A new lower bound on the independence number of a graph

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Abstract

For a given connected graph G on n vertices and m edges, we prove that its independence number $\alpha(G)$ is at least $\left((2m+n+2) - ((2m+n+2)^2 - 16n^2)^{½}\right)/8$.

Keywords : independence number, min algorithm, connected graph.

Introduction

Let G=(V,E) be a connected graph G on $n=|V|$ vertices and $m=|E|$ edges.

For a subgraph H of G and for a vertex $i \in V(H)$, let $d_H(i)$ be the degree of i in H and let $N_H(i)$ be its neighbourhood in H. Let $\delta(H)$ and $\Delta(H)$ be the minimum degree and the maximum degree of H, respectively. A subset X of V is called independent if its vertices are mutually non-adjacent. The independence number α (G) is the largest cardinality among all independent sets of G.

The problem of finding an independent set of maximum cardinality is know to be NP –complete[1]. Some approximation algorithms was designed to tackle this problem, among them, the well know MIN algorithm [4], which can be implemented in time linear in n and m :

G_1 :=G, j :=1

While $V(G_j) \neq \emptyset$ do

Begin

Choose $i_j \in V(G_j)$ with $d_{G_j}(i_j)=\delta(G_j)$, delete $\{i_j\} \cup N_{G_j}(i_j)$ to obtain G_{j+1} and set

j :=j+1 ;

End ;

k :=j-1

Let k_{MIN} be the smallest k the algorithm MIN provides for a given connected graph G.

Harant [3] proved that $\alpha(G) \geq k_{MIN} \geq ((2m+n+1) - ((2m+n+1)^2 - 4n^2)^{1/2})/2$.

The purpose of the present note is to improve this lower bound.

Lemma : For a given connected graph G on n vertices and m edges,

$\alpha(G) \geq ((2m+n+2) - ((2m+n+2)^2 - 16n^2)^{1/2})/8$.

The proof starts with the inequality (1) proved by Harant [3] :

$$k_{MIN} \geq n^2 / \left(2m+n-\sum_{i \in I_n} (d_G(i) - \delta(G_j))\right) \quad (1)$$

and uses a variation of the one given by Halldorson [2].

For j=1,..., k_{MIN}, let $d_{G_j}(i_j)$ be the degree in the remaining graph of the j–th vertice choosed at the j–th iteration of the algorithm MIN. The number of vertices deleted in the j–th iteration is thus $1+ d_{G_j}(i_j)$ and the sum of the degrees of the $1+ d_{G_j}(i_j)$ vertices deleted is at least $(1+ d_{G_j}(i_j))d_{G_j}(i_j)$. Thus the number of edges removed in the j–th iteration is at least $(1+ d_{G_j}(i_j))d_{G_j}(i_j)/2$.

Let X be an independent set of G of maximum cardinality α , and let k_j be the number of vertices among the $1+ d_{G_j}(i_j)$ vertices deleted in the j–th iteration that are also contained in X.

Then $\sum_{j=1}^{k_{MIN}} k_j = \alpha$

Since X is edgeless, and G is connected then the number of edges removed in the j–th iteration (j=1,..., k_{MIN} -1) is at least :

$$\binom{1+ d_{G_j}(i_j)}{2} + \binom{k_j}{2} + 1$$

(for $j=1,\dots,k_{MIN}-1$, there is at least one edge between $N_{G_j}(i_j)$ and G_{j+1}, because G is supposed connected).

In the k_{MIN}–th iteration, at least

$$\binom{1+d_{G_j}(i_{k_{MIN}})}{2}+\binom{k_{k_{MIN}}}{2}$$

edges are removed.

Hence we obtain the following inequality :

$$m\geq \sum_{j=1}^{k_{MIN}-1} \left(\binom{1+d_{G_j}(i_j)}{2}+\binom{k_j}{2}+1\right) +\binom{1+d_{G_j}(i_{k_{MIN}})}{2}+\binom{k_{k_{MIN}}}{2}$$

then :

$$2m\geq 2k_{MIN}-2+\sum_{j=1}^{k_{MIN}} \left(\left(1+d_{G_j}(i_j)\right) d_{G_j}(i_j)\right)+\sum_{j=1}^{k_{MIN}} k_j +\sum_{j=1}^{k_{MIN}} (k_j)^2$$

consequently :

$$2m\geq 4k_{MIN}-2 + \sum_{j=1}^{k_{MIN}}\left((1+d_{G_j}(i_j))\, d_{G_j}(i_j)\right) \qquad (2)$$

On the other hand :

Since $\forall (j,j') \in\{1,\dots,k_{MIN}\}, j\neq j' \Rightarrow \left(\{i_j\}\cup N_{G_j}(i_j)\right)\cap\left(\{i_{j'}\}\cup N_{G_{j'}}(i_{j'})\right) =\emptyset$ and

$$I_n=\bigcup_{j=1}^{k_{MIN}} \left(\{i_j\}\cup N_{G_j}(i_j)\right) =\{1,\dots,n\}$$

then

$$\sum_{i\in I_n}\delta(G_j)= \sum_{j=1}^{k_{MIN}} \sum_{i\in \{i_j\}\cup N_{G_j}(i_j)} \delta(G_j)$$

and

$$\sum_{i\in I_n} \delta(G_j)= \sum_{j=1}^{k_{MIN}} (1+ d_{G_j}(i_j))\delta(G_j) \le \sum_{j=1}^{k_{MIN}} \left((1+ d_{G_j}(i_j))d_{G_j}(i_j)\right)$$

thus

$$\sum_{i\in I_n}(d_G(i)- \delta(G_j))=2m-\sum_{i\in I_n}\delta(G_j) \ge 2m-\sum_{j=1}^{k_{MIN}} \left((1+ d_{G_j}(i_j))d_{G_j}(i_j)\right)$$

by using inequality (2) we have :

$$\sum_{i\in I_n}(d_G(i)- \delta(G_j)) \ge 4k_{MIN}-2$$

then inequality (1) implies :

$$k_{MIN} \ge n^2/(2m+n+2-4k_{MIN})$$

and consequently :

$$k_{MIN} \ge \left((2m+n+2) -((2m+n+2)^2-16n^2)^{1/2}\right)/8.$$

Conclusion

This note presented an improved lower bound on the independence number of a connected graph, and as a future work, it will be proved that this bound is optimale for a particular class of graphs.

References :

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