Stability of error bounds for convex constraint systems in Banach spaces

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Abstract

This paper studies stability of error bounds for convex constraint systems in Banach spaces. We show that certain known sufficient conditions for local and global error bounds actually ensure error bounds for the family of functions being in a sense small perturbations of the given one. A single inequality as well as semi-infinite constraint systems are considered.

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1 Introduction

In this paper we continue our study started in [19] of stability of error bounds of convex constraint systems under data perturbations. For the summary of the theory of error bounds and its various applications to sensitivity analysis, convergence analysis of algorithms, penalty functions methods in mathematical programming the reader is referred to the survey papers by Azé [2], Lewis & Pang [16], Pang [22], as well as the book by Auslender & Teboule [1].

For an extended real-valued function $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ on a Banach space X the *error bound* property is defined by the inequality

$$d(x, S_f) \le c[f(x)]_+, \tag{1}$$

where S_f denotes the lower level set of f:

$$S_f := \{x \in X : \ f(x) \le 0\}, \tag{2}$$

$c \ge 0$, and the notation $\alpha_+ := \max(\alpha, 0)$ is used.

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Given an $\bar{x} \in X$ with $f(\bar{x}) = 0$ we say that f admits an (local) *error bound* at $\bar{x}$ if there exist reals $c \geq 0$ and $\delta > 0$ such that (1) holds for all $x \in B_\delta(\bar{x})$. The *best bound* – the exact lower bound of all such c – coincides with $[\mathrm{Er}\, f(\bar{x})]^{-1}$, where

$$\mathrm{Er}\, f(\bar{x}) := \liminf_{\substack{x \to \bar{x} \\ f(x) > 0}} \frac{f(x)}{d(x, S(f))} \tag{3}$$

is the *error bound modulus* [8]) (also known as *conditioning rate* [23]) of f at $\bar{x}$. Thus, f admits an error bound at $\bar{x}$ if and only if $\mathrm{Er}\, f(\bar{x}) > 0$.

If (1) holds for some $c \geq 0$, and all $x \in X$ then we say that f admits a *global error bound*. In this case, the *best bound* – the exact lower bound of all such c – coincides with $[\mathrm{Er}\, f]^{-1}$, where

$$\mathrm{Er}\, f := \inf_{f(x) > 0} \frac{f(x)}{d(x, S(f))} \tag{4}$$

is the global *error bound modulus*. (In [3, 4] the last constant is denoted $\sigma_0(f)$.)

A huge literature deals with criteria for the error bound property in terms of various derivative-like objects defined either in the primal space (directional derivatives, slopes, etc.) or in the dual space (different kinds of subdifferentials) [3–6, 8–11, 13, 14, 16–18, 20–23, 25–27]. The convex case has attracted a special attention starting with the pioneering work by Hoffman [12] on error bounds for systems of affine functions, see [3, 6, 9, 16].

If f is a lower semicontinuous convex function the following conditions are known to provide sufficient criteria for the error bound property.

- *Local criteria*:

  (L1) $\liminf\limits_{x \to \bar{x},\, f(x) > f(\bar{x})} d(0, \partial f(x)) > 0$;

  (L2) $0 \notin \mathrm{Bdry}\, \partial f(\bar{x})$.

- *Global criteria*:

  (G1) $\inf\limits_{f(x) > 0} d(0, \partial f(x)) > 0$;

  (G2) $\inf\limits_{f(x) = 0} d(0, \mathrm{Bdry}\, \partial f(x)) > 0$.

The following implications are true:

$$(L2) \quad \Rightarrow \quad (L1) \qquad\qquad (G2) \quad \Rightarrow \quad (G1),$$

while conditions (L1) and (G1) are actually necessary and sufficient for the corresponding error bound properties – see Theorems 1 and 4 below.

We show in Theorems 2 and 5 that the stronger conditions (L2) and (G2) characterize stronger properties than just the existence of local or global error bounds for f, namely, they guarantee respectively the local or global error bound property for the family of functions being in a sense small perturbations of f.

In this paper we consider also semi-infinite constraint systems of the form

$$f_t(x) \leq 0 \quad \text{for all} \quad t \in T, \tag{5}$$

where T is a compact, possibly infinite, Hausdorff space, $f_t : X \to \mathbb{R}$, $t \in T$, are given continuous functions such that $t \mapsto f_t(x)$ is continuous on T for each $x \in X$, and establish similar characterizations of stability of local and global error bounds with respect to perturbations of the functions f_t – see Theorems 3 and 6.

The organization of the paper is simple: besides the current introductory section it contains two more sections devoted to local and global error bounds respectively.

If not specified otherwise, we consider extended real-valued functions on a Banach space X. The class of all lower semicontinuous proper convex functions on X will be denoted $\Gamma_0(X)$. $B_\delta(\bar{x})$ is the closed ball with center at $\bar{x}$ and radius δ. B^* denotes the dual unit ball. For a set Q, the notations $\operatorname{int} Q$ and $\operatorname{Bdry} Q$ mean the interior and the boundary of Q respectively.

2 Stability of local error bounds

In this section we discuss relationships between the local error bound criteria (L1) and (L2) and establish conditions for stability of local error bounds for the constraint systems (2) and (5).

Theorem 1. *Let $f \in \Gamma_0(X)$, $f(\bar{x}) = 0$. Consider the following properties:*

(i). f admits an error bound at $\bar{x}$, that is, $\operatorname{Er} f(\bar{x}) > 0$*;*

(ii). $\tau(f, \bar{x}) := \liminf\limits_{x \to \bar{x},\, f(x) > f(\bar{x})} d(0, \partial f(x)) > 0$*;*

(iii). $\varsigma(f, \bar{x}) := d(0, \operatorname{Bdry} \partial f(\bar{x})) > 0$*;*

(iv). $0 \notin \partial f(\bar{x})$*;*

(v). $0 \in \operatorname{int} \partial f(\bar{x})$.

Each of the properties (ii)–(v) is sufficient for the error bound property (i). Moreover,

(a). $\varsigma(f, \bar{x}) \le \tau(f, \bar{x}) = \operatorname{Er} f(\bar{x})$*;*

(b). [(iv) or (v)] $\Leftrightarrow$ (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i).

Proof. (a). We first prove the inequality $\varsigma(f, \bar{x}) \le \tau(f, \bar{x})$. If $0 \in \operatorname{int} \partial f(\bar{x})$ then

$$f(x) \ge \langle v^*, x - \bar{x} \rangle \quad \text{for all} \quad v^* \in \varsigma(f, \bar{x}) B^*,\ x \in X,$$

and consequently,

$$f(x) \ge \varsigma(f, \bar{x}) \|x - \bar{x}\| \quad \text{for all} \quad x \in X.$$

On the other hand,

$$-f(x) \ge \langle x^*, \bar{x} - x \rangle \quad \text{for all} \quad x \in X,\ x^* \in \partial f(x).$$

Adding the last two inequalities together we obtain

$$\langle x^*, x - \bar{x} \rangle \ge \varsigma(f, \bar{x}) \|x - \bar{x}\| \quad \text{for all} \quad x \in X,\ x^* \in \partial f(x),$$

and consequently, $\|x^*\| \geq \varsigma(f,\bar{x})$ if $x^* \in \partial f(x)$ and $x \neq \bar{x}$. Hence, $\tau(f,\bar{x}) \geq \varsigma(f,\bar{x})$.

If $0 \notin \operatorname{int} \partial f(\bar{x})$ then $\varsigma(f,\bar{x}) = d(0, \partial f(\bar{x}))$, and consequently, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x^*\| \geq \varsigma(f,\bar{x}) - \varepsilon$ for all $x^* \in \partial f(x)$ and $x \in B_\delta(\bar{x})$. It follows that $\tau(f,\bar{x}) \geq \varsigma(f,\bar{x})$.

The next step is to show that $\operatorname{Er} f(\bar{x}) \leq \tau(f,\bar{x})$. Consider any $x \in X$ with $f(x) > 0$ and any $x^* \in \partial f(x)$. By definition of the subdifferential,

$$f(u) - f(x) \geq \langle x^*, u - x \rangle \quad \text{for all} \quad u \in X.$$

In particular,

$$-f(x) \geq \langle x^*, u - x \rangle \geq -\|x^*\| \, \|u - x\| \quad \text{for all} \quad u \in S_f,$$

and consequently,

$$\frac{f(x)}{d(x, S(f))} \leq \|x^*\|,$$

which immediately implies the inequality $\operatorname{Er} f(\bar{x}) \leq \tau(f,\bar{x})$.

The proof of the opposite inequality $\tau(f,\bar{x}) \leq \operatorname{Er} f(\bar{x})$ is a typical example of the application of the Ekeland variational principle [7]. Suppose that $\operatorname{Er} f(\bar{x}) < \alpha$. We are going to show that $\tau(f,\bar{x}) \leq \alpha$. Choose a $\beta \in (\tau(f,\bar{x}), \alpha)$. By definition (3), for any $\delta > 0$ there exists an $x \in B_{\delta/2}(\bar{x})$ such that

$$0 < f(x) < \beta d(x, S(f)).$$

Consider a lower semicontinuous function $g : X \to \mathbb{R}_\infty$ given by $g(u) = [f(u)]_+$. It holds $g(u) \geq 0$ for all $u \in X$ and $g(x) = f(x) < \beta d(x, S(f))$. By Ekeland's theorem, there exists an $\hat{x} \in X$ such that $\|\hat{x} - x\| \leq (\beta/\alpha) d(x, S(f))$ and

$$g(u) - g(\hat{x}) + \alpha \|u - \hat{x}\| \geq 0 \quad \text{for all} \quad u \in X.$$

Since $\|\hat{x} - x\| < d(x, S(f))$ and f is lower semicontinuous, we have $g(u) = f(u)$ for all u near x, and it follows from the last inequality that $\|x^*\| \leq \alpha$ for some $x^* \in \partial f(\hat{x})$. Besides, $\|\hat{x} - \bar{x}\| < 2\|x - \bar{x}\| \leq \delta$. Hence, $\tau(f,\bar{x}) \leq \alpha$.

(b). The equivalence [(iv) or (v)] $\Leftrightarrow$ (iii) is obvious. The chain (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i) follows from (a). □

Remark 1. Constant $\tau(f,\bar{x})$ providing a necessary and sufficient characterization of the local error bound property is also known as the *strict outer subdifferential slope* $\overline{|\partial f|}^{>}(\bar{x})$ of f at $\bar{x}$ [8]. Criterion (ii) was used in [14, Theorem 2.1 (c)], [21, Corollary 2 (ii)], [23, Theorem 4.12], [26, Theorem 3.1]. Criterion (iii) was used in [9, Corollary 3.4], [10, Theorem 4.2]. The equality $\operatorname{Er} f(\bar{x}) = \tau(f,\bar{x})$ seems to be well known.

The inequality in (a) and the implication (iii) $\Rightarrow$ (ii) in (b) in Theorem 1 can be strict.

Example 1. $f(x) \equiv 0$, $x \in \mathbb{R}$. Obviously $0 \in \operatorname{Bdry} \partial f(\bar{x})$, $\varsigma(f,\bar{x}) = 0$, while $\tau(f,\bar{x}) = \infty$ for any $\bar{x} \in \mathbb{R}$.

Example 2. $f(x) = 0$ if $x \leq 0$, and $f(x) = x$ if $x > 0$. Then $\partial f(0) = [0,1]$ and $0 \in \operatorname{Bdry} \partial f(0)$, $\varsigma(f,\bar{x}) = 0$, while $\tau(f,0) = 1$.

Thus, condition (iii) in Theorem 1 is in general stronger than each of the equivalent conditions (i) and (ii). It characterizes a stronger property than just the existence of a local error bound for f at $\bar{x}$, namely, it guaranties the local error bound property for the family of functions being small perturbations of f.

Definition 1. Let $f(\bar{x}) < \infty$ and $\varepsilon \geq 0$. We say that $g : X \to \mathbb{R}_\infty$ is an *ε-perturbation* of f near $\bar{x}$ and write $g \in \operatorname{Ptb}(f, \bar{x}, \varepsilon)$ if $g(\bar{x}) = f(\bar{x})$ and

$$\limsup_{x \to \bar{x}} \frac{|g(x) - f(x)|}{\|x - \bar{x}\|} \leq \varepsilon. \tag{6}$$

Obviously, if $g \in \operatorname{Ptb}(f, \bar{x}, \varepsilon)$ then $f \in \operatorname{Ptb}(g, \bar{x}, \varepsilon)$.

Remark 2. If the functions are continuous at $\bar{x}$ then condition (6) implies $g(\bar{x}) = f(\bar{x})$. The last requirement can be dropped from Definition 1 if condition (6) is replaced by a more general one:

$$\limsup_{x \to \bar{x}} \frac{|(g(x) - f(x)) - (g(\bar{x}) - f(\bar{x}))|}{\|x - \bar{x}\|} \leq \varepsilon. \tag{7}$$

In this case, a perturbation function does not have to coincide with the given one at the point of reference. In fact, the difference $\alpha := g(\bar{x}) - f(\bar{x})$ can be arbitrarily large. However, this seemingly more general case can be easily reduced to the above one: if a function g satisfies (7) then the function $x \mapsto g(x) - \alpha$ satisfies (6).

Note also that neither g nor f in the above definition are assumed convex. The characterization below is (partially) in terms of Fréchet subdifferentials which in the convex case reduce to subdifferentials in the sense of convex analysis.

Proposition 1. *Let f be convex, $f(\bar{x}) < \infty$, and $\varepsilon \geq 0$. If $g \in \operatorname{Ptb}(f, \bar{x}, \varepsilon)$ then*

(i). $\partial g(\bar{x}) \subseteq \partial f(\bar{x}) + \varepsilon B^*$;

(ii). $d(0, \partial g(\bar{x})) \geq d(0, \partial f(\bar{x})) - \varepsilon$;

(iii). $d(0, \operatorname{Bdry} \partial g(\bar{x})) \geq d(0, \operatorname{Bdry} \partial f(\bar{x})) - \varepsilon$.

Proof. (i). Let $x^* \in \partial g(\bar{x})$. Then, by definition of the Fréchet subdifferential and by (6), for any $\xi > 0$ there exists a $\delta > 0$ such that

$$\frac{g(x) - g(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq -\frac{\xi}{2},$$
$$\frac{g(x) - f(x)}{\|x - \bar{x}\|} \leq \varepsilon + \frac{\xi}{2}$$

for all x, $0 < \|x - \bar{x}\| \leq \delta$. Subtracting the second inequality from the first one and recalling that $g(\bar{x}) = f(\bar{x})$ we obtain

$$\frac{f(x) + \varepsilon\|x - \bar{x}\| - f(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq -\xi$$

for all x, $0 < \|x - \bar{x}\| \leq \delta$, and consequently $x^* \in \partial(f + \varepsilon\|\cdot - \bar{x}\|)(\bar{x}) = \partial f(\bar{x}) + \varepsilon B^*$.

(ii) follows immediately from (i).

(iii). If $\varepsilon \geq d(0, \operatorname{Bdry} \partial f(\bar{x}))$ the assertion is trivial. Let $\varepsilon < d(0, \operatorname{Bdry} \partial f(\bar{x}))$. Then due to (i) zero is either outside both $\partial f(\bar{x})$ and $\partial g(\bar{x})$ or inside both of them. In the first case, the assertion coincides with (ii), while in the second one, it follows from (i). □

The next theorem shows that condition (iii) in Theorem 1 provides a characterization of the "combined" error bound property for the family of ε-perturbations of f near $\bar{x}$.

Theorem 2. *Let $f \in \Gamma_0(X)$, $f(\bar{x}) = 0$, $\varepsilon > 0$. The following assertions hold true:*

(i). $\operatorname{Er} g(\bar{x}) \geq \varsigma(f, \bar{x}) - \varepsilon$ *for any* $g \in \Gamma_0(X) \cap \operatorname{Ptb}(f, \bar{x}, \varepsilon)$*;*

(ii). if $0 \in \operatorname{Bdry} \partial f(\bar{x})$ *then the function* $g \in \Gamma_0(X) \cap \operatorname{Ptb}(f, \bar{x}, \varepsilon)$ *defined by*

$$g(u) := f(u) + \varepsilon \|u - \bar{x}\|, \quad u \in X, \tag{8}$$

satisfies $\operatorname{Er} g(\bar{x}) \leq \varepsilon$*;*

(iii). if $\dim X < \infty$ *and* $0 \in \operatorname{Bdry} \partial f(\bar{x})$ *then there exists an* $x^* \in \varepsilon B^*$ *such that the function* $g \in \Gamma_0(X) \cap \operatorname{Ptb}(f, \bar{x}, \varepsilon)$ *defined by*

$$g(u) := f(u) + \langle x^*, u - \bar{x} \rangle, \quad u \in X, \tag{9}$$

satisfies $\operatorname{Er} g(\bar{x}) \leq \varepsilon$*.*

Proof. (i). If $g \in \Gamma_0(X) \cap \operatorname{Ptb}(f, \bar{x}, \varepsilon)$ then, due to Proposition 1 (iii) and Theorem 1 (a), we have $\operatorname{Er} g(\bar{x}) \geq \varsigma(g, \bar{x}) \geq \varsigma(f, \bar{x}) - \varepsilon$.

(ii). Let $0 \in \operatorname{Bdry} \partial f(\bar{x})$, $\xi > 0$. Then

$$f(u) \geq 0 \quad \text{for all } u \in X,$$

and there exists an $u^* \in (\xi/2)B^*$ such that $u^* \notin \partial f(\bar{x})$, that is, there is a $y \in B_{2\xi/3}(\bar{x})$ such that

$$f(y) < \langle u^*, y - \bar{x} \rangle \leq (\xi/2)\|y - \bar{x}\|.$$

Obviously, $y \neq \bar{x}$. By virtue of the Ekeland variational principle [7], we can select an $x \in X$ satisfying $\|x - y\| \leq \|y - \bar{x}\|/2 \leq \xi/3$, such that the function $u \mapsto f(u) + \xi\|u - x\|$ attains its minimum at x. Hence $x \in B_\xi(\bar{x}) \setminus \{\bar{x}\}$ and $0 \in \partial f(x) + \xi B^*$, that is, there exists a $v^* \in \partial f(x)$, such that $\|v^*\| \leq \xi$. Then the function $g \in \Gamma_0(X)$ defined by (8) obviously satisfies

$$g \in \operatorname{Ptb}(f, \bar{x}, \varepsilon),$$
$$g(x) \geq \varepsilon\|x - \bar{x}\| > 0,$$
$$d(0, \partial g(x)) \leq \varepsilon + \xi.$$

As $\xi > 0$ can be chosen arbitrarily small, thanks to Theorem 1 (a), $\operatorname{Er} g(\bar{x}) = \tau(g, \bar{x}) \leq \varepsilon$.

(iii). Let $\dim X < \infty$ and $0 \in \operatorname{Bdry} \partial f(\bar{x})$. Setting $\xi = 1/k$ in the above proof of (ii) we obtain sequences $\{x_k\} \subset X$ and $\{v_k^*\} \subset X^*$ such that

$$f(x_k) \geq 0, \ 0 < \|x_k - \bar{x}\| \leq 1/k,$$
$$v_k^* \in \partial f(x_k), \ \|v_k^*\| \leq 1/k.$$

Without loss of generality $(x_k-\bar{x})/\|x_k-\bar{x}\| \to z$, $\|z\|=1$. Choose an $x^* \in X^*$ such that $\|x^*\| = \langle x^*, z\rangle = \varepsilon$. Then $\langle x^*, x_k-\bar{x}\rangle > 0$ for all sufficiently large k. It follows that for such k the function $g \in \Gamma_0(X)$ defined by (9) satisfies

$$g(x_k) > 0,\ d(0, \partial g(x_k)) \le \varepsilon + 1/k.$$

By virtue of Theorem 1 (a), $\mathrm{Er}\, g(\bar{x}) = \tau(g, \bar{x}) \le \varepsilon$.

□

The last assertion of the theorem providing a statement in terms of a perturbation by a linear term is important when dealing with semi-infinite linear constraint systems [19].

Given a function $f \in \Gamma_0(X)$ with $f(\bar{x}) = 0$ and a number $\varepsilon \ge 0$ denote

$$\mathrm{Er}\,\{\mathrm{Ptb}\,(f, \bar{x}, \varepsilon)\}(\bar{x}) := \inf_{g \in \Gamma_0(X) \cap \mathrm{Ptb}\,(f,\bar{x},\varepsilon)} \mathrm{Er}\, g(\bar{x}). \tag{10}$$

This number characterizes the error bound property for the whole family of convex ε-perturbations of f near $\bar{x}$. Obviously, $\mathrm{Er}\,\{\mathrm{Ptb}\,(f, \bar{x}, \varepsilon)\}(\bar{x}) \le \mathrm{Er}\, f(\bar{x})$ for any $\varepsilon \ge 0$.

Corollary 2.1. *Let $f \in \Gamma_0(X)$, $f(\bar{x}) = 0$, $\varepsilon > 0$. The following assertions hold true:*

(i). $\mathrm{Er}\,\{\mathrm{Ptb}\,(f, \bar{x}, \varepsilon)\}(\bar{x}) \ge \varsigma(f, \bar{x}) - \varepsilon$*;*

(ii). if $0 \in \mathrm{Bdry}\, \partial f(\bar{x})$ *then* $\mathrm{Er}\,\{\mathrm{Ptb}\,(f, \bar{x}, \varepsilon)\}(\bar{x}) = 0$.

Due to Corollary 2.1 (i), condition $0 \notin \mathrm{Bdry}\, \partial f(\bar{x})$ is sufficient for the error bound property of the family of ε-perturbations of f as long as $\varepsilon < \varsigma(f, \bar{x})$. If $0 \in \mathrm{Bdry}\, \partial f(\bar{x})$ then, due to Corollary 2.1 (ii), none family of ε-perturbations possesses the error bound property.

Corollary 2.2. *Let $f \in \Gamma_0(X)$, $f(\bar{x}) = 0$. The following properties are equivalent:*

(i). there exists an $\varepsilon > 0$ such that $\mathrm{Er}\,\{\mathrm{Ptb}\,(f, \bar{x}, \varepsilon)\}(\bar{x}) > 0$*;*

(ii). $0 \notin \mathrm{Bdry}\, \partial f(\bar{x})$.

We consider now a semi-infinite constraint system (5), where T is a compact, possibly infinite, Hausdorff space, $f_t : X \to \mathbb{R}$, $t \in T$, are given continuous functions such that $t \mapsto f_t(x)$ is continuous on T for each $x \in X$.

System (5) is equivalent to the single inequality

$$f(x) \le 0$$

in terms of the continuous function $f : X \to \mathbb{R}$ defined by

$$f(x) := \sup_{t \in T} f_t(x). \tag{11}$$

Stability of error bounds criterion for system (5) with respect to perturbations of the function (11) is given by Theorem 2. We are looking here for stability criteria with respect to perturbations of the original family of functions $\{f_t\}_{t \in T}$.

Consider another family of continuous functions $g_t : X \to \mathbb{R}$, $t \in T$, such that $t \mapsto g_t(x)$ is continuous on T for each $x \in X$, and the corresponding function $g : X \to \mathbb{R}$ defined by

$$g(x) := \sup_{t\in T} g_t(x).$$

Given $\bar{x} \in X$ and $\varepsilon \geq 0$, the following conditions can qualify to be extensions to families of functions of the ε-perturbation property introduced in Definition 1:

(C1) $\displaystyle\limsup_{x\to\bar{x}} \sup_{t\in T} \frac{|g_t(x) - f_t(x)|}{\|x - \bar{x}\|} \leq \varepsilon$;

(C2) $g(\bar{x}) = f(\bar{x})$ and $\displaystyle\limsup_{x\to\bar{x}} \sup_{t\in T} \frac{|(g_t(x) - f_t(x)) - (g_t(\bar{x}) - f_t(\bar{x}))|}{\|x - \bar{x}\|} \leq \varepsilon$;

(C3) $g(\bar{x}) = f(\bar{x})$ and $\displaystyle\sup_{x\in B_\delta(\bar{x})\setminus\{\bar{x}\},\, t\in T} \frac{|(g_t(x) - f_t(x)) - (g_t(\bar{x}) - f_t(\bar{x}))|}{\|x - \bar{x}\|} \leq \varepsilon$, where δ is a given positive number.

All above conditions are symmetric, and consequently $\{g_t\}_{t\in T}$ is an *ε-perturbation* of $\{f_t\}_{t\in T}$ near $\bar{x}$ (with respect to any of these conditions) if and only if $\{f_t\}_{t\in T}$ is an *ε-perturbation* of $\{g_t\}_{t\in T}$.

Since functions f_t and g_t are continuous, condition (C1) implies equality $g_t(\bar{x}) = f_t(\bar{x})$ for all $t \in T$, and consequently equality $g(\bar{x}) = f(\bar{x})$ and the inequality in (C2). Hence (C1) $\Rightarrow$ (C2). Obviously (C3) $\Rightarrow$ (C2) and conversely, if $\{g_t\}_{t\in T}$ is an *ε-perturbation* of $\{f_t\}_{t\in T}$ near $\bar{x}$ in the sense of (C2) then for any $\xi > \varepsilon$ there exists a $\delta > 0$ such that $\{g_t\}_{t\in T}$ is a *ξ-perturbation* of $\{f_t\}_{t\in T}$ in the sense of (C3).

Condition (C1) looks like a natural generalization of condition (6).

Proposition 2. *Let $\varepsilon > 0$. If $\{g_t\}_{t\in T}$ satisfies (C1) then $g \in \operatorname{Ptb}(f, \bar{x}, \varepsilon)$ and $\partial g(\bar{x}) \subseteq \partial f(\bar{x}) + \varepsilon B^*$.*

Proof. By (C1), we have $g(\bar{x}) = f(\bar{x})$, and for any $\varepsilon' > \varepsilon$ there exists a $\delta > 0$ such that

$$|g_t(x) - f_t(x)| \leq \varepsilon'\|x - \bar{x}\| \quad \text{for all } x \in B_\delta(\bar{x}),\ t \in T,$$

and consequently,

$$|g(x) - f(x)| \leq \varepsilon'\|x - \bar{x}\| \quad \text{for all } x \in B_\delta(\bar{x}).$$

Since ε' can be taken arbitrarily close to ε this condition implies (6). The inclusion follows from Proposition 1 (i). □

The following denotation is used in the sequel:

$$T_f(x) := \{t \in T :\ f_t(x) = f(x)\}. \tag{12}$$

Remark 3. Proposition 2 remains valid if instead of (C1) one employs the weaker set of conditions:

$$\limsup_{x\to\bar{x}} \sup_{t\in T_f(\bar{x})} \frac{|g_t(x) - f_t(x)|}{\|x - \bar{x}\|} \leq \varepsilon,$$

$$T_g(\bar{x}) = T_f(\bar{x}).$$

The requirement that $g_t(\bar{x}) = f_t(\bar{x})$ for all $t \in T$ (even for all $t \in T_f(\bar{x})$) can be too restrictive for applications. That is why conditions (C2) and (C3) can be of interest. Unlike condition (C1), these weaker properties are not sufficient in general to guarantee the conclusions of Proposition 2. Some additional assumptions are needed.

From now on in this section we limit ourselves to considering convex functions. We shall denote by $\mathcal{G}(X,T)$ the class of all families $\{f_t\}_{t\in T}$ of convex continuous functions $f_t : X \to \mathbb{R}$ such that $t \mapsto f_t(x)$ is continuous on T for each $x \in X$. For $\{f_t\}_{t\in T} \in \mathcal{G}(X,T)$ the convex continuous function f and the set $T_f(x)$ are defined by (11) and (12) respectively.

Under the assumptions made, if $\{f_t\}_{t\in T} \in \mathcal{G}(X,T)$ then the directional derivative $f'(\bar{x};\cdot)$ of f at $\bar{x}$ is given by the formula (see, for instance, [15, Theorem 4.2.3], [24, Proposition 4.5.2]):

$$f'(\bar{x};x) = \sup_{t\in T_f(\bar{x})} f'_t(\bar{x};x), \quad x \in X. \tag{13}$$

Given two families $\{f_t\}_{t\in T}, \{g_t\}_{t\in T} \in \mathcal{G}(X,T)$ denote

$$\alpha_{f,g}(x) := f(x) - \inf_{t\in T_g(x)} f_t(x). \tag{14}$$

Obviously $\alpha_{f,g}(x) \geq 0$, and $\alpha_{f,g}(x) = 0$ if and only if $T_g(x) \subseteq T_f(x)$. Note that in general $\alpha_{g,f}(x) \neq \alpha_{f,g}(x)$.

Proposition 3. *Let $\varepsilon > 0$ and families $\{f_t\}_{t\in T}, \{g_t\}_{t\in T} \in \mathcal{G}(X,T)$ satisfy condition (C2). The following assertions hold true:*

(i). $\partial g(\bar{x}) \subseteq \bigcup_{\delta>0} \bigcap_{0<\rho<\delta} \left[\bigcup_{x\in B_\rho(\bar{x})} \partial f(x) + (\varepsilon + \alpha_{f,g}(\bar{x})/\rho)B^* \right]$*;*

(ii). if $\alpha_{f,g}(\bar{x}) = 0$, that is $T_g(\bar{x}) \subseteq T_f(\bar{x})$, then $\partial g(\bar{x}) \subseteq \partial f(\bar{x}) + \varepsilon B^*$;

(iii). if condition (C3) is satisfied with some $\delta > 0$ then

$$\partial g(\bar{x}) \subseteq \bigcap_{0<\rho<\delta} \left[\bigcup_{x\in B_\rho(\bar{x})} \partial f(x) + (\varepsilon + \alpha_{f,g}(\bar{x})/\rho)B^* \right];$$

(iv). if in (iii) $\delta > \epsilon := \sqrt{\alpha_{f,g}(\bar{x})}$ then $\partial g(\bar{x}) \subseteq \bigcup_{x\in B_\epsilon(\bar{x})} \partial f(x) + (\varepsilon + \epsilon)B^*$.

Proof. We first prove (ii). It follows from (C2) that

$$|g'_t(\bar{x};x) - f'_t(\bar{x};x)| \leq \varepsilon\|x\| \quad \text{for all } x \in X,\ t \in T,$$

and consequently (using (13) and inclusion $T_g(\bar{x}) \subseteq T_f(\bar{x})$)

$$g'(\bar{x};x) = \sup_{t\in T_g(\bar{x})} g'_t(\bar{x};x) \leq \sup_{t\in T_f(\bar{x})} f'_t(\bar{x};x) + \varepsilon\|x\| = f'(\bar{x};x) + \varepsilon\|x\| \quad \text{for all } x \in X.$$

The conclusion follows immediately.

(iii). If $\alpha_{f,g}(\bar{x}) = 0$ then the assertion follows from (ii). Let $\alpha_{f,g}(\bar{x}) > 0$ and $u^* \in \partial g(\bar{x})$. By (C3),

$$\sup_{t\in T} |(g_t(x) - f_t(x)) - (g_t(\bar{x}) - f_t(\bar{x}))| \le \varepsilon \|x - \bar{x}\| \quad \text{for all } x \in B_\delta(\bar{x}).$$

Then for any $x \in B_\delta(\bar{x})$ one has

$$\begin{aligned} \langle u^*, x - \bar{x}\rangle \le g'(\bar{x}; x - \bar{x}) &= \sup_{t\in T_g(\bar{x})} g'_t(\bar{x}; x - \bar{x}) \\ &\le \sup_{t\in T_g(\bar{x})} (g_t(x) - g_t(\bar{x})) \le \sup_{t\in T_g(\bar{x})} (f_t(x) - f_t(\bar{x})) + \varepsilon\|x - \bar{x}\|. \end{aligned}$$

At the same time,

$$\sup_{t\in T_g(\bar{x})} (f_t(x) - f_t(\bar{x})) \le \sup_{t\in T_g(\bar{x})} f_t(x) - f(\bar{x}) + \alpha_{f,g}(\bar{x}) \le f(x) - f(\bar{x}) + \alpha_{f,g}(\bar{x}).$$

Hence, the continuous convex function

$$\varphi(x) := f(x) - \langle u^*, x - \bar{x}\rangle + \varepsilon\|x - \bar{x}\|, \quad x \in X,$$

satisfies

$$\varphi(\bar{x}) \le \inf_{x\in B_\delta(\bar{x})} \varphi(x) + \alpha_{f,g}(\bar{x}).$$

By virtue of the Ekeland variational principle, for any $\rho \in (0, \delta)$ we can find an $\hat{x} \in B_\rho(\bar{x})$ such that

$$\varphi(x) + (\alpha_{f,g}(\bar{x})/\rho)\|x - \hat{x}\| \ge \varphi(\hat{x}) \quad \text{for all } x \in B_\delta(\bar{x}).$$

Since $\rho < \delta$ it follows that

$$0 \in \partial\varphi(\hat{x}) + (\alpha_{f,g}(\bar{x})/\rho)B^* = \partial f(\hat{x}) - u^* + (\varepsilon + \alpha_{f,g}(\bar{x})/\rho)B^*.$$

Thus, $u^* \in \partial f(\hat{x}) + (\varepsilon + \alpha_{f,g}(\bar{x})/\rho)B^*$.

Assertion (i) follows from (iii) since condition (C2) implies (C3) with a greater ε and some $\delta > 0$.

If $\alpha_{f,g}(\bar{x}) = 0$ then assertion (iv) coincides with (ii), otherwise it is a particular case of (iii) with $\rho = \epsilon$. □

Remark 4. Analyzing the proof of Proposition 3 (iii) one can easily notice that it remains true if the inequality in condition (C3) is replaced by a weaker one:

$$\sup_{x\in B_\delta(\bar{x})\setminus\{\bar{x}\},\, t\in T_g(\bar{x})} \frac{|(g_t(x) - f_t(x)) - (g_t(\bar{x}) - f_t(\bar{x}))|}{\|x - \bar{x}\|} \le \varepsilon.$$

Furthermore, since the assertion establishes a "one-sided relation" (inclusion), it is sufficient to require a one-sided estimate:

$$\sup_{x\in B_\delta(\bar{x})\setminus\{\bar{x}\},\, t\in T_g(\bar{x})} \frac{(g_t(x) - f_t(x)) - (g_t(\bar{x}) - f_t(\bar{x}))}{\|x - \bar{x}\|} \le \varepsilon.$$

Similarly, the inequality in condition (C2) can be replaced by the following one:

$$\limsup_{x\to\bar{x}} \sup_{t\in T_g(\bar{x})} \frac{(g_t(x) - f_t(x)) - (g_t(\bar{x}) - f_t(\bar{x}))}{\|x - \bar{x}\|} \le \varepsilon.$$

Remark 5. The number $\alpha_{f,g}(\bar{x})$ in Proposition 3 is determined by the values $f(\bar{x})$ and $f_t(\bar{x})$. Note that it also depends on the other family of functions $\{g_t\}$ since the infimum in its definition is taken over $t \in T_g(\bar{x})$. If $g(\bar{x}) = f(\bar{x})$ which is a part of any of the conditions (C1), (C2), and (C3), then $\alpha_{f,g}(\bar{x})$ can be rewritten equivalently as

$$\alpha_{f,g}(\bar{x}) = \sup_{t \in T_g(\bar{x})} (g_t(\bar{x}) - f_t(\bar{x})).$$

Proposition 3 yields certain relations between distances from zero to subdifferentials of g and f and their boundaries.

Proposition 4. *Let the conditions of Proposition 3 be satisfied. The following assertions hold true:*

(i). if $0 \in \partial g(\bar{x})$ *and* $T_g(\bar{x}) \subseteq T_f(\bar{x})$ *then* $d(0, \operatorname{Bdry} \partial g(\bar{x})) \le d(0, \operatorname{Bdry} \partial f(\bar{x})) + \varepsilon$;

(ii). $d(0, \partial g(\bar{x})) \ge \inf_{\delta > 0} \sup_{0<\rho<\delta} \left[\inf_{x \in B_\rho(\bar{x})} d(0, \partial f(x)) - \alpha_{f,g}(\bar{x})/\rho \right] - \varepsilon$;

(iii). if condition (C3) is satisfied with some $\delta > 0$ *then*

$$d(0, \partial g(\bar{x})) \ge \sup_{0<\rho<\delta} \left[\inf_{x \in B_\rho(\bar{x})} d(0, \partial f(x)) - \alpha_{f,g}(\bar{x})/\rho \right] - \varepsilon;$$

(iv). if in (iii) $\delta > \epsilon := \sqrt{\alpha_{f,g}(\bar{x})}$ *then* $d(0, \partial g(\bar{x})) \ge \inf_{x \in B_\epsilon(\bar{x})} d(0, \partial f(x)) - \varepsilon - \epsilon$;

(v). if $0 \notin \partial f(\bar{x})$ *then for sufficiently small* δ *the subdifferentials in (iii) and (iv) can be replaced by their boundaries.*

Proof. (i). Denote $r = d(0, \operatorname{Bdry} \partial g(\bar{x}))$. If $r \le \varepsilon$ the assertion is trivial. If $r > \varepsilon$ then $rB^* \subseteq \partial g(\bar{x})$ and, due to Proposition 3 (ii), $(r - \varepsilon)B^* \subseteq \partial f(\bar{x})$. Hence $0 \in \operatorname{int} \partial f(\bar{x})$ and $d(0, \operatorname{Bdry} \partial f(\bar{x})) \ge r - \varepsilon$.

The estimates in (ii), (iii), and (iv) follow from Proposition 3 (i), (iii), and (iv) respectively.

(v). If $0 \notin \partial f(\bar{x})$ then $0 \notin \partial f(x)$ and consequently $d(0, \partial f(x)) = d(0, \operatorname{Bdry} \partial f(x))$ for all x near $\bar{x}$. If the estimate in (iii) (or (iv)) is nontrivial, that is the right-hand side of the corresponding inequality is positive, then it implies $d(0, \partial g(\bar{x})) > 0$, and consequently $0 \notin \partial g(\bar{x})$ and $d(0, \partial g(\bar{x})) = d(0, \operatorname{Bdry} \partial g(x))$. □

In Proposition 4 (i) the subdifferentials of f and g are computed at $\bar{x}$. In all other assertions in Proposition 4 the subdifferentials of f are computed at nearby points, which is not exactly what is needed for establishing stability of error bounds estimates. Fortunately Proposition 4 (iv) allows us to establish the desired estimate in terms of $\partial f(\bar{x})$.

Proposition 5. *Let* $\{f_t\}_{t \in T} \in \mathcal{G}(X, T)$*. Then for any* $\xi > 0$ *and* $\delta > 0$ *there exists an* $\varepsilon > 0$ *such that for all* $\{g_t\}_{t \in T} \in \mathcal{G}(X, T)$ *satisfying condition (C3) and*

$$\alpha_{f,g}(\bar{x}) \le \varepsilon \tag{15}$$

it holds

$$d(0, \partial g(\bar{x})) \ge d(0, \partial f(\bar{x})) - \xi.$$

Proof. Let $\xi > 0$ be given. Due to the upper semicontinuity of the subdifferential mapping, there exists an $\eta > 0$ such that $d(0, \partial f(x)) > d(0, \partial f(\bar{x})) - \xi/3$ for all $x \in B_\eta(\bar{x})$. Take a positive $\varepsilon < \min(\xi/3, \xi^2/9, \delta^2, \eta^2)$. If the family $\{g_t\}_{t\in T} \in \mathcal{G}(X, T)$ satisfies the assumptions of the proposition then $\epsilon := \sqrt{\alpha_{f,g}(\bar{x})} < \min(\delta, \eta, \xi/3)$ and it follows from Proposition 4 (iv) that

$$d(0, \partial g(\bar{x})) \geq \inf_{x\in B_\epsilon(\bar{x})} d(0, \partial f(x)) - \varepsilon - \epsilon \geq d(0, \partial f(\bar{x})) - \xi.$$

□

The following theorem gives a characterization of the stability of local error bounds for system (5).

Theorem 3. *Let $\{f_t\}_{t\in T} \in \mathcal{G}(X, T)$ and $f(\bar{x}) = 0$. The following assertions hold true:*

(i). If $0 \leq \tau < \varsigma(f, \bar{x})$, $\delta > 0$ then there exists an $\varepsilon > 0$ such that for any $\{g_t\}_{t\in T} \in \mathcal{G}(X, T)$ satisfying

(a) condition (C2) and $T_f(\bar{x}) \subseteq T_g(\bar{x})$ if $0 \in \operatorname{int} \partial f(\bar{x})$, or

(b) conditions (C3) and (15), otherwise,

one has $\operatorname{Er} g(\bar{x}) \geq \tau$.

(ii). If $\varsigma(f, \bar{x}) = 0$ then for any $\varepsilon > 0$ there exists a family $\{g_t\}_{t\in T} \in \mathcal{G}(X, T)$ satisfying condition (C1) such that $\operatorname{Er} g(\bar{x}) \leq \varepsilon$.

Proof. (i). Let $0 \leq \tau < \varsigma(f, \bar{x})$, $\{g_t\}_{t\in T} \in \mathcal{G}(X, T)$. By the definition of $\varsigma(f, \bar{x})$, $0 \notin \operatorname{Bdry} \partial f(\bar{x})$, that is, either $0 \in \operatorname{int} \partial f(\bar{x})$ or $0 \notin \partial f(\bar{x})$.

If $0 \in \operatorname{int} \partial f(\bar{x})$, then it is sufficient to take $\varepsilon = \varsigma(f, \bar{x}) - \tau$. Indeed, by the definition of $\varsigma(f, \bar{x})$ one has $\varsigma(f, \bar{x})B^* \subseteq \partial f(\bar{x})$. If condition (C2) is satisfied and $T_f(\bar{x}) \subseteq T_g(\bar{x})$ then, by Proposition 3 (ii), $\varsigma(f, \bar{x})B^* \subseteq \partial g(\bar{x}) + \varepsilon B^*$. It follows that $\tau B^* \subseteq \partial g(\bar{x})$, and consequently, by Theorem 1, $\operatorname{Er} g(\bar{x}) \geq \varsigma(g, \bar{x}) \geq \tau$.

Suppose now $0 \notin \partial f(\bar{x})$. Then $\varsigma(f, \bar{x}) = d(0, \partial f(\bar{x}))$. Take any $\xi \in (0, \varsigma(f, \bar{x}) - \tau)$. Proposition 5 implies the existence of an $\varepsilon > 0$ such that for any $\{g_t\}_{t\in T} \in \mathcal{G}(X, T)$ satisfying conditions (C3) and (15) it holds

$$d(0, \partial g(\bar{x})) \geq d(0, \partial f(\bar{x})) - \xi > \tau.$$

Hence $0 \notin \partial g(\bar{x})$ and, by Theorem 1, $\operatorname{Er} g(\bar{x}) \geq d(0, \partial g(\bar{x})) > \tau$.

(ii). By Theorem 2 (ii), for any $\varepsilon > 0$ there exists a $g \in \Gamma_0(X) \cap \operatorname{Ptb}(f, \bar{x}, \varepsilon)$, given by (9), such that $\operatorname{Er} g(\bar{x}) < \varepsilon$. Since f is continuous, g is continuous too. For $t \in T$ and $x \in X$, set $g_t(x) = f_t(x) + g(x) - f(x)$. Then $\{g_t\}_{t\in T} \in \mathcal{G}(X, T)$, $g(x) = \sup_{t\in T} g_t(x)$, and condition (C1) is satisfied. □

Remark 6. Due to the equivalent representation of $\alpha_{f,g}(\bar{x})$ formulated in Remark 5, condition (15) in Theorem 3 is equivalent to the following one

$$\sup_{t\in T_g(\bar{x})} (g_t(\bar{x}) - f_t(\bar{x})) \leq \varepsilon.$$

The last inequality is obviously ensured by a stronger condition from [19, Theorem 3]:

$$\sup_{t\in T} |g_t(\bar{x}) - f_t(\bar{x})| \leq \varepsilon.$$

The next corollary strengthens [19, Theorem 3].

Corollary 3.1. *Let $\{f_t\}_{t\in T} \in \mathcal{G}(X,T)$, and $f(\bar{x}) = 0$. The following properties are equivalent:*

(i). there exists an $\varepsilon > 0$ such that $\operatorname{Er} g(\bar{x}) \geq \varepsilon$ for any $\{g_t\}_{t\in T} \in \mathcal{G}(X,T)$ satisfying the conditions in Theorem 3 (i);

(ii). $0 \notin \operatorname{Bdry} \partial f(\bar{x})$.

Remark 7. The inclusion $T_f(\bar{x}) \subseteq T_g(\bar{x})$ in Theorem 3 (b) cannot be dropped – see [19, Remark 5].

3 Stability of global error bounds

In this section, we deal with the error bound property of the set $S_f = \{x \in X : f(x) \leq 0\}$ without relating it to a particular point $\bar{x} \in S_f^= := \{x \in X : \; f(x) = 0\}$. The next theorem represents a nonlocal analog of Theorem 1.

Theorem 4. *Let $f \in \Gamma_0(X)$, $S_f \neq \emptyset$. Consider the following properties:*

(i). f admits a global error bound, that is, $\operatorname{Er} f > 0$*;*

(ii). $\tau(f) := \inf\limits_{f(x)>0} d(0, \partial f(x)) > 0$*;*

(iii). $\varsigma(f) := \inf\limits_{f(x)=0} d(0, \operatorname{Bdry} \partial f(x)) > 0$*;*

(iv). $\inf\limits_{f(x)=0} d(0, \partial f(x)) > 0$*;*

(v). $0 \in \operatorname{int} \partial f(x)$ *for some x such that $f(x) = 0$.*

Each of the properties (ii)–(v) is sufficient for the error bound property (i). Moreover,

(a). $\varsigma(f) \leq \tau(f) = \operatorname{Er} f$*;*

(b). [(iv) or (v)] $\Leftrightarrow$ (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i).

Proof. The equality in (a) is well known (see [4, Theorem 3.1]), [27, Theorem 7].

If $0 \in \operatorname{int} \partial f(\bar{x})$ for some $\bar{x} \in S_f^=$ then $S_f = \{\bar{x}\}$, $\varsigma(f) = \varsigma(f, \bar{x})$, and $\varsigma(f)B^* \subseteq \partial f(\bar{x})$. It follows that

$$f(x) \geq \varsigma(f)\|x - \bar{x}\|.$$

On the other hand, if $x^* \in \partial f(x)$ for some $x \neq \bar{x}$ then

$$-f(x) \geq \langle x^*, \bar{x} - x \rangle.$$

Adding the last two inequalities together we obtain

$$\langle x^*, x - \bar{x} \rangle \geq \varsigma(f)\|x - \bar{x}\|.$$

Hence, $\|x^*\| \geq \varsigma(f)$, and consequently, $\tau(f) \geq \varsigma(f)$.

If $0 \notin \operatorname{int} \partial f(u)$ for all $u \in S_f^=$ then $\varsigma(f) = \inf_{f(u)=0} d(0, \partial f(u))$ and the inequality in (a) follows from [4, Theorem 3.2].

The equivalence [(iv) or (v)] $\Leftrightarrow$ (iii) in (b) is obvious, while the other one (ii) $\Leftrightarrow$ (i) and the implication (iii) $\Rightarrow$ (ii) follow from (a). □

Examples 1 and 2 in Section 2 are also applicable to global error bounds to show that inequality (a) and implication (iii) $\Rightarrow$ (ii) in Theorem 4 can be strict. Theorem 4 (a) guarantees that $\|x^*\| \geq \varsigma(f)$ for any $x^* \in \partial f(x)$ with $f(x) > 0$. The next *Asymptotic qualification condition* $(\mathcal{A}QC)$ ensures the limiting form of this estimate also for elements of $\partial f(x)$ at some points x with $f(x) < 0$.

$(\mathcal{A}QC)$ $\liminf_{k\to\infty} \|x_k^*\| \geq \varsigma(f)$ for any sequences $x_k \in X$ with $f(x_k) < 0$ and $x_k^* \in \partial f(x_k)$, $k = 1, 2, \ldots$, satisfying

(a) ether the sequence $\{x_k\}$ is bounded and $\lim_{k\to\infty} f(x_k) = 0$,

(b) or $\lim_{k\to\infty} \|x_k\| = \infty$ and $\lim_{k\to\infty} f(x_k)/\|x_k\| = 0$.

Note that if $\dim X < \infty$ and S_f is closed then part (a) of $(\mathcal{A}QC)$ automatically implies $\liminf_{k\to\infty} \|x_k^*\| \geq \varsigma(f)$ and consequently $(\mathcal{A}QC)$ coincides with the asymptotic qualification condition introduced in [19].

Proposition 6. *Let $f \in \Gamma_0(X)$, $S_f \neq \emptyset$, and $0 \notin \operatorname{int} \partial f(x)$ for all $x \in S_f^=$. Suppose that $(\mathcal{A}QC)$ holds true. Then*

$$\varsigma(f) = \sup_{\varepsilon>0} \inf_{f(x)>-\varepsilon\|x-x_0\|-\varepsilon} d(0, \partial f(x))$$

for any $x_0 \in X$.

Proof. Since $0 \notin \operatorname{int} \partial f(x)$ for all $x \in S_f^=$, thanks to Theorem 4(a) we have

$$\varsigma(f) = \inf_{f(x)=0} d(0, \partial f(x)) = \inf_{f(x)\geq 0} d(0, \partial f(x)).$$

Hence,

$$\sup_{\varepsilon>0} \inf_{f(x)>-\varepsilon\|x-x_0\|-\varepsilon} d(0, \partial f(x)) \leq \varsigma(f). \tag{16}$$

Consider sequences $\varepsilon_k \downarrow 0$, $x_k \in X$, and $x_k^* \in \partial f(x_k)$, $k = 1, 2, \ldots$, such that

$$-\varepsilon_k\|x_k - x_0\| - \varepsilon_k < f(x_k) < 0. \tag{17}$$

If $\{x_k\}$ is bounded then it follows from (17) that $f(x_k) \to 0$, and consequently part (a) of $(\mathcal{A}QC)$ is satisfied. If $\{x_k\}$ is unbounded we can assume that $\|x_k\| \to \infty$. Then (17) implies that $\lim_{k\to\infty} f(x_k)/\|x_k\| = 0$, that is, part (b) of $(\mathcal{A}QC)$ is satisfied. In both cases, thanks to $(\mathcal{A}QC)$, $\liminf_{k\to\infty} \|x_k^*\| \geq \varsigma(f)$. Together with (16) this proves the assertion. □

Condition (iii) in Theorem 4 corresponds to the existence of a global error bound for a family of functions being small perturbations of f.

Definition 2. Let $S_f \neq \emptyset$ and $\varepsilon \geq 0$. We say that $g : X \to \mathbb{R}_\infty$ is an *ε-perturbation* of f relative to $x_0 \in \operatorname{dom} f$ and write $g \in \operatorname{Ptb}_{x_0}(f, \varepsilon)$ if $S_g \neq \emptyset$ and

$$|g(x_0) - f(x_0)| \leq \varepsilon, \tag{18}$$

$$\sup_{x \neq u} \frac{|(g(x) - f(x)) - (g(u) - f(u))|}{\|x - u\|} \leq \varepsilon. \tag{19}$$

Condition (19) constitutes the Lipschitz continuity (with modulus ε) property of the difference $g - f$. Unlike the case of Definition 1, this condition does not involve the reference point x_0 which participates only in Condition (18).

Clearly $g \in \operatorname{Ptb}_{x_0}(f, \varepsilon) \Rightarrow g + f(x_0) - g(x_0) \in \operatorname{Ptb}(f, x_0, \varepsilon)$. Basically an ε-perturbation is a result of a small shift and a small rotation of the original function. Note that if there is an $x \in X$ such that $f(x) < 0$ (in particular, if $0 \notin \partial f(\bar{x})$ for some $\bar{x} \in S_f^=$), then condition $S_g \neq \emptyset$ in Definition 2 is satisfied automatically for a sufficiently small ε.

Theorem 5. *Let $f \in \Gamma_0(X)$, $S_f \neq \emptyset$, and $x_0 \in \operatorname{dom} f$. The following assertions hold true:*

(i). Suppose that either $0 \in \operatorname{int} \partial f(\bar{x})$ for some $\bar{x} \in S_f^=$ or $(\mathcal{AQC})$ holds. If $0 \leq \tau < \varsigma(f)$ then there exists an $\varepsilon > 0$ such that $\operatorname{Er} g \geq \tau$ for any $g \in \Gamma_0(X) \cap \operatorname{Ptb}_{x_0}(f, \varepsilon)$.

(ii). Suppose that $\varsigma(f) = 0$. For $\varepsilon > 0$ and $\bar{x} \in S_f^=$ set

$$\xi = \varepsilon \min(1/2, \|x_0 - \bar{x}\|^{-1}), \tag{20}$$

$$g(u) := f(u) + \xi\|u - \bar{x}\|, \quad u \in X. \tag{21}$$

Then $g \in \Gamma_0(X) \cap \operatorname{Ptb}_{x_0}(f, \varepsilon)$ and $\operatorname{Er} g \leq \varepsilon$.

(iii). Suppose that $\dim X < \infty$ and $\varsigma(f) = 0$. For $\varepsilon > 0$ and $\bar{x} \in S_f^=$ let $\xi > 0$ be defined by (20). Then there exists an $x^* \in \xi B^*$ such that the function g in assertion (ii) can be replaced by the following one:

$$g(u) := f(u) + \langle x^*, u - \bar{x} \rangle, \quad u \in X. \tag{22}$$

Proof. (i). Let $0 \leq \tau < \varsigma(f)$. If $0 \in \operatorname{int} \partial f(\bar{x})$ for some $\bar{x} \in S_f^=$, then $S_f = \{\bar{x}\}$, and $\varsigma(f) = \varsigma(f, \bar{x})$. Take an $\varepsilon \in (0, \varsigma(f) - \tau)$. If $g \in \Gamma_0(X) \cap \operatorname{Ptb}_{x_0}(f, \varepsilon)$ then $g - g(\bar{x}) \in \operatorname{Ptb}(f, \bar{x}, \varepsilon)$, and it follows from Proposition 1 (iii) that $0 \in \operatorname{int} \partial g(\bar{x})$ and $\varsigma(g - g(\bar{x})) = \varsigma(g, \bar{x}) \geq \varsigma(f) - \varepsilon > \tau$. If $x \neq \bar{x}$ and $x^* \in \partial g(x)$ then, due to Theorem 4, $\|x^*\| > \tau$. Since, by assumption, $S_g \neq \emptyset$, applying Theorem 4 again, we conclude that $\operatorname{Er} g \geq \tau$.

Let $0 \notin \operatorname{int} \partial f(x)$ for all $x \in S_f^=$, $(\mathcal{AQC})$ holds true, and $\tau' \in (\tau, \varsigma(f))$. Then it follows from Proposition 6 that there exists an $\varepsilon \in (0, \tau' - \tau]$ such that $\|x^*\| \geq \tau'$ if $x^* \in \partial f(x)$ and $f(x) > -\varepsilon\|x - x_0\| - \varepsilon$. Consider a function $g \in \Gamma_0(X) \cap \operatorname{Ptb}_{x_0}(f, \varepsilon)$. By definition,

$$f(x) \geq g(x) - \varepsilon\|x - x_0\| - \varepsilon \quad \text{for all} \quad x \in X.$$

Hence, if $g(x) > 0$ then $f(x) > -\varepsilon\|x - x_0\| - \varepsilon$, and consequently

$$d(0, \partial g(x)) \geq d(0, \partial f(x)) - \varepsilon \geq \tau' - \varepsilon \geq \tau.$$

The conclusion follows from Theorem 4.

(ii). Let $\varsigma(f)=0$, $\varepsilon>0$, and $\bar{x}\in S_f^=$. If the function $g\in\Gamma_0(X)$ is defined by (21) then $g-f$ is obviously Lipschitz continuous with modulus ξ and $|g(x_0)-f(x_0)|\le\varepsilon$. Hence $g\in \mathrm{Ptb}_{x_0}(f,\varepsilon)$. We need to show that $\mathrm{Er}\, g\le\varepsilon$.

By definition of $\varsigma(f)$, there exists a $y\in S_f^=$ and an $u^*\in \mathrm{Bdry}\,\partial f(y)$ such that $\|u^*\|<\varepsilon/2$. If it is possible to choose $y\ne\bar{x}$, then $g(y)>0$ and $\tau(g)\le\|u^*\|+\xi<\varepsilon$; thanks to Theorem 4, $\mathrm{Er}\, g<\varepsilon$. Otherwise, $\|u^*\|\ge\varepsilon/2$ for any $u^*\in \mathrm{Bdry}\,\partial f(y)$ with $y\in S_f^=\setminus\{\bar{x}\}$. Since $\varsigma(f)=0$ this means that $0\in \mathrm{Bdry}\,\partial f(\bar{x})$. Then, by Theorem 2 (ii), $\mathrm{Er}\, g\le \mathrm{Er}\, g(\bar{x})\le\xi\le\varepsilon$.

(iii). Let $\dim X<\infty$, $\varsigma(f)=0$, $\varepsilon>0$, and $\bar{x}\in S_f^=$. If in the above proof of (ii) it is possible to choose $y\ne\bar{x}$, then take $y^*\in X^*$ such that $\langle y^*,y-\bar{x}\rangle=\|y-\bar{x}\|$, $\|y^*\|=1$ and set $x^*=\xi y^*$; otherwise apply Theorem 2 (iii) instead of (ii). In both cases, if the function $g\in\Gamma_0(X)$ is defined by (22), then $g\in \mathrm{Ptb}_{x_0}(f,\varepsilon)$ and $\mathrm{Er}\, g\le\varepsilon$. □

Given a function $f\in\Gamma_0(X)$ with $S_f\ne\emptyset$, a point $x_0\in\mathrm{dom}\, f$, and a number $\varepsilon\ge 0$ denote

$$\mathrm{Er}\,\{\mathrm{Ptb}_{x_0}(f,\varepsilon)\}:=\inf_{g\in\Gamma_0(X)\cap \mathrm{Ptb}_{x_0}(f,\varepsilon)}\mathrm{Er}\, g. \tag{23}$$

This number characterizes the error bound property for the whole family of convex ε-perturbations of f relative to x_0. Obviously, $\mathrm{Er}\,\{\mathrm{Ptb}_{x_0}(f,\varepsilon)\}\le \mathrm{Er}\, f$ for any $x_0\in\mathrm{dom}\, f$ and $\varepsilon\ge 0$.

Corollary 5.1. *Let $f\in\Gamma_0(X)$, $S_f\ne\emptyset$, and $x_0\in\mathrm{dom}\, f$. The following assertions hold true:*

(i). Suppose that either $0\in\mathrm{int}\,\partial f(\bar{x})$ for some $\bar{x}\in S_f^=$ or $(\mathcal{A}QC)$ holds. Then $\sup\limits_{\varepsilon>0}\mathrm{Er}\,\{\mathrm{Ptb}_{x_0}(f,\varepsilon)\}\ge\varsigma(f)$.

(ii). If $\varsigma(f)=0$ then $\sup\limits_{\varepsilon>0}\mathrm{Er}\,\{\mathrm{Ptb}_{x_0}(f,\varepsilon)\}=0$.

Due to Corollary 5.1 (i), under the assumption that either $0\in\mathrm{int}\,\partial f(\bar{x})$ for some $\bar{x}\in S_f^=$ or $(\mathcal{A}QC)$ holds, condition $\varsigma(f)>0$ is sufficient for the error bound property of the family of ε-perturbations of f if $\varepsilon>0$ is sufficiently small. If $\varsigma(f)=0$ then, due to Corollary 5.1 (ii), the "uniform" error bound property does not hold.

Corollary 5.2. *Let $f\in\Gamma_0(X)$ and $S_f\ne\emptyset$. Suppose that either $0\in\mathrm{int}\,\partial f(\bar{x})$ for some $\bar{x}\in S_f^=$ or $(\mathcal{A}QC)$ holds. The following properties are equivalent:*

(i). for any $x_0\in\mathrm{dom}\, f$ there exists an $\varepsilon>0$ such that $\mathrm{Er}\,\{\mathrm{Ptb}_{x_0}(f,\varepsilon)\}>0$;

(ii). for some $x_0\in\mathrm{dom}\, f$ there exists an $\varepsilon>0$ such that $\mathrm{Er}\,\{\mathrm{Ptb}_{x_0}(f,\varepsilon)\}>0$;

(iii). $\varsigma(f)>0$.

The next theorem gives a characterization of the stability of global error bounds for the infinite convex constraint system (5). Along with the family of continuous functions $\{f_t\}_{t\in T}$ we consider the function $f:X\to\mathbb{R}_\infty$ and set valued mapping $T_f:X\rightrightarrows T$ defined by (11) and (12) respectively. To formulate stability criteria we need another family of continuous functions $\{g_t\}_{t\in T}$ together with the corresponding mappings g and T_g. Recall that $\mathcal{G}(X,T)$ denotes the class of all families $\{f_t\}_{t\in T}$ of convex continuous functions $f_t : X \to \mathbb{R}$ such that $t \mapsto f_t(x)$ is continuous on T for each $x \in X$. If $\{f_t\}_{t\in T} \in \mathcal{G}(X,T)$ then f is obviously convex continuous too. If $\{f_t\}_{t\in T} \in \mathcal{G}(X,T)$ and $\{g_t\}_{t\in T} \in \mathcal{G}(X,T)$ then $\alpha_{f,g}(x)$ is defined by (14).

Theorem 6. *Let $\{f_t\}_{t\in T} \in \mathcal{G}(X,T)$, $S_f \neq \emptyset$, and $x_0 \in X$. The following assertions hold true:*

(i). If $0 \le \tau < \varsigma(f)$ then there exists an $\varepsilon > 0$ such that for any $\{g_t\}_{t\in T} \in \mathcal{G}(X,T)$ satisfying $S_g \neq \emptyset$ and one of the following two groups of conditions:

(a) if $0 \in \operatorname{int} \partial f(\bar{x})$ for some $\bar{x} \in S_f^=$ then

$$\limsup_{x\to\bar{x}} \sup_{t\in T} \frac{|(g_t(x) - f_t(x)) - (g_t(\bar{x}) - f_t(\bar{x}))|}{\|x - \bar{x}\|} \le \varepsilon, \tag{24}$$

$$T_f(\bar{x}) \subseteq T_g(\bar{x}) \tag{25}$$

(b) if $0 \notin \partial f(x)$ for all $x \in S_f^=$ then $(\mathcal{AQC})$ holds and

$$\sup_{x\neq u,\, t\in T} \frac{|(g_t(x) - f_t(x)) - (g_t(u) - f_t(u))|}{\|x - u\|} \le \varepsilon, \tag{26}$$

$$\sup_{t\in T} |g_t(x_0) - f_t(x_0)| \le \varepsilon, \tag{27}$$

one has $\operatorname{Er} g \ge \tau$.

(ii). If $\varsigma(f) = 0$ then for any $\varepsilon > 0$ there exists a family $\{g_t\}_{t\in T} \in \mathcal{G}(X,T)$, satisfying (25)–(27), such that $\operatorname{Er} g < \varepsilon$.

Proof. (i). Let $0 \le \tau < \varsigma(f)$, $\{g_t\}_{t\in T} \in \mathcal{G}(X,T)$. By the definition of $\varsigma(f)$, two cases are possible.

(a) $\underline{0 \in \operatorname{int} \partial f(\bar{x}) \text{ for some } \bar{x} \in S_f^=}$. Then $S_f = \{\bar{x}\}$, $\varsigma(f) = \varsigma(f,\bar{x})$, and $\varsigma(f)B^* \subseteq \partial f(\bar{x})$. Let $\varepsilon \in (0, \varsigma(f) - \tau)$. If condition (24) is satisfied then the family $\{g_t - g(\bar{x})\}_{t\in T} \in \mathcal{G}(X,T)$ satisfies condition (C2) formulated in Section 2. If additionally condition (25) is satisfied then, by Proposition 3 (ii), $\varsigma(f)B^* \subseteq \partial g(\bar{x}) + \varepsilon B^*$. It follows that $(\varsigma(f)-\varepsilon)B^* \subseteq \partial g(\bar{x})$, and consequently, $S_{g-g(\bar{x})} = \{\bar{x}\}$ and $\varsigma(g-g(\bar{x})) \ge \varsigma(f) - \varepsilon > \tau$. By Theorem 4, $\tau(g - g(\bar{x})) \ge \varsigma(g - g(\bar{x})) > \tau$. Since $S_g \neq \emptyset$, we have $g(\bar{x}) \le 0$, and consequently,

$$\operatorname{Er} g = \tau(g) = \inf_{g(x)>0} d(0, \partial g(x)) \ge \inf_{g(x)>g(\bar{x})} d(0, \partial g(x)) = \tau(g - g(\bar{x})) > \tau.$$

(b) $\underline{0 \notin \partial f(u) \text{ for all } u \in S_f^=}$. Then $\varsigma(f) = \inf_{f(u)=0} d(0, \partial f(u))$. For any $u \in X$ and $t \in T_g(u)$ we have

$$g(u) - f(u) = g_t(u) - \sup_{s\in T} f_s(u) \le g_t(u) - f_t(u) \le |g_t(u) - f_t(u)|,$$

and consequently

$$|g(u) - f(u)| \le \sup_{t\in T} |g_t(u) - f_t(u)|. \tag{28}$$

By definition (14),

$$\begin{aligned}\alpha_{f,g}(u) &= f(u) - \inf_{t\in T_g(u)} f_t(u)\\ &= f(u) - g(u) + \sup_{t\in T_g(u)} (g_t(u) - f_t(u)) \le 2\sup_{t\in T} |g_t(u) - f_t(u)|. \quad (29)\end{aligned}$$

Let $\tau' \in (\tau, \varsigma(f))$. If $(\mathcal{A}QC)$ holds true then it follows from Proposition 6 that, for a sufficiently small $\xi > 0$, one has $\|u^*\| \ge \tau'$ if $u^* \in \partial f(u)$ and $f(u) > -\xi(\|u - x_0\| + 1)$. Choose an $\varepsilon > 0$ satisfying $\sqrt{\varepsilon}(\tau+\varepsilon)+3\varepsilon \le \xi$ and $\varepsilon + 2\sqrt{\varepsilon} \le \tau' - \tau$.

Let $x^* \in \partial g(x)$ for some $x \in X$ with $g(x) > 0$. If condition (26) holds true then the family $\{g_t - g(x)\}_{t\in T} \in \mathcal{G}(X,T)$ satisfies condition (C3) (with $\bar{x} = x$) for any $\delta > 0$. It follows from Proposition 3 (iii) that

$$x^* \in \bigcup_{v\in B_\rho(x)} \partial f(v) + (\varepsilon + \alpha_{f,g}(x)/\rho)B^*, \quad (30)$$

where $\rho := \sqrt{\varepsilon}(\|x - x_0\| + 1)$. Thanks to (30), for any $v \in B_\epsilon(x)$, it holds

$$f(v) - f(x) \ge -(\|x^*\| + \varepsilon + \alpha_{f,g}(x)/\rho)\|v - x\| \ge -\rho(\|x^*\| + \varepsilon) + \alpha_{f,g}(x). \quad (31)$$

If, additionally, condition (27) holds true then it follows from (28) and (29) that

$$f(x) \ge g(x) - \sup_{t\in T} |g_t(x_0) - f_t(x_0)| - \varepsilon\|x - x_0\| > -\varepsilon(\|x - x_0\| + 1), \quad (32)$$

$$\alpha_{f,g}(x) \le 2\sup_{t\in T} |g_t(x_0) - f_t(x_0)| + 2\varepsilon\|x - x_0\| \le 2\varepsilon(\|x - x_0\| + 1). \quad (33)$$

Suppose that $\|x^*\| < \tau$. Then (31), (32), and (33) yield

$$f(v) > -[\sqrt{\varepsilon}(\tau + \varepsilon) + 3\varepsilon](\|x - x_0\| + 1) \ge -\xi(\|x - x_0\| + 1),$$
$$\varepsilon + \alpha_{f,g}(x)/\rho \le \varepsilon + 2\sqrt{\varepsilon} \le \tau' - \tau.$$

Hence, $\|u^*\| \ge \tau'$ for any $u^* \in \partial f(v)$ and, thanks to (30), $\|x^*\| \ge \tau$. By Theorem 4, $\operatorname{Er} g = \tau(g) \ge \tau$.

(ii). By Theorem 5 (ii), for any $\varepsilon > 0$ there exists a $g \in \Gamma_0(X) \cap \operatorname{Ptb}_{x_0}(f,\varepsilon)$, given by (22), such that $\operatorname{Er} g < \varepsilon$. Since f is continuous, g is continuous too. For $t \in T$ and $x \in X$, set $g_t(x) = f_t(x) + g(x) - f(x)$. Then $\{g_t\}_{t\in T} \in \mathcal{G}(X,T)$, $g(x) = \sup_{t\in T} g_t(x)$, $T_f(x) = T_g(x)$, and conditions (26), (27) are satisfied. □

The next corollary strengthens [19, Theorem 7].

Corollary 6.1. *Let $\{f_t\}_{t\in T} \in \mathcal{G}(X,T)$ and $S_f \ne \emptyset$. The following properties are equivalent:*

(i). for any $x_0 \in X$ there exists an $\varepsilon > 0$ such that $\operatorname{Er} g \ge \varepsilon$ for all $\{g_t\}_{t\in T} \in \mathcal{G}(X,T)$ satisfying the conditions in Theorem 6 (i);

(ii). for some $x_0 \in X$ there exists an $\varepsilon > 0$ such that $\operatorname{Er} g \ge \varepsilon$ for all $\{g_t\}_{t\in T} \in \mathcal{G}(X,T)$ satisfying the conditions in Theorem 6 (i);

(iii). $\varsigma(f) > 0$.

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