RC24938 (W1001-089) January 28, 2010
Computer Science

IBM Research Report

Intractability of Approximate Multi-dimensional Nonlinear Optimization on Independence Systems

Jon Lee
IBM Research Division
Thomas J. Watson Research Center
P.O. Box 218
Yorktown Heights, NY 10598, USA

Shmuel Onn
Technion - Israel Institute of Technology
Haifa, Israel

Robert Weismantel
ETH
Zürich, Switzerland

Intractability of approximate multi-dimensional nonlinear optimization on independence systems

Jon Lee Shmuel Onn Robert Weismantel

Abstract

We consider optimization of nonlinear objective functions that balance d linear criteria over n-element independence systems presented by linear-optimization oracles. For $d = 1$, we have previously shown that an r-best approximate solution can be found in polynomial time. Here, using an extended Erdős-Ko-Rado theorem of Frankl, we show that for $d = 2$, finding a ρn-best solution requires exponential time.

1 Introduction

Given system $S \subseteq \{0,1\}^n$, integer $d \times n$ matrix W, and function $f : \mathbb{Z}^d \to \mathbb{Z}$, consider the problem of minimizing the nonlinear composite function $f(Wx)$ over S, that is,

$$\min\{f(Wx) \ : \ x \in S\} \ . \tag{1}$$

This can be interpreted as multi-criteria optimization, where row W_i of W gives a linear function $W_i x$ representing the value of feasible point $x \in S$ under criterion i, and the objective-function value $f(Wx) = f(W_1 x, \ldots, W_d x)$ is a balancing of these d criteria.

We assume that we can do linear optimization over S to begin with; namely S is presented by a *linear-optimization oracle*, which when queried on $w \in \mathbb{Z}^n$, solves $\max\{wx \ : \ x \in S\}$. For restricted systems S, such as matroids and matroid intersections, or restricted functions f, such as concave functions, problem (1) can be solved in polynomial time (see [1, 2]). A comprehensive description of the state of the art on this area can be found in [5].

Here, we continue our investigation from [4] of problem (1), where S is an arbitrary *independence system*, that is, S is nonempty, and $x \leq y \in S$ with $x \in \{0,1\}^n$ imply $x \in S$.

A feasible point $x^* \in S$ is called an *r-best solution* of problem (1) if there are at most r better objective-function values attainable by other feasible points, that is,

$$|\{f(Wx) \ : \ f(Wx) < f(Wx^*) \, , \ x \in S\}| \ \leq \ r \ .$$

So it provides a suitable approximation to (1). In particular, a 0-best solution is optimal.

In [4], we considered the case of $d = 1$, that is, the problem $\min\{f(wx) : x \in S\}$ with $w \in \mathbb{Z}^n$. We showed that for any fixed positive integers $a_1, \ldots, a_p$ there is a polynomial-time algorithm that, given any $w \in \{a_1, \ldots, a_p\}^n$, provides an $r(a_1, \ldots, a_p)$-best solution to the problem, where $r(a_1, \ldots, a_p)$ is a constant related to Frobenius numbers of some of the a_i. In particular, for any $p = 2$ integers, $r(a_1, a_2) = \mathrm{F}(a)$ is the Frobenius number.

In this note, we consider the problem in dimension $d = 2$. We restrict attention to $2 \times n$ matrices W that are $\{0,1\}$-valued. Then the *image* of S under W satisfies

$$WS \ := \ \{Wx \,:\, x \in S\} \ \subseteq \ \{0,1,\ldots,n\}^2 \ . \tag{2}$$

Therefore, the problem of computing the optimal objective-function *value* of (1) is seemingly reducible to computing the image WS by checking if $y \in WS$ for each of the $(n+1)^2$ points y in $\{0,1,\ldots,n\}^2$, and determining the minimum value of f over WS. Unfortunately, this so called *fiber problem*, of checking if $y \in WS$, is computationally hard. In particular, already for S the set of (indicators of) matchings in a bipartite graph, over which linear optimization is easy, this problem includes as a special case the notorious *exact matching problem* whose complexity is long open (see [6]).

Here we show that there is a universal positive constant ρ such that, already for $d = 2$, matrix W each column of which is one of the two unit vectors in $\mathbb{Z}^2$, and a very simple explicit function f supported on $\{0,1,\ldots,n\}^2$, there is no polynomial-time algorithm that can produce even a ρn-best solution of problem (1) for every independence system $S \subseteq \{0,1\}^n$, let alone find a constant r-best or optimal solution. Our construction makes use of a beautiful extension of the classical Erdős-Ko-Rado theorem due to Frankl (see [3]).

It would be interesting to know whether our construction can be refined to shed some light on the exact matching and related open problems of [6], and whether other natural oracles for S could lead to polynomial-time solution of problem (1) in dimensions $d = 2$ and higher.

2 A ρn-best solution cannot be found in polynomial time

Theorem 2.1. *There exists a universal positive constant ρ such that no polynomial-time algorithm can compute a ρn-best solution of the 2-dimensional nonlinear optimization problem* $\min\{f(Wx) \,:\, x \in S\}$ *over every independence system $S \subseteq \{0,1\}^n$ presented by a linear-optimization oracle, with W an integer $2 \times n$ weight matrix each column of which is one of the unit vectors in $\mathbb{Z}^2$, and f an explicit function supported on $\{0,1,\ldots,n\}^2$.*

In fact, the following explicit statement holds. Let l be any positive integer with $l \geq 2^{10}$, $k := 7l$, $m := 8l^2$, $n := 2m$, and $\rho := \frac{1}{17}$. Let W be the $2 \times n$ matrix with first m columns the unit vector $\mathbf{1}_1$ and last m columns the unit vector $\mathbf{1}_2$. Define f on $\mathbb{Z}^2$ explicitly by

$$f(y) \ = \ f(y_1, y_2) \ := \ \begin{cases} (y_1 - k) - l(y_2 - k) - 1 & \text{if } k+1 \leq y_1, y_2 \leq k+l \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{3}$$

Then at least $2^{\frac{1}{4}\sqrt{n}}$ queries to the oracle of S are needed to compute a $\frac{1}{17}n$-best solution.

Proof. Let $l \geq 2^{10}$ be a positive integer, k, m, n, ρ and W as above, and f as in (3) above. It is more convenient here to work with set systems over ground set $N := \{1,\ldots,n\}$ rather than sets of vectors in $\{0,1\}^n$. As usual, vectors $x \in \{0,1\}^n$ are in bijection with subsets $X \subseteq N$, with corresponding elements satisfying $X = \mathrm{supp}(x)$ the support of x, and $x = \mathbf{1}_X$ being the indicator of X. So we replace each $S \subseteq \{0,1\}^n$ by the set system $\mathcal{S} := \{X = \mathrm{supp}(x) \,:\, x \in S\}$. Also, for $c \in \mathbb{Z}^n$ and $X \subseteq N$, we write

$cX := c\mathbf{1}_X$. Let $N_1 \uplus N_2 = N$ be the natural equipartition of the ground set defined by $N_1 := \{1, \ldots, m\}$ and $N_2 := \{m+1, \ldots, 2m\}$. For each subset $X \subseteq N$ of the ground set, we write $X_1 := X \cap N_1$, $X_2 := X \cap N_2$, with $X = X_1 \uplus X_2$ the naturally induced partition of X.

The image of $X = X_1 \uplus X_2$ is denoted by $WX := W\mathbf{1}_X$ and is equal to $(|X_1|, |X_2|)$. The image of a set system $\mathcal{S}$ over N is $W\mathcal{S} := \{WX : X \in \mathcal{S}\}$. We use several set systems over N, defined as follows. First, for each pair of integers $0 \leq y_1, y_2 \leq m$, let

$$\mathcal{S}_{y_1,y_2} \ := \ \{X = X_1 \uplus X_2 \ : \ |X_1| = y_1 \, , \ |X_2| = y_2\} \ .$$

Next, let

$$\mathcal{S}^* \ := \ \{X \ : \ (|X_1|, |X_2|) \leq (m, k) \ \text{ or } \ (|X_1|, |X_2|) \leq (k, m)\} \ .$$

Then $\mathcal{S}^*$ is an independence system whose image is given by

$$W\mathcal{S}^* \ = \ \left\{(y_1, y_2) \in \mathbb{Z}_+^2 \ : \ (y_1, y_2) \leq (m, k) \ \text{ or } \ (y_1, y_2) \leq (k, m)\right\} \ .$$

Moreover, the objective-function value of every $X \in \mathcal{S}^*$, and hence in particular of every ρn-best solution of the minimization problem over $\mathcal{S}^*$, satisfies $f(WX) = 0$.

Next, for each $Y \in \mathcal{S}_{k+l,k+l}$, let

$$\mathcal{S}_Y \ := \ \mathcal{S}^* \ \cup \ \{X \ : \ X \subseteq Y\} \ .$$

Then $\mathcal{S}_Y$ is also an independence system, with image

$$W\mathcal{S}_Y \ = \ W\mathcal{S}^* \ \uplus \ \{(y_1, y_2) \ : \ (k+1, k+1) \leq (y_1, y_2) \leq (k+l, k+l)\} \ .$$

Moreover, the objective-function values of the points in $\mathcal{S}_Y \setminus \mathcal{S}^*$, whose images lie in $W\mathcal{S}_Y \setminus W\mathcal{S}^*$, attain exactly all $l^2 = \frac{1}{16}n > \rho n$ values $-1, -2, \ldots, -l^2$, and so the value of every ρn-best solution of the minimization problem over $\mathcal{S}_Y$ satisfies $f(WX) \leq -1$.

For each vector $c \in \mathbb{Z}^n$ and each pair $1 \leq i_1, i_2 \leq l$, let

$$\mathcal{T}_{i_1,i_2}(c) \ := \ \{Z \in \mathcal{S}_{k+i_1,k+i_2} \ : \ cZ > \max\{cX \ : \ X \in \mathcal{S}^*\}\} \ .$$

Claim: For every $c \in \mathbb{Z}^n$ and every pair $1 \leq i_1, i_2 \leq l$, we have

$$|\mathcal{T}_{i_1,i_2}(c)| \ \leq \ \binom{m}{l}\binom{m}{k+l} \ .$$

Proof of Claim: Consider any pair $U = U_1 \uplus U_2 \, , V = V_1 \uplus V_2 \in \mathcal{T}_{i_1,i_2}(c)$. We now show that either $|U_1 \cap V_1| \geq k+1$ or $|U_2 \cap V_2| \geq k+1$. Suppose, indirectly, otherwise. Let

$$\begin{aligned} X \ &:= \ (U_1 \cap V_1) \uplus (U_2 \cup V_2), \\ Y \ &:= \ (U_1 \cup V_1) \uplus (U_2 \cap V_2). \end{aligned}$$

Then $|U_1 \cap V_1| \leq k$ and $|U_2 \cup V_2| \leq m$ imply $X \in \mathcal{S}^*$, and $|U_1 \cup V_1| \leq m$ and $|U_2 \cap V_2| \leq k$ imply $Y \in \mathcal{S}^*$. We then obtain the following contradiction,

$$0 \ < \ cU - cX \ = \ c(U_1 \setminus V_1) - c(V_2 \setminus U_2) \ = \ cY - cV \ < \ 0.$$

So indeed, for every pair $U = U_1 \uplus U_2\,, V = V_1 \uplus V_2 \in \mathcal{T}_{i_1,i_2}(c) \subseteq \mathcal{S}_{k+i_1,k+i_2}$, either $|U_1 \cap V_1| \geq k+1$ or $|U_2 \cap V_2| \geq k+1$. Therefore, we can now apply the extended Erdős-Ko-Rado theorem for direct products of Frankl [3, Theorem 2], which implies

$$\frac{|\mathcal{T}_{i_1,i_2}(c)|}{|\mathcal{S}_{k+i_1,k+i_2}|} \leq \max\left\{\binom{m-(k+1)}{(k+i_1)-(k+1)}\Big/\binom{m}{k+i_1}\,,\binom{m-(k+1)}{(k+i_2)-(k+1)}\Big/\binom{m}{k+i_2}\right\}$$

from which it is easy to conclude that, as claimed,

$$|\mathcal{T}_{i_1,i_2}(c)| \;\leq\; \binom{m}{l}\binom{m}{k+l}\;.$$

We continue with the proof of our theorem. As $k = 7l$, $m = 8l^2$ and $l \geq 2$, we get

$$\binom{m}{k+l}\Big/\binom{m}{l}^3 \;=\; \binom{8l^2}{8l}\Big/\binom{8l^2}{l}^3 \;\geq\; \binom{4l^2}{8l}^{8l}\Big/(8l^2)^{3l} \;\geq\; (2^{-9}l)^{2l}\;.$$

Therefore

$$|\mathcal{S}_{k+l,k+l}| \;=\; \binom{m}{k+l}\binom{m}{k+l} \;\geq\; (2^{-9}l)^{2l}\binom{m}{l}^3\binom{m}{k+l}\;.$$

Consider any algorithm attempting to obtain a ρn-best solution to the nonlinear optimization problem over any system $\mathcal{S}$, and let $c^1,\ldots,c^q \in \mathbb{Z}^n$ be the sequence of queries to the oracle of $\mathcal{S}$ made by the algorithm. For each pair $1 \leq i_1, i_2 \leq l$ and each $Z \in \mathcal{T}_{i_1,i_2}(c^p)$, the number of $Y \in \mathcal{S}_{k+l,k+l}$ containing Z, and hence satisfying $Z \in \mathcal{S}_Y$, is

$$\binom{m-(k+i_1)}{l-i_1}\binom{m-(k+i_2)}{l-i_2} \;\leq\; \binom{m}{l}^2\;.$$

So the number of $Y \in \mathcal{S}_{k+l,k+l}$ containing some Z that lies in some $\mathcal{T}_{i_1,i_2}(c^p)$ is at most

$$\sum_{p=1}^{q}\sum_{i_1=1}^{l}\sum_{i_2=1}^{l}\binom{m}{l}^2|\mathcal{T}_{i_1,i_2}(c^p)| \;\leq\; ql^2\binom{m}{l}^3\binom{m}{k+l}\;.$$

Therefore, if the number of oracle queries satisfies $q < l^{-2}(2^{-9}l)^{2l}$, then there exists some $Y \in \mathcal{S}_{k+l,k+l}$ that does not contain any Z in any $\mathcal{T}_{i_1,i_2}(c^p)$. This means that any $Z \in \mathcal{S}_Y$ satisfies $c^pZ \leq \max\{c^pX : X \in \mathcal{S}^*\}$. Hence, whether the linear-optimization oracle presents $\mathcal{S}^*$ or $\mathcal{S}_Y$, on each query c^p it can reply with some $X^p \in \mathcal{S}^*$ attaining

$$c^pX^p \;=\; \max\{c^pX \;:\; X \in \mathcal{S}^*\} \;=\; \max\{c^pX \;:\; X \in \mathcal{S}_Y\}\;.$$

So the algorithm cannot tell whether the oracle presents $\mathcal{S}^*$ or $\mathcal{S}_Y$, whether the image is $W\mathcal{S}^*$ or $W\mathcal{S}_Y$, and whether the objective-function value of every ρn-best solution is zero or negative, let alone compute any ρn-best solution. Therefore, with $l \geq 2^{10}$, every algorithm that can produce a ρn-best solution for the 2-dimensional nonlinear optimization problem (1) over every system $\mathcal{S}$ must make at least an exponential number

$$q \;\geq\; l^{-2}(2^{-9}l)^{2l} \;\geq\; l^{-2}2^{2l} \;>\; 2^l \;=\; 2^{\frac{1}{4}\sqrt{n}}$$

of queries to the oracle presenting $\mathcal{S}$ and therefore cannot run in polynomial time. □

References

[1] Berstein, Y., Lee, J., Maruri-Aguilar, H., Onn, S., Riccomagno, E., Weismantel, R., Wynn, H.: Nonlinear matroid optimization and experimental design. *SIAM Journal on Discrete Mathematics* 22:901–919 (2008)

[2] Berstein, Y., Lee, J., Onn, S., Weismantel, R.: Parametric nonlinear discrete optimization over well-described sets and matroid intersections. *Mathematical Programming* (to appear)

[3] Frankl, P.: An Erdős-Ko-Rado theorem for direct products. *European Journal of Combinatorics* 17:727–730 (1996)

[4] Lee, J., Onn, S., Weismantel, R.: Approximate nonlinear optimization over weighted independence systems. *SIAM Journal on Discrete Mathematics* 23:1667–1681 (2009)

[5] Onn, S.: Nonlinear Discrete Optimization: An Algorithmic Theory. Zürich Lectures in Advanced Mathematics, European Mathematical Society, 1–143, in process.

[6] Papadimitriou, C.H., Yanakakis, M.: The complexity of restricted spanning tree problems. *Journal of the ACM* 29:285–309 (1982)

Jon Lee
IBM T.J. Watson Research Center, Yorktown Heights, USA

Shmuel Onn
Technion - Israel Institute of Technology, Haifa, Israel

Robert Weismantel
ETH, Zürich, Switzerland