A Note on Complexity of Traveling Tournament Problem

Rishiraj Bhattacharyya*

Abstract

Sports league scheduling problems have gained considerable amount of attention in recent years due to its huge applications and challenges. Traveling Tournament problem, proposed by Trick et. al. (2001), is a problem of scheduling round robin leagues which minimizes the total travel distance maintaining some constraints on consecutive home and away matches. No good algorithm is known to tackle this problem. Many issues including complexity of the problem is open. In this article we show that traveling tournament problem without the constraint of consecutive home and away matches is NP-hard. We hope our technique will be useful in proving the hardness of original TTP.

1 Introduction

Professional Sports Tournaments are major economic activities around the world. They draw attention of millions of people across the globe. The organizers and the broadcasters invest a lot of money in these events. The schedule is a very important aspect of these tournaments. On one hand, there are multiple decision makers; the teams, the broadcasters, the organizers and the government. On the other hand, a scheduler has to take care of multiple objectives and constraints (duration, logistics, fairness etc.). For these challenges, problems of scheduling sports tournaments have gained considerable amount of attention in recent years among the Operations Research community.

Some of the most popular tournaments are round robin leagues. In a round robin league of n(even) teams; each team plays against all other teams. In a single round robin tournament the teams play exactly once against each other team. The schedule consists of $n-1$ rounds. In every round, $\frac{n}{2}$ matches are played with each team playing exactly one match. The match is organized at the stadium of one of the teams playing the match. The corresponding team is said to play a home match and for the other team, it is an away match. In a double round robin league, each team plays against all other teams twice; one home match and one away match. So the tournament schedule has $2(n-1)$ rounds. The teams have to travel to play the away matches. If a team plays two consecutive away matches, it goes directly from one city to the next without returning to its home. Before the tournament starts, all the teams are at the home city and they return to home after the tournament is over.

Now one aspect of organizing a tournament is the cost due to travel. Many of the major tournaments, like the Champion's League in football, is organized across the countries. The players become tired and bored in a long trip. On the other hand, frequent travel incurs extra cost in logistics. Minimizing the total traveling cost maintaining a fair balance between consecutive home and away matches for each team is a challenge to the organizers.The *Traveling Tournament Problem* (TTP) addresses the problem of minimizing the total travel distance in a double round robin league schedule where each team can play at least L and at most U consecutive home and away matches [1]. More formally TTP can be defined as follows:

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*Applied Statistics Unit, Indian Statistical Institute, 203 B.T. Road, Kolkata, email:rishi_r@isical.ac.in

- **[Input]**:
    - The number of teams n.
    - A distance matrix $D_{n \times n}$.
    - Two integers L and U.

- **[Output]**: A double round robin league schedule of n teams such that
    - For all teams, the number of consecutive home and away matches is between L and U inclusive.
    - The total distance traveled by all the teams is minimized.

Since its proposal in [1], TTP has been a notoriously difficult problem to solve. The instances of only 8 teams could be solved till date. In [2], authors describe an combined approach of integer programming and constraint programming. Authors of [3] explored a Lagrangian Relaxation approach. Heuristic techniques, like Simulated Annealing or Tabu Search was employed in [8][10]. Even instances with much relaxed conditions, such as where distances between any two pair of cities are equal, could not be exactly solved. On the other hand no hardness result of TTP is known. In fact, the main open question asked in [4], *Is Traveling Tournament Problem NP-hard ?*

In this paper, we prove that the Traveling Tournament Problem without the constraint of consecutive home and away matches is NP-hard. Specifically we show that if the teams are allowed to play any number of consecutive home and away matches, there is a reduction from an instance of $(1, 2)$- Traveling Salesman Problem; problem of finding a TSP tour in a graph where cost of every edge is either 1 or 2.

Overview of Our Method

We start from an observation, due to [1], that if only one team had to move then the optimal solution to TTP would have been the Traveling Salesman Tour of the cities. But when more than one team has to move, synchronizing their travel seems to be the bottleneck of any previous attempts. To solve TSP using an oracle to solve TTP, We place $n - 1$ teams in one city c (let us call these teams *good* teams) and one team in each of the remaining $n - 1$ cities. Now all the good teams can travel outside c in the same rounds, and play their away matches without returning home. Note that, such a schedule will be feasible when U is set to be n(the number of cities). So we shall fix the parameter in such a way while showing a reduction. This will imply that traveling cost of each team from c is exactly the Traveling Salesman tour of the graph. On the other hand by bounding the weight of each edge we can bound the traveling cost of other teams. Although this bound may not be very tight, but increasing number of teams in c will make this difference very small compared to the overall cost.

Now, to lower bound the total traveling cost of the tournament, we observe that the traveling cost of any team has to be at least the cost of TSP tour of the cities. Next we construct a graph from the given instance that amplifies the cost of Traveling Salesman tour of original instance. In other words, if the input instance has a TSP tour of cost K, the constructed graph will have a TSP tour of cost nK. So if the optimal TSP tour of the instance become $K + 1$, the optimal tour in the constructed graph will be $nK + n$. Now if we can design an instance with a double round robin league schedule where traveling cost of each team is less than $nK + n$; (K is the cost of TSP tour in the input instance), we can reduce the problem of TSP to TTP.

Rest of the paper is organized as follows. In the next section, we introduce the notations used in the paper and formally define the decision problems. In Section 3 we describe the constructions and the associated

results needed for the reduction. Finally in Section 4 we describe the formal reduction from $(1,2)$-TSP to TTP.

2 Notation and Preliminaries

Throughout the paper we follow the following notations.

- $[t] = \{1, 2, \cdots, t\}$; the set of first t natural numbers.

- $G = (V, E)$ is a complete weighted graph without self-loops or parallel edges. V is the set of vertices and E is the set of edges.

We formulate the decision problem of Traveling Tournament Problem as follows:

- **Problem**: Traveling Tournament Problem

- **Instance**: An even integer n; A $n \times n$ matrix D; Two integers L and U; An integer K

- **Question**: Is there a double round robing league schedule of n teams located at venues represented by D such that the number of consecutive home or away matches is between L and U, with total distance traveled by all the teams is at most K?

We represent an instance of this problem by $TTP(n, D, L, U, K)$. Without loss of generality we can assume D is a metric as any real life distance matrix will be a metric. To show that TTP is $NP\text{-}hard$, we show a reduction from a variant of Traveling Salesman Problem; called $(1,2)$-Traveling Salesman Problem $((1,2)$-TSP). Then decision problem of $(1,2)$-TSP is defined as follows

- **Problem**: $(1,2)$- Traveling Salesman Problem

- **Instance**: An integer set $C = \{c_1, c_2, \cdots, c_n\}$ of n cities; Distance $d(c_i, c_j) \in \{1, 2\}$ for all pair $c_i, c_j \in C$; An integer K.

- **Question**: Is there a Traveling salesman tour of C with cost at most K. In other terms is there a permutation $\pi : [n] \to [n]$ such that

$$\sum_{i=1}^{n-1} d(c_{\pi(i)}, c_{\pi(i+1)}) + d(c_{\pi(n)}, c_{\pi(1)}) \leq K$$

It is easy to check that $(1,2)$-TSP is NP-hard. One can construct a reduction from $Hamiltonian\ Cycle$ problem[5] as follows. Let $G = (V, E)$ be an instance of Hamiltonian-Cycle problem with $|V| = n$. One can construct an instance (C, d, K) of $(1,2)$-TSP by making $C = V$ and $K = n$. Finally $d(v_i, v_j)$ is 1 if $(v_i, v_j) \in E$ and 2 otherwise. Now G has a Hamiltonian cycle in G iff there is a tour of C with cost at most n.
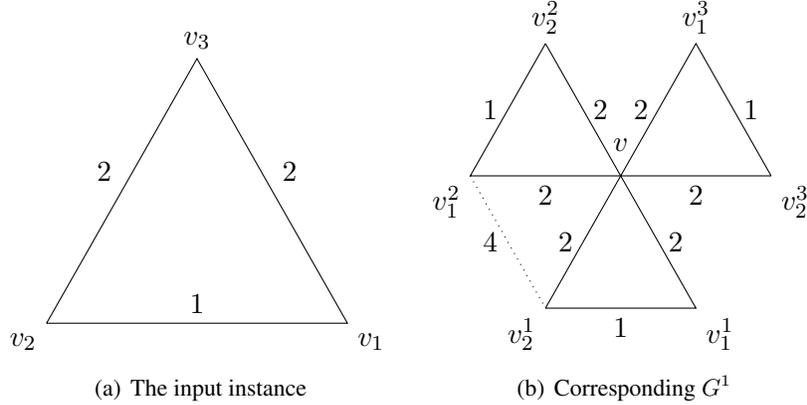
Figure 1: (a): The input instance of TSP, (b): Construction of G^1

3 Construction

Before proving the reduction we need to make some construction. Let (n, D, K) be an instance of $(1, 2)$-TSP. We denote the n cities by $c_1, c_2, \cdots, c_n$. We view the instance as a complete weighted graph $G = (V, E)$ with $|V|$ and $d(v_i, v_j) = D(c_i, c_j)$ for all $i, j \in [n]$ where $d : E \to \{1, 2\}$ is the cost function of the edges. Our objective is to construct a graph G^1 from G such that optimal Traveling Salesman tour of G has cost K iff Traveling Salesman tour of G^1 is of cost nK. Such a graph can easily be constructed in the following way.

Fix a vertex $v \in V$. Consider n distinct copies of G and contract all copies of v into one vertex. So the new graph has $n^2 - n + 1$ vertices. We call v the central vertex of G^1. The cost of the edges are set as follows. If two vertices u, w belongs to same copy of G, then $d'(u, w) = d(u, w)$. Otherwise, $d'(u, w) = d(u, v) + d(v, w)$.

In Figure 1(b) we show the construction of G^1 from the input instance of Figure 1(a). We took three copies of G and contracted all the copies of v_3 to one vertex u. We only showed one edge between different copies to keep the image neat. It is easy to check the following result

Lemma 3.1 G^1 has a traveling salesman tour of of cost nK iff D has a traveling salesman tour of cost K.

4 Reduction

In this section we show the formal Reduction from $(1, 2)$-TSP to TTP. Let (n, d, K) be the input instance of $(1, 2)$-TSP. Construct the graph G^1 from the instance as described in Section 3. Recall that, G^1 has $n^2 - n + 1$ vertices. Let the vertices of G^1 are denoted by $v_1, v_2, \cdots, v_m$ where $m = n^2 - n + 1$. We construct a graph H from G^1 by adding another vertex u to G^1. Each edge (u, v_i) has weight $1000(m+1)^4$.

Now, we construct the corresponding TTP instance. There total $n' = 10(m+1)^2$ teams of which 2 teams are at the central vertex v of G^1, $(m-1)$ teams are at the remaining vertices of G^1 (each of the vertex of G^1, other than the central vertex u has exactly one team) and rest $10(m + 1)^2 - (m + 1) = (m + 1)(10m + 9)$ teams are at vertex u. Let D' denote the distance matrix of graph H. We set $L = 1$ and $U = n'$ which implies that there is effectively no constraint on the number consecutive home and away matches. In Lemma 4.1, we show an upper bound of the total traveling cost of optimum schedule of TTP-$(n', D', 1, n')$

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Lemma 4.1 *If the input instance (n, D) with $n > 5$ has a traveling salesman tour of cost K, then TTP($n', D', 1, n'$) has a feasible schedule of cost at most $20000(m+1)^6 + 10m(m+1)(nK+1)$.*

Proof.   Let τ be a Traveling Salesman tour in (n, D), the input instance of TSP, with cost K. By Lemma 3.1, G^1 has a traveling salesman tour, τ' of cost nK. We shall construct a feasible schedule of the instance TTP($n', D', 1, n'$) with total traveling cost at most $20000(m+1)^6 + 10m(m+1)(nK+1)$. First we divide the teams in $10(m+1)$ groups, each of size $m+1$. All the teams from G^1 are in same group, say Group 1. For rest of the teams (teams at vertex u), groups are formed arbitrarily. It is easy to see that a tournament schedule for any team consists of $2m$ matches (m home match and m away match) against teams from their own group and $2(m+1)(10m+9)$ matches against teams of other groups. In our schedule, in first $2m$ rounds, teams will play all the matches against the teams from own group. Rest of the matches will be played between the groups.

First we need a feasible double round robin schedule of $m+1$ teams. The following lemma, proved in [9] [11], shows that such a schedule is indeed possible.

Lemma 4.2 *There exist a single round robin league schedule of n teams for all even n.*

In our case, the number of teams in each group is $m+1 = n(n-1)+2$, which is an even number. Moreover $L = 1$ and $U = n' > n$, ensures that any double round robin league is feasible in our case. Now, one can can easily construct a double round robin league schedule by repeating the schedule of Lemma 4.2, with home-away reversed.

The total traveling cost of these $2m$ rounds can be bounded in the following way. All the matches of teams, other than of group 1, are at vertex u. So the teams at vertex u do not need to travel in these rounds, hence no traveling cost is incurred by them. By triangle inequality in graph G^1, the traveling cost of each team of group 1, is at most twice the weights of edges incident on the corresponding vertex. Recall that G^1 has m vertices with two teams at u and one team at each of other vertices. Hence the total cost of traveling cost of team at vertex v_i of group 1 is $\leq 2 \sum_{j=1, j \neq i}^{m+1} w(v_i, v_j) \leq 8m$. So total cost incurred in the first $2m$ rounds is at most $8m(m+1)$.

Rest of the matches will be played among the groups. So we shall view the rest of the schedule as a *dummy* tournament of $10(m+1)$ teams where team i represents group i. A match of such a tournament, played between group i and group j, is actually $m+1$ rounds of matches where each team of group i plays against each member of group j. Moreover if group i plays the home matches, then all members of group i will play the home matches in these m rounds.

First we consider any single round tournament of $10(m+1)$ teams. In all the matches of this $10m+9$ rounds, we assign away matches for team 1. By the argument in the previous paragraph, this implies $(10m+9)(m+1)$ away matches for each team of group 1 at vertex u. As all other teams are at vertex u itself, all the matches of these $(10m+9)(m+1)$ rounds are played at vertex u.

Recall that at the end of first $2m$ rounds $(m+1)$ teams of group 1 were at some vertices (not necessarily at their home) in G^1 and all other teams were at u. In the next $(10m+9)(m+1)$ rounds, only the teams of group 1 travel to vertex u. As the cost of an edge from any vertex of G^1 to the vertex u is $1000(m+1)^4$, the total cost of the travel of these rounds is $1000(m+1)^4 \times (m+1) = 1000(m+1)^5$.

For the last $(10m+9)(m+1)$ rounds we repeat the same schedule between the groups with home away reversed. So all the teams of group 1 play home matches in these rounds. Every other group come to G^1 and play their away matches. Recall that each group play exactly m matches at G^1 in m consecutive rounds. As there are m vertices in G^1, the visiting teams need to travel. To illustrate the travel pattern, suppose group i is visiting group 1. First every team plays with their corresponding team; i.e. team 1 of group i plays at

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home of team 1 of group 1, team 2 of group i plays at home of team 2 of group 1 and so on. In the next round the teams of group i move along τ'.

To calculate the cost of these $(10m + 9)(m + 1)$ rounds, note that the teams of group 1 needs to return to their corresponding home vertex at G^1. This travel costs $1000(m+1)^4 \times (m+1) = 1000(m+1)^5$. Next each group go to G^1 and move along the traveling salesman tour. Observe that each team travels exactly m edges in G^1. So total cost of travel for group i

$$
\begin{aligned}
&\leq \quad 2 * 1000(m+1)^5 + (m+1)nK - nK \\
&= \quad 2000(m+1)^5 + mnK
\end{aligned}
$$

As $2(10m + 9)(m + 1) + 2m = 2 * (10m^2 + 20m + 10 - 1) = 2(10(m + 1)^2 - 1)$, these rounds finishes the schedule. So total traveling cost of the schedule is at most $(10m+9)(2000(m+1)^5 + mnK) + 2000(m + 1)^5 + 8m(m + 1) = 20000(m + 1)^6 + (10m + 9)mnK + 8m(m + 1) < 20000(m + 1)^6 + 10m(m + 1)nK + 8m(m + 1) < 20000(m + 1)^6 + 10m(m + 1)(nK + 1)$. ∎

Lemma 4.3 *If the input instance (n, D) with $n > 5$ has no traveling salesman tour of cost at most K, then $TTP(n', D', 1, n')$ has no feasible schedule of cost at most $20000(m + 1)^6 + 10m(m + 1)(nK + 1)$.*

Proof.   For any schedule, traveling cost of every team is at least the cost of the traveling salesman tour of the cities. Suppose the optimal TSP tour of the input instance is of cost $K + k$ where $k \geq 1$. By lemma 3.1, the optimal traveling salesman tour of G^1 is of cost $n(K + k)$. Recall that each edge of G^1 has cost at most 4. The cost of edges joining u with any vertex of G^1 is $2000(m + 1)^4$ Hence the optimal traveling salesman tour of H is of cost at least $2000(m + 1)^4 + n(K + k) - 4$. So the cost of the tournament will be at least $10(m+1)^2(2000(m+1)^5 + n(K+k) - 4) = 20000(m+1)^6 + 10(m+1)^2nK + 10(m+1)^2(nk-4)$. Now for $n > 5$, the total traveling cost of the tournament is more than $20000(m + 1)^6 + 10m(m + 1)(nK + 1)$. ∎

Using Lemma 4.1 and Lemma 4.3 we get the following result

Theorem 4.4 *Traveling Tournament problem without any constraint on consecutive home and away matches is NP-Hard.*

5   Conclusion

In this note we considered the hardness of Traveling Tournament Problem. We showed that even without the constraint on consecutive home or away matches traveling tournament problem remains NP-Hard. We hope our method will be useful to prove the hardness of TTP arising from practical situations where the upper bound of consecutive home or away matches are fixed constant like 3.

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