol{W}) = (\boldsymbol{X}, \boldsymbol{X}\boldsymbol{X}^T)$ is a feasible solution of the the SDP (35) with the same objective value $\hat{f}_0(\boldsymbol{V}, \boldsymbol{W}) = f_0(\boldsymbol{X})$.

Assume that an approximate global optimal solution $\widetilde{\boldsymbol{X}} \in F_0$ with the objective value $\tilde{f}_0 = f_0(\widetilde{\boldsymbol{X}})$ is available or that an upper bound $\tilde{f}_0$ for the optimal value of (34) is known. Let $F = \{\boldsymbol{X} \in F_0 : f_0(\boldsymbol{X}) \leq \tilde{f}_0\}$. In both cases, the semialgebraic set F contains all optimal solutions. To compute an ellipsoidal set containing F, we utilize the framework of the SDP relaxation described above. Let $\boldsymbol{M} \in \mathbb{S}^m_+$, and define

$$\widehat{C} \;=\; \left\{ (\boldsymbol{V}, \boldsymbol{W}) \in \widehat{F}_0 : \hat{f}_0(\boldsymbol{V}, \boldsymbol{W}) \leq \tilde{f}_0 \right\}.$$

We know that $(\boldsymbol{X}, \boldsymbol{X}\boldsymbol{X}^T) \in \widehat{C}$ if $\boldsymbol{X} \in F$. Now we can apply (ii) of Theorem 2.3 to $\widehat{C}$. Let $(\widehat{\boldsymbol{Z}}, \widehat{\boldsymbol{W}})$ be a maximum solution of the problem (4) with the objective value $\hat{\gamma} = \zeta(\widehat{\boldsymbol{Z}}, \widehat{\boldsymbol{W}})$. Then $\boldsymbol{V} \in E(\widehat{\boldsymbol{Z}}, \hat{\gamma})$ for every $(\boldsymbol{V}, \boldsymbol{W}) \in \widehat{C}$; hence $F \subset E(\widehat{\boldsymbol{Z}}, \hat{\gamma})$.

The QOP (34) and its SDP relaxation problem (35) share coefficient matrices $\begin{pmatrix} \boldsymbol{G}_k & \boldsymbol{A}_k^T \\ \boldsymbol{A}_k & \boldsymbol{Q}_k \end{pmatrix}$ $(k = 0, 1, 2, \ldots, p)$. If these matrices are sparse in (34), then the sparsity is inherited to the *domain space sparsity* , introduced in the paper [17], of (35), and we can reduce the size of (35) effectively by apply the d-space conversion methods proposed there. In such a case, if we take an $\boldsymbol{M} \in \mathbb{S}^m_+$ which does not destroy the domain sparsity, we can efficiently solve the problem (4). To describe a class $\mathcal{M}_+$ of such matrices, we consider a graph $G(\mathcal{N}, \mathcal{E})$ with the node set $\mathcal{N} = \{1, 2, \ldots, m\}$ and the edge set

$$\mathcal{E} = \{(i, j) : i < j,\ [Q_k]_{ij} \neq 0 \text{ for some } k \in \{0, 1, 2, \ldots, p\}\},$$

which represents the aggregated sparsity pattern of the matrices $\boldsymbol{Q}_k$ $(k = 0, 1, \ldots, p)$. Let $G(\mathcal{N}, \overline{\mathcal{E}})$ be a chordal extension of $G(\mathcal{N}, \mathcal{E})$, and we define

$$\widetilde{\mathcal{E}} = \{(i, j) \in \mathcal{N} \times \mathcal{N} : i = j,\ (i, j) \in \overline{\mathcal{E}} \text{ or } (j, i) \in \overline{\mathcal{E}}\}.$$

Then we can describe $\mathcal{M}_+$ as $\mathcal{M}_+ = \left\{ \boldsymbol{M} \in \mathbb{S}^m_+ : M_{ij} = 0 \text{ if } (i, j) \notin \widetilde{\mathcal{E}} \right\}$. We note that any diagonal matrix with nonnegative entries belongs to $\mathcal{M}_+$.

4.2 The sensor network localization problem with exact distance

We consider a sensor network localization problem with n sensors and $n_a = \bar{n} - n$ anchors. Let $\rho > 0$ be a radio range, which determines the set $\mathcal{N}^\rho_x$ for pairs of sensors p and q such that their (Euclidean) distance d_{pq} is not greater than ρ, and the set $\mathcal{N}^\rho_a$ for pairs of a sensor p and an anchor r such that their distance d_{pr} does not exceed ρ;

$$\left.\begin{array}{rcl} \mathcal{N}^\rho_x & = & \{(p, q) : 1 \leq p < q \leq n, \|\boldsymbol{x}_p - \boldsymbol{x}_q\| \leq \rho\}, \\ \mathcal{N}^\rho_a & = & \{(p, r) : 1 \leq p \leq n,\ n+1 \leq r \leq \bar{n}, \|\boldsymbol{x}_p - \boldsymbol{a}_r\| \leq \rho\}, \end{array}\right\}$$

where $\boldsymbol{x}_p \in \mathbb{R}^\ell$ denotes unknown location of sensor p and $\boldsymbol{a}_r \in \mathbb{R}^\ell$ known location of anchor r. Cases where $\ell = 2$ or $\ell = 3$ are of practical interest. Let $\mathcal{N}_x$ be a subset of $\mathcal{N}^\rho_x$ and $\mathcal{N}_a$ a subset of $\mathcal{N}^\rho_a$. Then the unknown locations of sensors are characterized by the system of equations:

$$d^2_{pq} = \|\boldsymbol{x}_p - \boldsymbol{x}_q\|^2 \;\; (p, q) \in \mathcal{N}_x,\ d^2_{pr} = \|\boldsymbol{x}_p - \boldsymbol{a}_r\|^2 \;\; (p, r) \in \mathcal{N}_a.$$

For given $\boldsymbol{a}_r$ $(n+1 \leq r \leq \bar{n})$, d_{pq} $((p, q) \in \mathcal{N}_x)$, d_{pr} $((p, r) \in \mathcal{N}_a)$, we want to find the unknown sensors' locations $\boldsymbol{x}_p$ $(1 \leq p \leq n)$ of this system.

To apply the Biswas-Ye SDP relaxation [1] to the above system of equations, we introduce an $m \times \ell$ matrix variable $\boldsymbol{X} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)^T$, and rewrite it as

$$\left.\begin{array}{lcl} \boldsymbol{X}_{p.}\boldsymbol{X}_{p.}^T - \boldsymbol{X}_{p.}\boldsymbol{X}_{q.}^T - \boldsymbol{X}_{q.}\boldsymbol{X}_{p.}^T + \boldsymbol{X}_{q.}\boldsymbol{X}_{q.}^T - d_{pq}^2 & = & 0 \quad (p,q) \in \mathcal{N}_x, \\ \boldsymbol{X}_{p.}\boldsymbol{X}_{p.}^T - \boldsymbol{X}_{p.}\boldsymbol{a}_r - \boldsymbol{a}_r^T\boldsymbol{X}_{p.} + \boldsymbol{a}_r^T\boldsymbol{a}_r - d_{pr}^2 & = & 0 \quad (p,r) \in \mathcal{N}_a, \end{array}\right\} \tag{36}$$

where $\boldsymbol{X}_{p.}$ denotes the pth row of the matrix $\boldsymbol{X}$ or $\boldsymbol{x}_p^T$. Let $F_0 \subset \mathbb{R}^{m\times\ell}$ denote the solution set of (36). The functions involved in the left side of the system of equations are quadratic in $\boldsymbol{X} \in \mathbb{R}^{m\times\ell}$, and can be written in the form of (33). Thus, if we set $f_0(\boldsymbol{X}) = 0$ for every $\boldsymbol{X} \in \mathbb{R}^{m\times\ell}$, we can reduce the sensor network localization problem to a special case of the QOP (34) with J consisting of the zero vector in the $\#\mathcal{N}_x + \#\mathcal{N}_a$ dimensional space, and the Biswas-Ye SDP relaxation [1] of F_0 coincides with $\widehat{F}_0$ of the form

$$\left\{ (\boldsymbol{V}, \boldsymbol{W}) \in \mathbb{R}^{m\times\ell} \times \mathbb{S}^m : \begin{array}{lcl} W_{pp} + W_{qq} - 2W_{pq} - d_{pq}^2 & = & 0 \quad (p,q) \in \mathcal{N}_x, \\ \|\boldsymbol{a}_r\|^2 - 2\sum_{i=1}^{\ell} V_{pi}a_{ri} + W_{pp} - d_{pr}^2 & = & 0 \quad (p,r) \in \mathcal{N}_a, \\ \boldsymbol{O} \preceq \begin{pmatrix} \boldsymbol{I}_\ell & \boldsymbol{V}^T \\ \boldsymbol{V} & \boldsymbol{W} \end{pmatrix} & & \end{array} \right\} \tag{37}$$

Since the objective function is identically zero, we can set $\tilde{f}_0 = 0$. Hence we take $F = F_0$ and $\widehat{C} = \widehat{F}_0$.

It was shown in the paper [1] that if the system (36) of equations is uniquely localizable then the SDP relaxation $\widehat{F}_0 = \widehat{C}$ of its solutions set $F_0 = F$ consists of a unique $(\boldsymbol{V}, \boldsymbol{W}) \in \mathbb{R}^{n\times\ell} \times \mathbb{S}^n$, which satisfies $\boldsymbol{W} = \boldsymbol{V}\boldsymbol{V}^T$, and $\boldsymbol{X} = \boldsymbol{V}$ is a unique solution of (36). See Theorem 2 of [1]. We are interested in cases where (36) is not uniquely localizable. In such a case, the ellipsoidal set $E(\widehat{\boldsymbol{Z}}, \hat{\gamma})$ presented in the previous subsection provides an error bound for the SDP solution, where $(\widehat{\boldsymbol{Z}}, \widehat{\boldsymbol{W}})$ denotes a maximum solution of the problem (4) and $\hat{\gamma} = \zeta(\widehat{\boldsymbol{Z}}, \widehat{\boldsymbol{W}})$ the maximum objective value. If we take $\boldsymbol{M} \in \mathbb{S}^m_+$ to be the identity matrix, then we know that the unknown locations $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$ satisfies that

$$\left(\frac{1}{m} \sum_{p=1}^{m} \|\widehat{\boldsymbol{Z}}_{p.}^T - \boldsymbol{x}_p\|^2 \right)^{1/2} \leq \left(\frac{1}{m}\hat{\gamma} \right)^{1/2}.$$

Here the left hand side is often called the root mean square distance (rmsd). In general, the rmsd is available only when the true locations of sensors are known. Our method computes an upper bound for the rmsd.

Table 5 shows numerical results on a 2-dimensional sensor network localization problem with 1000 sensors and 100 anchors randomly distributed in the unit square region $[0,1] \times [0,1]$, and Table 6 numerical results on a 3-dimensional sensor network localization problem with 500 sensors and 50 anchors randomly distributed in the unit cube region $[0,1]^3$. For the numerical experiment, we construct SDP relaxation problems by SFSDP [16], which implemented a sparse variant [15] of the Biswas-Ye SDP relaxation for the sensor network localization problem, and employed SDPA [6] to solve the resulting SDP. We used 2×2.8GHz Quad-Core Intel Xeon with 4GB memory for numerical experiments. The column Elapsed

ρ	Elapsed time	Rmsd	$(\hat{\gamma}/m)^{1/2}$
0.05	5.0	2.50e-2	3.36e-2
0.06	11.2	7.30e-3	1.09e-2
0.07	5.8	2.14e-4	1.96e-3
0.08	4.7	1.56e-4	1.25e-3
0.09	2.2	3.93e-7	5.11e-4

Table 5: A 2-dimensional sensor network localization problem with randomly distributed 1000 sensors and 100 anchors in $[0, 1] \times [0, 1]$.

ρ	Elapsed time	Rmsd	$(\hat{\gamma}/m)^{1/2}$
0.18	71.3	2.34e-2	3.12e-2
0.20	41.5	8.85e-3	1.38e-2
0.22	32.3	3.88e-3	5.27e-3
0.24	20.5	5.16e-6	7.08e-4

Table 6: A 3-dimensional sensor network localization problem with randomly distributed 500 sensors and 50 randomly generated anchors in $[0, 1]^3$.

time denotes the elapsed time for solving SDPs by SDPA. We observe in both tables that as we take a larger radio range, both rmsd and its upper bound $(\hat{\gamma}/m)^{1/2}$ get smaller.

Let p be fixed. If we take the $m \times m$ matrix $\boldsymbol{M}$ with the (p, p)th element $M_{pp} = 1$ and 0 elsewhere, then $\zeta(\boldsymbol{V}, \boldsymbol{W}) = W_{pp} - \boldsymbol{V}_{p.}\boldsymbol{V}_{p.}^T$. In this case, we know that

$$\|(\widehat{\boldsymbol{Z}}^p_{p.})^T - \boldsymbol{x}_p\| \leq (\hat{\gamma}_p)^{1/2}.$$

Here $(\widehat{\boldsymbol{Z}}^p, \widehat{\boldsymbol{W}}^p)$ denotes a maximum solution of the problem (4) with the objective function $\zeta(\boldsymbol{V}, \boldsymbol{W}) = W_{pp} - \boldsymbol{V}_{p.}\boldsymbol{V}_{p.}^T$ and $\hat{\gamma}_p = \zeta(\widehat{\boldsymbol{Z}}^p, \widehat{\boldsymbol{W}}^p)$ the maximum objective value. It was shown in the paper [1] that if $(\boldsymbol{V}, \boldsymbol{W})$ lies in the relative interior of the solution set of (37) and $\zeta(\boldsymbol{V}, \boldsymbol{W}) = W_{pp} - \boldsymbol{V}_{p.}\boldsymbol{V}_{p.}^T = 0$ then $\boldsymbol{V}_{p.}^T$ attains the unknown location $\boldsymbol{x}_p$ exactly. The inequality above strengthens this result.

Figures 2 shows numerical results on a 2-dimensional sensor network localization problem with 500 sensors distributed randomly in the unit square region $[0, 1] \times [0, 1]$ and 4 anchors placed at the corner of the region. We took the radio range $\rho = 0.09$. The upper left figure shows the graph induced from the underlying sensor network. Some nodes have only two adjacent nodes, so that their locations are not determined uniquely through the system of equations (36). The upper right figure shows solutions of the SDP relaxation (37) which we computed by applying SFSDP with SDPA.We observe some deviations between true and computed locations of sensors. For each sensor p, we applied the ellipsoidal enclosing method with taking the $m \times m$ matrix $\boldsymbol{M}$ with the (p, p)th element $M_{pp} = 1$ and 0 elsewhere. The figure below shows that the deviation $\|\boldsymbol{x}_p - (\widehat{\boldsymbol{Z}}^p_{p.})^T\|$ between true and computed locations is bounded by $(\hat{\gamma}_p)^{\frac{1}{2}}$. We note that the problem (4) with setting the objective function $\zeta(\boldsymbol{V}, \boldsymbol{W}) = W_{pp} - \boldsymbol{V}_{p.}\boldsymbol{V}_{p.}^T$ was solved (in about 5 seconds by SDPA) repeatedly for all $p = 1, 2 \ldots, 500$.

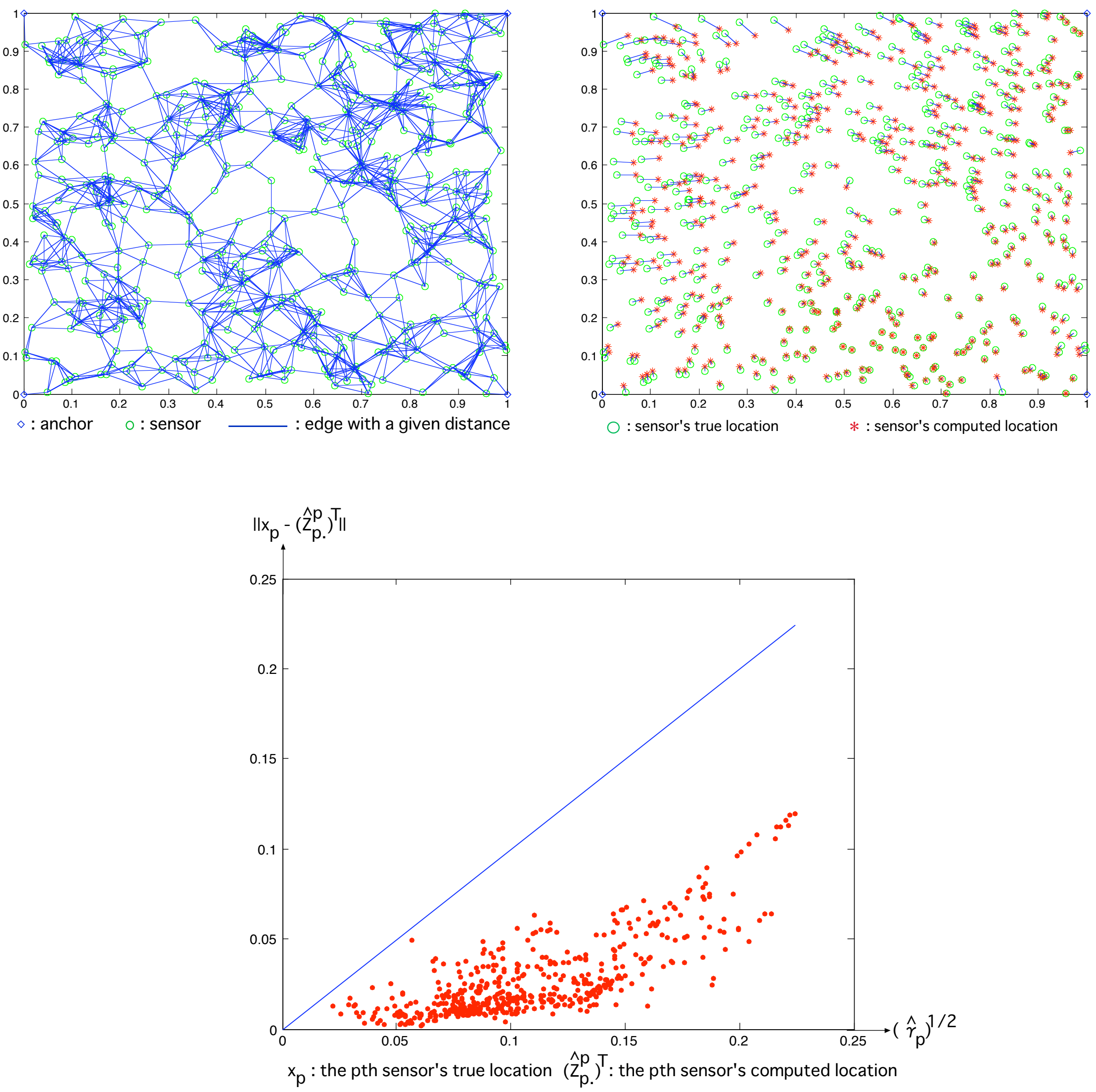

Figure 2: A 2-dimensional sensor network localization problem with randomly distributed 500 sensors in $[0, 1] \times [0, 1]$, 4 anchors at the corner and $\rho = 0.09$.

5 Concluding discussion

We have proposed a numerical method based on lifting and SDP relaxation for computing an enclosing ellipsoidal set $E(\widehat{\boldsymbol{z}}, \hat{\gamma}) = \{\boldsymbol{x} \in \mathbb{R}^m : (\boldsymbol{x} - \widehat{\boldsymbol{z}})^T \boldsymbol{M} (\boldsymbol{x} - \widehat{\boldsymbol{z}}) \leq \hat{\gamma}\}$ of a given semialgebraic subset F of $\mathbb{R}^m$. Here $\boldsymbol{M}$ denotes an $m \times m$ positive semidefinite matrix freely chosen in advance. The quality of Lasserre's hierarchy SDP relaxation [24] and its sparse variant [38] depend on the description of F in terms of polynomial inequalities. We can often improve the quality by tightening lower and upper bounds for variables and/or adding polynomial valid inequalities [38, Section 5.5].We note that an enclosing ellipsoidal set $E(\widehat{\boldsymbol{Z}}, \hat{\gamma})$ of F induces a quadratic valid inequality $(\boldsymbol{x} - \widehat{\boldsymbol{z}})^T \boldsymbol{M} (\boldsymbol{x} - \widehat{\boldsymbol{z}}) \leq \hat{\gamma}$ for F. We did numerical experiments on a successive ellipsoidal enclosing method which applied an SDP relaxation to the semialgebraic set F at each iteration after replacing the old quadratic valid inequality induced from an enclosing ellipsoidal set by a new one. The method generated a sequence of enclosing ellipsoidal sets $\{\widehat{C}^p \ (p = 1, 2, \dots)\}$ of F such that $\widehat{C}^p \supset \widehat{C}^{p+1}$. We observed that the enclosing ellipsoid $\widehat{C}^p$ shrank for a few iterations in some instances but it did not shrank at all in some other instances. More detailed analysis on such successive ellipsoidal enclosing methods will be the subject of further researches.

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