

MULTIDISCIPLINARY FREE MATERIAL OPTIMIZATION

J. HASLINGER*, M. KOČVARA†, G. LEUGERING‡, AND M. STINGL§

Abstract. We present a mathematical framework for the so-called multidisciplinary free material optimization (MDFMO) problems, a branch of structural optimization in which the full material tensor is considered as a design variable. We extend the original problem statement by a class of generic constraints depending either on the design or on the state variables. Among the examples are local stress or displacement constraints. We show the existence of optimal solutions for this generalized FMO problem and discuss convergent approximation schemes based on the finite element method.

Key words. structural optimization, material optimization, H-convergence, semidefinite programming, nonlinear programming

AMS subject classifications. 74B05, 74P05, 90C90, 90C30, 90C22

1. Introduction. Free material optimization (FMO) is a branch of structural optimization. The underlying FMO model was introduced in [4] and further studied in several papers such as, for example, [3, 23]. The design variable in FMO is represented by the full elastic stiffness tensor that can vary from point to point. The method is supported by powerful optimization and numerical techniques which allow scenarios with complex bodies, fine finite-element meshes and many load cases. FMO has been successfully used for conceptual design of aircraft components; the most prominent example is the design of ribs in the leading edge of Airbus A380 [11].

The basic FMO model has certain limitations, though. For example, structures may fail due to high stresses, or due to lack of stability (see [12, 13] for further discussion). In order to prevent this undesirable behavior, additional requirements have to be taken into account in the FMO model. Typically, such modifications lead to additional constraints on the set of admissible materials and/or the set of admissible displacements. These constraints usually destroy the favorable mathematical structure of the original problem (see [12, 13]).

As a consequence, the standard theorems ensuring the existence of optimal solutions and convergence of appropriate approximation schemes fail to hold. In particular, it turns out that in order to prove the existence of optimal solutions of the extended FMO problem, completely different mathematical tools have to be applied. The tool used in our paper is the so-called H-convergence introduced by Murat and Tartar [20, 15], a concept originally invented in the context of homogenized materials.

In the first part of the paper we briefly list the classic FMO results and discuss various ways of proving the existence of optimal solutions. We then formulate a class of multidisciplinary FMO problems (MDFMO) and use H-convergence in order to prove the existence of optimal solutions under reasonable assumptions. Then, in Section 3, we propose an approximation scheme for continuous setting of MDFMO problems, which is based on the discretization of the design space by piecewise constant functions. We further prove that a

*Department of Numerical Mathematics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic (Jaroslav.Haslinger@mff.cuni.cz),.

†School of Mathematics, University of Birmingham, Birmingham B15 2TT, UK, and Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 08 Praha 8, Czech Republic (kocvara@maths.bham.ac.uk)

‡Department Mathematics, Applied Mathematics II, University of Erlangen, Martensstr. 3, 91058 Erlangen, Germany (leugering@am.uni-erlangen.de)

§Department Mathematics, Applied Mathematics II, University of Erlangen, Martensstr. 3, 91058 Erlangen, Germany (stingl@am.uni-erlangen.de)

subsequence of optimal solutions of the discrete problems converges to a solution of the original problem in an appropriate sense. Finally, we discuss convergence of fully discretized optimization problems, and propose a discretization based on the finite element method.

Throughout this paper we use the following notation: We denote by \mathbb{S}^N the space of symmetric $N \times N$ matrices equipped with the standard inner product $\langle \cdot, \cdot \rangle$ defined by $\langle A, B \rangle := \text{Tr}(AB)$ for any $A, B \in \mathbb{S}^N$, where $\text{Tr}(C)$ denotes the trace of the matrix $C \in \mathbb{S}^N$. We further denote by \mathbb{S}_+^N the cone of all positive semidefinite matrices in \mathbb{S}^N and use the abbreviation $A \succcurlyeq 0$ for matrices $A \in \mathbb{S}_+^N$. Moreover, for $A, B \in \mathbb{S}^N$, we say that $A \succcurlyeq B$ if $A - B \succcurlyeq 0$, and similarly for $A \preccurlyeq B$. Finally, we denote by $\mathcal{L}_\infty(\Omega, \mathbb{S}^n)$ and $\mathcal{L}_\infty(\Omega, \mathbb{R}^n)$, respectively, the spaces of matrix-valued and vector-valued functions which are componentwise in $\mathcal{L}_\infty(\Omega)$.

2. The mathematical model and existence of optimal solutions.

2.1. The basic FMO model. Material optimization deals with optimal design of elastic structures, where the design variables are material properties. The material can even vanish in certain areas, thus the so-called topology optimization (see, e.g., [5]) can be considered a special case of material optimization.

Consider an elastic body occupying an N -dimensional bounded domain $\Omega \subset \mathbb{R}^N$ with a Lipschitz boundary, where $N \in \{2, 3\}$. By $u(x) \in \mathbb{R}^N$ we denote the displacement vector at a point x , and by

$$e_{ij}(u(x)) = \frac{1}{2} \left(\frac{\partial u_i(x)}{\partial x_j} + \frac{\partial u_j(x)}{\partial x_i} \right), \quad i, j = 1, \dots, N$$

the linearized strain tensor. We assume that our system is governed by linear Hooke's law, i.e., the stress is a linear function of the strain

$$\sigma_{ij}(x) = E_{ijkl}(x) e_{kl}(u(x)) \quad (\text{in tensor notation}),$$

where E is the elastic (plane-stress for $N = 2$) stiffness tensor. The symmetries of E allow us to write the 2nd order tensors e and σ as vectors in $\mathbb{R}^{\bar{N}}$, with $\bar{N} = N(N+1)/2$, for instance we obtain for $N = 2$:

$$e = (e_{11}, e_{22}, \sqrt{2}e_{12})^\top \in \mathbb{R}^3, \quad \sigma = (\sigma_{11}, \sigma_{22}, \sqrt{2}\sigma_{12})^\top \in \mathbb{R}^3.$$

Correspondingly, the 4th order tensor E can be written as a symmetric $\bar{N} \times \bar{N}$ matrix. Assuming again $N = 2$, the corresponding matrix reads as:

$$E = \begin{pmatrix} E_{1111} & E_{1122} & \sqrt{2}E_{1112} \\ & E_{2222} & \sqrt{2}E_{2212} \\ \text{sym.} & & 2E_{1212} \end{pmatrix}. \quad (2.1)$$

In this notation, Hooke's law is expressed by $\sigma(x) = E(x)e(u(x))$. In the rest of the paper we will use this simplified notation. To avoid confusion with the stiffness matrix introduced later, we will call E the *material matrix*.

For given external load functions $f \in \mathcal{L}_2(\Gamma; \mathbb{R}^N)$, $g \in \mathcal{L}_2(\Omega; \mathbb{R}^N)$ we consider the following boundary value problem of linear elasticity:

$$\left. \begin{array}{l} \text{Find } u \in H^1(\Omega; \mathbb{R}^N) \text{ such that} \\ \left. \begin{array}{ll} -\text{div}(\sigma) = g & \text{in } \Omega \\ \sigma \cdot n = f & \text{on } \Gamma \\ u = 0 & \text{on } \Gamma_0 \\ \sigma = E \cdot e(u) & \text{in } \Omega. \end{array} \right\} \end{array} \right\} \quad (2.2)$$

In what follows, we assume that $g = 0$, i.e. the volume forces are neglected. Here Γ and Γ_0 are open disjoint subsets of $\partial\Omega$. The corresponding weak form of (2.2) reads as:

Find $u \in \mathcal{V}$ such that (2.3)

$$\int_{\Omega} \langle E(x)e(u(x)), e(w(x)) \rangle dx = \int_{\Gamma} f(x) \cdot w(x) ds \quad \forall w \in \mathcal{V},$$

where $\mathcal{V} = \{w \in H^1(\Omega; \mathbb{R}^N) \mid w = 0 \text{ on } \Gamma_0\}$ reflects the Dirichlet boundary conditions. Below we will use the abbreviate notation

$$a_E(w, z) := \int_{\Omega} \langle E(x)e(w(x)), e(z(x)) \rangle dx \quad (2.4)$$

for the bilinear form on the left hand side of (2.3) to denote that the system (2.3) will be parametrized by E . In *free material optimization*, the design variable is the material matrix E which is a function of the space variable x (see [4]). The only constraint on E is that it is physically reasonable, i.e., that E is symmetric and positive semidefinite. This gives rise to the following definition of the feasible set

$$\mathcal{E}_0 := \left\{ E \in \mathcal{L}_{\infty}(\Omega, \mathbb{S}^N) \mid E \succcurlyeq 0 \text{ a.e. in } \Omega \right\}. \quad (2.5)$$

This choice of \mathcal{E}_0 is due to the fact that we want to allow material/no-material situations. A frequently used measure of the stiffness of the material matrix is its trace. In order to avoid arbitrarily stiff material, we add pointwise stiffness restrictions of the form $\text{Tr}(E) \leq \bar{\rho}$, where $\bar{\rho} > 0$ is given. Moreover, we restrict the total stiffness by the constraint $v(E) \leq \bar{v}$. Here $v(E)$ is defined as $\int_{\Omega} \text{Tr}(E) dx$ and $\bar{v} > 0$ is an upper bound on overall resources¹. Accordingly, we define the *set of admissible materials* as

$$\mathcal{E} := \{E \in \mathcal{E}_0 \mid \text{Tr}(E) \leq \bar{\rho} \text{ a.e. in } \Omega, v(E) \leq \bar{v}\}.$$

Note that these assumptions do not necessarily imply the uniform ellipticity of the bilinear form a_E . To this end we define

$$u_E \in \mathcal{V} : u_E := \arg \inf_{u \in \mathcal{V}} \left\{ \frac{1}{2} a_E(u, u) - \int_{\Gamma} f \cdot u \, ds \right\}. \quad (2.6)$$

We are now able to present the *minimum compliance single-load FMO problem*

$$\begin{aligned} & \inf_{E \in \mathcal{E}} c(E) & (2.7) \\ & \text{subject to} \\ & u_E \text{ satisfies (2.6),} \end{aligned}$$

where $c(E) := \int_{\Gamma} f \cdot u_E \, ds$. This objective, the so-called compliance functional, measures how well the structure can carry the load f .

2.2. Various ways how to prove the existence of solutions. Problem (2.7) can be (up to a constant factor) rephrased as the following saddle-point problem:

$$\inf_{E \in \mathcal{E}} \sup_{u \in \mathcal{V}} -\Pi(E, u). \quad (2.8)$$

¹The total stiffness is often interpreted as a volume, analogously to topology optimization. That is why we call the constraint a ‘volume constraint’, as it sounds better than ‘constraint on overall resources’

Here $\Pi(E, u)$ is the total potential energy of the deformed body given by

$$\Pi(E, u) = \frac{1}{2}a_E(u, u) - \int_{\Gamma} f \cdot u \, ds.$$

The existence theory in [3, 23, 14] is based on classic saddle-point arguments applied to this rewritten problem. The existence proof in [14] guarantees not only the existence of an optimal material E^* , but also the existence of an associated displacement field u solving (2.3) for $E := E^*$. As long as no explicit access to the state variable u (or σ) is needed, the same argumentation remains valid if the basic problem setting is extended by convex constraint functionals that are weakly-* lower-semicontinuous in the design variable E . An example of such an additional constraint is the minimal eigenvalue function defined in [17]. Alternatively, the existence of optimal solutions can be proved by means of the following closed formula for the compliance functional in (2.7):

$$c(E) = \sup_{u \in \mathcal{V}} -\Pi(E, u). \quad (2.9)$$

Now the existence of optimal solutions follows directly from the facts that

- the set \mathcal{E} is weakly-* compact in $\mathcal{L}^\infty(\Omega, \mathbb{S}^N)$ [3],
- the function $\sup_{u \in \mathcal{V}} -\Pi(E, u)$ is weakly-* lower-semicontinuous [22],

see, for instance, [8, Theorem II, 1.4]. At first glance this argumentation seems to be attractive, as unlike the saddle-point approach no convexity of the cost functional and admissible set is required. On the other hand, any information on the displacement field associated with the optimal material is lost. In the worst case this means that the optimal solution is physically meaningless. The situation becomes even more involved when explicit knowledge of the state variables is needed, for instance, when problem (2.7) is extended by state constraints. In this case none of the arguments above can be used. The reason is that once the state variable u is constrained, the equivalence to the saddle-point problem (2.8) or the problem formulation arising from the closed-form compliance (2.9) function is lost (see, e.g. Appendix A in [10]). A viable alternative seems to be the regularization of the set \mathcal{E} :

$$\mathcal{E}^\varepsilon := \{E \in \mathcal{E} \mid E \succ \varepsilon I_{\bar{N}} \text{ a.e. in } \Omega\}, \quad (2.10)$$

where ε is a small positive number and $I_{\bar{N}}$ the unit matrix in $\mathbb{S}^{\bar{N}}$. Then it is possible—due to the uniqueness of the solution to (2.3) for each $E \in \mathcal{E}^\varepsilon$ —to consider pairs of the design and state variables $(E, u) \in \mathcal{E}^\varepsilon \times \mathcal{V}$ such that u is a solution of (2.3) associated with E . Now it is well known (see e.g. [1]) that for each sequence of pairs $(E_n, u_n) \in \mathcal{E}^\varepsilon \times \mathcal{V}$ ($\varepsilon > 0$ being fixed) one can find a subsequence converging to a limit pair $(\bar{E}, \bar{u}) \in \mathcal{E}^\varepsilon \times \mathcal{V}$ in the sense of the weak-* topology in \mathcal{E}^ε and the weak topology in \mathcal{V} . It is not, however, true that the limit state \bar{u} is a solution of the limiting state equation associated with \bar{E} .

2.3. H-convergence. A usual way how to overcome the difficulty mentioned at the end of the previous section is to make use of H-convergence, going back to Tartar [20] and Murat and Tartar [15]. In order to do so, we define another set of admissible materials

$$\mathcal{E}^{\alpha, \beta} := \left\{ E \in \mathcal{L}^\infty(\Omega, \mathbb{S}^{\bar{N}}) \mid \alpha I_{\bar{N}} \preccurlyeq E \preccurlyeq \beta I_{\bar{N}} \text{ a.e. in } \Omega \right\}, \quad (2.11)$$

where $0 < \alpha < \beta$ are given. Using this set, the definition of H-convergence is as follows (cf. Definition 1.4.1, [1]):

DEFINITION 2.1. *A sequence of admissible materials $\{E_n\}$ in $\mathcal{E}^{\alpha, \beta}$ is said to H-converge to an H-limit E^* if, for any right hand side $g \in \mathcal{L}_2(\Omega; \mathbb{R}^N)$ and $f \in \mathcal{L}_2(\Gamma; \mathbb{R}^N)$, the sequence*

$\{u_n\}$ of solutions to (2.2) with $E := E_n$ satisfies

$$\begin{aligned} u_n &\rightharpoonup u^* \text{ weakly in } \mathcal{V} \\ \sigma_n := E_n e(u_n) &\rightharpoonup \sigma^* := E^* e(u^*) \text{ weakly in } \mathcal{L}_2(\Omega; \mathbb{R}^N), \end{aligned}$$

i.e. $u^* \in \mathcal{V}$ is the solution of

$$\begin{aligned} -\operatorname{div}(E^* e(u)) &= g && \text{in } \Omega \\ E^* e(u) \cdot n &= f && \text{on } \Gamma \\ u &= 0 && \text{on } \Gamma_0. \end{aligned}$$

Subsequently, we use notation $E_n \xrightarrow{H} E^*$. We will also use standard notation $E_n \xrightarrow{*} \bar{E}$ for weak-* convergence of the sequence $\{E_n\}$ to \bar{E} . Now, on the basis of this definition, one can prove *H-compactness* of $\mathcal{E}^{\alpha, \beta}$ (cf. Theorem 1.4.2, [1]).

THEOREM 2.2. *For any sequence $\{E_n\}$ in $\mathcal{E}^{\alpha, \beta}$ there exists a subsequence, still denoted by $\{E_n\}$, and a ('homogenized') $E^* \in \mathcal{E}^{\alpha, \beta}$ such that $\{E_n\}$ H-converges to E^* .*

REMARK 2.3. *Originally, the definition of the underlying set of admissible materials used in H-convergence for elastic systems is different from (2.11), cf. formula (1.120) in [1]. It can be shown, however, that both definitions are fully equivalent in the sense of (2.1).*

REMARK 2.4. *The concept of H-convergence has been originally introduced for PDEs subject to Dirichlet boundary conditions only. However, it is known that all important results remain valid in more general situations, such as, for instance, problems with mixed Dirichlet/Neumann conditions (see Proposition 1.4.6 in [1] or [9]).*

2.4. The existence proof based on H-convergence. Next we want to apply Theorem 2.2 in order to give an alternative existence proof for the regularized version of problem (2.7) (among others) which uses the set \mathcal{E}^ε instead of \mathcal{E} . For this purpose, we use the following result.

LEMMA 2.5. *The set \mathcal{E}^ε is H-compact.*

Proof. Setting $\alpha = \varepsilon$ and $\beta = \bar{\rho}/\bar{N}$, we obtain

$$\mathcal{E}^\varepsilon \subset \mathcal{E}^{\alpha, \beta}.$$

Given a sequence $\{E_n\}$ in \mathcal{E}^ε , due to the compactness of $\mathcal{E}^{\alpha, \beta}$ with respect to weak-* convergence and Theorem 2.2 we can pass to a subsequence denoted by the same symbol that weakly-* converges to a limit material \bar{E} and H-converges to an H-limit E^* . Then it is known from [1, Proposition 1.4.9] that

$$E^* \preceq \bar{E}.$$

The latter inequality implies

$$\operatorname{Tr}(E^*) \leq \operatorname{Tr}(\bar{E}) \text{ a.e. in } \Omega \quad \text{and} \quad \int_{\Omega} \operatorname{Tr}(E^*) \, dx \leq \int_{\Omega} \operatorname{Tr}(\bar{E}) \, dx. \quad (2.12)$$

As \mathcal{E}^ε is closed in the weak-* topology, we know that

$$\operatorname{Tr}(\bar{E}) \leq \bar{\rho} \text{ a.e. in } \Omega \quad \text{and} \quad \int_{\Omega} \operatorname{Tr}(\bar{E}) \, dx \leq \bar{v}$$

and therefore we conclude from (2.12)

$$\operatorname{Tr}(E^*) \leq \bar{\rho} \text{ a.e. in } \Omega \quad \text{and} \quad \int_{\Omega} \operatorname{Tr}(E^*) \, dx \leq \bar{v}.$$

Consequently, \mathcal{E}^ε is closed in the sense of H-convergence and hence is H-compact. \square

Now we consider a class of cost functionals

$$J : \mathcal{E}^\varepsilon \times \mathcal{V} \rightarrow \mathbb{R} \quad (2.13)$$

with the following property:

$$\left. \begin{array}{l} E_n \xrightarrow{H} E, \quad E_n, E \in \mathcal{E}^\varepsilon \\ v_n \rightharpoonup v \text{ in } \mathcal{V} \end{array} \right\} \implies \liminf_{n \rightarrow \infty} J(E_n, v_n) \geq J(E, v), \quad (2.14)$$

i.e. cost functionals that are lower semicontinuous w.r.t. the design E in the topology induced by H-convergence (referred to as H-lower-semicontinuous below) and lower semicontinuous w.r.t. the state in the sense of the weak topology in \mathcal{V} . The following existence theorem can be proved exactly as Theorem 1.2 in [7,].

THEOREM 2.6. *The regularized free material optimization problem*

$$\inf_{E \in \mathcal{E}^\varepsilon} J(E, u_E) \quad (2.15)$$

with J satisfying (2.14) and $u_E \in \mathcal{V}$ solving (2.3) has at least one solution.

Typical examples satisfying assumption (2.14) are

- the compliance functional:

$$J(E, u_E) := c(E) \quad (2.16)$$

- the tracking functional:

$$J(E, u_E) := \|u_E - u_0\|_{(\mathcal{L}^2(\Omega))^N}^2 \text{ with } u_0 \in \mathcal{V} \text{ given} \quad (2.17)$$

- stress functional:

$$J(E, u_E) := \int_{\Omega} \sigma_E(x)^\top \cdot M \sigma_E(x) \, dx, \quad (2.18)$$

where M is the von Mises matrix and $\sigma_E = Ee(u_E)$, for instance.

2.5. Extensions: design dependent functionals. Next we want to introduce a general class of functionals which are H-lower-semicontinuous w.r.t. the design variable E . Following [21] we consider functionals of the form

$$\begin{aligned} \Phi : \mathcal{E}^\varepsilon &\rightarrow \mathbb{R} \\ E &\mapsto \int_{\Omega} \varphi(E(x)) \, dx \end{aligned} \quad (2.19)$$

where $\varphi : \mathbb{S}^{\bar{N}} \rightarrow \mathbb{R}$ is monotone in the sense

$$A \preceq B \Rightarrow \varphi(A) \leq \varphi(B), \quad A, B \in \mathbb{S}^{\bar{N}}. \quad (2.20)$$

Then the following proposition relates weakly-* lower-semicontinuity of the functional Φ to lower-semicontinuity with respect to H-convergence (see [2, Theorem 2]).

PROPOSITION 2.7. *Let φ be continuous and nondecreasing in the sense of (2.20). Let further Φ defined by (2.19) be weakly-* lower-semicontinuous. Then Φ is also H-lower-semicontinuous.*

An example of Φ satisfying the assumptions of Proposition 2.7 is, for instance, the functional

$$v(E) = \int_{\Omega} \text{Tr} E \, dx.$$

Note that, thanks to Proposition 2.7, any weakly-* lower semicontinuous functional $\Phi : \mathcal{E}^\varepsilon \rightarrow \mathbb{R}$ of the form (2.19) with (2.20) satisfies (2.14) for the regularized FMO problem (2.15). Thus one can add to the feasible set \mathcal{E}^ε any constraint of the type

$$\Phi(E) \leq C,$$

$C \in \mathbb{R}$ given, without losing the existence result of Theorem 2.6.

2.6. Extensions: state constraints. The goal of this section is to extend the regularized FMO problem (2.15) by state constraints of the type

$$g_I(u_E) \leq C_u \quad \text{or} \quad g_{II}(\sigma_E) \leq C_\sigma \quad (C_u, C_\sigma \in \mathbb{R} \text{ given})$$

with some weakly lower-semicontinuous functionals g_I, g_{II} . In order to do so, we define the solution map $S : \mathcal{E}^\varepsilon \rightarrow \mathcal{V}$ that assigns each admissible material $E \in \mathcal{E}^\varepsilon$ the unique solution $u_E = S(E)$ of the state equation (2.3). Then we can also write $\sigma_E = Ee(S(E))$. Next we assume that for each element E_n of the sequence $\{E_n\}$, the corresponding state variables $u_n = S(E_n), \sigma_n = E_n e(S(E_n))$ satisfy the constraints $g_I(u_n) \leq C_u$ and $g_{II}(\sigma_n) \leq C_\sigma$. We may assume without loss of generality that the sequence $\{E_n\}$ H-converges to some E in \mathcal{E}^ε . Then we know from the definition of H-convergence that

- $\{u_n\}$ converges weakly to $u \in \mathcal{V}$ where $u = S(E)$
- $\{\sigma_n\}$ converges weakly to $\sigma \in \mathcal{L}^2(\Omega; \mathbb{R}^N)$ where $\sigma = Ee(S(E))$.

Now weak lower-semicontinuity of g_I and g_{II} implies that the state constraints hold for the limiting states as well, i.e. $g_I(u) \leq C_u$ and $g_{II}(\sigma) \leq C_\sigma$ and we can formulate the following theorem.

THEOREM 2.8. *Let g_I and g_{II} be weakly lower-semicontinuous functionals of the state variables u_E and σ_E , respectively. Then the set*

$$\mathcal{E}^{\varepsilon, g_I, g_{II}} := \{E \in \mathcal{E}^\varepsilon \mid g_I(u_E) \leq C_u, g_{II}(\sigma_E) \leq C_\sigma\},$$

is H-compact.

Proof. The H-compactness follows immediately from the H-compactness of \mathcal{E}^ε and the H-closedness of $\mathcal{E}^{\varepsilon, g_I, g_{II}}$ which was outlined above. \square

Below we list some constraint functionals satisfying the assumption of Theorem 2.8:

- *Linear displacement constraints* of the form

$$\int_{\Omega} d(x) \cdot u_E(x) \, dx \leq C,$$

where $d \in \mathcal{L}^2(\Omega; \mathbb{R}^N)$.

- *tracking type displacement constraints* of the form

$$\|u_E - u_0\|_{(\mathcal{L}^2(\Omega))^N}^2 \leq C,$$

with $u_0 \in \mathcal{V}$ given.

- *Integral stress constraints* of the form

$$\int_{\omega} \sigma_E(x)^\top \cdot M \sigma_E(x) \, dx \leq C,$$

where $\omega \subset \Omega$ and M is either the unit or the von Mises matrix.

Note that, due to the properties of the trace operator, both types of the displacement constraints may also be formulated for the boundary displacements only.

We conclude this section by formulating the multidisciplinary free material optimization problem (\mathcal{P}) whose discretization will be discussed in the next section:

$$\inf_{E \in \mathcal{E}^\varepsilon} J(E, u) \quad (2.21)$$

subject to

$$u = S(E), \quad (2.22)$$

$$g_I(u) \leq C_u, \quad g_{II}(\sigma) \leq C_\sigma, \quad \sigma = Ee(S(E)), \quad (2.23)$$

where $g_I : \mathcal{V} \rightarrow \mathbb{R}$, $g_{II} : \mathcal{L}^2(\Omega; \mathbb{R}^{\bar{N}}) \rightarrow \mathbb{R}$ and $C_u, C_\sigma \in \mathbb{R}$. If J, g_I, g_{II} satisfy the assumptions formulated above, then problem (\mathcal{P}) has a solution.

3. Discretization and convergence analysis in MDFMO. This section is devoted to a two-level discretization of (\mathcal{P}) followed by a convergence analysis. In the first level only the design set \mathcal{E}^ε will be discretized while the continuous setting of the state problem will be kept. In the second level we add a discretization of the state problem to get a fully discrete scheme. Convergence results will be established separately for state constrained and unconstrained problems.

3.1. Discretization of the design set. In order to construct inner approximations $\{\mathcal{E}_\kappa^\varepsilon\}$, $\kappa \rightarrow 0+$, of \mathcal{E}^ε , we follow closely [7].

Let $\{\mathcal{T}_\kappa\}$, $\kappa \rightarrow 0+$, be a family of partitions of $\bar{\Omega}$ into mutually disjoint subsets $\Omega_i \subset \Omega$, $i = 1, \dots, m := m(\kappa)$:

$$\bar{\Omega} = \bigcup_{i=1}^m \bar{\Omega}_i, \quad (3.1)$$

$$\max_i \text{diam}(\Omega_i) \leq \kappa. \quad (3.2)$$

With any such \mathcal{T}_κ we associate the following subset of \mathcal{E}^ε :

$$\mathcal{E}_\kappa^\varepsilon = \left\{ E \mid E_i := E|_{\Omega_i} \in P_0(\Omega_i), E_i \succcurlyeq \varepsilon I_{\bar{N}}, \text{Tr}(E_i) \leq \bar{\rho}, \quad i = 1, \dots, m; \right. \\ \left. \sum_{i=1}^m \text{Tr}(E_i)|_{\Omega_i} \leq \bar{v} \right\}, \quad (3.3)$$

where $|\Omega_i| = \text{meas } \Omega_i$. $\mathcal{E}_\kappa^\varepsilon$ consists of all material matrices $E \in \mathcal{E}^\varepsilon$ that are piecewise constant over the partition \mathcal{T}_κ .

The *first level approximation* $(\mathcal{P})_\kappa$ of (2.21)-(2.23) reads as follows:

$$\inf_{E_\kappa \in \mathcal{E}_\kappa^\varepsilon} J(E_\kappa, u) \quad (3.4)$$

subject to

$$u = S(E_\kappa), \quad (3.5)$$

$$g_I(u) \leq C_u, \quad g_{II}(\sigma) \leq C_\sigma, \quad \sigma = E_\kappa e(S(E_\kappa)). \quad (3.6)$$

In $(\mathcal{P})_\kappa$ we use the discrete set $\mathcal{E}_\kappa^\varepsilon$ defined by (3.3) but the state $u \in \mathcal{V}$ satisfies the equation (2.3) with $E := E_\kappa$. In what follows we shall suppose that all assumptions on J , g_I and g_{II} which guarantee the existence of an optimal solution E_κ^* to $(\mathcal{P})_\kappa$ are satisfied for any $\kappa > 0$.

3.2. Convergence analysis. Next we shall examine if there is any relation between solutions of (\mathcal{P}) and $(\mathcal{P})_\kappa$ when $\kappa \rightarrow 0+$. We start with problem (\mathcal{P}) without state constraints. Its discrete form is given by (3.4) and (3.5). The following result plays the key role in our analysis.

PROPOSITION 3.1. *The system $\{\mathcal{E}_\kappa^\varepsilon\}$, $\kappa \rightarrow 0+$, with $\mathcal{E}_\kappa^\varepsilon$ defined by (3.3) is dense in \mathcal{E}^ε in the following sense: for any $E \in \mathcal{E}^\varepsilon$ there exists a sequence $\{E_\kappa\}$, $E_\kappa \in \mathcal{E}_\kappa^\varepsilon$, such that*

$$E_\kappa \rightarrow E, \kappa \rightarrow 0+ \text{ in } (\mathcal{L}^p(\Omega))^{\bar{N} \times \bar{N}} \quad \forall p \in [1, \infty). \quad (3.7)$$

Proof. We define E_κ by

$$E_\kappa|_{\Omega_i} = \frac{1}{|\Omega_i|} \int_{\Omega_i} E(x) \, dx, \quad \Omega_i \in \mathcal{T}_\kappa$$

i.e., E_κ is the \mathcal{L}^2 -projection of E onto the space of piecewise constant functions over \mathcal{T}_κ . For such a sequence (3.7) is satisfied. Moreover E_κ satisfies all the constraints characterizing the set \mathcal{E}^ε . Hence $E_\kappa \in \mathcal{E}_\kappa^\varepsilon$. \square

COROLLARY 3.2. *Let $\{E_\kappa\}$, $E_\kappa \in \mathcal{E}_\kappa^\varepsilon$, satisfy (3.7). Then ([16])*

$$u_\kappa := S(E_\kappa) \rightarrow u := S(E) \text{ in } \mathcal{V}, \kappa \rightarrow 0+.$$

In addition to (2.14) we shall suppose that J is continuous in the following sense:

$$\left. \begin{array}{l} E_\kappa \rightarrow E \text{ in } (\mathcal{L}^2(\Omega))^{\bar{N} \times \bar{N}} \\ v_\kappa \rightarrow v \text{ in } \mathcal{V} \text{ as } \kappa \rightarrow 0+ \end{array} \right\} \implies \lim_{\kappa \rightarrow 0+} J(E_\kappa, v_\kappa) = J(E, v). \quad (3.8)$$

THEOREM 3.3. *Let the cost functional J satisfy (2.14) and (3.8). Then for any sequence $\{(E_\kappa^*, u_\kappa^*)\}$, $u_\kappa^* = S(E_\kappa^*)$, of optimal pairs of $(\mathcal{P})_\kappa$ one can find a subsequence $\{(E_{\kappa_j}^*, u_{\kappa_j}^*)\}$ such that*

$$\left. \begin{array}{l} E_{\kappa_j}^* \xrightarrow{H} E^* \\ u_{\kappa_j}^* \rightarrow u^* \text{ in } \mathcal{V} \text{ as } j \rightarrow \infty. \end{array} \right\} \quad (3.9)$$

Moreover, (E^, u^*) is an optimal solution of (\mathcal{P}) . Any accumulation point of $\{(E_\kappa^*, u_\kappa^*)\}$ in the sense of (3.9) possesses this property.*

Proof. The existence of a subsequence satisfying (3.9) with $u^* = S(E^*)$ follows from Lemma 2.5, the fact that $\mathcal{E}_\kappa^\varepsilon \subset \mathcal{E}^\varepsilon$ for any $\kappa > 0$ and the definition of H-convergence. Let $\bar{E} \in \mathcal{E}^\varepsilon$, be arbitrary and $\{\bar{E}_{\kappa_j}\}$, $\bar{E}_{\kappa_j} \in \mathcal{E}_{\kappa_j}^\varepsilon$ be a sequence with the property (3.7). From the definition of $(\mathcal{P})_\kappa$ it follows that

$$J(E_{\kappa_j}^*, u_{\kappa_j}^*) \leq J(\bar{E}_{\kappa_j}, \bar{u}_{\kappa_j}), \quad \bar{u}_{\kappa_j} = S(\bar{E}_{\kappa_j}).$$

Therefore

$$J(E^*, u^*) \leq \liminf_{j \rightarrow \infty} J(E_{\kappa_j}^*, u_{\kappa_j}^*) \leq \lim_{j \rightarrow \infty} J(\bar{E}_{\kappa_j}, \bar{u}_{\kappa_j}) = J(\bar{E}, \bar{u}),$$

where $\bar{u} = S(\bar{E})$, making use of (2.14), (3.8), (3.9) and Consequence 3.2. \square

Examples of cost functionals satisfying (2.14) and (3.8) are the compliance functional (2.16), the tracking functional (2.17), and the stress functional (2.18).

3.3. Discretization of the state equation. In what follows we shall suppose that the parameter $\kappa > 0$ characterizing the discretization of $\mathcal{E}_\kappa^\varepsilon$ is fixed. In order to discretize the state equation (2.3) we introduce a family of finite-dimensional subspaces $\{\mathcal{V}_h\}$, $\mathcal{V}_h \subset \mathcal{V}$, $\forall h > 0$, which is dense in \mathcal{V} :

$$\forall v \in \mathcal{V} \quad \exists \{v_h\}, v_h \in \mathcal{V}_h \quad : \quad v_h \rightarrow v \text{ in } \mathcal{V}, \quad h \rightarrow 0+. \quad (3.10)$$

Let $E_\kappa \in \mathcal{E}_\kappa^\varepsilon$ be given. We use the Galerkin approximation of (2.3):

$$\left. \begin{array}{l} \text{Find } u_h \in \mathcal{V}_h \text{ such that} \\ a_{E_\kappa}(u_h, w_h) = \int_\Gamma f \cdot w_h ds \quad \forall w_h \in \mathcal{V}_h. \end{array} \right\} \quad (3.11)$$

This problem has a unique solution $u_h := u_h((E_\kappa))$. This enables us to define the solution map $S_h : \mathcal{E}_\kappa^\varepsilon \mapsto \mathcal{V}_h$ by $u_h := S_h((E_\kappa))$, $E_\kappa \in \mathcal{E}_\kappa^\varepsilon$.

The *second level approximation* $(\mathcal{P})_{\kappa h}$ of (3.4) and (3.5) reads as follows:

$$\left. \begin{array}{l} \inf_{E_\kappa \in \mathcal{E}_\kappa^\varepsilon} J(E_\kappa, u_h) \\ \text{subject to} \\ u_h = S_h(E_\kappa). \end{array} \right\} \quad (3.12)$$

Suppose that h and κ are fixed and the cost functional J is *lower-semicontinuous* on $\mathcal{E}_\kappa^\varepsilon \times \mathcal{V}_h$. Using the classic compactness arguments one has the following existence result.

PROPOSITION 3.4. *Problem (3.12) has a solution.*

To study the convergence properties of the proposed discretization, let us consider a fixed $\kappa > 0$, $h \rightarrow 0+$, and denote by $(E_{\kappa h}^*, u_h^*)$, $u_h^* = S_h(E_{\kappa h}^*)$, an optimal solution pair of (3.12). Then one can find subsequences $\{E_{\kappa h_j}^*\}$, $\{u_{h_j}^*\}$ such that

$$\left. \begin{array}{l} E_{\kappa h_j}^* \rightarrow E_\kappa^* \in \mathcal{E}_\kappa^\varepsilon \quad \text{in } (\mathcal{L}^\infty(\Omega))^{\bar{N} \times \bar{N}} \\ u_{h_j}^* \rightarrow u_\kappa^* \quad \text{in } \mathcal{V}, \quad j \rightarrow \infty \end{array} \right\} \quad (3.13)$$

using that all $E_{\kappa h}^*$, $\kappa > 0$ fixed, belong to the same finite dimensional space and $\{u_h^*\}$ is bounded in \mathcal{V} . Let $\bar{w} \in \mathcal{V}$ be given and $\{\bar{w}_h\}$, $\bar{w}_h \in \mathcal{V}_h$, be such that (see (3.10))

$$\bar{w}_h \rightarrow \bar{w} \text{ in } \mathcal{V}, \quad h \rightarrow 0+. \quad (3.14)$$

Letting $j \rightarrow \infty$ in

$$a_{E_{\kappa h_j}^*}(u_{h_j}^*, \bar{w}_{h_j}) = \int_\Gamma f \cdot \bar{w}_{h_j} ds$$

we obtain

$$a_{E_\kappa^*}(u_\kappa^*, \bar{w}) = \int_\Gamma f \cdot \bar{w} ds$$

making use of (3.13) and (3.14), i.e. $u_\kappa^* = S(E_\kappa^*)$. From (3.13) it also follows that $u_{h_j}^* \rightarrow u_\kappa^*$ in \mathcal{V} , $j \rightarrow \infty$. If J satisfies (3.8) then using the same approach as in Theorem 3.3 one can show that (E_κ^*, u_κ^*) is an optimal pair for (\mathcal{P}_κ) . We have just proved the following result.

THEOREM 3.5. *Let J satisfy (3.8). Then for any sequence $\{(E_{\kappa h}^*, u_h^*)\}$, of optimal pairs of $(\mathcal{P})_{\kappa h}$, $\kappa > 0$ fixed, one can find a subsequence $\{(E_{\kappa h_j}^*, u_{h_j}^*)\}$ such that*

$$\left. \begin{array}{l} E_{\kappa h_j}^* \rightarrow E_\kappa^* \in \mathcal{E}_\kappa^\varepsilon \quad \text{in } (\mathcal{L}^\infty(\Omega))^{\bar{N} \times \bar{N}} \\ u_{h_j}^* \rightarrow u_\kappa^* \quad \text{in } \mathcal{V}, \quad j \rightarrow \infty. \end{array} \right\} \quad (3.15)$$

Moreover $(E_{\kappa}^*, u_{\kappa}^*)$ is an optimal pair of $(\mathcal{P})_{\kappa}$. Any accumulation point of $\{(E_{\kappa h}^*, u_h^*)\}$ in the sense of (3.15) possesses this property.

REMARK 3.6. Arguing as in [1, p. 83] one can find a filter of indices such that

$$E_{\kappa_j h_j}^* \xrightarrow{H} E^*, \quad j \rightarrow \infty$$

where E^* is a solution of (2.21) and (2.22), provided that J satisfies (2.14) and (3.8).

3.4. The constrained case. Rather than discretizing the state constrained MDFMO problem (\mathcal{P}) given by (2.21)-(2.23) directly, we approximate it by a penalty method.

For this purpose we define a penalty functional $j : \mathbb{R} \rightarrow \mathbb{R}$ associated with the state constraints $g_I(u_E) \leq C_u$ and $g_{II}(\sigma_E) \leq C_{\sigma}$, respectively. In the sequel we shall assume that the following assumptions hold for j :

$$j \in \mathcal{C}(\mathbb{R}, \mathbb{R}), \quad j(t) = 0 \quad \forall t \leq 0, \quad t_1 \leq t_2 \Rightarrow j(t_1) \leq j(t_2) \quad \forall t_1, t_2 \in \mathbb{R}. \quad (3.16)$$

Then, instead of (\mathcal{P}) , we consider a family of problems (\mathcal{P}^{γ}) ($\gamma > 0$):

$$\min_{E \in \mathcal{E}^{\varepsilon}} J^{\gamma}(E, u_E), \quad (3.17)$$

where

$$J^{\gamma}(E, u_E) := J(E, u_E) + \frac{1}{\gamma} (j(g_I(u_E)) + j(g_{II}(\sigma_E))),$$

$u_E = S(E)$ and $\sigma_E = Ee(u_E)$. The following result follows immediately from (2.14), (3.16) and the definition of H-convergence.

PROPOSITION 3.7. *The functional J^{γ} satisfies (2.14) for all $\gamma > 0$.*

COROLLARY 3.8.

1. *Problem (\mathcal{P}^{γ}) has a solution for all $\gamma > 0$.*
2. *If in addition J^{γ} , $\gamma > 0$ fixed, satisfies also (3.8) then all approximation results stated in Theorems 3.3 and 3.5 hold true for (\mathcal{P}^{γ}) as well.*

Now let $\{\gamma_j\}$ be a sequence of penalty parameters tending to zero as $j \rightarrow \infty$. Then we derive the following relation between (\mathcal{P}) and (\mathcal{P}^{γ_j}) , $j \rightarrow \infty$.

THEOREM 3.9. *Let $\{(E_j^*, u_j^*)\}$ be a sequence of optimal pairs of (\mathcal{P}^{γ_j}) . Then one can find a subsequence $\{(E_{j_k}^*, u_{j_k}^*)\}$ and elements $E^* \in \mathcal{E}^{\varepsilon, g_I, g_{II}}$ and $u^* = S(E^*)$ such that*

$$\left. \begin{array}{l} E_{j_k}^* \xrightarrow{H} E^* \\ u_{j_k}^* \rightharpoonup u^* \text{ in } \mathcal{V} \quad \text{as } k \rightarrow \infty. \end{array} \right\} \quad (3.18)$$

Moreover (E^*, u^*) is an optimal pair of (\mathcal{P}) . Any accumulation point of $\{(E_j^*, u_j^*)\}$ in the sense of (3.18) has this property.

Proof. From Lemma 2.5 and the definition of H-convergence follows the existence of a subsequence $(E_{j_k}^*, u_{j_k}^*)$ and a pair $\{E^*, u_{E^*}\} \in \mathcal{E}^{\varepsilon} \times \mathcal{V}$ such that

$$\left. \begin{array}{l} E_{j_k}^* \xrightarrow{H} E^* \\ u_{\kappa_j}^* \rightharpoonup u_{E^*} \text{ in } \mathcal{V} \\ \sigma_{\kappa_j}^* \rightharpoonup \sigma_{E^*} \text{ in } (\mathcal{L}^2(\Omega))^N, \quad k \rightarrow \infty, \end{array} \right\} \quad (3.19)$$

where $u_{\kappa_j}^* = S(E_{\kappa_j}^*)$, $\sigma_{\kappa_j}^* = E_{\kappa_j}^* e(u_{\kappa_j}^*)$, $u_{E^*} = S(E^*)$ and $\sigma_{E^*} = E^* e(u^*)$. Now using the definition of $(\mathcal{P}^{\gamma_{j_k}})$ one has:

$$J^{\gamma_{j_k}}(E_{j_k}^*, u_{j_k}^*) \leq J^{\gamma_{j_k}}(E, u_E) \quad \forall E \in \mathcal{E}^{\varepsilon}$$

implying that

$$J(E_{j_k}^*, u_{j_k}^*) + \frac{1}{\gamma_{j_k}} (j(g_I(u_{j_k}^*)) + j(g_{II}(\sigma_{j_k}^*))) \leq J(\bar{E}, u_{\bar{E}}) \quad (3.20)$$

holds for any $\bar{E} \in \mathcal{E}^{\varepsilon, g_I, g_{II}}$ as follows from (3.16). Hence

$$0 \leq j(g_I(u_{j_k}^*)) + j(g_{II}(\sigma_{j_k}^*)) \leq \gamma_{j_k} (J(\bar{E}, u_{\bar{E}}) - J(E_{j_k}^*, u_{j_k}^*)).$$

If $\gamma_{j_k} \rightarrow 0+$, $k \rightarrow \infty$ then

$$j(g_I(u_{j_k}^*)) + j(g_{II}(\sigma_{j_k}^*)) \rightarrow 0, \quad k \rightarrow \infty. \quad (3.21)$$

But $j \circ g_I$ and $j \circ g_{II}$ are weakly lower-semicontinuous so that

$$\liminf_{k \rightarrow \infty} (j(g_I(u_{j_k}^*)) + j(g_{II}(\sigma_{j_k}^*))) \geq j(g_I(u_{E^*})) + j(g_{II}(\sigma_{E^*})) \geq 0.$$

From this and (3.21) we see that $j(g_I(u_{E^*})) + j(g_{II}(\sigma_{E^*})) = 0$, i.e. $E^* \in \mathcal{E}^{\varepsilon, g_I, g_{II}}$. Finally letting $k \rightarrow \infty$ in (3.20) and using (2.14) and (3.16) we arrive at

$$\begin{aligned} J(E^*, u_{E^*}) &\leq \liminf_{k \rightarrow \infty} J(E_{j_k}^*, u_{j_k}^*) \\ &\leq \liminf_{k \rightarrow \infty} J^{\gamma_{j_k}}(E_{j_k}^*, u_{j_k}^*) \\ &\leq J(\bar{E}, u_{\bar{E}}) \quad \forall \bar{E} \in \mathcal{E}^{\varepsilon, g_I, g_{II}}. \end{aligned}$$

Thus (E^*, u_{E^*}) is an optimal solution pair of (\mathcal{P}) . \square

In Section 4 we will consider the following choices of J^γ :

- *Compliance functional with penalized displacement constraints*

$$J^\gamma(E, u_E) := c(E) + \frac{1}{\gamma} \sum_{i=1,2,\dots,n_{dis}} j\left(\int_{\Omega} d_i(x) \cdot u_E(x) dx - c_i\right), \quad (3.22)$$

where $d_i \in \mathcal{L}^2(\Omega; \mathbb{R}^N)$ and $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n_{dis}$, define n_{dis} linear displacement constraints.

- *Compliance functional with penalized von-Mises stress constraints*

$$J^\gamma(E, u_E) := c(E) + \frac{1}{\gamma} \sum_{i=1,2,\dots,m} j(\|\sigma_E\|_{M,(\mathcal{L}^2(\Omega))_i}^2 - s_\sigma |\Omega_i|), \quad (3.23)$$

where $\|\sigma_E\|_{M,(\mathcal{L}^2(\Omega_i))}^2 := \int_{\Omega_i} \sigma_E(x)^\top M \sigma_E(x) dx$ with

$$M = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix},$$

$s_\sigma \in \mathbb{R}$ is a given stress bound on $\Omega_i \subset \Omega$ for all $i = 1, 2, \dots, m$.

4. Numerical solution of MDFMO problems.

4.1. The discrete MDFMO problem in algebraic form. In this section we derive fully algebraic formulations of problems (3.12) and (3.17). The Galerkin approximation (3.11) is realized by a standard finite element approach. For this purpose, $\bar{\Omega}$ is partitioned into quadrilateral elements $\Omega_i \subset \Omega$, $i = 1, \dots, m$. For the sake of simplicity of notation, we use the same partitioning for the discretization of the design space (see Section 3.1) and the state space. We further assume that the space \mathcal{V}_h is spanned by continuous functions that are bilinear (in 2D) or trilinear (in 3D) on every element. Such a function can be written as $u(x) = \sum_{i=1}^n u_i \vartheta_i(x)$ where u_i is the value of u at the i^{th} node and ϑ_i is the basis function associated with the i^{th} node (for details, see [6]). Thus, each function $u \in \mathcal{V}_h$ can be identified with its nodal displacement vector $\mathbf{u} \in \mathbb{R}^{\hat{n}}$, where $\hat{n} = N \cdot n$.

The discrete state equation reads as

$$A(\mathbf{E})\mathbf{u} = \mathbf{f}, \quad (4.1)$$

where $A(\mathbf{E})$ is the stiffness matrix arising from the discretization of the bilinear form a_E and \mathbf{f} is the load vector. The discretized MDFMO problem (3.17) in algebraic form becomes:

$$\min_{\mathbf{E}=(E_1, E_2, \dots, E_m) \in (\mathcal{S}^N)^m} \hat{J}^\gamma(E, \mathbf{u}_{\mathbf{E}}), \quad (4.2)$$

subject to

$$\begin{aligned} \mathbf{u}_{\mathbf{E}} &= A(\mathbf{E})^{-1}\mathbf{f}, \\ E_i &\succcurlyeq \varepsilon I_{\bar{N}}, \quad i = 1, \dots, m, \\ \text{Tr}(E_i) &\leq \bar{\rho}, \quad i = 1, \dots, m, \\ \sum_{i=1}^m \text{Tr}(E_i) |\Omega_i| &\leq \bar{v}. \end{aligned}$$

In what follows we will use the following choices of \hat{J}^γ , which are discrete counterparts of J^γ defined by (3.22) and (3.23):

- *Discretized compliance functional with penalized displacement constraints*

$$\hat{J}^\gamma(\mathbf{E}, \mathbf{u}_{\mathbf{E}}) := \frac{1}{2} \mathbf{f}^\top \mathbf{u}_{\mathbf{E}} + \frac{1}{\gamma} \sum_{i=1,2,\dots,n_{dis}} \max(0, \mathbf{d}_i^\top \mathbf{u}_{\mathbf{E}} - c_i)^2, \quad (4.3)$$

where $\mathbf{d}_i \in \mathbb{R}^{\hat{n}}$ and $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n_{dis}$.

- *Discretized compliance functional with penalized von-Mises stress constraints*

$$\hat{J}^\gamma(\mathbf{E}, \mathbf{u}_{\mathbf{E}}) := \frac{1}{2} \mathbf{f}^\top \mathbf{u}_{\mathbf{E}} + \frac{1}{\gamma} \sum_{i=1,2,\dots,m} \max\left(0, \sum_{k=1}^G \|\sigma_{i,k}\|_M^2 - s_\sigma |\Omega_i|\right)^2, \quad (4.4)$$

where $\sigma_{i,k}$ is the discretized stress tensor σ_E associated with the element Ω_i evaluated in the k -th Gauss integration point and $\|\sigma_{i,k}\|_M^2 := \sigma_{i,k}^\top M \sigma_{i,k}$, $k = 1, 2, \dots, G$ (see [13] for details).

Note that we used the penalty function $j : \mathbb{R} \rightarrow \mathbb{R}$, $t \mapsto \max(0, t)^2$ in both cases.

4.2. A numerical algorithm for the solution of discrete MDFMO problems. Problem (4.2) for a fixed penalty parameter $\gamma > 0$ is a large-scale nonlinear semidefinite program. Recently, an efficient algorithm for the solution of problems of this type has been proposed in [19, 18]. The new algorithm is based on the sequential convex programming (SCP) concept

and leads to an efficient implementation in the computer code PENSOP (see [19]). In the core of the method, a generally non-convex semidefinite program is replaced by a sequence of sub-problems, in which nonlinear constraints and objective functions defined in matrix variables are approximated by block separable convex models. In order to solve the multidisciplinary problem (4.2) numerically, we combine this idea with a simple penalty strategy, yielding the following algorithm:

Algorithm I.

- (1) Choose an initial penalty factor $\gamma_0 > 0$, set $i := 0$.
- (2) Solve problem (4.2) for $\gamma = \gamma_i$ by the SCP method. Denote the approximate solution by \mathbf{E}^i .
- (3) Convergence test: Compute the KKT-error $\varepsilon_{KKT}(\mathbf{E}^i)$. If $\varepsilon_{KKT}(\mathbf{E}^i) \leq 10^{-4}$, STOP.
- (4) Update the penalty parameter γ by the formula $\gamma_{i+1} = 3\gamma_i$.
- (5) Set $i := i + 1$ and go to (2).

The whole algorithm has been implemented in the software PLATO-N, a platform for the solution of large-scale topology and free material design problems (see project website www.plato-n.org for details). All test examples were run on a single core of a standard PC with a tact frequency 2.83 GHz and 8 Gbyte memory.

4.3. Numerical results. We consider two model examples. In both examples the following values of the bounds are used in (4.2):

$$\bar{v} := 0.333|\Omega|, \quad \bar{\rho} = 1, \quad \varepsilon = 10^{-4}.$$

Example 1. We consider a cubic design domain which is loaded in the vertical direction in the center of its top and bottom surfaces and fixed close to the corners of the bottom surface (see Fig. 4.1). The cube is discretized by 27.000 finite elements. First, we minimize the compliance of the structure without assuming any additional constraints. Fig. 4.2 shows:

- the optimal material density computed by the trace of the material tensor on every element (a);
- the principal material orientation computed via the eigenvector associated with the principal eigenvector of the Voigt tensor (b);
- the deformation of the loaded body (c).

One can observe that the whole body deforms in the direction of the applied vertical load.

Now we add a number of the displacement constraints by using (4.3), in order to force the loaded nodes on the bottom of the structure to deform in the direction opposite to the applied load. We solve the resulting multidisciplinary problem by Algorithm I. The results are depicted in Fig. 4.3. It can be clearly seen that the loaded structure deforms in the desired direction. Because the applied volume bound is the same in both cases, one can expect that the compliance of the optimal structure becomes worse after adding the displacement constraints. Precisely that is seen in Fig. 4.3c: the loaded nodes on the top surface are deformed much more in the direction of the load compared to the unconstrained case. Statistics for both numerical experiments are summarized in Table 4.1.

Example 2. In our second example we consider a three-dimensional L-shaped geometry. The design domain clamped at the bottom is loaded by a vertical load on the right hand side of the structure (see Fig. 4.4). This time the design space is discretized by approximately 10.000 finite elements. Again we first minimize the compliance of the structure without any additional constraint. Fig. 4.5 shows:

- the optimal material density computed by the trace of the material tensor on every element together with the deformation of the body (a);
- the principal material orientation (b);

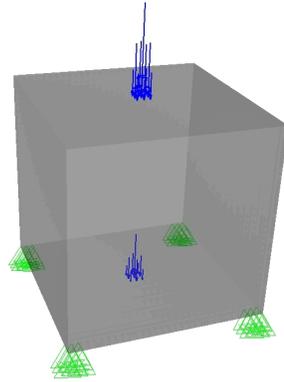


FIG. 4.1. *Ex.1 – geometry and forces*

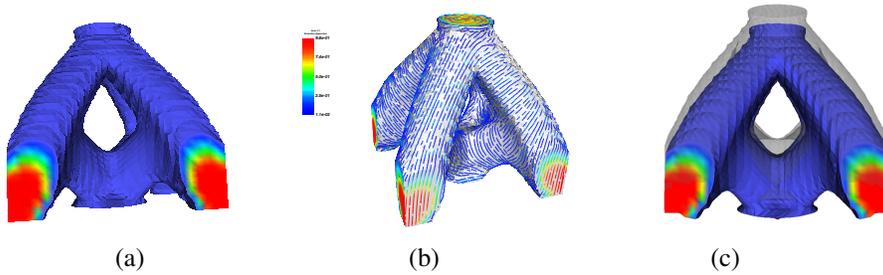


FIG. 4.2. *Ex.1 – no displacement constraints; material density (a) / principal material orientation (b) / deformed body (c)*

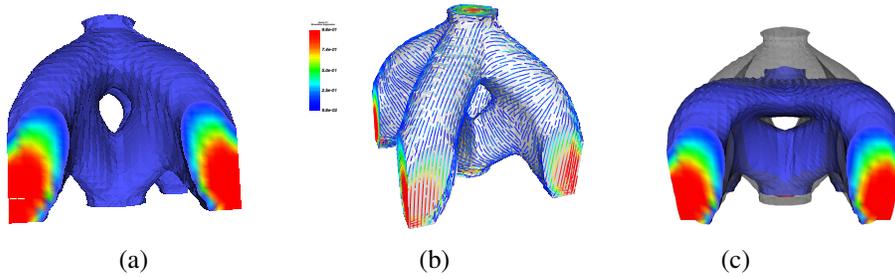
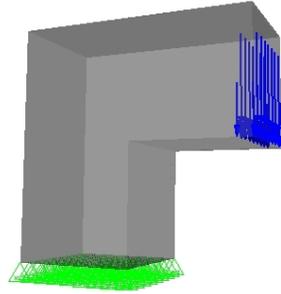


FIG. 4.3. *Ex.1 – with displacement constraints; material density (a) / principal material orientation (b) / deformed body (c)*

TABLE 4.1
Ex.1 – statistics.

	compliance (scaled)	outer iterations	inner iterations	time in seconds
no displ. constr.	0.606	1	250	11,130
displ. constr.	8.01	8	776	35,640

outer iterations: number of iterations required by Algorithm I
inner iterations: total number of iterations reported by PENSOP

FIG. 4.4. *Ex.2 – geometry and forces*TABLE 4.2
Ex.2 – statistics.

	compliance (scaled)	maximal stress (scaled)	outer iterations	inner iterations	time in seconds
no stress constr.	2.007	8.1	1	133	1,680
stress constr.	2.425	2.5	7	770	18,250

- the von Mises stress distribution (c).

Stress concentration appears at the sharp edge connecting the horizontal with the vertical bar (see Fig. 4.5c).

Now, in order to avoid the stress concentration, we add the stress constraints (4.4) on every element in the design domain. We solve the resulting problem by Algorithm I. The results are outlined in Fig. 4.6. The first observation is that the compliance of the structure is worse by approximately 22.5 percent in the stress constrained case (see Fig. 4.6a). On the other hand it can be seen that the stress concentration can be completely avoided (see Fig. 4.6c). Moreover Fig.4.6d indicates that the stress constraints become active in wide parts of the design domain (activity is indicated by the red color). This is in sharp contrast to the previous experiment, where the stress concentration appeared only in very few elements. Fig. 4.6b provides an explanation how the stress reduction is achieved: it is seen that the material forms an arch like structure close to the sharp edge. Statistics for both, constrained and unconstrained case are summarized in Table 4.2.

5. Acknowledgement. The paper was finished during the visit of J. Haslinger at the University of Erlangen. This work was partly supported by the EU Commission in the Sixth Framework Program, Project No. 30717 PLATO-N, by the DFG Research Unit 894 and DFG Excellence cluster *Engineering of advanced materials* (MS, GL) and by the grants A100750802 (JH, MK) and MSM0021620839 (JH) of the Czech Academy of Sciences.

REFERENCES

- [1] F. Allaire. *Shape Optimization by the Homogenization Method*. Springer-Verlag New-York, 2002.
- [2] C. Barbarosie and S. Lopez. Study of the cost functional for free material design problems. Mathematics Preprints Pre-2007-003, Universidade de Lisboa, Centro de Matematica e Aplicacoes Fundamentais, 2007.
- [3] A. Ben-Tal, M. Kočvara, A. Nemirovski, and J. Zowe. Free material design via semidefinite programming. The multi-load case with contact conditions. *SIAM J. Optimization*, 9:813–832, 1997.
- [4] M. P. Bendsøe, J. M. Guades, R. B. Haber, P. Pedersen, and J. E. Taylor. An analytical model to predict optimal material properties in the context of optimal structural design. *J. Applied Mechanics*, 61:930–937, 1994.

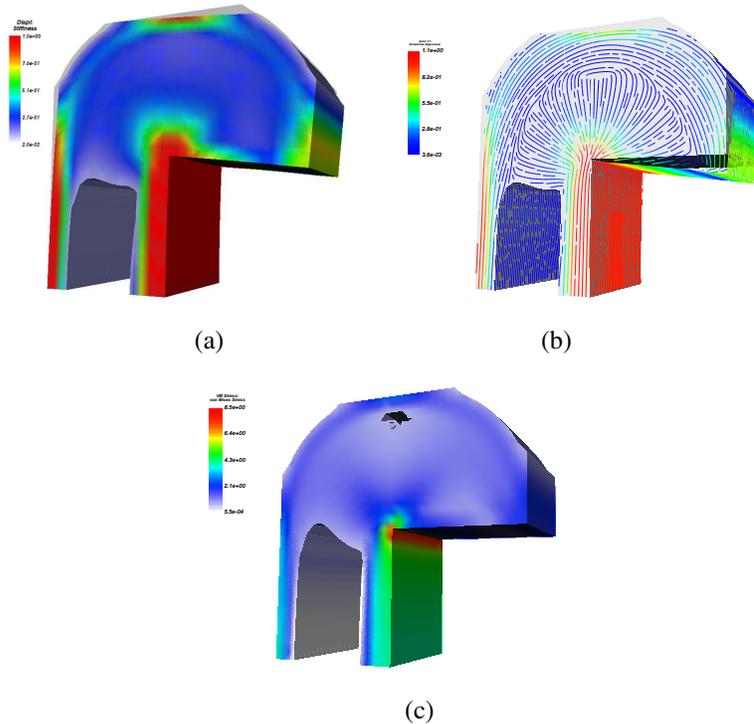


FIG. 4.5. Ex.2 – no stress constraints; material density & deformed geometry (a) / principal material orientation (b) / stress distribution (c)

- [5] M. P. Bendsøe and O. Sigmund. *Topology Optimization. Theory, Methods and Applications*. Springer-Verlag, Heidelberg, 2002.
- [6] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland, Amsterdam, New York, Oxford, 1978.
- [7] J. Dvořák, J. Haslinger, and M. Miettinen. Homogenization/optimal shape design approach in material distribution problems. part i: The scalar case. *Advances in Math. Sci. and Appl.*, 9:665–694, 1999.
- [8] I. Ekeland and T. Turnbull. *Infinite-Dimensional Optimization and Convexity*. University of Chicago Press, 1983.
- [9] B. Gustafsson and J. Mossino. A criterion for h -convergence in elasticity. *Asymptotic Analysis*, 51:247–269, 2007.
- [10] M. Kočvara. Topology optimization problems with displacement constraints: A bilevel approach. *Structural Optimization*, 14:256–263, 1997.
- [11] M. Kočvara, M. Stingl, and J. Zowe. Free material optimization: recent progress. *Optimization*, 57:79–100, 2008.
- [12] M. Kočvara and M. Stingl. Solving nonconvex SDP problems of structural optimization with stability control. *Optimization Methods and Software*, 19(5):595–609, 2004.
- [13] M. Kočvara and M. Stingl. Free material optimization: Towards the stress constraints. *Structural and Multidisciplinary Optimization*, 33(4-5):323–335, 2007.
- [14] J. Mach. Finite element analysis of free material optimization problem. *Applications of Mathematics*, 49:285–307(23), 2004.
- [15] F. Murar and L. Tartar. H-convergence. In *Seminaire d'Analyse Fonctionnelle et Numerique de l'Universite d'Alger*. Mimeographed notes, 1978.
- [16] S. Spagnolo. Convergence in energy for elliptic operators. In B. Hubbard, editor, *Numerical solutions of partial differential equations III Synspade 1975*, pages 469–482. Academic Press New York, 1976.
- [17] M. Stingl, G. Leugering, and M. Kočvara. Free material optimization with fundamental eigenfrequency constraints. *SIAM J Optim*, 20(1):524–547, 2009.
- [18] M. Stingl, G. Leugering, and M. Kočvara. A new non-linear semidefinite programming algorithm with an application to multidisciplinary free material optimization. *International Series of Numerical Mathematics*,

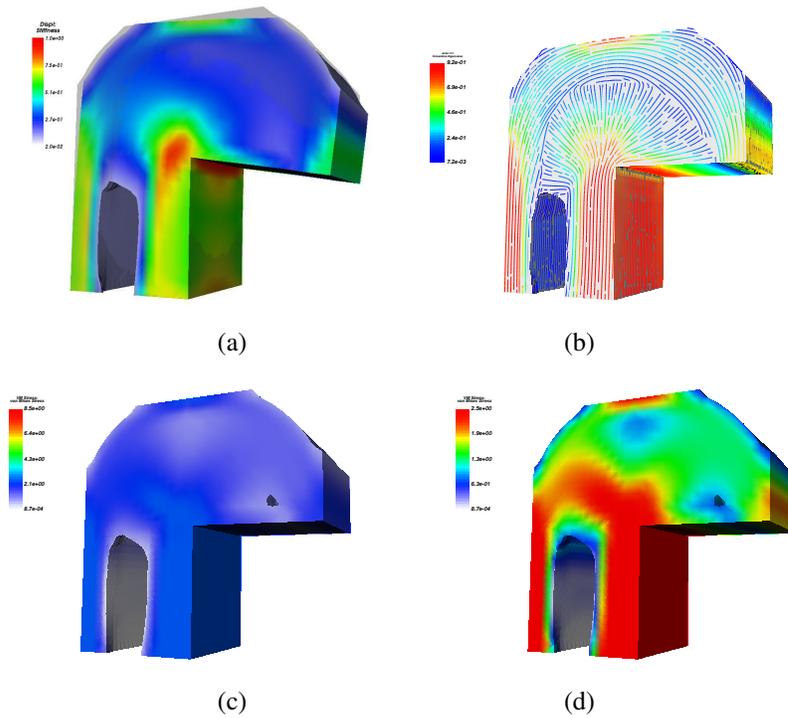


FIG. 4.6. Ex.2 – with stress constraints; material density & deformed geometry (a) / principal material orientation (b) / stress distribution (c) / stress distribution - active set (d)

133:275–295, 2009.

- [19] M. Stingl, G. Leugering, and M. Kočvara. A sequential convex semidefinite programming algorithm with an application to multiple-load free material optimization. *SIAM J Optim*, 20(1):130–155, 2009.
- [20] L. Tartar. Quelques remarques sur l’homogénéité. In *Japan–France Seminar 1976: Functional Analysis and Numerical Analysis*, pages 469–482. Japan Society for the Promotion of Sciences, 1978.
- [21] L. Tartar. Compensated compactness and applications to partial differential equations. In: *Knops, R.J. (Ed.), Nonlinear Anal. and Mech., Pitman, London*, pages 136–212, 1979.
- [22] R. Werner. *Free Material Optimization*. PhD thesis, Institute of Applied Mathematics II, Friedrich-Alexander University of Erlangen-Nuremberg, 2000.
- [23] J. Zowe, M. Kočvara, and M. P. Bendsøe. Free material optimization via mathematical programming. *Math. Prog., Series B*, 79:445–466, 1997.