MINIMIZING IRREGULAR CONVEX FUNCTIONS: ULAM STABILITY FOR APPROXIMATE MINIMA

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Abstract. The main concern of this article is to study Ulam stability of the set of ε-approximate minima of a proper lower semicontinuous convex function bounded below on a real normed space X, when the objective function is subjected to small perturbations (in the sense of Attouch & Wets). More precisely, we characterize the class all proper lower semicontinuous convex functions bounded below such that the set-valued application which assigns to each function the set of its ε-approximate minima is Hausdorff upper semicontinuous for the Attouch-Wets topology when the set $\mathcal{C}(X)$ of all the closed and nonempty convex subsets of X is equipped with the uniform Hausdorff topology. We prove that a proper lower semicontinuous convex function bounded below has Ulam-stable ε-approximate minima if and only if the boundary of any of its sublevel sets is bounded.

1. Statement of the problem

In the sequel, $(X, \|\cdot\|)$ will be a real normed linear space with closed unit ball B_X and origin θ_X; the closed unit ball and origin of the dual space $X^\star$ will be denoted by $B_{X^\star}$ and $\theta_{X^\star}$, respectively. We will often be forced to consider finite products of normed spaces, e.g., $X \times \mathbb{R}$, and in such spaces, the box norm will be understood. We denote the space of proper lower semicontinuous extended-real-valued convex mappings defined on X by $\Gamma_0(X)$.

An important concept for our study is that of the Attouch-Wets topology τ_{AW} on $\mathcal{C}(X)$, the class of all the closed and convex subsets of X, which is nothing but the topology of the uniform convergence on bounded sets applied to the distance functionals to the sets from $\mathcal{C}(X)$; moreover this topology is metrizable [9]. This topology has been defined in different ways and we refer to section 2 for its historical presentation as it can be found for instance in Attouch's book [3].

More precisely, we consider the Attouch-Wets topology on $\Gamma_0(X)$: given a sequence $(f_n)_{n\in\mathbb{N}}$ in $\Gamma_0(X)$, we say that it Attouch-Wets converges to f provided the sequence $(\operatorname{epi} f_n)_{n\in\mathbb{N}}$ of the epigraphs of functions f_n is convergent to $\operatorname{epi} f$ in the Attouch-Wets topology of the space $\mathcal{C}(X \times \mathbb{R})$. Our study

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is particulary focused on $(\Gamma_b(X),\, \tau_{AW})$, the subspace of those functions in $\Gamma_0(X)$ that are bounded below, endowed with the Attouch-Wets topology. Let us also note $(\mathcal{C}(X),\, \tau_H)$ the topology induced on $\mathcal{C}(X)$ by the uniform Hausdorff metric.

Finally, for any fixed positive value ε, we define the application $\mathfrak{A}_\varepsilon$ from $(\Gamma_b(X),\, \tau_{AW})$ to $(\mathcal{C}(X),\, \tau_H)$, as being the set-valued mapping which assigns to every $f \in \Gamma_b(X)$ the nonempty set

$$\mathfrak{A}_\varepsilon(f) = \{x \in X :\, f(x) \leq \inf_X f + \varepsilon\}$$

of all the ε-approximate minima of f.

This article focuses on the study of the upper semi-continuity of $\mathfrak{A}_\varepsilon$. A function $f \in \Gamma_b(X)$ such that $\mathfrak{A}_\varepsilon$ is Hausdorff upper semicontinuous at f is said to have a *Ulam stable ε-approximate minima*. Our main objective is to characterize the subclass of $\Gamma_b(X)$ formed of those functions at which $\mathfrak{A}_\varepsilon$ is Hausdorff upper semicontinuous.

Despite its abstract appearance, this type of stability turns out to be essential in numerical optimization, namely in answering the natural question of defining the largest class of functionals $f \in \Gamma_b(X)$ for which minimization algorithms exist.

Thus, it is not surprising that this topic has been intensively studied over the last twenty years. Subsection 1.3 provides a very brief overview of some of the most important results in this field obtained by Asplund, Attouch, Beer, Beeer & Lucchetti, Moreau, Rockafellar & Wets and many others. From the above mentionned results it is not hard to deduce that Ulam stability of approximate minima is achieved for all the coercive functions $f \in \Gamma_0(X)$. Moreover, several convincing examples prove that Ulam stability may fail outside the coercive setting. All this facts suggest that coercivity and Ulam stability are closely connected notions.

The reason why we re-address such an well-explored subject, is the unexpected (and, to our best knowledge, new) fact that Ulam stability holds true also for some non-coercive functions $f \in \Gamma_b(X)$, since easily observed the mapping $\mathfrak{A}_\varepsilon$ is Hausdorff upper semicontinuous at any constant function, and also is upper semi-continous everywhere on $\Gamma_b(\mathbb{R})$.

Thus, it appears that Ulam stability is a property shared by three apparently dissimilar type of $\Gamma_b(X)$-functionals: the class of all mappings $f \in \Gamma_0(X)$ which are coercive, the constant functions, and $\Gamma_b(\mathbb{R})$ (that is when $X = \mathbb{R}$).

The main objective of the present article is to provide an explanation to this bizarre situation. To this respect, we characterize (Theorem 1) the mappings $f \in \Gamma_b(X)$ with Ulam-stable approximate minima as being those functions $f \in \Gamma_b(X)$ for which the boundary of every sublevel set is bounded, and we remark (Proposition 4) that a $\Gamma_0(X)$-functional has all its sublevel

sets with a bounded boundary if and only if it is coercive, constant, or defined on the real axis.

Accordingly, the key concept for Ulam stability is the boundedness of the boundary of any sublevel set of the function, and not, as the emphasis classically put on coerciveness might suggest, the boundedness of the sublevel sets themselves. The interpretation of Ulam stability as ensuring the existence of a minimizing algorithm allows us now to state that $f \in \Gamma_b(X)$ may be numerically minimized as long as the boundary of each of its sublevel sets is bounded.

1.1. **Ulam stability of the approximate minima.** "It seems desirable in many mathematical formulations of physical problems to add still another requirement to the well known desiderata of Hadamard of existence, uniqueness and continuity of the dependence of solutions on the initial parameters. Specifically [...] the solutions should vary continuously even when the operator itself is subject to "small" variations. [...] Speaking descriptively, the question is: when is it true that solutions of two problems in the calculus of variations which corresponds to "close" physical data must be close to each other? [28]".

Throughout this contribution, the above mentionned type of stability will be referred to as Ulam stability, that shall not be confused with Hyers-Ulam stability for functional equations (see [23] for an recent account of the latter notion).

Ulam stability is crucial when applied to the minimization of a function $f \in \Gamma_b(X)$. Indeed, in order to correctly work, most (if not all) of the numerical minimizing algorithms require enough regularity on the function to be minimized. This regularity in general fails in real world applications.

In order to overcome this difficulty and be able to minimize also less regular functions in the class $\Gamma_b(X)$, it is customary to use one of the available regularizing techniques (such as the rolling ball technique [22], the Lipschitzian regularization (see, e.g., [17], [18], [7], [10], [5], [15], [16]), the Moreau-Yosida regularization (see, e.g., [6] , [4], [20], [29]), or the robust regularization (see, e.g., [8], [26]) and construct in this way a sequence $(f_n)_{n\in\mathbb{N}}$ in $\Gamma_b(X)$ "very regular" (typically coercive and two times continuously differentiable) converging to f; usually, such a sequence $(f_n)_{n\in\mathbb{N}}$ is called a regularizing sequence for f.

For every function f_n and error bound $\varepsilon > 0$, it is now possible to use a minimizing algorithm and compute an ε-approximate minimizer of f_n, that is a vector $x_{n,\varepsilon}$ such that

$$f_n\left(x_{n,\varepsilon}\right) - \min_X f_n \leq \varepsilon; \tag{1}$$

following Zolezzi [31], the sequence $(x_{n,\varepsilon})_{n\in\mathbb{N}}$ is called an ε-asymptotically minimizing sequence.

Of course, this technique is of some use only if the ε-asymptotically minimizing sequence actually approaches the set of ε-approximate minima of f, that is if

$$\lim_{n\to\infty}\left(\operatorname{dist} x_{n,\varepsilon},\ \ \{x\in X:\ f(x)-\inf_X f\le\varepsilon\}\right)=0. \tag{2}$$

This article is concerned with the class of those mappings in $\Gamma_b(X)$ for which the above-described technique works for one of the regularizing techniques ordinarily in use. In agreement with the analysis undertaken in [11, pages 796-797], we are interested in the case when relation (2) holds for all the sequences $(f_n)_{n\in\mathbb{N}}, f_n\in\Gamma_b(X)$ Attouch-Wets converging (see Section 2 for a detailed account of this notion) to f, as the Attouch-Wets convergence appears as the main common feature of all the above-mentionned regularizing sequences.

More precisely, we seek for all $f\in\Gamma_b(X)$ such that, for every sequence $(f_n)_{n\in\mathbb{N}}, f_n\in\Gamma_b(X)$ Attouch-Wets converging to f, and every error bound $\varepsilon>0$, it holds that

$$(\varepsilon-\operatorname{argmin} f_n)\subset\ (\varepsilon-\operatorname{argmin} f)+\delta_n\,B_X \tag{3}$$

for some sequence $\delta_n>0$ converging to zero (recall that B_X stands for the closed unit ball of X).

Let us remark that, as the Attouch-Wets topology is metrizable, relation (3) says in fact that the set-valued application

$$\mathfrak{A}_\varepsilon:(\Gamma_b(X),\tau_{AW})\to\ (\mathcal{C}(X),\,\tau_H)\,,$$

defined as

$$\Gamma_b(X)\ \ni\ g\ \to\ \mathfrak{A}_\varepsilon(g)=\varepsilon-\operatorname{argmin} g\ \in\ \mathcal{C}(X), \tag{4}$$

is Hausdorff upper semicontinuous at f.

Our interest for the upper semi-continuity of the above-defined application is thus motivated by the fact that, although potentially irregular, any $\Gamma_b(X)$- functional f at which the mapping $\mathfrak{A}_\varepsilon$ is Hausdorff upper semicontinuous may still be minimized. Indeed, it is enough to apply a minimizing algorithm to any regularizing sequence of f, as the ε-asymptotically minimizing sequence thus obtained must approach the ε-approximate minima of the function f to be minimized.

1.2. **Ulam stability for the minima of a convex mapping.** A closely related (and obvioulsy stronger) notion is the concept of Ulam-stable minima, meaning that any asymptotically minimizing sequence $x_n\in\ \varepsilon_n-$ argmin f_n approaches the minima of f, i.e.,

$$\lim_{n\to\infty}\operatorname{dist}(x_n,\ \operatorname{argmin} f)=0,$$

for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of error bounds going to zero and any sequence $(f_n)_{n\in\mathbb{N}}$ Attouch-Wets converging to f (see for instance Beer & Luchetti [11, Theorem 4.1]).

It is easy to observe that the minima of f is Ulam-stable if and only if f has an Ulam-stable ε-approximate minima for some $\varepsilon > 0$ as well as a sharp minima (in Ferris' s sense, [14], [13]). Accordingly, once the characterization of functions with an Ulam-stable ε-approximate minima is provided, one can easily deduce a characterization of $\Gamma_b(X)$- functionals with Ulam-stable minima.

1.3. **Classical results on Ulam stabilty of the approximate minima.** As customary, we note $X^\star$, the topological dual of X and $\langle \cdot, \cdot \rangle : X^\star \times X \to \mathbb{R}$ the duality pairing between $X^\star$ and X. Let us first recall that, for each $f \in \Gamma_0(x)$, the ε-subfifferential of f at

$$x \in \text{Dom } f = \{x \in X \,:\, f(x) < +\infty\}$$

is the set

$$\partial_\varepsilon f(x) = \{x \in X^\star \,:\, \langle x^\star, y - x \rangle \leq f(y) - f(x) + \varepsilon \quad \forall y \in X\},$$

while the Fenchel-Legendre conjugate $f^\star : X^\star \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$f^\star(x^\star) = \sup_{x \in X} \left(\langle x^\star, x \rangle - f(x) \right).$$

Note that $\partial_\varepsilon f^\star(0)$ coincides with the set of ε-approximate minima of f. It is thus possible to interpret a classical theorem by Moreau (see, [21, Proposition 11.3]) in the sense of our analysis; this result, initially published in [19], establishes the upper semi-continuity of the approximate subdifferential on the interior of the effective domain of every $\Gamma_0(X)$-functional (see also the important article of Asplund and Rockafellar [1, Proposition 5]). Therefore, Moreau's result can be viewed as a partial answer to the Ulam-stability question. Indeed, the above-cited theorem literary means that the restriction of the set-valued mapping defined at relation (4) to the subset

$$\{f - y^\star \in \Gamma_b(X) : y^\star \in X^\star\}$$

of $\Gamma_b(X)$ is Hausdorff upper semicontinuous at f provided that f is coercive.

A more detailed account of Ulam stability was achieved by Beer and Lucchetti in their well known article [11]. It is an easy consequence of their central result [11, Theorem 3.6] (and we can only agree with the authors when they write "*we anticipate that our next theorem will have numerous applications*" [11, page 802]), that any coercive $\Gamma_0(X)$ functional admits an Ulam-stable ε-approximate minima for any positive value $\varepsilon > 0$.

It is therefore legitimate to consider that the result establishing Ulam stability of the approximate minima of any coercive $\Gamma_0(X)$-functional defined on a real normed space X, although (at our best knowledge) never stated in this form, is implicit in the mathematical literature.

1.4. **Scope and plan of the article.** It may thus appear as a real surprise that this classical result does not completely solve the Ulam stability problem for approximate minima. It is however a simple exercice to show that any $\Gamma_b(X)$-functional has an Ulam-stable ε-approximate minima for any $\varepsilon > 0$ provided that $X = \mathbb{R}$.

Moreover, the upper semi-continuity is always achieved when the set of ε-approximate minima of f is the whole underlying space X, that is when f is constant.

We have accordingly identified three completely dissimilar classes of $\Gamma_b(X)$ -functionals, each of them composed uniquely from mappings with Ulam-stable ε-approximate minima. It is thus natural to ask if, given X, a normed space of dimension larger than one, it is possible to find a $\Gamma_b(X)$-functional with Ulam-stable ε-approximate minima which is neither constant nor coercive, and in the event of a negative answer, to investigate why Ulam-stability is a feature shared, by and only by, coercive, constant and one-dimensional $\Gamma_b(X)$ functionals.

All these questions are answered in this note. On one hand we prove that the coercive, constant and one-dimensional $\Gamma_0(X)$ functionals are exactly those $\Gamma_0(X)$-mappings admiting sublevel sets with a bounded boundary (see Proposition 4, Section 2). On the other hand we prove that, if the boundary of a sublevel set of a $\Gamma_b(X)$-functional f is unbounded, then the mapping defined in (4) is not Hausdorff upper semicontinuous at f (Proposition 5, Section 3).

Finally, we are in a position to prove the main result of our article, Theorem 1, Section 3, stating that the approximate minima of a $\Gamma_b(X)$- functional is Ulam stable if and only if the boundary of any of its sublevel sets is bounded.

2. Closed and convex sets with a bounded boundary

Let us recall the definition of the Hausdorff distance between two sets from $\mathcal{C}(X)$, the class containing all the closed and nonempty convex subsets of a normed space $(X, \|\cdot\|)$:

$$\mathrm{d_H}(C_1, C_2) = \max\left(e(C_1, C_2),\, e(C_2, C_1)\right) \tag{5}$$

where

$$e(C_1, C_2) = \sup_{x_1 \in C_1} \inf_{x_2 \in C_2} \|x_1 - x_2\|,$$

is the excess of C_1 over C_2. Endowed with this metric, $\mathcal{C}(X)$ becomes a metric space, denoted hereafter by $(\mathcal{C}(X), \tau_H)$.

This topology turns out to be too fine for its applications to variational problems, in the sense that many sequences of sets which "intuitively" must converge, are not Hausdorff converging. Hence, several coarser topologies have been introduced. Among them, a prominent place is held by the

Attouch-Wets topology (sometimes called the bounded Haussdorff topology): a net $(C_i)_{i\in I} \subset \mathcal{C}(X)$ Attouch-Wets converges to $C \in \mathcal{C}(X)$ if and only if it converges with respect to any of the distances d_ρ, $\rho > 0$, where

$$\mathrm{d}_\rho(C_1, C_2) = \max\left(e(C_1 \cap \rho\, B_X,\, C_2),\, e(C_2 \cap \rho\, B_X, C_1)\right).$$

We shall denote this topology by $(\mathcal{C}(X),\, \tau_{AW})$; we also use the notation $(\Gamma_0(X),\, \tau_{AW})$ for the topology induced on $\Gamma_0(X)$ by the one-to-one mapping

$$\mathrm{epi}:\, \Gamma_0(X) \to\, (C(X \times \mathbb{R}), \tau_H)\,,$$

where $\mathrm{epi}\, f$ is the epigraph of the $\Gamma_0(X)$-functional f.

Let us also mention that the Attouch-Wets topology is metrizable, for instance by using the mapping

$$\mathrm{d}_{AW}(C_1,\, C_2) = \sum_{i=1}^{+\infty} \frac{\mathrm{d}_i\,(C_1,\, C_2)}{2^i\,(1 + \mathrm{d}_i\,(C_1,\, C_2))};$$

hence, we may use sequences instead of nets in studying this topology.

2.1. **Attouch-Wets versus uniform Hausdorff topologies.** Consider τ_1 and τ_2, two topologies on the same partially ordered set $(M,\, \leq)$; an important issue in studying the interplay between τ_1 and τ_2 is to characterize all the points $x \in M$ at which the identity mapping $\iota : (M, \tau_1) \to (M, \tau_2)$ is continuous, as well as lower or Hausdorff upper semicontinuous.

In the case of the topological spaces $(\mathcal{C}(X),\, \tau_{AW})$ and $(\mathcal{C}(X),\, \tau_H)$, both defined on the partially ordered set $((\mathcal{C}(X),\, \subset)$, it can be observed that the three following statements are equivalent:
i) the mapping ι is continuous at $C \in \mathcal{C}(X)$;
ii) the mapping ι is lower semi-continuous at $C \in \mathcal{C}(X)$;
iii) the set C is bounded.

The present subsection addresses the remaining problem of the upper semi-continuity of the mapping ι.

Let us first provide a technical characterization of the class of closed and convex sets with a bounded boundary.

Lemma 1. *Let C be a closed convex subset of a normed space $(X, \|\cdot\|)$. The two following sentences are equivalent:*
i) the boundary of C is unbounded;
ii) there is an unbounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that

$$a \leq \mathrm{dist}(x_n,\, C) \leq b \quad \forall\, n \in \mathbb{N} \tag{6}$$

for some values a and b such that $0 < a < b$.

Proof of Lemma 1: i) ⇒ ii). Since $\partial\, C$ is an unbounded set, it is possible to pick $(y_n)_{n\in\mathbb{N}} \subset \partial\, C$ such that

$$\|y_n\| \geq n + 4 \qquad \forall\, n \in \mathbb{N}. \tag{7}$$

Since each y_n belongs to the boundary of C, it is possible to find an element $z_n \in X \setminus C$ such that

$$\|y_n - z_n\| \leq 1. \tag{8}$$

Set $\mu_n = \mathrm{dist}(z_n, C)$ (of course, $0 < \mu_n \leq 1$), and pick $v_n \in C$ such that

$$\|z_n - v_n\| \leq 2\,\mu_n. \tag{9}$$

We claim that relation (6) holds for

$$x_n = v_n + \frac{z_n - v_n}{\mu_n}.$$

Indeed, $x_n = (x_n - v_n) + (v_n - z_n) + (z_n - y_n) + y_n$, so

$$\|x_n\| \leq \|y_n\| - \|x_n - v_n\| - \|v_n - z_n\| - \|z_n - y_n\|;$$

from relations (7), (8) and (9) it yields that

$$\|x_n\| \leq n + 2 - \|x_n - v_n\| \quad \forall\, n \in \mathbb{N}. \tag{10}$$

Remark that $x_n - v_n = \frac{z_n - v_n}{\mu_n}$, so relation (9) implies that

$$\|x_n - v_n\| \leq 2 \quad \forall\, n \in \mathbb{N}. \tag{11}$$

Combined with relation (10), the above inequality shows that

$$\|x_n\| \leq n \quad \forall\, n \in \mathbb{N}; \tag{12}$$

on the other hand, from relation (11) it results that

$$\mathrm{dist}(x_n,\, C) \leq 2 \quad \forall\, n \in \mathbb{N}. \tag{13}$$

To the purpose of estimating the lower bound of the distance between x_n and C, let us pick $u \in C$. Set $w = v_n + \mu_n(u - v_n)$; being a convex set, C contains w, so

$$\|z_n - w\| \geq \mathrm{dist}(z_n,\, C) = \mu_n. \tag{14}$$

Remark that $u = v_n + \frac{w - v_n}{\mu_n}$, and hence that $x_n - u = \frac{z_n - w}{\mu_n}$; we deduce from relation (14) that

$$\|x_n - u\| = \frac{\|z_n - w\|}{\mu_n} \geq 1 \qquad \forall u \in C,\, n \in \mathbb{N}. \tag{15}$$

Let us now use relation (12) to prove that the sequence $(x_n)_{n\in\mathbb{N}}$ is unbounded, and combine relations (13) and (15) to obtain relation (6) for $a = 1$ and $b = 2$.

ii) $\Rightarrow$ *i)* Suppose instead that the boundary of C is bounded, that is

$$\partial\, C \subset \kappa\, B_X \tag{16}$$

for some $\kappa > 0$.

The following general fact is easy to prove: (as usual, we denote by $\overset{\circ}{S}$ the topological interior of the set S).

Lemma 2. *Let X be a topological vector space and C be a closed subset of X of non-void interior. Any line segment of ends $x \in \overset{\circ}{C}$ and $y \in X \setminus C$ meets the boundary of C.*

Among the many useful but simple consequences of Lemma 2, let us mention the following statement.

Lemma 3. *Let X be a normed vector space and C be a closed subset of X. Then, for any $x \in X \setminus C$ it holds that*

$$\operatorname{dist}(x, C) = \operatorname{dist}(x, \partial C),$$

where by ∂S we mean the topological boundary of the set S.

The first inequality in relation (6) proves that $x_n \notin C$, so we may use Lemma 3, and deduce that $\operatorname{dist}(x_n, C) = \operatorname{dist}(x_n, \partial C)$; combine this relation with the second inequality in relation (6) to obtain that

$$\operatorname{dist}(x_n,\, \partial C) \leq b \qquad \forall n \in \mathbb{N}; \tag{17}$$

from relations (16) and (17) it yields

$$\|x_n\| \leq \kappa + b \qquad \forall n \in \mathbb{N},$$

fact which contradicts the unboundedness of the sequence $(x_n)_{n\in\mathbb{N}}$, and the proof is complete. $\triangle$

Remark 1. *The convexity assumption on C is essential in establishing that the implication $i) \Rightarrow ii)$ holds true. Indeed, the closed subset*

$$C = \left\{ (x, y) :\, |x\, y| \leq \frac{1}{2} \right\}$$

of the normed space $\left(\mathbb{R}^2, \|(x, y)\| = \sqrt{x^2 + y^2}\right)$ possesses an unbounded boundary. Straightforward calculations prove that

$$\operatorname{dist}\left((x, y),\, C\right) \leq \frac{1}{2\sqrt{x^2 + y^2}} \quad \forall (x, y) \in \mathbb{R}^2,\ (x, y) \neq (0, 0).$$

Hence, no sequence fulfilling relation (6) may exist.

The following result uses Lemma 1 to establish a sufficient condition for the upper semi-continuity of the operator ι.

Proposition 1. *The identity mapping*

$$\iota :\ (\mathcal{C}(X),\, \tau_{AW}) \to (\mathcal{C}(X),\, \tau_H)$$

is Hausdorff upper semicontinuous at C provided that the boundary of C is bounded.

Proof of Proposition 1: Let us consider a closed and convex set C at which the mapping ι is not Hausdorff upper semicontinuous.

An equivalent way to express the fact that the mapping ι is not Hausdorff upper semicontinuous at C is to state the existence of a sequence $(C_n)_{n\in\mathbb{N}} \subset \mathcal{C}(X)$ Attouch-Wets converging to C, and of a value $\gamma > 0$ such that

$$\mathrm{e}\,(C_n,\, C) \geq \gamma \qquad \forall n \in \mathbb{N}. \tag{18}$$

Our aim is to construct an unbounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that

$$\gamma \leq \mathrm{dist}(x_n, C) \leq\, 2\,\gamma \qquad \forall n \in \mathbb{N}. \tag{19}$$

Let us pick $\overline{x} \in C$; since the sequence $(C_n)_{n\in\mathbb{N}}$ converges in the Attouch-Wets sense to C, it results that $\mathrm{dist}(\overline{x}, C_n)$ goes to 0. Thus, by passing, if necessary, to a subsequence, we may state that there is $y_n \in C_n$ such that $\|\overline{x} - y_n\| \leq \gamma$.

On the other hand, according to relation (18), let us pick $z_n \in C_n$ such that $\mathrm{dist}(z_n, C) \geq \gamma$.

As the function

$$[0,1] \ni \lambda \to \mathfrak{D}_n(\lambda) = \mathrm{dist}\,(\lambda\, y_n +\, (1-\lambda) z_n,\; C) \in \mathbb{R}$$

is continuous, $\mathfrak{D}_n(0) \geq \gamma$ and $\mathfrak{D}_n(1) \leq \gamma$, it follows that there is a value $\lambda_n \in [0,1]$ such that

$$\mathfrak{D}_n(\lambda_n) = \gamma. \tag{20}$$

We claim that the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n = \lambda_n\, y_n + (1-\lambda_n) z_n$ is unbounded and fulfills relation (19).

Indeed, relation (20) directly implies relation (19). Moreover, since $(C_n)_{n\in\mathbb{N}}$ Attouch-Wets converges to C, it holds that for any bounded sequence, say $(w_n)_{n\in\mathbb{N}}$ of elements from C_n, the sequence $\mathrm{dist}(w_n, C)$ goes to 0. Relation (19) proves that this is not the case for the sequence $(x_n)_n$, and thus that this sequence is necessarily unbounded.

The conclusion of Proposition 1 is now a simple consequence of the implication $ii) \Rightarrow ii)$ of Lemma 1. $\triangle$

A first step in proving the converse of Proposition 1 is the following result, which provides a sistematic manner to construct sequences of closed and convex sets Attouch-Wets converging to a given set.

Lemma 4. *Let C be a closed and convex subset of a normed space $(Y\|\cdot\|)$ and set $C_n := C \cap n\, B_Y$, the intersection of C with the closed ball of center θ_Y and radius n. For any sequence $(y_n)_{n\in\mathbb{N}} \subset Y$, denote by A_n the convex hull of the union between the singleton $\{y_n\}$ and the set C_n.*

Then $A_n \in \mathcal{C}(X)$, and the sequence $(A_n)_{n\in\mathbb{N}} \subset C(Y)$ Attouch-Wets converges to C provided that

$$\|y_n\| \geq\, n^2 \quad \text{and} \quad \mathrm{dist}\,(y_n,\, C) \leq\, k \quad \forall\, n \in \mathbb{N}, \tag{21}$$

where $k > 0$ is a postive real number.

Proof of Lemma 4: As easily seen, the convex hull of the union between a singleton and a bounded closed and convex set is again a bounded closed and convex set; in particular, this fact implies that $A_n \in C(Y)$ for all $n \in \mathbb{N}$.

In order to prove that the sequence $(A_n)_{n\in\mathbb{N}}$ Attouch-Wets converges to C, it is sufficient to show that

$$\mathrm{d}_\rho(A_n, C) \leq \rho\,\frac{2(k+1)}{n} \quad \forall\, \rho \in \mathbb{R},\, n \in \mathbb{N} \text{ s.t. } 0 < \rho \leq n. \tag{22}$$

Let us consider $\rho \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $0 < \rho \leq n$. It is obvious that $C \cap \rho\, B_Y \subset A_n$, so

$$e(C \cap \rho\, B_Y, A_n) = 0 \quad \forall\, \rho \in \mathbb{R},\, n \in \mathbb{N} \text{ s.t. } 0 < \rho \leq n. \tag{23}$$

It remains to estimate $e(A_n \cap \rho\, B_Y, C)$. To this end, pick $x \in A_n \cap \rho\, B_Y$; accordingly, $\|x\| \leq \rho$, and there are $\lambda(x) \in [0,1]$, and $y(x) \in C_n$ such that $x = \lambda(x)\, y_n + (1-\lambda(x))y(x)$.

Recall that $\mathrm{dist}\,(y_n,\, C) \leq k$, to deduce that there exists $z_n \in C$ such that

$$\|y_n - z_n\| \leq k+1 \qquad n \in \mathbb{N}. \tag{24}$$

As C is a convex set, it follows that the convex combination $z(x) = \lambda(x)\, z_n + (1-\lambda(x))y(x)$ belongs to C. Obviously,

$$\|x - z(x)\| = \|\lambda(x)(y_n - z_n)\| = \lambda(x)\|y_n - z_n\|,$$

so, in view of relation (24) it follows that

$$\|x - z(x)\| \leq \lambda(x)(k+1). \tag{25}$$

Finally, recall that $\|x\| \leq \rho$; as

$$\|x\| = \|\lambda(x)\, y_n + (1-\lambda(x))y(x)\| \geq \lambda(x)\|y_n\| - (1-\lambda(x))\|y(x)\|,$$

it follows that

$$\lambda(x)\,(\|y_n\| + \|y(x)\|) \leq \rho + \|y(x)\|.$$

But $\|y_n\| \geq n^2$; the previous inequality gives thus the upper estimate

$$\lambda(x) \leq \frac{\rho + \|y(x)\|}{n^2 + \|y(x)\|}.$$

On one hand, since $n \in \mathbb{N}$, it holds that $n \leq n^2$. Hence, $0 < \rho \leq n^2$ and therefore, the mapping

$$\mathbb{R}_+ \ni s \to \frac{\rho + s}{n^2 + s} \in \mathbb{R}_+$$

is increasing. On the other hand, we know that $\|y(x)\| \leq n$; accordingly,

$$\lambda(x) \leq \frac{\rho + n}{n^2 + n} = \frac{2}{n}\,\frac{n^2 + \rho\, n}{n^2 + (n+2)\, n},$$

and, since $0 < \rho \leq n < n+2$, it yields that

$$\lambda(x) \leq \frac{2}{n} \qquad \forall\, x \in A_n \cap \rho\, B_X. \tag{26}$$

Combining relations (25) and (26) we deduce that

$$\|x - z(x)\| \leq \frac{2(k+1)}{n} \qquad \forall\, x \in A_n \cap \rho\, B_X.$$

Consequently,

$$\inf_{c \in C} \|x - c\| \leq \|x - z(x)\| \leq \frac{2(k+1)}{n} \qquad \forall\, x \in A_n \cap \rho\, B_X,$$

and therefore,

$$\sup_{x \in A_n \cap \rho\, B_X} \left(\inf_{c \in C} \|x - c\| \right) \leq \frac{2(k+1)}{n}.$$

We have thus proved that

$$e(A_n \cap \rho\, B_X,\, C) \leq \frac{2(k+1)}{n} \quad \forall\ \rho \in \mathbb{R},\, n \in \mathbb{N} \text{ s.t. } 0 < \rho \leq n; \tag{27}$$

the desired inequality (22) stems from relations (23) and (27). △

We have now all the ingredients to prove the converse of Proposition 1, that is a necessary upper semi-continuity condition for ι.

Proposition 2. *If the identity mapping*

$$\iota :\ (\mathcal{C}(X),\, \tau_{AW}) \ \to\ (\mathcal{C}(X),\, \tau_H)$$

is Hausdorff upper semicontinuous at C, then the boundary of C is bounded.

Proof of Proposition 2: Let $D \in \mathcal{C}(X)$ be a set at which the mapping ι is Hausdorff upper semicontinuous, and assume instead that D has an unbounded boundary.

Apply implication $i) \Rightarrow ii)$ from Lemma 1 to D, and deduce that there is an unbounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$ and two values a and b such that for any $n \in \mathbb{N}$ it holds that $a \leq \operatorname{dist}(x_n, D) \leq b$.

As the sequence $(x_n)_{n\in\mathbb{N}}$ is unbounded, take a subsequence, still denoted by $(x_n)_n$ such that $\|x_n\| \geq n^2$. Define D_n as the convex hull of the point x_n and of the set $D \cap n\, B_X$; apply Lemma 4 for $X = Y$, $D = C$, $x_n{=}y_n$ and $b = k$ to deduce that the sequence $(D_n)_{n\in\mathbb{N}}$ Attouch-Wets converges to D. On the other hand,

$$\mathrm{e}(D_n, D) \geq \operatorname{dist}(x_n, D) \geq a \qquad \forall\, n \in \mathbb{N}.$$

Hence, the mapping i is not Hausdorff upper semicontinuous at D, a contradiction. △

2.2. **A characterization of closed and convex sets with a bounded boundary.** Propositions 1 and 2 motivate our interest for the study of closed and convex sets with a bounded boundary, as their class turns out to be a key notion when studying the interplay between the topologies $(\mathcal{C}(X),\, \tau_H)$ and $(\mathcal{C}(X),\, \tau_{AW})$.

The following result allows us to completely characterize the family of closed and convex sets with a bounded boundary, as well as $\Gamma_0(X)$- functionals with sublevel sets possessing this property.

Proposition 3. *Let X be a normed vector space of dimension greater than one, and let C be an unbounded and proper closed and convex subset of X. Then, the boundary of C is unbounded.*

Proof of Proposition 3: Being unbounded, the set C is nonempty; being proper, it does not coincides with X. Recall that the only two subsets of X that are simultaneously open and closed are $\emptyset$ and X, to deduce that C is not an open set, and thus that its boundary $\partial\, C$ is nonempty. It is thus possible to pick $\overline{x} \in \partial\, C$.

Let us assume instead that the boundary of C is bounded. It means that $\partial\, C \subset \kappa\, B_X$ for some $\kappa > 0$. As C is an unbounded set, there is $\overline{y} \in C$ such that $\|\overline{y}\| \geq 2\,\kappa$; of course, $\overline{y}$ belongs to the interior of C.

Using a well known geometrical consequence of the Hahn-Banach theorem ([30, Theorem 1.1.5]) we deduce the existence of some $f \in X^\star$ and $\alpha \in \mathbb{R}$ such that

$$\sup_{x \in \kappa\, B_X} \langle f, x\rangle < \alpha < \langle f, \overline{y}\rangle\,.$$

Recall that the dimension of X is greater than one; accordingly, the dimension of the hyperplane $f^{-1}(0)$ is greater or equal to one, meaning that there is a non-null vector $v \in X$ such that $\langle f, v\rangle = 0$.

We claim that the line $\overline{x}+\mathbb{R}\, v$ is completely contained within the boundary of C. In order to prove this result, let us first remark that $\overline{y} + \mathbb{R}\, v \subset C$; indeed, if a point of the line $\overline{y} + \mathbb{R}\, v$ is not in C, then Lemma 2 implies that $\overline{y} + \mathbb{R}\, v$ meets $\partial\, C$, fact which is impossible, since

$$\langle f, \overline{y} + \mu\, v\rangle = \langle f, \overline{y}\rangle > \alpha > \sup_{x \in \kappa\, B_X} \langle f, x\rangle \geq \sup_{x \in \partial\, C} \langle f, x\rangle\,.$$

Invoke the well known result (originally proved by Steinitz [27] for euclidean spaces, but valid with no modification in any topological vector space) stating that, if a half-line is contained in a closed and convex set, then its translate to any point of the set is still included in it, to deduce that the line $\overline{x} + \mathbb{R}\, v$ is completely contained within the set C.

Finally, use the standard result ([30, ii), Theorem 1.1.2]) stating that the interior of a line segment is completely contained in the interior of C provided that the segment is included in C and meets $\overset{\circ}{C}$, to infer that a line

contained in C either completely misses $\overset{\circ}{C}$, or is a part of $\overset{\circ}{C}$. As

$$[\overline{x} + \mathbb{R}\, v \ni \overline{x} \in \partial\, C] \qquad \text{and} \qquad \left[\partial\, C \cap \overset{\circ}{C} = \emptyset\right],$$

it follows that the line $\overline{x}+\mathbb{R}\, v$ cannot be entirely contained in $\overset{\circ}{C}$, establishing our claim.

Of course, the fact that the boundary of C contains a line contradicts our initial assumption on the boundedness of $\partial\, C$, and proves in this way Proposition 3. $\triangle$

As the boundary of X is empty, the boundary of a bounded set is bounded (is a part of the set), and a boundary of a convex subset of the real axis is composed of at most two points, it follows that the boundaries of all the three above-mentionned classes of sets are bounded. We have thus proved the following characterization of the subfamily of $\mathcal{C}(X)$ consisting of sets with a bounded boundary.

Proposition 4. *The boundary of a closed and convex subset C of a normed space X is bounded if and only if at least one of the following three statements holds true:*

s_1) *the underlying space is the real axis, that is $X = \mathbb{R}$;*
s_2) *the set C itself is bounded;*
s_3) *the set C coincides with the whole underlying space, $C = X$.*

Accordingly, all the sublevel sets of a $\Gamma_0(X)$- functional have bounded boundaries, if and only if at least one of the following three statements holds true:

f_1) *the underlying space is the real axis, that is $X = \mathbb{R}$;*
f_2) *the mapping f is coercive;*
f_3) *the mapping f is constant.*

3. A characterization of Ulam stability of approximate minima

The last section of the article is concerned with the main object of our study, namely the upper semi-continuity of the operator $\mathfrak{A}_\varepsilon$. Let us first establish a $\Gamma_0(X)$-version of the technical Lemma 4.

3.1. A sequence of Attouch-Wets converging functions.

A standard way to associate an extended real-valued function to a subset S of the product space $(X \times \mathbb{R},\, \|\cdot,\cdot\|_B)$ (here $\|\cdot,\cdot\|_B$ is the standard box norm on $X \times \mathbb{R}$, $\|x, s\|_B = \max\left(\|x\|,\, |s|\right))$ is to consider k_S, the lower-boundary function to S. This application is defined (see [24, Theorem 5.3], or [26, 2.1, page 46]), as the unique extended-real-valued function $k_S : X \to X \cup \{-\infty,\, +\infty\}$ such that

$$\operatorname{epi}(k_S) = S \cup \{0\} \times \mathbb{R}_+;$$

more precisely,

$$X \ni u \to k_S(u) = \inf\{s \in \mathbb{R} : (x,s) \in S\} \in X \cup \{-\infty, +\infty\}.$$

As easily observed, when S is a convex subset of $X \times \mathbb{R}$, then k_S is a convex function. However, assuming that S is closed and convex does not automatically implies that k_S belongs to $\Gamma_0(X)$ as, on one hand, k_S may take the value $-\infty$, and on the other, its epigraph may not be a closed subset of $X \times \mathbb{R}$.

A classical result yields that, when S is a closed and convex subset of $X \times \mathbb{R}$, the function k_S belongs to $\Gamma_0(X)$ if and only if k_S takes somewhere a finite value (in other words, if the vector $(0,-1)$ does not belong to the recession cone of S).

Let us now consider a sequence $(S_n)_{n\in\mathbb{N}} \subset X \times \mathbb{R}$ which Attouch-Wets converges; even if k_{A_n} exists for every n, it may happen that the sequence $(k_{A_n})_{n\in\mathbb{N}}$ is not Attouch-Wets convergent in $\Gamma_0(X)$ (take for instance $X = \mathbb{R}$ and $A_n = \{(x,\, n\,x) : x \in \mathbb{R}\}$).

The following result provides a very general frame in which the Attouch-Wets convergence of a sequence of sets $(S_n)_{n\in\mathbb{N}}$ implies the Attouch-Wets convergence of the sequence $(k_{S_n})_{n\in\mathbb{N}}$.

Lemma 5. *Let $(S_n)_{n\in\mathbb{N}} \subset X \times \mathbb{R}$ be a sequence of sets Attouch-Wets converging to the epigraph of a functional $f \in \Gamma_0(X)$. Then:*
i) for n large enough, the functions k_{S_n} belong to $\Gamma_0(X)$;
ii) the sequence $(k_{S_n})_{n\in\mathbb{N}}$ Attouch-Wets converges to f.

Proof of Lemma 5: Let us pick $x_0 \in X$ such that $f(x_0) \in \mathbb{R}$. It is well known (and easy to prove) that a functional $f \in \Gamma_0(X)$ is bounded below on bounded sets; thus, for any $c > 0$, there is $m_c \in \mathbb{R}$ such that

$$f(x) \geq m_c + 1 \qquad \|x - x_0\| \leq c + 1. \tag{28}$$

An important step in proving both statements *i)* and *ii)* is to establish that mappings k_{A_n} are uniformly bounded below on bounded sets. More precisely, we must prove that there is an integer $n_c \in \mathbb{N}$ such that

$$k_{A_n}(x) \geq m_c \ \text{ for all } x \text{ s.t. } \|x_0 - x\| \leq c, \text{ and for all } n \geq n_c. \tag{29}$$

To this respect, remark that relation (28) implies that

$$\operatorname{dist}((x,s), \operatorname{epi} f) \geq 1 \qquad \forall\, (x,s) \in (x_0 + c\,B_X) \times [m_c - 1,\, m_c]. \tag{30}$$

As $(S_n)_{n\in\mathbb{N}}$ Attouch-Wets converges to $\operatorname{epi} f$, it results that $\mathrm{d}_\delta(S_n, \operatorname{epi} f)$ goes to 0 for every $\delta > 0$, and in particular for $\delta = \|x_0\| + c + m_c + 1$. Hence, there exists $p_c \in \mathbb{N}$ such that

$$\operatorname{dist}((x,s),\, \operatorname{epi} f) < 1 \quad \forall (x,s) \in S_n \cap (\|x_0\| + c + m_c + 1)\, B_{X\times\mathbb{R}},\ \forall\, n \geq p_c;$$

as $(x_0 + c\,B_X) \times [m_c - 1,\, m_c] \subset (\|x_0\| + c + m_c + 1)\, B_{X\times\mathbb{R}}$, the previous relation implies that, for any $n \geq p_c$ it holds that

$$\operatorname{dist}((x,s),\, \operatorname{epi} f) < 1 \ \forall (x,s) \in S_n \cap (x_0 + c\,B_X) \times [m_c - 1,\, m_c]. \tag{31}$$

From relations (30) and (31) it yields that

$$S_n \cap (x_0 + c\, B_X) \times [m_c - 1,\, m_c] = \emptyset \qquad \forall\, n \geq p_c. \tag{32}$$

On the other hand, let us remark that from relation (28) it follows that $f(x_0) \geq m_c + 1$, which means that the closed and convex set

$$K_c = (x_0 + c\, B_X) \times [m_c,\, +\infty[$$

is a neighborhood of $(x_0, f(x_0))$. As $\mathrm{dist}((x_0, f(x_0)),\, S_n)$ goes to 0 (recall that $(x_0, f(x_0)) \in \mathrm{epi} f$ and that $(S_n)_{n\in\mathbb{N}}$ Attouch-Wets converges to $\mathrm{epi}\, f$), we deduce that, for n large enough, the sets S_n must meet K_c; in other words, there is $q_c \in \mathbb{N}$ such that relation

$$S_n \cap (x_0 + c\, B_X) \times [m_c,\, +\infty[\neq \emptyset \tag{33}$$

holds true for any $n \geq q_c$.

Combining relations (32) and (33), and taking into account the convexity of S_n it follows that

$$S_n \cap (x_0 + c\, B_X) \times]-\infty, m_c - 1[= \emptyset \qquad n \geq \max(p_c, q_c); \tag{34}$$

from relations (32) and (34) it stems that the desired relation (29) holds provided that $n_c = \max(p_c, q_c)$.

In order to prove the statement $i)$, let $n \geq n_c = \max(p_c, q_c)$; as $n \geq q_c$, relation (33) says that the set $S_n \cap K_c$ is non-void, so we may pick $(x_n, s_n) \in S_n \cap K_c$. Since $(x_n, s_n) \in S_n$, the definition of the mapping k_{S_n} implies that $k_{S_n}(x_n) \leq s_n$; on the other hand, we know that $(x_n, s_n) \in K_c$, so $\|x_0 - x_n\| \leq c$. We may thus apply relation (29) for $x = x_n$ and deduce that $k_{S_n}(x_n) \geq m_c$.

We have proved that $m_c \leq k_{S_n}(x_n) \leq s_n$ provided that $n \geq n_c = \max(p_c, q_c)$. In other words, the value of k_{S_n} at x_n is finite, and let us recall that, given a closed and convex set A, the mapping k_A belongs to $\Gamma_0(X)$ if and only if it takes a finite value. Accordingly, the statement $i)$ is valid whenever $n \geq n_c = \max(p_c, q_c)$.

Let us now address statement $ii)$. Recall that for any sets A, B and C such that $C \subset B$, it holds that $\mathrm{e}(A, B) \leq \mathrm{e}(A, C)$, to conclude that

$$\mathrm{e}\,(\mathrm{epi} f \cap \delta\, B_{X\times\mathbb{R}}, \mathrm{epi}\, k_{A_n}) \leq \mathrm{e}\,(\mathrm{epi} f \cap \delta\, B_{X\times\mathbb{R}}, A_n) \quad \forall \delta > 0. \tag{35}$$

The sequence $(A_n)_{n\in\mathbb{N}}$ Attouch-Wets converges to $\mathrm{epi}\, f$, so

$$\lim_{n\to\infty} \mathrm{e}\,(\mathrm{epi} f \cap \delta\, B_{X\times\mathbb{R}}, A_n) = 0 \quad \forall \delta > 0;$$

use relation (35) to obtain that

$$\lim_{n\to\infty} \mathrm{e}\,(\mathrm{epi} f \cap \delta\, B_{X\times\mathbb{R}}, \mathrm{epi}\, k_{A_n}) = 0 \quad \forall \delta > 0. \tag{36}$$

Proving that $(\mathrm{e}\,(\mathrm{epi}\, k_{A_n} \cap \delta\, B_{X\times\mathbb{R}}, \mathrm{epi} f))_{n\in\mathbb{N}}$ also converges to 0 is more difficult, as the set $\mathrm{epi}\, k_{A_n} \cap \delta\, B_{X\times\mathbb{R}}$ is larger than $A_n \cap \delta\, B_{X\times\mathbb{R}}$. It is however

possible to prove that, for every $\delta > 0$ there is a value $\beta(\delta) > 0$ such that

$$\text{epi}\, k_{A_n} \cap \delta\, B_{X\times\mathbb{R}} \subset \text{epi} k_{A_n \cap \beta(\delta)\, B_{X\times\mathbb{R}}} \quad \forall n \geq n_c. \tag{37}$$

To this respect, consider $n \geq n_c$ and $(x, s) \in \text{epi}\, k_{A_n} \cap \delta\, B_{X\times\mathbb{R}}$. Hence, $\|x\| \leq \delta$; thus

$$\|x_0 - x\| \leq \|x_0\| + \|x\| \leq \|x_0\| + \delta.$$

It is then possible to apply relation (29) for $c = \|x_0\| + \delta$, and establish that

$$m_{\|x_0\|+\delta} \leq k_{A_n}(x).$$

Accordingly,

$$\|x, k_{A_n}(x)\|_B \leq \|x\| + |k_{A_n}(x)| \leq \delta + |m_{\|x_0\|+\delta}|.$$

We have therefore proved that relation (37) holds true, provided that $\beta(\delta) = \delta + |m_{\|x_0\|+\delta}|$.

Combining the inequality $\text{e}(B, A) \leq \text{e}(C, A)$, which obviously holds for any sets A, B and C such that $C \subset B$ and relation (37), we deduce that

$$\text{e}\left(\text{epi}\, k_{A_n} \cap \delta\, B_{X\times\mathbb{R}}, \text{epi} f\right) \leq \text{e}\left(\text{epi}\, k_{A_n \cap \beta(\delta)\, B_{X\times\mathbb{R}}}, \text{epi} f\right). \tag{38}$$

Let $(x, s) \in \text{epi}\, k_{A_n \cap \beta(\delta)\, B_{X\times\mathbb{R}}}$; clearly, $s \geq k_{A_n \cap \beta(\delta)\, B_{X\times\mathbb{R}}}(x)$, so, for any $(y, t) \in \text{epi} f$ it holds that $(y, t + s - k_{A_n \cap \beta(\delta)\, B_{X\times\mathbb{R}}}(x)) \in \text{epi} f$, and

$$\|(x, k_{A_n \cap \beta(\delta)\, B_{X\times\mathbb{R}}}(x)) - (y, t)\|_B = \|(x, s) - (y, t + s - k_{A_n \cap \beta(\delta)\, B_{X\times\mathbb{R}}}(x))\|_B;$$

the previous relation implies that

$$\text{e}\left(\text{epi}\, k_{A_n \cap \beta(\delta)\, B_{X\times\mathbb{R}}}, \text{epi} f\right) \leq \text{e}\left(A_n \cap \beta(\delta)\, B_{X\times\mathbb{R}}, \text{epi} f\right),$$

and, as the reverse inequality trivially holds, we conclude that

$$\text{e}\left(\text{epi}\, k_{A_n \cap \beta(\delta)\, B_{X\times\mathbb{R}}}, \text{epi} f\right) = \text{e}\left(A_n \cap \beta(\delta)\, B_{X\times\mathbb{R}}, \text{epi} f\right). \tag{39}$$

Relations (38) and (39) allow us to deduce that

$$\text{e}\left(\text{epi}\, k_{A_n} \cap \delta\, B_{X\times\mathbb{R}}, \text{epi} f\right) \leq \text{e}\left(A_n \cap \beta(\delta)\, B_{X\times\mathbb{R}}, \text{epi} f\right);$$

since the sequence $(A_n)_{n\in\mathbb{N}}$ Attouch-Wets converges to epi f, it results that the sequence $(\text{e}\left(A_n \cap \beta(\delta)\, B_{X\times\mathbb{R}}, \text{epi} f\right))_{n\in\mathbb{N}}$ goes to zero, so

$$\lim_{n\to\infty} \text{e}\left(\text{epi}\, k_{A_n} \cap \delta\, B_{X\times\mathbb{R}}, \text{epi} f\right) = 0 \quad \forall \delta > 0. \tag{40}$$

Relations (36) and (40) complete the proof of statement *ii*), and hence of Lemma 5. △

3.2. **The main result.** The following Proposition provides the main technical result of this article.

Proposition 5. *Let $f \in \Gamma_b(X)$ and $\eta > 0$. The application $\mathfrak{A}_\varepsilon$ is not Hausdorff upper semicontinuous at f for any $\varepsilon < \eta$, provided that the sublevel set $L_\eta = \{x \in X : f(x) \leq \inf_X f + \eta\}$ of f has an unbounded boundary.*

Proof of Proposition 5: Let $f \in \Gamma_b(X)$ such that the boundary of the sublevel set L_η is unbounded. The proof will proceed through the construction of a sequence $(f_n)_{n\in\mathbb{N}} \subset \Gamma_b(X)$, Attouch-Wets converging to f and satisfying:

$$\operatorname{dist}(x_n,\, L_\eta) \geq 1. \qquad \forall n \in \mathbb{N}, \tag{41}$$

for some $x_n \in \operatorname{argmin} f_n$.

In order to construct the sequence $(f_n)_{n\in\mathbb{N}}$ fulfilling the requirements of relation (41), let us apply the implication $i) \Rightarrow ii)$ from Lemma 1 to the closed and convex set L_η with unbounded boundary, and deduce that there is an unbounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that

$$1 \leq \operatorname{dist}(x_n,\, L_\eta) \leq 2.$$

Using, if necessary, a subsequence, (still denoted by $(x_n)_n$, as no confusion may occur), we may assume that $\|x_n\| \geq n^2$.
Since, $\operatorname{dist}(x_n,\, L_\eta) \leq 2$, and denoting by F the epigraph of f, we have

$$\operatorname{dist}\left(\left(x_n, \left(\inf_X f\right) - 1\right),\, F\right) \leq\ \eta + 3. \tag{42}$$

Next, set F_n for the convex hull of the singleton $(x_n, (\inf_X f) - 1)$ and the intersection between F and the ball of center $(\theta_X, 0)$ and radius n of $X \times \mathbb{R}$.

We claim that the sequence of lower-boundary functions $f_n = k_{F_n}$ fulfills relation (41).

As both the singleton $(x_n, (\inf_X f) - 1)$ and the epigraph of f belong to the half-space $X \times [(\inf_X f) - 1, +\infty[$, the same holds also for F_n; accordingly,

$$\inf_X f_n \geq \left(\inf_X f\right) - 1. \tag{43}$$

Recall that the set F_n contains the singleton $(x_n, (\inf_X f) - 1)$, and that f_n is the lower-boundary function of F_n to infer that

$$f_n(x_n) \leq \left(\inf_X f\right) - 1. \tag{44}$$

From relations (43) and (44) we obtain:

$$\min_X f_n = \left(\inf_X f\right) - 1 \quad \text{and} \quad x_n \in \operatorname{argmin}(f_n).$$

Finally, as $1 \leq \operatorname{dist}(x_n,\, F_\eta)$, we have constructed a sequence $(f_n)_{n\in\mathbb{N}}$ satisfying relation (41).

Let us conclude by establishing that the sequence $(f_n)_{n\in\mathbb{N}}$ Attouch-Wets converges to f.

Let us first apply Lemma 4 when the normed space $(X\times\mathbb{R},\, \|(\cdot,\cdot)\|_B)$ stands for $(Y, \|\cdot\|)$, F stands for the closed and convex set C with unbounded boundary, $((x_n, (\inf_X f)-1))_n$ stands for the unbounded sequence $(x_n)_{n\in\mathbb{N}}$, and $\eta+3$ stands for a (see relation (42)), to deduce that the sequence $(F_n)_{n\in\mathbb{N}}$ Attouch-Wets converges to F.

Finally, let us apply Lemma 5 to deduce that the sequence $(k_{F_n})_{n\in\mathbb{N}}$ Attouch-Wets converges to f.

Using the previous construction, we are now in position to conclude the proof of Proposition 5. For every $\varepsilon<\eta$ it follows that

$$\begin{aligned}\mathfrak{A}_\varepsilon(f) &= \left\{x\in X:\, f(x)\le \left(\inf_X f\right)+\varepsilon\right\}\\ &\subset \left\{x\in X:\, f(x)\le \left(\inf_X f\right)+\eta\right\} = L_\eta;\end{aligned}$$

thus, for any $S\in\mathcal{C}(X)$ it holds that

$$\begin{aligned}\mathrm{e}\,(S,\,\mathfrak{A}_\varepsilon(f)) &= \sup_{u\in S}\inf_{v\in\mathfrak{A}_\varepsilon(f)}\|u-v\|\\ &\ge \sup_{u\in S}\inf_{v\in L_\eta}\|u-v\|\\ &\ge \mathrm{dist}\,(u, L_\eta)\quad \forall u\in S.\end{aligned}\tag{45}$$

Since

$$x_n\in \mathrm{argmin} f_n\subset \mathfrak{A}_\varepsilon(f_n),\quad \forall\varepsilon>0,\,\forall\, n\in\mathbb{N},$$

it is possible to take $S=\mathfrak{A}_\varepsilon(f_n)$ and $u=x_n$ in relation (45), and therefore to deduce that

$$\mathrm{e}\,(\mathfrak{A}_\varepsilon(f_n),\,\mathfrak{A}_\varepsilon(f))\ge \mathrm{dist}(x_n,\,L_\eta)\quad \forall\varepsilon\in]0,\eta],\,\forall\, n\in\mathbb{N}.\tag{46}$$

According to relations (41) and (46), it follows that $\mathfrak{A}_\varepsilon$ fails to be upper semicontinuous at at f, proving in this way Proposition 5. △

The theorem coming next demonstrates that we can completely characterize the subclass of $\Gamma_b(X)$ of those functions with Ulam-stable ε-approximate minima.

Theorem 1. *Let $(X,\|\cdot\|)$ be a normed space. The application $\mathfrak{A}_\varepsilon$ is Hausdorff upper semicontinuous at f for any $\varepsilon>0$ if and only if the boundary of every sublevel set of f is bounded.*

Proof of Theorem 1: The "only if" part is an obvious consequence of Proposition 5. Let us move on to the "if" part.

Consider a function $f\in\Gamma_b(X)$ which possesses only bounded boundary sublevel sets. Choose any $\varepsilon>0$, and a sequence $(f_n)_{n\in\mathbb{N}}\subset\Gamma_b(X)$ Attouch-Wets converging to f. We have to prove that

$$\lim_{n\to\infty}\mathrm{e}\,(\mathfrak{A}_\varepsilon(f_n),\mathfrak{A}_\varepsilon(f))=0.\tag{47}$$

A standard argument yields that

$$\limsup_{n\to\infty} \inf_X (f_n) \leq \inf_X f. \tag{48}$$

Indeed, let us pick, for every $\alpha > 0$, a point $x_\alpha \in X$ such that

$$f(x_\alpha) \leq (\inf_X f) + \alpha.$$

As the sequence $(f_n)_{n\in\mathbb{N}}$ Attouch-Wets converges to f, it is possible to pick $(x_{\alpha,n}, s_{\alpha,n})_n \in \operatorname{epi} f_n$ such that $(x_{\alpha,n})_{n\in\mathbb{N}}$ converges to x_α and $(s_{\alpha,n})_{n\in\mathbb{N}}$ to $f(x_\alpha)$. Since $(x_{\alpha,n}, s_{\alpha,n})_n \in \operatorname{epi} f_n$, then

$$\inf_X (f_n) \leq f(x_{\alpha,n}) \leq s_{\alpha,n},$$

and therefore

$$\limsup_{n\to\infty} \inf_X (f_n) \leq \lim_{n\to\infty} s_{\alpha,n} = f(x_\alpha) \leq (\inf_X f) + \alpha \quad \forall \alpha > 0.$$

Hence relation (48) is established.

As a consequence of relation (48), it results that the sequence $(\omega_n)_{n\in\mathbb{N}}$ defined by

$$\omega_n = \max\left(\inf_X (f_n) - \inf_X f,\, 0\right)$$

converges to 0. Invoking the sum theorem [9, Theorem 7.4.5, page 260] we claim that the sequence $(f_n - \omega_n)_{n\in\mathbb{N}}$ Attouch-Wets converges to f. Next, using this fact and applying a theorem by Beer & Luchetti [11, Theorem 3.6] we deduce that the sequence of level sets

$$L_{\varepsilon,n} = \{x \in X : f_n(x) \leq \omega_n + \inf_X f + \varepsilon\}$$

Attouch-Wets converges to

$$L_\varepsilon = \{x \in X : f(x) \leq \inf_X f + \varepsilon\}.$$

Since by assumption the boundary of L_ε is bounded, applying Proposition 1 we deduce the upper semi-continuity at L_ε of the mapping ι. Accordingly,

$$\lim_{n\to\infty} \mathrm{e}\,(L_{\varepsilon,n}, L_\varepsilon) = 0. \tag{49}$$

Finally, remarking that $\inf_X (f_n) \leq \omega_n + \inf_X f$, we deduce that $\mathfrak{A}_\varepsilon(f_n) \subset L_{\varepsilon,n}$; as, on the other hand, $\mathfrak{A}_\varepsilon(f) = L_\varepsilon$ the desired relation (47) follows from relation (49) and the proof is established. △

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