THE MATRICIAL RELAXATION OF A LINEAR MATRIX INEQUALITY

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Abstract. Given linear matrix inequalities (LMIs) L_1 and L_2 it is natural to ask:

(Q_1) when does one dominate the other, that is, does $L_1(X) \succeq 0$ imply $L_2(X) \succeq 0$?

(Q_2) when are they mutually dominant, that is, when do they have the same solution set?

The matrix cube problem of Ben-Tal and Nemirovski [B-TN02] is an example of LMI domination. Hence such problems can be NP-hard. This paper describes a natural relaxation of an LMI, based on substituting matrices for the variables x_j. With this relaxation, the domination questions (Q_1) and (Q_2) have elegant answers, indeed reduce to constructible semidefinite programs. As an example, to test the strength of this relaxation we specialize it to the matrix cube problem and obtain essentially the relaxation given in [B-TN02]. Thus our relaxation could be viewed as generalizing it.

Assume there is an X such that $L_1(X)$ and $L_2(X)$ are both positive definite, and suppose the positivity domain of L_1 is bounded. For our "matrix variable" relaxation a positive answer to (Q_1) is equivalent to the existence of matrices V_j such that

$$L_2(x) = V_1^* L_1(x) V_1 + \cdots + V_\mu^* L_1(x) V_\mu. \qquad (A_1)$$

As for (Q_2) we show that L_1 and L_2 are mutually dominant if and only if, up to certain redundancies described in the paper, L_1 and L_2 are unitarily equivalent. Algebraic certificates for positivity, such as (A_1) for linear polynomials, are typically called Positivstellensätze. The paper goes on to derive a Putinar-type Positivstellensatz for polynomials with a cleaner and more powerful conclusion under the stronger hypothesis of positivity on an underlying bounded domain of the form $\{X \mid L(X) \succeq 0\}$.

An observation at the core of the paper is that the relaxed LMI domination problem is equivalent to a classical problem. Namely, the problem of determining if a linear map τ from a subspace of matrices to a matrix algebra is "completely positive". Complete positivity is one of the main techniques of modern operator theory and the theory of operator algebras. On one hand it provides tools for studying LMIs and on the other hand, since completely positive maps are not so far from representations and generally are more tractable than their merely positive counterparts, the theory of completely positive maps provides perspective on the difficulties in solving LMI domination problems.

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1. Introduction and the statement of the main results

In this section we state most of our main results of the paper. We begin with essential definitions.

1.1. Linear pencils and LMI sets.

For symmetric matrices $A_0, A_1, \dots, A_g \in \mathbb{SR}^{d\times d}$, the expression

$$L(x) = A_0 + \sum_{j=1}^{g} A_j x_j \in \mathbb{SR}^{d\times d}\langle x\rangle \tag{1.1}$$

in noncommuting variables x, is a **linear pencil**. If $A_0 = I$, then L is **monic**. If $A_0 = 0$, then L is a **truly linear pencil**. The truly linear part $\sum_{j=1}^{g} A_j x_j$ of a linear pencil L as in (1.1) will be denoted by $L^{(1)}$.

Given a block column matrix $X = \mathrm{col}(X_1, \dots, X_g) \in (\mathbb{SR}^{n\times n})^g$, the **evaluation** $L(X)$ is defined as

$$L(X) = A_0 \otimes I_n + \sum A_j \otimes X_j \in \mathbb{SR}^{dn\times dn}. \tag{1.2}$$

The tensor product in this expressions is the usual (Kronecker) tensor product of matrices. We have reserved the tensor product notation for the tensor product of matrices and have eschewed the strong temptation of using $A\otimes x_\ell$ in place of Ax_ℓ when x_ℓ is one of the variables.

Let L be a linear pencil. Its **matricial linear matrix inequality (LMI) set** (also called a **matricial positivity domain**) is

$$\mathcal{D}_L := \bigcup_{n\in\mathbb{N}} \{X \in (\mathbb{SR}^{n\times n})^g \mid L(X) \succeq 0\}. \tag{1.3}$$

Let

$$\mathcal{D}_L(n) = \{X \in (\mathbb{SR}^{n\times n})^g \mid L(X) \succeq 0\} = \mathcal{D}_L \cap (\mathbb{SR}^{n\times n})^g, \tag{1.4}$$

$$\partial\mathcal{D}_L(n) = \{X \in (\mathbb{SR}^{n\times n})^g \mid L(X) \succeq 0,\ L(X) \not\succ 0\}, \tag{1.5}$$

$$\partial\mathcal{D}_L = \bigcup_{n\in\mathbb{N}} \partial\mathcal{D}_L(n). \tag{1.6}$$

The set $\mathcal{D}_L(1) \subseteq \mathbb{R}^g$ is the feasibility set of the semidefinite program $L(X) \succeq 0$ and is called a **spectrahedron** by algebraic geometers.

We call $\mathcal{D}_L$ **bounded** if there is an $N \in \mathbb{N}$ with $\|X\| \le N$ for all $X \in \mathcal{D}_L$. We shall see later below (Proposition 2.4) that $\mathcal{D}_L$ is bounded if and only if $\mathcal{D}_L(1)$ is bounded.

1.2. Main results on LMIs.

Here we state our main theorems giving precise algebraic characterizations of (matricial) LMI domination. While the main theme of this article is that matricial LMI domination problems are more tractable than their traditional scalar counterparts, the reader interested only in algorithms for the scalar setting can proceed to the following subsection, §1.3, and then onto Section 4.

Theorem 1.1 (Linear Positivstellensatz)**.** *Let $L_j \in \mathbb{SR}^{d_j\times d_j}\langle x\rangle$, $j=1,2$, be monic linear pencils and assume $\mathcal{D}_{L_1}$ is bounded. Then $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$ if and only if there is a $\mu \in \mathbb{N}$ and an isometry $V \in \mathbb{R}^{\mu d_1 \times d_2}$ such that*

$$L_2(x) = V^*\big(I_\mu \otimes L_1(x)\big)V. \tag{1.7}$$

Suppose $L \in \mathbb{SR}^{d\times d}\langle x\rangle$,

$$L = I + \sum_{j=1}^{g} A_j x_j$$

is a monic linear pencil. A subspace $\mathcal{H} \subseteq \mathbb{R}^d$ is *reducing for* L if $\mathcal{H}$ reduces each A_j; i.e., if $A_j\mathcal{H} \subseteq \mathcal{H}$. Since each A_j is symmetric, it also follows that $A_j\mathcal{H}^\perp \subseteq \mathcal{H}^\perp$. Hence, with respect to the decomposition $\mathbb{R}^d = \mathcal{H} \oplus \mathcal{H}^\perp$, L can be written as the direct sum,

$$L = \tilde{L} \oplus \tilde{L}^\perp = \begin{bmatrix} \tilde{L} & 0 \\ 0 & \tilde{L}^\perp \end{bmatrix}, \quad \text{where} \quad \tilde{L} = I + \sum_{j=1}^{g} \tilde{A}_j x_j,$$

and $\tilde{A}_j$ is the restriction of A_j to $\mathcal{H}$. (The pencil $\tilde{L}^\perp$ is defined similarly.) If $\mathcal{H}$ has dimension ℓ, then by identifying $\mathcal{H}$ with $\mathbb{R}^\ell$, the pencil $\tilde{L}$ is a monic linear pencil of size ℓ. We say that $\tilde{L}$ is a *subpencil* of L. If moreover, $\mathcal{D}_L = \mathcal{D}_{\tilde{L}}$, then $\tilde{L}$ is a *defining subpencil* and if no proper subpencil of $\tilde{L}$ is defining subpencil for $\mathcal{D}_L$, then $\tilde{L}$ is a *minimal defining (sub)pencil*.

Theorem 1.2 (Linear Gleichstellensatz)**.** *Let $L_j \in \mathbb{SR}^{d\times d}\langle x\rangle$, $j=1,2$, be monic linear pencils with $\mathcal{D}_{L_1}$ bounded. Then $\mathcal{D}_{L_1} = \mathcal{D}_{L_2}$ if and only if minimal defining pencils $\tilde{L}_1$ and $\tilde{L}_2$ for $\mathcal{D}_{L_1}$ and $\mathcal{D}_{L_2}$ respectively, are unitarily equivalent. That is, there is a unitary matrix U such that*

$$\tilde{L}_2(x) = U^*\tilde{L}_1(x)U. \tag{1.8}$$

An observation at the core of these results is that the relaxed LMI domination problem is equivalent to the problem of determining if a linear map τ from a subspace of matrices to a matrix algebra is *completely positive.*

1.3. Algorithms for LMIs.

Of widespread interest is determining if

$$\mathcal{D}_{L_1}(1) \subseteq \mathcal{D}_{L_2}(1), \tag{1.9}$$

or if $\mathcal{D}_{L_1}(1) = \mathcal{D}_{L_2}(1)$. For example, the paper of Ben-Tal and Nemirovski [B-TN02] exhibits simple cases where determining this is NP-hard. We explicitly give (in Section 4.1) a certain semidefinite program whose feasibility is equivalent to $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$. Of course, if $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$, then $\mathcal{D}_{L_1}(1) \subseteq \mathcal{D}_{L_2}(1)$. Thus our algorithm is a type of relaxation of the problem (1.9). The algorithms in this section can be read immediately after reading Section 1.3.

We also have an SDP algorithm (Section 4.4) easily adapted from the first to determine if $\mathcal{D}_L$ is bounded, and what its "radius" is. Proposition 2.4 shows that $\mathcal{D}_L$ is bounded if and only if $\mathcal{D}_L(1)$ is bounded. Thus our algorithm definitively tells if $\mathcal{D}_L(1)$ is a bounded set; in addition it yields an upper bound on the radius of $\mathcal{D}_L(1)$.

In Section 4.5 we specialize our relaxation to solve a matricial relaxation of the classical matrix cube problem, finding the biggest matrix cube contained in $\mathcal{D}_L$. It turns out, as shown in Section 5, that our matricial relaxation is essentially that of [B-TN02]. Thus the our LMI inclusion relaxation could be viewed as a generalization of theirs, indeed a highly canonical one, in light of the precise correspondence to classical complete positivity theory shown in §3. A potential advantage of our relaxation is that there are possibilities for strengthening it, presented generally in Section 4.2 and illustrated on the matrix cube in Section 5.2.

Finally, given a matricial LMI set $\mathcal{D}_L$, Section 4.6 gives an algorithm to compute the linear pencil $\tilde{L} \in \mathbb{SR}^{d\times d}\langle x\rangle$ with smallest possible d satisfying $\mathcal{D}_L = \mathcal{D}_{\tilde{L}}$.

1.4. **Positivstellensatz.** Algebraic characterizations of polynomials p which are positive on $\mathcal{D}_L$ are called Positivstellensätze and are classical for polynomials on $\mathbb{R}^g$. This theory underlies the main approach currently used for global optimization of polynomials, cf. [Las09, Par03]. The generally noncommutative techniques in this paper lead to a cleaner and more powerful commutative Putinar-type Positivstellensatz [Put93] for p strictly positive on a bounded spectrahedron $\mathcal{D}_L(1)$. In the theorem which follows, $\mathbb{SR}^{d\times d}[y]$ is the set of symmetric $d \times d$ matrices with entries from $\mathbb{R}[y]$, the algebra of (commutative) polynomials with coefficients from $\mathbb{R}$. Note that an element of $\mathbb{SR}^{d\times d}[y]$ may be identified with a polynomial (in commuting variables) with coefficients from $\mathbb{SR}^{d\times d}$.

Theorem 1.3. *Suppose $L \in \mathbb{SR}^{d\times d}[y]$ is a monic linear pencil and $\mathcal{D}_L(1)$ is bounded. Then for every symmetric matrix polynomial $p \in \mathbb{R}^{\ell\times\ell}[y]$ with $p|_{\mathcal{D}_L(1)} \succ 0$, there are $A_j \in \mathbb{R}^{\ell\times\ell}[y]$, and $B_k \in \mathbb{R}^{d\times\ell}[y]$ satisfying*

$$p = \sum_j A_j^* A_j + \sum_k B_k^* L B_k. \tag{1.10}$$

We also consider symmetric (matrices of) polynomials p in noncommuting variables with the property that $p(X)$ is positive definite for all X in a bounded matricial LMI set $\mathcal{D}_L$; see Section 6. For such noncommutative (NC) polynomials (and for even more general algebras of polynomials, see Section 7) we obtain a Positivstellensatz (Theorem 6.1) analogous to (1.10). In the case that the polynomial p is linear, this Positivstellensatz reduces to Theorem 1.1, which can be regarded as a “Linear Positivstellensatz”. For perspective we mention that the proofs of our Positivstellensätze actually rely on the linear Positivstellensatz. For experts we point out that the key reason LMI sets behave better is that the quadratic module associated to a monic linear pencil L with bounded $\mathcal{D}_L$ is archimedean.

1.5. **Outline.** The paper is organized as follows. Section 2 collects a few basic facts about linear pencils and LMIs. In Section 3, inclusion and equality of matricial LMI sets are characterized and our results are then applied in the algorithmic Section 4. Section 5 gives some further details about matricial relaxations of the matrix cube problem. The last two sections give algebraic certificates for polynomials to be positive on LMI sets.

2. Preliminaries on LMIs

This section collects a few basic facts about linear pencils and LMIs.

Proposition 2.1. *If L is a linear pencil and $\mathcal{D}_L$ contains 0 as an interior point, i.e., $0 \in \mathcal{D}_L \smallsetminus \partial\mathcal{D}_L$, then there is a monic pencil $\widehat{L}$ with $\mathcal{D}_L = \mathcal{D}_{\widehat{L}}$.*

Proof. As $0 \in \mathcal{D}_L$, $L(0) = A_0$ is positive semidefinite. Since $0 \notin \partial\mathcal{D}_L$, $A_0 \succeq \varepsilon A_j$ for some small $\varepsilon \in \mathbb{R}_{>0}$ and all j. Let $V = \operatorname{Ran} A_0 \subseteq \mathbb{R}^d$, and set

$$\widetilde{A}_j := A_j|_V \quad \text{for} \quad j = 0, 1, \dots, g.$$

Clearly, $\widetilde{A}_0 : V \to V$ is invertible and thus positive definite. We next show that $\operatorname{Ran} A_0$ contains $\operatorname{Ran} A_j$ for $j \geq 1$. If $x \perp \operatorname{Ran} A_0$, i.e., $A_0 x = 0$, then $0 = x^* A_0 x \geq \pm \varepsilon x^* A_j x$ and hence $x^* A_j x = 0$. Since $A_0 + \varepsilon A_j \geq 0$ and $x^*(A_0 + \varepsilon A_j)x = 0$ it follows that $(A_0 + \varepsilon A_j)x = 0$, and since $A_0 x = 0$, we finally conclude that $A_j x = 0$, i.e., $x \perp \operatorname{Ran} A_j$. Consequently, $\widetilde{A}_j : V \to V$ are all symmetric and $\mathcal{D}_L = \mathcal{D}_{\widetilde{L}}$ for $\widetilde{L} = \widetilde{A}_0 + \sum_{j=1}^g \widetilde{A}_j x_j$.

To build $\widehat{L}$, factor $\widetilde{A}_0 = B^* B$ with B invertible and set

$$\widehat{A}_j := B^{-*} \widetilde{A}_j B^{-1} \quad \text{for} \quad j = 0, \dots, g.$$

The resulting pencil $\widehat{L} = I + \sum_{j=1}^g \widehat{A}_j x_j$ is monic and $\mathcal{D}_L = \mathcal{D}_{\widehat{L}}$. ■

Our primary focus will be on the matricial LMI sets $\mathcal{D}_L$. If the spectrahedron $\mathcal{D}_L(1) \subseteq \mathbb{R}^g$ does not contain interior points, then (as it is a convex set) it is contained in a proper affine subspace of $\mathbb{R}^g$. By reducing the number of variables we arrive at a new pencil whose spectrahedron *does* have an interior point. By a translation we can ensure that 0 is an interior point. Then Proposition 2.1 applies and yields a monic linear pencil with the same matricial LMI set. This reduction enables us to concentrate only on monic linear pencils in the sequel.

Lemma 2.2. *Let $L \in \mathbb{SR}^{d\times d}\langle x\rangle$ be a linear pencil with $\mathcal{D}_L$ bounded, and let $\widehat{L} \in \mathbb{SR}^{n\times n}\langle x\rangle$ be another linear pencil. Set $s := n(1+g)$. Then:*

(1) *$\widehat{L}|_{\mathcal{D}_L} \succ 0$ if and only if $\widehat{L}|_{\mathcal{D}_L(s)} \succ 0$;*
(2) *$\widehat{L}|_{\mathcal{D}_L} \succeq 0$ if and only if $\widehat{L}|_{\mathcal{D}_L(s)} \succeq 0$.*

Proof. In both statements the direction ($\Rightarrow$) is obvious. If $\widehat{L}|_{\mathcal{D}_L} \not\succ 0$, there is an ℓ, $X \in \mathcal{D}_L(\ell)$ and $v = \oplus_{j=1}^n v_j \in (\mathbb{R}^\ell)^n$ with

$$\langle \widehat{L}(X)v, v\rangle \leq 0.$$

Let

$$\mathcal{K} := \operatorname{span}\big(\{X_i v_j \mid i = 1, \dots, g,\ j = 1, \dots, n\} \cup \{v_j \mid j = 1, \dots, n\}\big).$$

Clearly, $\dim \mathcal{K} \leq s$. Let P be the orthogonal projection of $\mathbb{R}^\ell$ onto $\mathcal{K}$. Then

$$\langle \widehat{L}(PXP)v, v\rangle = \langle \widehat{L}(X)v, v\rangle \leq 0.$$

Since $PXP \in \mathcal{D}_L(s)$, this proves (1). The proof of (2) is the same. ■

Lemma 2.3. *Let L be a linear pencil. Then*

$$\mathcal{D}_L \text{ is bounded} \quad \Leftrightarrow \quad \mathcal{D}_L\big((1+g)^2\big) \text{ is bounded.}$$

Proof. Given a positive $N \in \mathbb{N}$, consider the monic linear pencil

$$\mathcal{J}_N(x) = \frac{1}{N}\begin{bmatrix} N & x_1 & \cdots & x_g \\ x_1 & N & & \\ \vdots & & \ddots & \\ x_g & & & N \end{bmatrix} = \frac{1}{N}\begin{bmatrix} N & x^* \\ x & NI_g \end{bmatrix} \in \mathbb{SR}^{(g+1)\times(g+1)}\langle x\rangle.$$

Note that $\mathcal{D}_L$ is bounded if and only if for some $N \in \mathbb{N}$, $\mathcal{J}_N|_{\mathcal{D}_L} \succeq 0$. The statement of the lemma now follows from Lemma 2.2. ∎

To the linear pencil L we can also associate its **matricial ball**

$$\mathcal{B}_L := \bigcup_{n\in\mathbb{N}} \{X \in (\mathbb{SR}^{n\times n})^g \mid \|L(X)\| \leq 1\} = \{X \mid I - L(X)^2 \succeq 0\}.$$

Observe that $\mathcal{B}_L = \mathcal{D}_{L'}$ for

$$L' = \begin{bmatrix} I & L \\ L & I \end{bmatrix}. \tag{2.1}$$

Proposition 2.4. *Let L be a linear pencil. Then:*

(1) *$\mathcal{D}_L$ is bounded if and only if $\mathcal{D}_L(1)$ is bounded;*
(2) *$\mathcal{B}_L$ is bounded if and only if $\mathcal{B}_L(1)$ is bounded.*

Proof. (1) The implication ($\Rightarrow$) is obvious. For the converse suppose $\mathcal{D}_L$ is unbounded. By Lemma 2.3, this means $\mathcal{D}_L(N)$ is unbounded for some $N \in \mathbb{N}$. Then there exists a sequence $(X^{(k)})$ from $(\mathbb{SR}^{N\times N})^g$ such that $\|X^{(k)}\| = 1$ and a sequence $t_k \in \mathbb{R}_{>0}$ tending to ∞ such that $L(t_k X^{(k)}) \succeq 0$. A subsequence of $(X^{(k)})$ converges to $X = (X_1, \ldots, X_g) \in (\mathbb{SR}^{N\times N})^g$ which also has norm 1. For any t, $tX^{(k)} \to tX$ and for k big enough, $tX^{(k)} \in \mathcal{D}_L$ by convexity. So X satisfies $L(tX) \succeq 0$ for all $t \in \mathbb{R}_{\geq 0}$.

There is a nonzero vector v so that $\langle X_i v, v\rangle \neq 0$ for at least one i. Then with $Z := (\langle X_1 v, v\rangle, \ldots, \langle X_g v, v\rangle) \in \mathbb{R}^g \setminus \{0\}$, and V denoting the map $V : \mathbb{R} \to \mathbb{R}^N$ defined by $Vr = rv$,

$$L(tZ) = (I \otimes V)^* L(tX)(I \otimes V)$$

is nonnegative for all $t > 0$, so $\mathcal{D}_L(1)$ is unbounded.

To conclude the proof observe that (2) is immediate from (1) using (2.1). ∎

A linear pencil L is **nondegenerate**, if it is one-one in that $L(X) = L(Y)$ implies $X = Y$ for all $n \in \mathbb{N}$ and $X, Y \in (\mathbb{SR}^{n\times n})^g$. In particular, a truly linear pencil L is nondegenerate if and only if $L(X) \neq 0$ for $X \neq 0$.

Lemma 2.5. *For a linear pencil $L(x) = A_0 + \sum_{j=1}^g A_j x_j$ the following are equivalent:*

(i) *L is nondegenerate;*

(ii) $L(Z) = L(W)$ *implies* $Z = W$ *for all* $Z, W \in \mathbb{R}^g$*;*
(iii) *the set* $\{A_j \mid j = 1, \dots, g\}$ *is linearly independent;*
(iv) $L^{(1)}$ *is nondegenerate.*

Proof. Clearly, (i) $\Leftrightarrow$ (iv). Also, (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious. For the remaining implication (iii) $\Rightarrow$ (i), assume $L(X) = L(Y)$ for some $X, Y \in (\mathbb{SR}^{n\times n})^g$. Equivalently, $L^{(1)}(X - Y) = 0$. Note that $L^{(1)}(X - Y)$ equals $\sum_{j=1}^{g}(X_j - Y_j) \otimes A_j$ modulo the canonical shuffle. If this expression equals 0, then the linear independence of the $A_1, \dots, A_g$ (applied entrywise) implies $X = Y$. ∎

Proposition 2.6. *Let* $L = I + \sum_{j=1}^{g} A_j x_j \in \mathbb{SR}^{d\times d}\langle x\rangle$ *be a monic linear pencil and let* $L^{(1)}$ *denote its truly linear part. Then:*

(1) $\mathcal{B}_{L^{(1)}}$ *is bounded if and only if* $L^{(1)}$ *is nondegenerate;*
(2) *if* $\mathcal{D}_L$ *is bounded then* $\{I, A_j \mid j = 1, \dots, g\}$ *is linearly independent; the converse fails in general.*

Proof. (1) Suppose $L^{(1)}$ is not nondegenerate, say $\sum_{j=1}^{g} z_j A_j = 0$ for some $z_j \in \mathbb{R}$. Then with $Z = (z_1, \dots, z_g) \in \mathbb{R}^g$ we have $tZ \in \mathcal{B}_{L^{(1)}}$ for every t, so $\mathcal{B}_{L^{(1)}}$ is not bounded. Let us now prove the converse. First, if $\mathcal{B}_{L^{(1)}}$ is unbounded, then by Proposition 2.4, $\mathcal{B}_{L^{(1)}}(1)$ is unbounded. So suppose $\mathcal{B}_{L^{(1)}}(1)$ is unbounded. Then there exists a sequence $(Z^{(k)})$ from $\mathbb{R}^g$ such that $\|Z^{(k)}\| = 1$ and a sequence $t_k \in \mathbb{R}_{>0}$ tending to ∞ such that $\|L^{(1)}(t_k Z^{(k)})\| \le 1$. A subsequence of $(Z^{(k)})$ converges to $Z \in \mathbb{R}^g$ which also has norm 1; however, $\|L^{(1)}(Z)\| = 0$ and thus $L^{(1)}$ is degenerate.

For (2) assume

$$\lambda + \sum_j x_j A_j = 0 \tag{2.2}$$

with $\lambda, x_j \in \mathbb{R}$. We may assume $x_j \neq 0$ for at least one index j. Let $Z = (x_1, \dots, x_g) \neq 0$. If $\lambda = 0$, then $L(tZ) = I$ is positive semidefinite for all $t \in \mathbb{R}$. Thus $\mathcal{D}_L$ is not bounded.

Now let $\lambda \in \mathbb{R}$ be nonzero. Then $L(Z/\lambda) = 0$. Thus, $L(tZ/\lambda) \succeq 0$ for all $t < 0$, showing $\mathcal{D}_L$ is unbounded.

The converse of (2) fails in general. For instance, if the A_j are positive semidefinite, then $\mathcal{D}_L$ contains $(\mathbb{R}_{\ge 0})^g$ and thus cannot be bounded. ∎

3. Matricial LMI sets: Inclusion and Equality

Given L_1 and L_2 monic linear pencils

$$L_j(x) = I + \sum_{\ell=1}^{g} A_{j,\ell} x_\ell \in \mathbb{SR}^{d_j\times d_j}\langle x\rangle, \quad j = 1, 2, \tag{3.1}$$

we shall consider the following two inclusions for matricial LMI sets:

$$\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}; \tag{3.2}$$

$$\partial \mathcal{D}_{L_1} \subseteq \partial \mathcal{D}_{L_2}. \tag{3.3}$$

Equation (3.2) is equivalent to: for all $n \in \mathbb{N}$ and $X \in (\mathbb{SR}^{n\times n})^g$,

$$L_1(X) \succeq 0 \quad \Rightarrow \quad L_2(X) \succeq 0.$$

Similarly, (3.3) can be rephrased as follows:

$$L_1(X) \succeq 0 \quad \text{and} \quad L_1(X) \not\succ 0 \quad \Rightarrow \quad L_2(X) \succeq 0 \quad \text{and} \quad L_2(X) \not\succ 0.$$

In this section we characterize precisely the relationship between L_1 and L_2 satisfying (3.2) and (3.3). Section 3.1 handles (3.2) and gives a Positivstellensatz for linear pencils. Section 3.3 shows that "minimal" pencils L_1 and L_2 satisfying (3.3) are the same up to unitary equivalence.

Example 3.1. By Lemma 2.2 it is enough to test condition (3.2) on matrices of some fixed (large enough) size. It is, however, not enough to test on $X \in \mathbb{R}^g$. For instance, let

$$\Delta(x_1,x_2) = I + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} x_2 = \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{bmatrix} \in \mathbb{SR}^{3\times 3}\langle x\rangle$$

and

$$\Gamma(x_1,x_2) = I + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_2 = \begin{bmatrix} 1+x_1 & x_2 \\ x_2 & 1-x_1 \end{bmatrix} \in \mathbb{SR}^{2\times 2}\langle x\rangle.$$

Then

$$\begin{aligned} \mathcal{D}_\Delta &= \{(X_1,X_2) \mid 1 - X_1^2 - X_2^2 \succeq 0\}, \\ \mathcal{D}_\Delta(1) &= \{(X_1,X_2) \in \mathbb{R}^2 \mid X_1^2 + X_2^2 \le 1\}, \\ \mathcal{D}_\Gamma(1) &= \{(X_1,X_2) \in \mathbb{R}^2 \mid X_1^2 + X_2^2 \le 1\}. \end{aligned}$$

Thus $\mathcal{D}_\Delta(1) = \mathcal{D}_\Gamma(1)$. On one hand,

$$\left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{3}{4} \\ \frac{3}{4} & 0 \end{bmatrix} \right) \in \mathcal{D}_\Delta \smallsetminus \mathcal{D}_\Gamma,$$

so $\Delta(X_1,X_2) \succeq 0$ does not imply $\Gamma(X_1,X_2) \succeq 0$. On the other hand, $\Gamma(X_1,X_2) \succeq 0$ does imply $\Delta(X_1,X_2) \succeq 0$. We shall prove this later below, see Example 3.4. ☐

We now introduce subspaces to be used in our considerations:

$$\mathcal{S}_j = \operatorname{span}\{I, A_{j,\ell} \mid \ell = 1,\dots,g\} \subseteq \mathbb{SR}^{d_j\times d_j}. \tag{3.4}$$

Lemma 3.2. $\mathcal{S}_j = \operatorname{span}\{L_j(X) \mid X \in \mathbb{R}^g\}$.

The key tool in studying inclusions of matricial LMI sets is the mapping τ we now define.

Definition 3.3. Let L_1, L_2 be monic linear pencils as in (3.1). If $\{I, A_{1,\ell} \mid \ell = 1,\dots,g\}$ is linearly independent (e.g. $\mathcal{D}_{L_1}$ is bounded), we define the unital linear map

$$\tau : \mathcal{S}_1 \to \mathcal{S}_2, \quad A_{1,\ell} \mapsto A_{2,\ell}. \tag{3.5}$$

We shall soon see that, assuming (3.2), τ has a property called complete positivity, which we now introduce. Let $\mathcal{S}_j \subseteq \mathbb{R}^{d_j\times d_j}$ be unital linear subspaces invariant under the transpose, and $\phi : \mathcal{S}_1 \to \mathcal{S}_2$ a unital linear $*$-map. For $n \in \mathbb{N}$, ϕ induces the map

$$\phi_n = I_n \otimes \phi : \mathbb{R}^{n\times n} \otimes \mathcal{S}_1 = \mathcal{S}_1^{n\times n} \to \mathcal{S}_2^{n\times n}, \quad M \otimes A \mapsto M \otimes \phi(A),$$

called an **ampliation** of ϕ. Equivalently,

$$\phi_n\left(\begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn}\end{bmatrix}\right) = \begin{bmatrix} \phi(T_{11}) & \cdots & \phi(T_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(T_{n1}) & \cdots & \phi(T_{nn})\end{bmatrix}$$

for $T_{ij} \in \mathcal{S}_1$. We say that ϕ is **k-positive** if ϕ_k is a positive map. If ϕ is k-positive for every $k \in \mathbb{N}$, then ϕ is **completely positive**. If ϕ_k is an isometry for every k, then ϕ is **completely isometric**.

Example 3.4 (Example 3.1 revisited). The map $\tau : \mathcal{S}_2 \to \mathcal{S}_1$ in our example is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Consider the extension of τ to a unital linear $*$-map $\psi : \mathbb{R}^{2\times 2} \to \mathbb{R}^{3\times 3}$, defined by

$$E_{11} \mapsto \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_{12} \mapsto \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \; E_{21} \mapsto \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \; E_{22} \mapsto \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(Here E_{ij} are the 2×2 matrix units.) Now we show the map ψ is completely positive. To do this, we use its Choi matrix defined as

$$C = \begin{bmatrix} \psi(E_{11}) & \psi(E_{12}) \\ \psi(E_{21}) & \psi(E_{22}) \end{bmatrix}. \tag{3.6}$$

[Pau02, Theorem 3.14] says ψ is completely positive if and only if $C \succeq 0$. We will use the Choi matrix again in Section 4 for computational algorithms. To see that C is positive semidefinite, note

$$C = \frac{1}{2}W^*W \quad \text{for} \quad W = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}.$$

Now ψ has a very nice representation:

$$\psi(S) = \frac{1}{2}V_1^*SV_1 + \frac{1}{2}V_2^*SV_2 = \frac{1}{2}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^* \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{3.7}$$

for all $S \in \mathbb{R}^{2\times 2}$. (Here $V_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, thus $W = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$.) In particular,

$$2\Delta(x,y) = V_1^*\Gamma(x,y)V_1 + V_2^*\Gamma(x,y)V_2. \tag{3.8}$$

Hence $\Gamma(X_1, X_2) \succeq 0$ implies $\Delta(X_1, X_2) \succeq 0$, i.e., $\mathcal{D}_\Gamma \subseteq \mathcal{D}_\Delta$.

The formula (3.8) illustrates our linear Positivstellensatz which is the subject of the next subsection. The construction of the formula in this example is a concrete implementation of the theory leading up to the general result that is presented in Corollary 3.7. □

3.1. **The map τ is completely positive: Linear Positivstellensatz.** We begin by equating n-positivity of τ with inclusion $\mathcal{D}_{L_1}(n) \subseteq \mathcal{D}_{L_2}(n)$. Then we use the complete positivity of τ to give an algebraic characterization of pencils L_1, L_2 producing an inclusion $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$.

Theorem 3.5. *Let*

$$L_j(x) = I + \sum_{\ell=1}^{g} A_{j,\ell} x_\ell \in \mathbb{SR}^{d_j \times d_j}\langle x\rangle, \quad j = 1, 2$$

be monic linear pencils and assume the matricial LMI set $\mathcal{D}_{L_1}$ is bounded. Let $\tau : \mathcal{S}_1 \to \mathcal{S}_2$ be the unital linear map $A_{1,\ell} \mapsto A_{2,\ell}$.

(1) *τ is n-positive if and only if $\mathcal{D}_{L_1}(n) \subseteq \mathcal{D}_{L_2}(n)$;*
(2) *τ is completely positive if and only if $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$;*
(3) *τ is completely isometric if and only if $\partial\mathcal{D}_{L_1} \subseteq \partial\mathcal{D}_{L_2}$,*

We remark that the binding condition (3.3) used in (3) implies (3.2) used in (2) under the boundedness assumption; see Proposition 3.9. The proposition says that the relaxed domination problem (see the abstract) can be restated in terms of complete positivity, under a boundedness assumption. Conversely, suppose $\mathcal{D}$ is a unital (self-adjoint) subspace of $\mathbb{SR}^{d\times d}$ and $\tau : \mathcal{D} \to \mathbb{SR}^{d'\times d'}$ is completely positive. Given a basis $\{I, A_1, \dots, A_g\}$ for $\mathcal{D}$, let $B_j = \tau(A_j)$. Let

$$L_1 = I + \sum A_j x_j, \quad L_2 = I + \sum B_j x_j.$$

The complete positivity of τ implies, if $L_1(X) \succeq 0$, then $L_2(X) \succeq 0$ and hence $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$. Hence the completely positive map τ (together with a choice of basis) gives rise to an LMI domination.

To prove the theorem we need a lemma.

Lemma 3.6. *Let $L = I + \sum_{j=1}^{g} A_j x_j \in \mathbb{SR}^{d\times d}\langle x\rangle$ be a monic linear pencil with bounded matricial LMI set $\mathcal{D}_L$. Then:*

(1) *if $\Lambda \in \mathbb{R}^{n\times n}$ and $X \in (\mathbb{SR}^{n\times n})^g$, and if*

$$S := I \otimes \Lambda + L^{(1)}(X) \tag{3.9}$$

is symmetric, then $\Lambda = \Lambda^*$;

(2) *if $S \succeq 0$, then $\Lambda \succeq 0$;*
(3) *if $\Lambda \in \mathbb{R}^{n\times n}$ and $X \in (\mathbb{SR}^{n\times n})^g$, and if*

$$T := \Lambda \otimes I + \sum_{j=1}^{g} X_j \otimes A_j \succeq 0 \tag{3.10}$$

then $\Lambda \succeq 0$.

Proof. To prove item (1), suppose

$$S = I \otimes \Lambda + \sum_{j=1}^{g} A_j \otimes X_j$$

is symmetric. Then $0 = S - S^* = I \otimes (\Lambda - \Lambda^*)$. Hence $\Lambda = \Lambda^*$.

For (2), if $\Lambda \not\succeq 0$, then there is a vector v such that $\langle \Lambda v, v\rangle < 0$. Consider the projection P onto $\mathbb{R}^d \otimes \mathbb{R}v$, and let $Y = (\langle X_j v, v\rangle)_{j=1}^g \in \mathbb{R}^g$. Then the corresponding compression

$$PSP = P(I \otimes \Lambda + L^{(1)}(X))P = I \otimes \langle \Lambda v, v\rangle + L^{(1)}(Y) \succeq 0,$$

which says that $L^{(1)}(Y) \succ 0$. This implies $0 \neq tY \in \mathcal{D}_L$ for all $t > 0$; contrary to $\mathcal{D}_L$ being bounded.

Finally, for (3), we note that T is, after applying a permutation (often called the canonical shuffle), of the form (3.9). Hence $\Lambda \succeq 0$ by (2). ∎

Proof of Theorem 3.5. In each of the three statements, the direction ($\Rightarrow$) is obvious. We focus on the converses.

Fix $n \in \mathbb{N}$. Suppose $T \in \mathcal{S}_1^{n\times n}$ is positive definite. Then T is of the form (3.10) for some $\Lambda \succeq 0$ and $X \in (\mathbb{SR}^{n\times n})^g$. By applying the canonical shuffle,

$$S = I \otimes \Lambda + \sum A_{1,j} \otimes X_j \succ 0.$$

If we change Λ to $\Lambda + \varepsilon I$, the resulting $T = T_\varepsilon$ is in $\mathcal{S}_1^{n\times n}$, so without loss of generality we may assume $\Lambda \succ 0$. Hence,

$$(I \otimes \Lambda^{-\frac{1}{2}})S(I \otimes \Lambda^{-\frac{1}{2}}) = I \otimes I + \sum A_{1,j} \otimes (\Lambda^{-\frac{1}{2}} X_j \Lambda^{-\frac{1}{2}}) \succ 0.$$

Condition (3.2) thus says that

$$I \otimes I + \sum A_{2,j} \otimes (\Lambda^{-\frac{1}{2}} X_j \Lambda^{-\frac{1}{2}}) \succeq 0.$$

Multiplying on the left and right by $I \otimes \Lambda^{\frac{1}{2}}$ shows

$$I \otimes \Lambda + \sum A_{2,j} \otimes X_j \succeq 0.$$

Applying the canonical shuffle again, yields

$$\tau(T_\varepsilon) = \Lambda \otimes I + \sum X_j \otimes A_{2,j} \succeq 0.$$

Thus we have proved, if $T_\varepsilon \in \mathcal{S}_1^{n\times n}$ and $T_\varepsilon \succ 0$, then $\tau(T_\varepsilon) \succeq 0$. An approximation argument now shows if $T \succeq 0$, then $\tau(T) \succeq 0$ and hence τ is n-positive proving (1). Now (2) follows immediately.

For (3), suppose $T \in \mathcal{S}_1^{n\times n}$ has norm one. It follows that

$$W = \begin{bmatrix} I & T \\ T^* & I \end{bmatrix} \succeq 0.$$

From what has already been proved, $\tau(W) \succeq 0$ and therefore $\tau(W)$ has norm at most one. Moreover, since W has a kernel, so does $\tau(W)$ and hence the norm of $\tau(T)$ is at least one. We conclude that τ is completely isometric. ∎

Corollary 3.7 (Linear Positivstellensatz)**.** *Let*

$$L_j(x) = I + \sum_{\ell=1}^{g} A_{j,\ell} x_\ell \in \mathbb{SR}^{d_j \times d_j}\langle x\rangle, \quad j = 1, 2$$

be monic linear pencils and assume $\mathcal{D}_{L_1}$ is bounded. If (3.2) holds, that is, if $L_1(X) \succeq 0$ implies $L_2(X) \succeq 0$ for all X, then there is $\mu \in \mathbb{N}$ and an isometry $V \in \mathbb{R}^{\mu d_1 \times d_2}$ such that

$$L_2(x) = V^* \big(I_\mu \otimes L_1(x)\big) V. \tag{3.11}$$

Conversely, if μ, V are as above, then (3.11) implies (3.2) holds.

Remark 3.8. Before turning to the proof of the Corollary, we pause for a couple of remarks.

(1) Equation (3.11) can be equivalently written as

$$L_2(x) = \sum_{j=1}^{\mu} V_j^* L_1(x) V_j, \tag{3.12}$$

where $V_j \in \mathbb{R}^{d_1 \times d_2}$ and $V = \operatorname{col}(V_1, \dots, V_\mu)$. Since $\sum_{j=1}^{\mu} V_j^* V_j = I_{d_2}$, V is an isometry. Moreover, μ can be uniformly bounded (see the proof of Corollary 3.7, or Choi's characterization [Pau02, Proposition 4.7] of completely positive maps between matrix algebras). In fact, $\mu \le d_1 d_2$.

(2) Corollary 3.7 can be regarded as a Positivstellensatz for linear (matrix valued) polynomials, a theme we expand upon later below. Indeed, (3.12) is easily seen to be equivalent to the more common statement

$$L_2(x) = B + \sum_{j=1}^{\eta} W_j^* L_1(x) W_j \tag{3.13}$$

for some positive semidefinite $B \in \mathbb{SR}^{d_2 \times d_2}$ and $W_j \in \mathbb{R}^{d_1 \times d_2}$.

If we worked over $\mathbb{C}$, the proof of Corollary 3.7 would proceed as follows. First invoke Arveson's extension theorem [Pau02, Theorem 7.5] to extend τ to a completely positive map ψ from $d_1 \times d_1$ matrices to $d_2 \times d_2$ matrices, and then apply the Stinespring representation theorem [Pau02, Theorem 4.1] to obtain

$$\psi(a) = V^* \pi(a) V, \quad a \in \mathbb{C}^{d_1 \times d_1} \tag{3.14}$$

for some unital $*$-representation $\pi : \mathbb{C}^{d_1 \times d_1} \to \mathbb{C}^{d_3 \times d_3}$ and isometry (since τ is unital) $V : \mathbb{C}^{d_1} \to \mathbb{C}^{d_3}$. As all representations of $\mathbb{C}^{d_1 \times d_1}$ are (equivalent to) a multiple of the identity representation, i.e., $\pi(a) = I_\mu \otimes a$ for some $\mu \in \mathbb{N}$ and all $a \in \mathbb{C}^{d_1 \times d_1}$, (3.14) implies (3.11).

However, in our case, the pencils L_j have real coefficients and we want the isometry V to have real entries as well. For this reason and to aid understanding of this and our algorithm Section S 4 we present a self-contained proof, keeping all the ingredients real.

We prepare for the proof by reviewing some basic facts about completely positive maps. This serves as a tutorial for LMI experts, who often are unfamiliar with complete positivity.

Linear functionals $\sigma : \mathbb{R}^{d_1\times d_1}\otimes\mathbb{R}^{d_2\times d_2} \to \mathbb{R}$ are in a one-one correspondence with mappings $\psi : \mathbb{R}^{d_1\times d_1} \to \mathbb{R}^{d_2\times d_2}$ given by

$$\langle \psi(E_{ij})e_a, e_b\rangle = \langle \psi(e_i e_j^*)e_a, e_b\rangle = \sigma(e_j e_i^* \otimes e_a e_b^*). \tag{3.15}$$

Here, with a slight conservation of notation, the e_i, e_j are from $\{e_1, \dots, e_{d_1}\}$ and e_a, e_b are from $\{e_a, \dots, e_{d_2}\}$ which are the standard basis for $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ respectively.

Now we verify that positive functionals σ correspond precisely to completely positive ψ and give a nice representation for such a ψ.

A positive functional $\sigma : \mathbb{R}^{d_1\times d_1} \otimes \mathbb{R}^{d_2\times d_2} = \mathbb{R}^{d_1d_2\times d_1d_2} \to \mathbb{R}$ corresponds to a positive semidefinite $d_1d_2 \times d_1d_2$ matrix C via

$$\sigma(Z) = \operatorname{tr}(ZC).$$

Express $C = (C_{pq})_{p,q=1}^{d_1}$ as a $d_1 \times d_1$ matrix with $d_2 \times d_2$ entries. Thus, the (a, b) entry of the (i, j) block entry of C is

$$(C_{ij})_{ab} = \langle C(e_j \otimes e_a), e_i \otimes e_b\rangle.$$

With $Z = (e_j \otimes e_a)(e_i \otimes e_b)^*$ observe that

$$\langle \psi(E_{ij})e_a, e_b\rangle = \sigma(e_j e_i^* \otimes e_a e_b^*) = \operatorname{tr}(ZC) = \langle C(e_j \otimes e_a), e_i \otimes e_b\rangle = \langle C_{ij}e_a, e_b\rangle.$$

Hence, given $S = (s_{ij}) = \sum_{i,j=1}^{d_1} s_{ij}E_{ij}$, by the linearity of ψ,

$$\psi(S) = \sum_{i,j} s_{ij}C_{ij}.$$

(The matrix C is the Choi matrix for ψ, illustrated earlier in (3.6).) The matrix C is positive and thus factors (over the reals) as W^*W. Expressing $W = (W_{ij})_{i,j=1}^{d_1}$ as a $d_1 \times d_1$ matrix with $d_2 \times d_2$ entries W_{ij},

$$C_{ij} = \sum_{\ell=1}^{d_1} W_{j\ell}^* W_{i\ell}.$$

Define $V_\ell = (W_{i\ell})$. Then we have σ positive implies

$$\psi(S) = \sum_{i,j=1}^{d_2} s_{ij}C_{ij} = \sum_{\ell=1}^{d_1}\sum_{i,j=1}^{d_2} W_{i\ell}^* s_{ij} W_{j\ell} = \sum_{\ell=1}^{d_1} V_\ell^* S V_\ell = V^*\big((I_{d_1} \otimes S) \otimes I_{d_2}\big)V, \tag{3.16}$$

where V denotes the column with ℓ-th entry V_ℓ. Hence ψ is completely positive.

Proof of Corollary 3.7. We now proceed to prove Corollary 3.7. Given τ as in Theorem 3.5, define a linear functional $\tilde\sigma : \mathcal{S}_1 \otimes \mathbb{R}^{d_2\times d_2} \to \mathbb{R}$ as in correspondence (3.15) by

$$\tilde\sigma(S \otimes Y) = \sum_{a,b} \langle Y e_b, e_a\rangle\langle \tau(S)e_b, e_a\rangle.$$

Suppose $Z = \sum S_k \otimes Y_k \in \mathcal{S}_1 \otimes \mathbb{R}^{d_2 \times d_2}$ is positive semidefinite and let $\mathbf{e} = \sum_{a=1}^{d_2} e_a \otimes e_a$. Since the map $\tau_{d_1} = I_{d_1} \otimes \tau$, called an ampliation of τ, is positive,

$$0 \le \langle \tau_{d_1}(Z)\mathbf{e}, \mathbf{e} \rangle = \tilde{\sigma}(Z).$$

Thus $\tilde{\sigma}$ is positive and hence extends to a positive mapping $\sigma : \mathbb{R}^{d_1 \times d_1} \otimes \mathbb{R}^{d_2 \times d_2} \to \mathbb{R}$ by the Krein extension theorem, which in turn corresponds to a completely positive mapping $\psi : \mathbb{R}^{d_1 \times d_1} \to \mathbb{R}^{d_2 \times d_2}$ as in (3.15). It is easy to verify that $\psi|_{\mathcal{S}_1} = \tau$. By the above,

$$\psi(S) = V^* \big((I_{d_1} \otimes S) \otimes I_{d_2} \big) V.$$

Since $\psi(I) = I$, it follows that $V^*V = I$. ∎

3.2. **Equal matricial LMI sets.** In this section we begin an analysis of the binding condition (3.3). We present an equivalent reformulation:

Proposition 3.9. *Let L_1, L_2 be monic linear pencils. If $\mathcal{D}_{L_1}$ is bounded and (3.3) holds, that is, if $\partial \mathcal{D}_{L_1} \subseteq \partial \mathcal{D}_{L_2}$, then $\mathcal{D}_{L_1} = \mathcal{D}_{L_2}$.*

The proof is an easy consequence of the following elementary observation on convex sets.

Lemma 3.10. *Let $C_1 \subseteq C_2 \subseteq \mathbb{R}^n$ be closed convex sets, $0 \in \operatorname{int} C_1 \cap \operatorname{int} C_2$. If $\partial C_1 \subseteq \partial C_2$ then $C_1 = C_2$.*

Proof. By way of contradiction, assume $C_1 \subsetneq C_2$ and let $a \in C_2 \smallsetminus C_1$. The interval $[0, a]$ intersects C_1 in $[0, \mu a]$ for some $0 < \mu < 1$. Then $\mu a \in \partial C_1 \subseteq \partial C_2$. Since $0 \in \operatorname{int} C_1$, C_1 contains a small disk $D(0, \varepsilon)$. Then $K := \operatorname{co}(D(0, \varepsilon) \cup \{a\})$ is contained in C_2 and $\mu a \in \operatorname{int} K \subseteq \operatorname{int} C_2$ contradicting $\mu a \in \partial C_2$. ∎

Proof of Proposition 3.9. Let $C_i := \mathcal{D}_{L_i}$, $i = 1, 2$. Then

$$\partial C_i = \{ X \in \mathcal{D}_{L_i} \mid L_i(X) \succeq 0, L_i(X) \not\succ 0 \}.$$

Since C_1 is closed and bounded, it is the convex hull of its boundary. Thus by (3.3), $C_1 \subseteq C_2$. Hence the assumptions of Lemma 3.10 are fulfilled and we conclude $C_1 = C_2$. ∎

Example 3.11. It is tempting to guess that $\mathcal{D}_{L_1} = \mathcal{D}_{L_2}$ implies L_1 and L_2 (or, equivalently, $L_1^{(1)}$ and $L_2^{(1)}$) are unitarily equivalent. In fact, in the next subsection we will show this to be true under a certain irreducibility-type assumption. However, in general this fails for the trivial reason that the direct sum of a representing pencil and an "unrestrictive" pencil is also representative.

Let L_1 be an arbitrary monic linear pencil (with $\mathcal{D}_{L_1}$ bounded) and

$$L_2(x) = I + \big(L_1^{(1)}(x) \oplus \frac{1}{2} L_1^{(1)}(x) \big) = \begin{bmatrix} I + L_1^{(1)}(x) & 0 \\ 0 & I + \frac{1}{2} L_1^{(1)}(x) \end{bmatrix} = \begin{bmatrix} L_1(x) & 0 \\ 0 & I + \frac{1}{2} L_1^{(1)}(x) \end{bmatrix}.$$

Then $\mathcal{D}_{L_1} = \mathcal{D}_{L_2}$ but L_1 and L_2 are obviously not unitarily equivalent. However,

$$L_1(x) = \begin{bmatrix} I \\ 0 \end{bmatrix}^* L_2(x) \begin{bmatrix} I \\ 0 \end{bmatrix}$$

in accordance with Corollary 3.7.

Another guess would be that under $\mathcal{D}_{L_1} = \mathcal{D}_{L_2}$, we have $p = 1$ in Corollary 3.7. However this example also refutes that. Namely, there is no isometry $V \in \mathbb{R}^{d_1 \times 2d_1}$ satisfying

$$\begin{bmatrix} L_1(x) & 0 \\ 0 & I + \frac{1}{2} L_1^{(1)}(x) \end{bmatrix} = L_2(x) = V^* L_1(x) V.$$

(Here L_1 is assumed to be a $d_1 \times d_1$ pencil.) □

3.3. **Minimal L representing $\mathcal{D}_L$ are unique: Linear Gleichstellensatz.** Let $L = I + \sum A_i x_i$ be a $d \times d$ monic linear pencil and $\mathcal{S} = \operatorname{span}\{I, A_\ell \mid \ell = 1, \dots, g\}$. In this subsection we explain how to associate a monic linear pencil $\tilde{L}$ to L with the following properties:

(a) $\mathcal{D}_{\tilde{L}} = \mathcal{D}_L$;
(b) $\tilde{L}$ is the minimal (with respect to the size of the defining matrices) pencil satisfying (a).

A pencil $\tilde{L} = I + \sum \tilde{A}_j x_j$ is a **subpencil** of L provided there is a nontrivial reducing subspace $\mathcal{H}$ for $\mathcal{S}$ such that $\tilde{A}_j = V^* A_j V$, where V is the inclusion of $\mathcal{H}$ into $\mathbb{R}^d$, where d is the size of the matrices A_j. The pencil L is **minimal** if there does not exist a subpencil $\tilde{L}$ such that $\mathcal{D}_L = \mathcal{D}_{\tilde{L}}$.

Theorem 3.12. *Suppose L and M are linear pencils of size $d \times d$ and $e \times e$ respectively. If $\mathcal{D}_L = \mathcal{D}_M$ is bounded and both L and M are minimal, then $d = e$ and there is a unitary $d \times d$ matrix U such that $U^* L U = M$; i.e., L and M are unitarily equivalent.*

In particular, all minimal pencils for a given matricial LMI set have the same size (with respect to the defining matrices) and this size is the smallest possible.

Example 3.13. Suppose L and M are only minimal with respect to the spectrahedra $\mathcal{D}_L(1)$ and $\mathcal{D}_M(1)$, respectively. Then $\mathcal{D}_L(1) = \mathcal{D}_M(1)$ does not imply that L and M are unitarily equivalent. For instance, let L and M be the two pencils studied in Example 3.1. Then both L and M are minimal, $\mathcal{D}_L(1) = \mathcal{D}_M(1)$, but L and M are clearly not unitarily equivalent. □

The remainder of this subsection is devoted to the proof of, and corollaries to, Theorem 3.12. We shall see how $\mathcal{D}_L$ is governed by the multiplicative structure (i.e., the C^*-algebra) $C^*(\mathcal{S})$ generated by $\mathcal{S}$ as well as the embedding $\mathcal{S} \hookrightarrow C^*(\mathcal{S})$. For this we borrow heavily from Arveson's noncommutative Choquet theory [Arv69, Arv08, Arv10] and to a lesser extent from the paper of the third author with Dritschel [DM05].

We start with a basics of real C^*-algebras needed in the proof of Theorem 3.12. First, the well-known classification result.

Proposition 3.14. *A finite dimensional real C^*-algebra is $*$-isomorphic to a direct sum of real $*$-algebras of the form $M_n(\mathbb{R})$, $M_n(\mathbb{C})$ and $M_n(\mathbb{H})$. (Here the quaternions $\mathbb{H}$ are endowed with the standard involution.)*

Proposition 3.15. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and let $\Phi : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ be a real $*$-isomorphism.*

(1) *If* $\mathbb{K} \in \{\mathbb{R}, \mathbb{H}\}$*, then there exists a unitary* $U \in M_n(\mathbb{K})$ *with* $\Phi(A) = U^*AU$ *for all* $A \in M_n(\mathbb{K})$*.*
(2) *For* $K = \mathbb{C}$*, there exists a unitary* $U \in M_n(\mathbb{C})$ *with* $\Phi(A) = U^*AU$ *for all* $A \in M_n(\mathbb{C})$ *or* $\Phi(A) = U^*\bar{A}U$ *for all* $A \in M_n(\mathbb{C})$*. (Here* $\bar{A}$ *denotes the entrywise complex conjugate of* A*.)*

Proof. In (1), $M_n(\mathbb{K})$ is a central simple $\mathbb{R}$-algebra. By the Skolem-Noether theorem [KMRT98, Theorem 1.4], there exists an invertible matrix $U \in M_n(\mathbb{K})$ with

$$\Phi(A) = U^{-1}AU \quad \text{for all} \quad A \in M_n(\mathbb{K}). \tag{3.17}$$

Since Φ is a $*$-isomorphism,

$$U^{-1}A^*U = \Phi(A^*) = \Phi(A)^* = \big(U^{-1}AU\big)^* = U^*A^*U^{-*},$$

leading to UU^* being central in $M_n(\mathbb{K})$. By scaling, we may assume $UU^* = I$, i.e., U is unitary.

(2) $\Phi(i)$ is central and a skew-symmetric matrix, hence $\Phi(i) = \alpha i$ for some $\alpha \in \mathbb{R}$. Moreover, $\Phi(i^2) = -1$ yields $\alpha^2 = 1$. So $\Phi(i) = i$ or $\Phi(i) = -i$. In the former case, Φ is a $*$-isomorphism over $\mathbb{C}$ and thus given by a unitary conjugation as in (1). If $\Phi(i) = -i$, then Φ composed with entrywise conjugation is a $*$-isomorphism over $\mathbb{C}$. Hence there is some unitary U with $\Phi(A) = U^*\bar{A}U$ for all $A \in M_n(\mathbb{C})$. ■

Remark 3.16. For $K \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, every real $*$-isomorphism $\Phi : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ lifts to a unitary conjugation isomorphism $M_{dn}(\mathbb{R}) \to M_{dn}(\mathbb{R})$, where $d = \dim_{\mathbb{R}} \mathbb{K}$. By Proposition 3.15, this is clear if $K \in \{\mathbb{R}, \mathbb{H}\}$. To see why this is true in the complex case we proceed as follows.

Consider the standard real presentation of complex matrices, induced by

$$\iota : \mathbb{C} \to M_2(\mathbb{R}), \quad a + i\,b \mapsto \begin{bmatrix} a & b \\ -b & a \end{bmatrix}. \tag{3.18}$$

If the real $*$-isomorphism $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is itself a unitary conjugation, the claim is obvious. Otherwise $\bar{\Phi}$ is conjugation by some unitary $U \in M_n(\mathbb{C})$ and thus has a natural extension to a $*$-isomorphism

$$\check{\Phi} : M_{2n}(\mathbb{R}) \to M_{2n}(\mathbb{R}), \quad A \mapsto \iota(U)^* A \iota(U).$$

Then

$$\hat{\Phi} : M_{2n}(\mathbb{R}) \to M_{2n}(\mathbb{R}), \quad A \mapsto \left(I_n \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)^{-1} \check{\Phi}(A) \left(I_n \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

is a unitary conjugation $*$-isomorphism of $M_{2n}(\mathbb{R})$ and restricts to Φ on $M_n(\mathbb{C})$.

Let K be the biggest two sided ideal of $C^*(\mathcal{S})$ such that the natural map

$$C^*(\mathcal{S}) \to C^*(\mathcal{S})/K, \quad a \mapsto \tilde{a} := a + K \tag{3.19}$$

is completely isometric on $\mathcal{S}$. K is called the **Šilov ideal** (also the boundary ideal) for $\mathcal{S}$ in $C^*(\mathcal{S})$. Its existence and uniqueness is nontrivial, see the references given above. The snippet

[Arv+] contains a streamlined, compared to approaches which use injectivity, presentation of the Šilov ideal based upon the existence of completely positive maps with the unique extension property. While this snippet, as well as all of the references in the literature of which we are aware, use complex scalars, the proofs go through with no essential changes in the real case.

A **central projection** P in $C^*(\mathcal{S})$ is a projection $P \in C^*(\mathcal{S})$ such that $PA = AP$ for all $A \in C^*(\mathcal{S})$ (alternately $PA = AP$ for all $A \in \mathcal{S}$). We will say that a projection Q **reduces** or is **a reducing projection for** $C^*(\mathcal{S})$ if $QA = AQ$ for all $A \in C^*(\mathcal{S})$. In particular, P is a central projection if P reduces $C^*(\mathcal{S})$ and $P \in C^*(\mathcal{S})$.

Proposition 3.17. *Let L be a $d \times d$ truly linear pencil and suppose $\mathcal{D}_L$ is bounded. Then L is minimal if and only if*

(1) *every minimal reducing projection Q is in fact in $C^*(\mathcal{S})$; and*
(2) *the Šilov ideal of $C^*(\mathcal{S})$ is (0).*

Proof. Assume (1) does not hold and let Q be a given minimal nonzero reducing projection for $C^*(\mathcal{S})$, which is not an element of $C^*(\mathcal{S})$. Let P be a given minimal nonzero central projection such that P dominates Q; i.e., $Q \preceq P$. By our assumption, $Q \neq P$.

Consider the real C^*-algebra $\mathcal{A} = C^*(\mathcal{S})P$ as a real $*$-algebra of operators on the range $\mathcal{H}$ of P. First we claim that the mapping $\mathcal{A} \ni A \mapsto AQ$ is one-one. If not, it has a nontrivial kernel J which is an ideal in $\mathcal{A}$. The subspace $\mathcal{K} = J\mathcal{H}$ reduces $\mathcal{A}$ and moreover, because of finite dimensionality, the projection R onto $\mathcal{K}$ is in fact in $\mathcal{A}$. Hence, R is a central projection. By minimality, $R = P$ or $R = (0)$. In the second case the mapping is one-one. In the first case, $J\mathcal{H} = \mathcal{H}$ and thus $J = C^*(\mathcal{S})P$; i.e., the mapping $C^*(\mathcal{S})P \ni A \mapsto AQ$ is identically zero. In this case, the mapping $C^*(\mathcal{S})P \ni A \mapsto A(I-Q)$ is completely isometric, contradicting the minimality of L. Hence the map $\mathcal{A} \ni A \mapsto AQ$ is indeed one-one.

Therefore, the mapping $C^*(\mathcal{S}) \ni A \mapsto A(I-P) + AQ$ is faithful and in particular completely isometric. Thus the restriction of our pencil to the span of the ranges of $I - P$ and Q produces a pencil L' with $\mathcal{D}_{L'} = \mathcal{D}_L$, but of lesser dimension. Thus, we have proved, if (1) does not hold, then L is not minimal.

It is clear that if the Šilov ideal of $C^*(\mathcal{S})$ is nonzero, then L is not minimal. Suppose $J \subseteq C^*(\mathcal{S})$ is an ideal and the quotient mapping $\sigma : \mathcal{S} \to C^*(\mathcal{S})/J$ is completely isometric. As before, let $\mathcal{K} = J\mathbb{R}^d$ (where the pencil L has size d). The projection P onto $\mathcal{K}$ is a central projection. Because for $S \in \mathcal{S}$ we have both $\sigma(S) = \sigma(S - SP)$, and σ is completely isometric, it follows that $S \mapsto S(I-P)$ is completely isometric. By the minimality of L, it follows that $P = 0$.

Conversely, suppose (1) and (2) hold. If L is not minimal, let $\tilde{L}$ denote a minimal subpencil with $\mathcal{D}_{\tilde{L}} = \mathcal{D}_L$, corresponding to a reducing subspace $\mathcal{K} \subsetneq \mathbb{R}^d$ for $\mathcal{S}$. Let Q denote the projection onto $\mathcal{K}$ and $\mathcal{T}$ denote $\{SQ \mid S \in \mathcal{S}\}$. Note that the equality $\mathcal{D}_{\tilde{L}} = \mathcal{D}_L$ says exactly that the mapping $\mathcal{S} \to \mathcal{T}$ given by $S \mapsto SQ$ is completely isometric. In particular, if R is the projection onto a reducing subspace which contains $\mathcal{K}$, then also $S \mapsto SR$ is completely isometric.

Let $P : \mathbb{R}^d \to \mathcal{K}'$ denote any minimal orthogonal projection onto a reducing subspace of $\mathcal{K}^\perp$. By (1), $P \in C^*(\mathcal{S})$, and hence $C^*(\mathcal{S})P$ is a (minimal) two-sided ideal of $C^*(\mathcal{S})$. On the other hand, $(I-P)$ is the projection onto a reducing subspace which contains $\mathcal{K}$ and hence $S \mapsto S(I-P)$ is completely isometric. Now let $S = (S_{i,j}) \in M_n(\mathcal{S})$ be given. If $T = (T_{i,j})(I_n \otimes P) \in M_n(C^*(\mathcal{S}))P$, then

$$\|S+T\| = \|S(I_n \otimes (I-P)) \oplus (S+T)(I_n \otimes P)\| \geq \|S(I_n \otimes (I-P))\| = \|S\|,$$

where the last equality comes from the fact that $S \mapsto S(I-P)$ is completely isometric and the inequality from the fact that the norm of a direct sum is the maximum of the norm of the summands. Of course choosing $T = S(I_n \otimes P)$ it follows that the norm of S in the quotient $C^*(\mathcal{S})/C^*(\mathcal{S})P$ is the same as $\|S\|$. Hence the induced map $\mathcal{S} \to C^*(\mathcal{S})/C^*(\mathcal{S})P$ is completely isometric and therefore $C^*(\mathcal{S})P$ is contained in the Šilov ideal of $\mathcal{S}$, contradicting (2). ∎

Proof of Theorem 3.12. Write $L = I + \sum A_j x_j$ and $M = I + \sum B_j x_j$ and let $C^*(\mathcal{S})$ and $C^*(\mathcal{T})$ denote the unital C^*-algebras generated by $\{A_1, \dots, A_g\}$ and $\{B_1, \dots, B_g\}$ respectively. By Proposition 3.17, both $C^*(\mathcal{S})$ and $C^*(\mathcal{T})$ are reduced relative to $\mathcal{S}$ and $\mathcal{T}$ respectively; i.e., the Šilov ideals for $\mathcal{S}$ and $\mathcal{T}$ respectively are (0).

Moreover, for $\mathcal{Q}$ and $\mathcal{P}$ maximal families of minimal nonzero reducing projections for $C^*(\mathcal{S})$ and $C^*(\mathcal{T})$ respectively, we use Proposition 3.17 to obtain

$$C^*(\mathcal{S}) = \oplus_{Q \in \mathcal{Q}} C^*(\mathcal{S})Q, \quad C^*(\mathcal{T}) = \oplus_{P \in \mathcal{P}} C^*(\mathcal{T})P.$$

For later use we note that a minimal ideal in these C^*-algebras is of the form $C^*(\mathcal{S})Q$ for $Q \in \mathcal{Q}$, and $C^*(\mathcal{T})P$ for $P \in \mathcal{P}$, respectively.

The unital linear $*$-map

$$\tau : \mathcal{S} \to \mathcal{T}, \quad A_j \mapsto B_j$$

is a completely isometric isomorphism by Theorem 3.5 and maps between reduced operator systems. By [Arv69, Theorem 2.2.5], τ is induced by a $*$-isomorphism

$$\rho : C^*(\mathcal{S}) \to C^*(\mathcal{T}).$$

Since ρ is an isomorphism of C^*-algebras and $C^*(\mathcal{S})P$ for $P \in \mathcal{P}$, is a minimal ideal,

$$\rho(C^*(\mathcal{S})P) = C^*(\mathcal{T})Q \tag{3.20}$$

for some $Q \in \mathcal{Q}$. The converse is true too. That is, for each $Q \in \mathcal{Q}$ there is a unique $P \in \mathcal{P}$ such that (3.20) holds. We conclude that $d = e$.

By Proposition 3.15 and Remark 3.16 we also conclude that the C^*-isomorphism $\rho : C^*(\mathcal{S})P \to C^*(\mathcal{T})Q$ must be implemented by a unitary mapping $\operatorname{Ran} P \to \operatorname{Ran} Q$. ∎

Corollary 3.18. *Let $L \in \mathbb{SR}^{d\times d}\langle x\rangle$ be a monic linear pencil with bounded $\mathcal{D}_L$ and $\tilde{L} \in \mathbb{SR}^{\ell\times\ell}\langle x\rangle$ its minimal pencil. Then there is a $(d-\ell)\times(d-\ell)$ monic linear pencil J satisfying $J|_{\mathcal{D}_L} \succeq 0$ and a unitary $U \in \mathbb{R}^{d\times d}$ such that*

$$L(x) = U^* \begin{bmatrix} \tilde{L}(x) & \\ & J(x) \end{bmatrix} U.$$

Proof. Easy consequence of the construction of $\tilde{L}$. ■

4. Computational algorithms

In this section we present several numerical algorithms using semidefinite programming (SDP) [WSV00], based on the theory developed in the preceding section. However, one can read and implement these algorithms without reading anything beyond Section 1.3 of the introduction. In each case, we first present the algorithm and then give the justification (which a user need not read). The following section, Section 5, provides comparisons and refinements of the matricial matrix cube algorithm of Subsection 4.5 below.

Given L_1 and L_2 monic linear pencils

$$L_j(x) = I + \sum_{\ell=1}^{g} A_{j,\ell} x_\ell \in \mathbb{SR}^{d_j \times d_j}\langle x \rangle, \quad j = 1, 2, \tag{4.1}$$

with bounded matricial LMI set $\mathcal{D}_{L_1}$, we present an algorithm, **the inclusion algorithm**, to test whether $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$. Of course this numerical test yields a sufficient condition for containment of the spectrahedra $\mathcal{D}_{L_1}(1) \subseteq \mathcal{D}_{L_2}(1)$. We refer the reader to Section 4.4 for a test of boundedness of LMI sets, which works both for commutative LMIs and matricial LMIs, and computes the radius of a matricial LMI set based on the basic inclusion algorithm. Subsection 4.2 contains a refinement of the basic inclusion algorithm, in the case that either L_1 or L_2 is a direct sum of pencils of smaller size. As an application, we then present a matricial version of the classical matrix cube problem in Section 4.5. Analysis of the matricial matrix cube algorithm are in Section 5 along with a comparison to the matrix cube algorithm of Ben-Tal and Nemirovski [B-TN02]. There further algorithms, which offer improved estimates, at the expense of additional computation, for the matrix cube problem are also discussed. The final subsection of this section gives a (generically successful) algorithm for computation of a minimal representing pencil and the Šilov ideal, these being the only algorithms whose statement is not self contained.

4.1. **Checking inclusion of matricial LMI sets.**

The inclusion algorithm

Given: $A_{1,\ell}$ and $A_{2,\ell}$ for $\ell = 1, \ldots, g$. Let $\alpha^\ell_{p,q}$ denote the (p,q) entry of $A_{1,\ell}$.

Solve the following (feasibility) SDP:

$$(c_{pq})_{p,q=1}^{d_1} := C \succeq 0, \qquad \sum_{p}^{d_1} c_{pp} = I_{d_2}, \qquad \forall \ell = 1, \ldots, g : \sum_{p,q}^{d_1} \alpha^\ell_{pq} c_{pq} = A_{2,\ell}, \tag{4.2}$$

for the unknown symmetric matrix C. Since each c_{pq} is a $\mathbb{R}^{d_2 \times d_2}$ matrix, the symmetric matrix C of unknown variables (reasonably termed the Choi matrix) is of size $d_1 d_2 \times d_1 d_2$ and there are $\frac{1}{2} d_1 d_2 (d_1 d_2 + 1)$ (scalar) unknowns and $\frac{1}{2}(1+g) d_2 (d_2+1)$ (scalar) linear equality constraints. This can be, in practice, solved numerically with standard SDP solvers. In the next subsection, we show that if L_1 has special structure, then the number of (C) variables can be reduced, sometimes dramatically.

Conclude: $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$ if and only if the SDP (4.2) is feasible, i.e., has a solution.

Justification. By Theorem 3.5 and Corollary 3.7, L_2 is positive semidefinite on $\mathcal{D}_{L_1}$ if and only if there is a completely positive unital map

$$\tau : \mathbb{R}^{d_1 \times d_1} \to \mathbb{R}^{d_2 \times d_2} \tag{4.3}$$

satisfying

$$\tau(A_{1,\ell}) = A_{2,\ell} \quad \text{for} \quad \ell = 1, \dots, g. \tag{4.4}$$

To determine the existence of such a map, consider the Choi matrix $C = \big(\tau(E_{ij})\big)_{i,j=1}^{d_1} \in (\mathbb{R}^{d_2 \times d_2})^{d_1 \times d_1}$ of τ. (Here, E_{ij} are the $d_1 \times d_1$ elementary matrices.) For convenience of notation we consider C to be a $d_1 \times d_1$ matrix with $d_2 \times d_2$ entries c_{ij}. This is the matrix C which appears in the algorithm. It is well-known that τ is completely positive if and only if C is positive semidefinite [Pau02, Theorem 3.14].

Note that we can write $A_{1,\ell} = \sum_{p,q} \alpha_{pq}^{\ell} E_{pq}$. Then $\tau(A_{1,\ell}) = \sum_{p,q} \alpha_{pq}^{\ell} \tau(E_{pq}) = \sum_{p,q} \alpha_{pq}^{\ell} c_{pq}$. This lays behind the last equation in (4.2). If a solution C to (4.2) has been obtained, then a Positivstellensatz-type certificate for the inclusion of the matricial LMI sets $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$ can be obtained; cf. Example 3.4 or the proof of Corollary 3.7. ∎

4.2. **LMIs which are direct sums of LMIs.** If either pencil L_j as in (4.1) is given as a direct sum of pencils, then the Choi matrix C in the inclusion algorithm can be chosen with many fewer unknowns, reflecting this structure. We start with L_1.

Proposition 4.1. *Suppose $L_1 = \oplus_{\mu=1}^{k} M_\mu$, where $M_1, \dots, M_k$ are monic linear pencils,*

$$M_\mu = I + \sum_{\ell=1}^{g} B_\ell^\mu x_\ell,$$

where the B_j^μ are of size $\delta_\mu \times \delta_\mu$. Thus, $A_{1,\ell} = \oplus_{\mu=1}^{k} B_\ell^\mu$. Let $\alpha_{pq}^{\ell,\mu}$ denote the (p,q) entry of B_ℓ^μ. Then, $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$ if and only if there exists a symmetric matrix $C = \oplus_{\mu=1}^{k} C^\mu$ such that

$$\begin{aligned} \forall \mu = 1, \dots, k: \qquad & C^\mu := (c_{pq}^\mu)_{p,q=1}^{\delta_\mu} \succeq 0, \\ \forall \ell = 1, \dots, g: \qquad & \sum_{\mu=1}^{k} \sum_{p,q=1}^{\delta_\mu} \alpha_{pq}^{\ell,\mu} c_{pq}^\mu = A_{2,\ell}, \\ & \sum_{\mu=1}^{k} \sum_{p=1}^{\delta_\mu} c_{pp}^\mu = I_{d_2}. \end{aligned} \tag{4.5}$$

Each c_{pq}^μ is an unknown $d_2 \times d_2$ matrix and $(c_{pq}^\mu)^* = (c_{qp}^\mu)$.

Proof. The inclusion $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$ is equivalent to the existence of a Choi matrix C satisfying the feasibility conditions (4.2) of the inclusion algorithm. Thus C is a $d_1 \times d_1$ block matrix with $d_2 \times d_2$ entries. On the other hand, $d_1 = \sum_\mu \delta_\mu$ and the matrix C can be viewed as a block matrix $C = (C_{i,j})_{i,j}^k$ where $C_{i,j}$ is $\delta_i \times \delta_j$ block matrix whose entries are $d_2 \times d_2$ matrices. Observe that for $i \neq j$, the entries of $C_{i,j}$ do not appear as part of the linear constraint in the inclusion algorithm - they are unconstrained because our direct sum structure forces certain α_{pq} to be zero.

Since $d_1 = \sum \delta_\mu$ and the matrix C is a $d_1 \times d_1$ block matrix with $d_2 \times d_2$ blocks and is positive semidefinite, hence there exist $d_1 \times \delta_\mu$ block matrices W_μ having $d_2 \times d_2$ entries such that C factors as

$$C = W^* W = \begin{bmatrix} W_1^* \\ \vdots \\ W_k^* \end{bmatrix} \begin{bmatrix} W_1 & \cdots & W_k \end{bmatrix}$$

Consider the set $\mathcal{C}$ of 2^{k-1} matrices of the form

$$\begin{bmatrix} W_1^* \\ \pm W_2^* \\ \pm W_3^* \\ \vdots \\ \pm W_k^* \end{bmatrix} \begin{bmatrix} W_1 & \pm W_2 & \pm W_3 & \cdots & \pm W_k \end{bmatrix}.$$

Each $\tilde{C} \in \mathcal{C}$ solves the inclusion algorithm; i.e., validates $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$. Hence the matrix $\hat{C}$ obtained by averaging over $\mathcal{C}$ also validates the inclusion. Noting that, because each off diagonal entry of $\hat{C}$ is the average of 2^{k-2} terms $W_i^* W_j$ with 2^{k-2} terms $-W_i^* W_j$, we get $\hat{C}$ is the block diagonal matrix with diagonal entries $W_j^* W_j$, which completes the proof. ∎

With the hypotheses of Proposition 4.1, the number of unknown variables in the LMI inclusion algorithm are greatly reduced. Indeed, from $\frac{1}{2}(d_1 d_2 + 1) d_1 d_2$, to

$$\frac{1}{2} \sum_{\mu=1}^{k} (d_2 \delta_\mu + 1) d_2 \delta_\mu.$$

The number of equality constraints is still $\frac{1}{2}(1+g) d_2 (d_2 + 1)$.

A reduction in both the number of variables and equality constraints occurs if L_2, the range linear pencil, in the inclusion algorithm has a direct sum structure.

Proposition 4.2. *In the inclusion algorithm, if the pencil L_2 is a direct sum; i.e., $L_2 = \oplus_{\mu=1}^k M_\mu$, where each*

$$M_\mu = I + \sum_1^g B_j^\mu x_j$$

is a monic linear pencil of size $\delta_\mu \times \delta_\mu$ *(so that* $\sum \delta_\mu = d_2$*), then* $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$ *if and only if there exists a symmetric matrix* $C = \oplus_{\mu=1}^k C^\mu$ *such that*

$$\begin{aligned} \forall \mu = 1,\dots,k: &\qquad C^\mu := (c^\mu_{pq})_{p,q=1}^{d_1} \succeq 0, \\ \forall \ell = 1,\dots,g, \quad \mu = 1,\dots,k: &\qquad \sum_{p,q=1}^{d_\mu} \alpha^\ell_{pq} c^\mu_{pq} = B^\mu_\ell, \\ \forall \mu = 1,\dots,k: &\qquad \sum_{p=1}^{d_\mu} c^\mu_{pp} = I_{\delta_\mu}. \end{aligned} \tag{4.6}$$

Each c^μ_{pq} *is an unknown* $\delta_\mu \times \delta_\mu$ *matrix and* $(c^\mu_{pq})^* = (c^\mu_{qp})$.

The count of unknowns is $\frac{1}{2}\sum_{\mu=1}^k (d_1\delta_\mu + 1)d_1\delta_\mu$ and of scalar equality constraints is $gk\delta_\mu(\delta_\mu+1) + k\delta_\mu(\delta_\mu+1)$.

4.3. **Tightening the relaxation.** There is a general approach to tightening the inclusion algorithm which relaxes $\mathcal{D}_{L_1}(1) \subseteq \mathcal{D}_{L_2}(1)$, and thus applies to the algorithms in the section, based upon the following simple lemma.

Lemma 4.3. *Suppose* L_1, L_2 *and* M *are linear pencils and let* $\hat{M} = L_1 \oplus M$*. If* $\mathcal{D}_{L_1}(1) \subseteq \mathcal{D}_M(1)$*, then*

$$\mathcal{D}_{\hat{M}} \subseteq \mathcal{D}_{L_1} \quad \text{and} \quad \mathcal{D}_{\hat{M}}(1) = \mathcal{D}_{L_1}(1).$$

In particular, if $\mathcal{D}_{\hat{M}} \subseteq \mathcal{D}_{L_2}$*, then* $\mathcal{D}_{L_1}(1) \subseteq \mathcal{D}_{L_2}(1)$.

Proof. The first part of the lemma is evident:

$$\mathcal{D}_{\hat{M}} = \mathcal{D}_{L_1} \cap \mathcal{D}_M \subseteq \mathcal{D}_{L_1}.$$

Likewise, $\mathcal{D}_{\hat{M}}(1) = \mathcal{D}_{L_1}(1) \cap \mathcal{D}_M(1) = \mathcal{D}_{L_1}(1)$, since $\mathcal{D}_{L_1}(1) \subseteq \mathcal{D}_M(1)$. For the last statement note that $\mathcal{D}_{\hat{M}} \subseteq \mathcal{D}_{L_2}$ implies $\mathcal{D}_{L_1}(1) = \mathcal{D}_{\hat{M}}(1) \subseteq \mathcal{D}_{L_2}(1)$. ∎

This lemma tells us applying our inclusion algorithm to $\hat{M}$ versus L_2 is at least as accurate as applying it to L_1 versus $\hat{M}$ and it quite possibly is more accurate. The lemma is used in the context of the matrix cube problem in Section 5.

4.4. **Computing the radius of matricial LMI sets.** Let L be a monic linear pencil,

$$L(x) = I + \sum_{\ell=1}^{g} A_\ell x_\ell \in \mathbb{SR}^{d\times d}\langle x\rangle. \tag{4.7}$$

We present an algorithm based on semidefinite programming to compute the radius of a matricial LMI set $\mathcal{D}_L$ (and at the same time check whether it is bounded). The idea is simply to use the test in Section 4.1 to check if $\mathcal{D}_L$ is contained in the ball of radius N. The smallest such N will be the matricial radius, and also an upper bound on the radius of the spectrahedron $\mathcal{D}_L(1)$.

Let

$$\mathcal{J}_N(x) = \frac{1}{N}\begin{bmatrix} N & x^* \\ x & NI_g \end{bmatrix} = I + \frac{1}{N}\sum_{j=1}^{g}(E'_{1,j+1} + E'_{j+1,1})x_j \in \mathbb{SR}^{(g+1)\times(g+1)}\langle x\rangle$$

be a monic linear pencil. Here E'_{ij} the $(g+1)\times(g+1)$ elementary matrix with a 1 in the (i,j) entry and zeros elsewhere.

Then $\mathcal{D}_L$ is bounded, and its matricial radius is $\leq N$, if and only if $\mathcal{D}_{\mathcal{J}_N} \supseteq \mathcal{D}_L$.

The matricial radius algorithm

Let $\alpha^\ell_{r,s}$ denote the (r,s) entry of A_ℓ, that is, $A_\ell = \sum_{r,s}\alpha^\ell_{rs}E_{rs}$. Solve the SDP (RM):

$\min b := \sum_{r,s}\alpha^1_{rs}(c_{rs})_{1,2}$ subject to

(RM$_1$) $(c_{rs})_{r,s=1}^d := C \succeq 0$,

(RM$_2$) $\sum\limits_{r=1}^{d} c_{rr} = I_{g+1}$,

(RM$_3$) $\forall \ell = 1,\dots,g,\ \forall p,q = 1,\dots,g+1$:

$$\sum_{r,s}\alpha^\ell_{rs}(c_{rs})_{p,q} = 0 \quad \text{for} \quad (p,q)\notin\{(1,\ell+1),\,(\ell+1,1)\},$$

(RM$_4$) $\sum_{r,s}\alpha^1_{rs}(c_{rs})_{1,2} = \sum_{r,s}\alpha^1_{rs}(c_{rs})_{2,1} = \sum_{r,s}\alpha^2_{rs}(c_{rs})_{1,3} = \sum_{r,s}\alpha^2_{rs}(c_{rs})_{3,1} = \cdots$
$= \sum_{r,s}\alpha^g_{rs}(c_{rs})_{1,g+1} = \sum_{r,s}\alpha^g_{rs}(c_{rs})_{g+1,1}$

for the unknown C; i.e., the d^2 unknown $(g+1)\times(g+1)$ matrices (c_{rs}). If the optimal value of (RM) is $b \in \mathbb{R}_{>0}$, then $\|X\| \leq \frac{1}{b}$ for all $X \in \mathcal{D}_L$, and this bound is sharp.

This SDP is always feasible (for $b = \sum_{r,s}\alpha^1_{rs}(c_{rs})_{1,2} = 0$). Clearly, $\mathcal{D}_L$ is bounded if and only if this SDP has a positive solution. In fact, any value of $b > 0$ obtained gives an upper bound of $\frac{1}{b}$ for the norm of an element in $\mathcal{D}_L$. The size of the (symmetric) matrix of unknown variables is $d(g+1)\times d(g+1)$ and there are $\frac{1}{2}(g^3+4g^2+3g+4)$ (scalar) linear constraints. To reduce the number of unknowns, solve the linear system of $\frac{1}{2}g(g^2+3g-2)$ equations given in (RM$_3$).

Checking boundedness of $\mathcal{D}_L(1)$ is a classical, fairly basic semidefinite programming problem. Indeed, given a nondegenerate monic linear pencil L, $\mathcal{D}_L(1)$ is bounded (equivalently, $\mathcal{D}_L$ is bounded) if and only if the following SDP is infeasible:

$$L^{(1)}(X) \succeq 0, \quad \operatorname{tr}\left(L^{(1)}(X)\right) = 1.$$

(Here, $L^{(1)}$ denotes the truly linear part of L.)

However, computing the radius of $\mathcal{D}_L(1)$ is harder. Thus our algorithm, yielding a convenient upper bound on the radius, might be of broad interest, motivating us to spend more time describing its implementation. The algorithm can be written entirely in a matricial form which is both elegant and easy to code in MATLAB or Mathematica. The matricial component of the algorithm is as follows. Let $\mathbf{e}_n$ denote the vector of length n with all ones, let $\mathbf{E}_n = \mathbf{e}_n \otimes \mathbf{e}_n^t$ be the $n \times n$ matrix of all ones. Then (RM_2) is (using $\bullet_{\mathrm{H}}$ for the Hadamard product)

$$\big(\mathbf{e}_{g+1} \otimes I_d\big)^t \big((I_d \otimes \mathbf{E}_{g+1}) \bullet_{\mathrm{H}} C\big)\big(\mathbf{e}_{g+1} \otimes I_d\big) = I_{g+1},$$

while the left hand side of (RM_3) can be presented as the (p,q) entry of

$$\big(\mathbf{e}_{g+1} \otimes I_d\big)^t \big((A_\ell \otimes \mathbf{E}_{g+1}) \bullet_{\mathrm{H}} C\big)\big(\mathbf{e}_{g+1} \otimes I_d\big).$$

Equations (RM_3) and (RM_4) give constraints on these matrices.

As an example we computed the matricial radius of an ellipse, which for the example we computed agrees with the scalar radius. The corresponding Mathematica notebook can be downloaded from

http://srag.fmf.uni-lj.si/preprints/ncLMI-supplement.zip

Justification. As in the previous subsection, we need to determine whether there is a completely positive unital map $\tau : \mathbb{R}^{d\times d} \to \mathbb{R}^{(g+1)\times(g+1)}$ satisfying $\tau(A_j) = \frac{1}{N}(E'_{1,j+1} + E'_{j+1,1})$ for some N. The Choi matrix here is $C = (\tau(E_{ij}))_{i,j} \in (\mathbb{R}^{(g+1)\times(g+1)})^{d\times d}$. Let $A_\ell = \sum_{r,s} \alpha_{rs}^\ell E_{rs}$. Then the linear constraints we need to consider say that

$$\tau(A_\ell) = \sum_{r,s} \alpha_{rs}^\ell c_{rs}$$

has all entries 0 except for the $(1, \ell+1)$ and $(\ell+1, 1)$ entries which are the same; indeed they are all equal to $\frac{1}{N}$. Thus we arrive at the feasibility SDP (RM) above. ■

4.5. **The matricial matrix cube problem.** This section describes our matricial matrix cube algorithm - a test for inclusion of the matricial matrix cube (as defined below) into a given LMI set. Variations on the algorithm and an analysis of the connection between this algorithm and the matrix cube algorithm of [B-TN02] is the subject of Section 5.

Let $L \in \mathbb{SR}^{d\times d}\langle x\rangle$ be a monic linear pencil as in (4.7). We present an algorithm that computes the size ρ of the biggest matricial cube contained in $\mathcal{D}_L$. That is, $\rho \in \mathbb{R}$ is the largest number with the following property: if $n \in \mathbb{N}$ and $X \in (\mathbb{SR}^{n\times n})^g$ satisfies $\|X_i\| \le \rho$ for all $i = 1, \dots, g$, then $X \in \mathcal{D}_L$. When X_i is in $\mathbb{R}^{1\times 1}$ this is the classical matrix cube problem (cf. Ben-Tal and Nemirovski [B-TN02]), which they show is NP-hard.

First we need an LMI which defines the cube. Let

$$\mathcal{C}_\rho(x) = \frac{1}{\rho}\Big(\big(\oplus_{j=1}^g \rho - x_j\big) \bigoplus \big(\oplus_{j=1}^g \rho + x_j\big)\Big) \in \mathbb{SR}^{2g\times 2g}\langle x\rangle.$$

Then $\mathcal{C}_\rho(x) = I + \frac{1}{\rho}\sum_{j=1}^g (E_{jj} - E_{g+j,g+j})x_j$, where $E_{i,j}$ is a the elementary $2g \times 2g$ matrix with a 1 in the (i,j) entry and zeros elsewhere, and

$$\mathcal{D}_{\mathcal{C}_\rho} = \bigcup_{n\in\mathbb{N}} \left\{X \in (\mathbb{SR}^{n\times n})^g \mid \ \|X_i\| \le \rho \text{ for all } i = 1, \dots, g\right\}.$$

This is a matricial cube. Our algorithm uses the test in Section 4.1 to compute the largest ρ with $\mathcal{D}_{\mathcal{C}_\rho} \subseteq \mathcal{D}_L$. It also takes advantage of the fact that $\mathcal{C}_\rho$ is a direct sum (of scalar-valued pencils) by using Proposition 4.1 with $k = 2g$, $\delta_\mu = 1$ and $d_2 = d$. This immediately gives rise to the following SDP:

$$\max \rho \quad \text{subject to}$$

$$
\begin{aligned}
&(\text{preMC}_1) \quad C^j \succeq 0, \quad j = 1, \dots, 2g\\
&(\text{preMC}_2) \quad \forall j = 1, \dots, g: \quad C^j - C^{g+j} = \rho A_j.\\
&(\text{preMC}_3) \quad \sum_{j=1}^{2g} C^j = I_d,
\end{aligned}
$$

Each of the $2g$ symmetric matrices C^j is in $\mathbb{SR}^{d\times d}$.

Next we make this algorithm more efficient by solving the equality constraints (preMC_2) to eliminate $C^{g+1}, \dots, C^{2g}$ and (preMC_3) to obtain

$$C^g = \frac{1}{2}\Big(I - 2\sum_{j=1}^{g-1} C^j + \rho \sum_{j=1}^{g} A_j\Big) \tag{4.8}$$

With this, the above SDP reduces to

The matricial matrix cube algorithm

$$\max \rho \quad \text{subject to}$$

$$
\begin{aligned}
&(\text{MC}_1) \quad C^j \succeq 0, \quad j = 1, \dots, g-1\\
&(\text{MC}_2) \quad C^j \succeq \rho A_j\\
&(\text{MC}_3) \quad I_d - 2\textstyle\sum_j^{g-1} C^j + \sum_j^{g-1} \rho A_j \pm \rho A_g \succeq 0,
\end{aligned}
$$

where each of the $g-1$ symmetric matrices C^j is in $\mathbb{SR}^{d\times d}$.

This SDP is always feasible (with $\rho = 0$). If its optimal value is $\rho > 0$, then $\mathcal{D}_{\mathcal{C}_\rho} \subseteq \mathcal{D}_L$, and the obtained upper bound for the size of the matricial cube is sharp. There are $\frac{1}{2}(g-1)d(d+1)$ variables and all of the linear equality constraints have been eliminated. There are $2g$ matrix inequality constraints.

Example 4.4. Consider finding the largest square embedded inside the unit disk. We consider the two pencils Δ, Γ from Example 3.1, each of which represents the unit disk, since $\mathcal{D}_\Delta(1) = \mathcal{D}_\Gamma(1) = \{(X_1, X_2) \in \mathbb{R}^2 \mid X_1^2 + X_2^2 \le 1\}$. It is clear that $\mathcal{D}_{\mathcal{C}_{\sqrt{2}/2}}(1)$ is the maximal square contained in the unit disk $\mathcal{D}_\Delta(1)$. Indeed the biggest matricial cube in $\mathcal{D}_\Delta$ is $\mathcal{D}_{\mathcal{C}_{\sqrt{2}/2}}$, but the biggest matricial cube in $\mathcal{D}_\Gamma$ is $\mathcal{D}_{\mathcal{C}_{\frac{1}{2}}}$. For details, see the Mathematica notebook available at

http://srag.fmf.uni-lj.si/preprints/ncLMI-supplement.zip

We will revisit this example in Section 5. □

Justification. A justification for the matrix cube algorithm based on the pre-algorithm has already been given. So it suffices to justify the pre-matricial matrix cube algorithm. Let $B_j := E_{j,j} - E_{g+j,g+j} \in \mathbb{SR}^{2g\times 2g}$. Taking advantage of the fact that $\mathcal{C}_\rho$ is the direct sum of $2g$ scalar linear pencils, we want to determine the biggest ρ for which there exists a completely positive unital map $\tau : \oplus_1^{2g}\mathbb{R}^{1\times 1} \to \mathbb{R}^{d\times d}$ satisfying $\tau(B_j) = \rho A_j$, $j = 1, \dots, g$. Suppose $C = \oplus_{j=1}^{2g}(c^j) \in \oplus_1^{2g}(\mathbb{R}^{d\times d})^{1\times 1}$ (because each c^j is a 1×1 block matrix whose entries are (symmetric) $d \times d$ matrices, there is no need for the indexing c^j_{pq}) is the corresponding Choi matrix as in Proposition 4.1. Then the linear constraint $\tau(B_j) = \rho A_j$ translates into $c^j_{1,1} - c^{g+j}_{1,1} = \rho A_j$ which is (preMC$_2$). ■

4.6. **Minimal pencils and the Šilov ideal.** This section describes an algorithm aimed at constructing from a given pencil L a pencil $\tilde{L}$ of minimal size with $\mathcal{D}_L = \mathcal{D}_{\tilde{L}}$.

4.6.1. *Minimal pencils.* Let L be a monic linear pencil,

$$L(x) = I + \sum_{\ell=1}^{g} A_\ell x_\ell \in \mathbb{SR}^{d\times d}\langle x\rangle \tag{4.9}$$

with bounded $\mathcal{D}_L$. We present a probabilistic algorithm based on semidefinite programming that computes a minimal pencil $\tilde{L}$ with the same matricial LMI set.

The two-step procedure goes as follows. In Step 1, one uses the decomposition of a semisimple algebra into a direct sum of simple algebras, a classical technique in computational algebra, cf. Friedl and Rońyal [FR85], Eberly and Giesbrecht [EG96], or Murota, Kanno, Kojima, and Kojima [MKKK10] for a recent treatment. This yields a unitary matrix $U \in \mathbb{R}^{d\times d}$ that simultaneously transforms the A_ℓ into block diagonal form, that is,

$$U^* A_\ell U = \oplus_{j=1}^{s} B_\ell^j \quad \text{for all} \quad \ell.$$

For each j, the set $\{I, B_1^j, \dots, B_g^j\}$ generates a simple real algebra. Define the monic linear pencils

$$L^j(x) = I + \sum_{\ell=1}^{g} B_\ell^j x_\ell, \quad L'(x) = U^* L(x) U = \oplus_{j=1}^{s} L^j(x).$$

Given ℓ, let $\tilde{L}_\ell = \oplus_{j\neq\ell} L^j$. If there is no ℓ such that

$$L^\ell|_{\mathcal{D}_{\tilde{L}_\ell}} \succeq 0,$$

(this can be tested using SDP as explained in Section 4.1) then the pencil is minimal. If there is such an ℓ remove the (one) corresponding block from L' to obtain a new pencil and repeat the process. Once we have no more redundant blocks in L', the obtained pencil $\tilde{L}$ is minimal, and satisfies $\mathcal{D}_{\tilde{L}} = \mathcal{D}_L$ by construction.

4.6.2. *Šilov ideal.* Thus subsection requires material from Section 3. Using our results from Section 3.3 (cf. Proposition 3.17) and Section 4.1, one can compute the Šilov ideal of a unital matrix algebra $\mathcal{A}$ generated by symmetric matrices $A_1, \dots, A_g \in \mathbb{SR}^{d\times d}$. Form the monic linear pencil

$$L = I + \sum A_\ell x_\ell \in \mathbb{SR}^{d\times d}\langle x\rangle,$$

and compute the minimal pencil

$$\tilde{L} = I + \sum \tilde{A}_\ell x_\ell$$

as in the previous subsection. If

$$\tilde{S} = \operatorname{span}\{I, \tilde{A}_\ell \mid \ell = 1, \dots, g\},$$

then the kernel of the canonical unital map

$$\mathcal{A} \to C^*(\tilde{S}), \quad A_\ell \mapsto \tilde{A}_\ell$$

is the Šilov ideal of $\mathcal{A}$.

5. More on the matrix cube problem

This section provides perspective on the inclusion algorithm by focusing on the matrix cube problem. The first subsection shows that the estimate based on the inclusion algorithm, namely the matricial matrix cube algorithm of Subsection 4.5, is essentially identical to that obtained by the algorithm of Ben-Tal and Nemirovski in [B-TN02].

Subsection 5.2, illustrates the tightening procedure of Lemma 4.3 on the matricial cube.

5.1. Comparison with the algorithm in [B-TN02].

Let $L = I + \sum_{\ell=1}^{g} A_\ell x_\ell \in \mathbb{SR}^{d\times d}\langle x\rangle$ be a monic linear pencil and recall the pencil $\mathcal{C}_\rho$ and its corresponding positivity domain $\mathcal{D}_{\mathcal{C}_\rho}$, the matricial cube. In [B-TN02] the *verifiable sufficient condition* for the inclusion $\mathcal{D}_\rho(1) \subseteq \mathcal{D}_L(1)$ is the following: Suppose there exist symmetric matrices $B_1, \dots B_g$ such that

$$B_j \succeq \pm A_j \text{ for all } j = 1, 2, \dots, g; \quad \text{and} \quad I - \rho \sum_j B_j \succeq 0 \tag{S}$$

holds. Then $\mathcal{D}_{\mathcal{C}_\rho}(1) \subseteq \mathcal{D}_L(1)$.

The following proposition says that the estimate of the largest cube contained in a given spectrahedron given by the matricial relaxation based upon the matricial matrix cube algorithm is the same as that based upon condition (S).

Proposition 5.1. *Given $\rho \in \mathbb{R}_{\geq 0}$, condition (S) holds if and only if $\mathcal{D}_{\mathcal{C}_\rho} \subseteq \mathcal{D}_L$. Moreover, there is an explicit formula for converting condition (S) to a feasible point for the matricial matrix cube algorithm and vice-versa.*

Proof. Suppose we have found the optimal ρ and the corresponding C^j for $j = 1, \dots, 2g$, in the matricial matrix cube algorithm. From (preMC$_2$) $\rho A_j = C^j - C^{g+j}$. Set

$$\rho B^j := C^j + C^{g+j}.$$

From (preMC$_1$),

$$\rho(B^j - A_j) = 2C^j \succeq 0 \quad \text{and} \quad \rho(B^j + A_j) = 2C^{j+g} \succeq 0. \tag{5.1}$$

Relation (preMC$_3$) gives $I = \rho \sum_j B^j$. Thus we see B^j and ρ satisfy (S).

Conversely, suppose B_j, ρ are a solution to (S). Solve (5.1) for C^j, C^{g+j}. It is straightforward to check that these C^j satisfy the conditions (preMC$_j$) for $j = 1, 2, 3$. ∎

Now that we know the estimate provided by our relaxation is the same as that of algorithm (S), we look at the computational cost. (S) has $\frac{1}{2}gd(d+1)$ unknowns and the number of $d \times d$ matrix inequality constraints is $(2g+1)$. As we saw, our matricial matrix cube algorithm had $\frac{1}{2}(g-1)d(d+1)$ unknowns and $2g$ matrix $(d \times d)$ inequality constraints, so the costs are a bit less than those of (S). However, (S) can be improved easily by the general trick in the following remark which removes an unknown and a constraint, thus making the cost of (S) the same as ours.

Remark 5.2. If, in the $A_{1,\ell}$ in the inclusion algorithm of Section 4.1 all have trace 0, then the condition $I_{d_2} - \sum_{p=1}^{d_1} c_{pp} = 0$ is equivalent to the inequality $\Delta = I_{d_2} - \sum_{p=1}^{d_1} c_{pp} \succeq 0$, since then the c_{pp} can, without harm, be replaced by $c_{pp} + \frac{\Delta}{d_1}$. When presented with the inequality form we could convert it to equality, then eliminate one variable by solving for it.

5.2. **The lattice of inclusion algorithm relaxations for the matrix cube.** A virtue of the general method of the inclusion algorithm based on matricial relaxations is that, as alluded to in Section 4.3, it allows tightening in order to improve estimates (with added cost). This subsection discusses and illustrates properties of this tightening procedure, mainly as an introduction to a topic that might merit further study. We do show in an example that tightening can produce an improved estimate.

5.2.1. *General theory.* The operator theory upon which this paper is based, when converted to the language of LMIs, contains a theory of matricial relaxations of a given LMI set and thus provides a general framework containing the tightening methods described in Section 4.3.

Suppose $S \subseteq \mathbb{R}^g$ is an LMI set; i.e., suppose there is a monic linear pencil Λ such that $S = \mathcal{D}_\Lambda(1)$. The collection $\mathcal{L}_S$ of all monic linear pencils L with $S = \mathcal{D}_L(1)$ is naturally ordered by inclusion. Namely, if $L, M \in \mathcal{L}_S$, then $L \geq M$ if, $\mathcal{D}_L(n) \subseteq \mathcal{D}_M(n)$ for every positive integer n. If Λ' is also a monic linear pencil and $S \subseteq S' = \mathcal{D}_{\Lambda'}$, then the matricial inclusion $\mathcal{D}_M \subseteq \mathcal{D}_{\Lambda'}$ implies the inclusion $\mathcal{D}_L \subseteq \mathcal{D}_{\Lambda'}$. Thus, the pencil L gives at least as good a test for the inclusion $S \subseteq S'$, than does M. Similarly, if $M', L' \in \mathcal{L}_{S'}$ and $L' \geq M'$, then M' gives at least as good a test for the inclusion of S into S' as does L'.

If $\mathcal{L}_S$ has a maximal element, denote it by $L_{\max}$ and similarly for $L_{\min}$. Generally, $\mathcal{L}_S$ will not have either a minimal or maximal element; however, it turns out for the matrix cube there is a minimal element. See Proposition 5.3 below. Further, in general by dropping the requirement that the pencils L have matrix coefficients and instead allowing for operator coefficients, it is possible to prove that $L_{\max}$ and $L_{\min}$ exists. (The discussion in [Pau02] on max and min operator space structures is easily seen to carry over to the present setting.)

Thus, though typically not practical, the matricial relaxation for the inclusion of the set $S = \mathcal{D}_\Lambda(1)$ into $T = \mathcal{D}_{\Lambda'}$ based upon using $L_{\max}$ in place of Λ produces the exact answer; whereas the matricial relaxation based upon $\mathcal{D}_{L_{\min}}$ produces the most conservative estimate over all possible matricial relaxations.

5.2.2. *$L_{\min}$ and $L_{\max}$ for the matrix cube.* From the next proposition it follows that for the matrix cube $\mathcal{D}_{\mathcal{C}_1}(1)$ the minimal pencil $L_{\min}$ is $\mathcal{C}_1$.

Proposition 5.3. *If M is a monic linear pencil and $\mathcal{D}_M(1) \subseteq \mathcal{D}_{\mathcal{C}_\rho}(1)$, then $\mathcal{D}_M \subseteq \mathcal{D}_{\mathcal{C}_\rho}$. In particular, if $\mathcal{D}_M(1) = \mathcal{D}_{\mathcal{C}_\rho}(1)$, and L is a monic linear pencil for which $\mathcal{D}_{\mathcal{C}_\rho} \subseteq \mathcal{D}_L$, then $\mathcal{D}_M \subseteq \mathcal{D}_L$. Hence if $\mathcal{D}_M(1) = \mathcal{D}_{\mathcal{C}_1}(1)$, then the inclusion $\mathcal{D}_M \subseteq \mathcal{D}_L$ is at least as good a test for $\mathcal{D}_L$ to contain the unit cube as the inclusion $\mathcal{D}_{\mathcal{C}_1} \subseteq \mathcal{D}_L$.*

Proof. Write $M = I - \sum A_j x_j$. The condition $\mathcal{D}_M(1) \subseteq \mathcal{D}_{\mathcal{C}_\rho}(1)$ implies, if

$$\frac{1}{\rho}\sum A_j x_j \preceq I, \quad \text{for some } x_j \in \mathbb{R},$$

then $|x_j| \leq 1$ for each $1 \leq j \leq g$.

Now suppose that $X = (X_1, \dots, X_g) \in (\mathbb{S}\mathbb{R}^{n\times n})^g$ and

$$\frac{1}{\rho}\sum A_j \otimes X_j \preceq \rho I.$$

For each vector f and unit vector x (of the appropriate sizes), it follows that

$$\frac{1}{\rho}\sum \langle A_j f, f\rangle \langle X_j x, x\rangle \leq \|f\|^2.$$

With x fixed, varying f shows that

$$\frac{1}{\rho}\sum A_j \langle X_j x, x\rangle \preceq I.$$

It now follows that $|\langle X_j x, x\rangle| \leq 1$ for each j and unit vector x. Hence, $\|X_j\| \leq 1$ for each j and hence $X \in \mathcal{D}_{\mathcal{C}_\rho}(n)$ and the proof is complete. ■

We have not computed $L_{\max}$ for the matrix cube ($g > 2$ variables), but we have for the matrix square ($g = 2$ variables) and found it to be a pencil with operator (infinite dimensional) coefficients. We do not give the calculation in this paper, rather we content ourselves with the simple, and natural, example below which suffices to show that there are in fact choices of M in Proposition 5.3 which do lead to improved estimates for the matrix cube problem and that $L_{\min}$ and $L_{\max}$ are different for the cube. Of course, any such improved estimate comes with additional computational cost; and, because it is in only two variables (where solving four LMIs gives the exact answer), the example is purely illustrative.

Given $\eta = (s, t) \in \mathbb{R}^2$ with $s^2 + t^2 = 1$, let

$$A_1(\eta) = \begin{bmatrix} s & 0 \\ 0 & -s \end{bmatrix}, \quad A_2(\eta) = \begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}, \qquad L_\eta = I + \sum A_j(\eta) x_j.$$

Recall, a unitary matrix which is symmetric is called a **signature matrix**. Up to scaling the $A_j(\eta)$ are signature matrices, and further $(A_1(\eta) \pm A_2(\eta))^2 = I$.

It is straightforward to show that $\mathcal{D}_{L_\eta}(1)$ contains the unit square; i.e., $\mathcal{D}_{\mathcal{C}_1}(1) \subseteq \mathcal{D}_{L_\eta}(1)$. Hence, with $M_\eta = \mathcal{C}_1 \oplus L_\eta$,

$$\mathcal{D}_{M_\eta}(1) = \mathcal{D}_{\mathcal{C}_1}(1)$$

and at the same time,

$$\mathcal{D}_{M_\eta} \subseteq \mathcal{D}_{\mathcal{C}_1}. \tag{5.2}$$

On the other hand, for $\eta \neq (\pm 1, 0)$ or $(0, \pm 1)$, it is possible to check by hand that $\mathcal{D}_{\mathcal{C}_1}(n) \not\subseteq \mathcal{D}_{L_\eta}(n)$ for each $n \geq 2$. Indeed, let $X = (X_1, X_2)$ denote the tuple of 2×2 matrices

$$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad X_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then $(X_1, X_2) \in \mathcal{D}_{\mathcal{C}_1}(2)$, but $(X_1, X_2) \notin \mathcal{D}_{L_\eta}(2)$. Hence, the inclusion in equation (5.2) is proper.

Another way to see the inclusion is proper is to verify that the extreme points $X = (X_1, X_2)$ of $\mathcal{D}_{\mathcal{C}_1}(n)$ are exactly the pairs of $n \times n$ signature matrices X_1, X_2. On the other hand, for $\eta \notin \{(\pm 1, 0), (0, \pm 1)\}$, the extreme points $X = (X_1, X_2)$ of $\mathcal{D}_{\mathcal{C}_1}(n)$ which are also in $\mathcal{D}_{L_\eta}(n)$ are precisely the pairs of $n \times n$ signature matrices X_1, X_2 which commute.

Example 5.4 (Example 4.4 revisited). Recall the pencil Γ from Example 3.1. In Example 4.4 we employed the matricial matrix cube relaxation $\mathcal{D}_{\mathcal{C}_\rho} \subseteq \mathcal{D}_\Gamma$ to obtain a lower bound of $\frac{1}{2}$ for the biggest square inside $\mathcal{D}_\Gamma(1)$. To tighten the relaxation we direct sum L_η to $\mathcal{C}_\rho$ to obtain a linear pencil M_η. Hand calculations for this problem tell us that with $\eta = (\sqrt{2}/2, \sqrt{2}/2)$ we obtain the exact relaxation $\mathcal{D}_{M_\eta} \subseteq \mathcal{D}_\Gamma$. However, suppose we did not know this and ask: will selecting η without much care give a reasonable improvement?

We made 100 runs with random η and considered the inclusion $\rho\mathcal{D}_{M_\eta} \subseteq \mathcal{D}_\Gamma$ for each η and found the average value for ρ to be approximately 0.6. This is a considerable improvement over 0.5 obtained in the untightened problem. □

6. Positivstellensätze on a matricial LMI set

We give an algebraic characterization of symmetric polynomials p in noncommuting variables with the property that $p(X)$ is positive definite for all X in a bounded matricial LMI set $\mathcal{D}_L$. The conclusion of this Positivstellensatz is stronger than previous ones because of the stronger hypothesis that $\mathcal{D}_L$ is an LMI set. If the polynomial p is linear, then an algebraic characterization is given by Theorem 1.1. We shall use the linear Positivstellensatz, Corollary 3.7, to prove that the quadratic module associated to a monic linear pencil L with bounded $\mathcal{D}_L$ is archimedean. Thereby we obtain a Putinar-type Positivstellensatz [Put93] without the unpleasant added "bounding term". In this section, for simplicity of presentation we stick to polynomials on the free $*$-algebra. Later in Section 7 we give this improved type of Positivstellensatz on general $*$-algebras, a few examples being commuting variables, free variables and free symmetric variables (this section). The material here is motivated by the study of positivity of matrix polynomials in *commuting* variables undertaken in [KS10]; see also [HM04].

To state and prove our next string of results, we need to introduce notation pertaining to words and polynomials in noncommuting variables.

6.1. **Words and NC polynomials.** Given positive integers n, d, d' and g, let $\mathbb{R}^{d'\times d}$ denote the $d' \times d$ matrices with real entries and $(\mathbb{R}^{n\times n})^g$ the set of g-tuples of real $n \times n$ matrices.

We write $\langle x\rangle$ for the monoid freely generated by $x = (x_1, \dots, x_g)$, i.e., $\langle x\rangle$ consists of **words** in the g noncommuting letters $x_1, \dots, x_g$ (including the empty word $\varnothing$ which plays the role of the identity 1). Let $\mathbb{R}\langle x\rangle$ denote the associative $\mathbb{R}$-algebra freely generated by x, i.e., the elements of $\mathbb{R}\langle x\rangle$ are polynomials in the noncommuting variables x with coefficients in $\mathbb{R}$. Its elements are called **(NC) polynomials**. An element of the form aw where $0 \neq a \in \mathbb{R}$ and $w \in \langle x\rangle$ is called a **monomial** and a its **coefficient**. Hence words are monomials whose coefficient is 1. Endow $\mathbb{R}\langle x\rangle$ with the natural **involution** fixing $\mathbb{R} \cup \{x\}$ pointwise. The involution reverses words. For example, $(3 - 2x_1^2x_2x_3)^* = 3 - 2x_3x_2x_1^2$.

6.1.1. *NC matrix polynomials.* More generally, for an abelian group R we use $R\langle x\rangle$ to denote the abelian group of all (finite) sums of **monomials** in $\langle x\rangle$. Besides $R = \mathbb{R}$, the most important example is $R = \mathbb{R}^{d'\times d}$ giving rise to NC matrix polynomials. If $d' = d$, i.e., $R = \mathbb{R}^{d\times d}$, then $R\langle x\rangle$ is an algebra, and admits an involution fixing $\{x\}$ pointwise and being the usual transposition on $\mathbb{R}^{d\times d}$. We also use $*$ to denote the canonical mapping $\mathbb{R}^{d'\times d}\langle x\rangle \to \mathbb{R}^{d\times d'}\langle x\rangle$.

A **matrix NC polynomial** is an NC polynomial with matrix coefficients, i.e., an element of $\mathbb{R}^{d'\times d}\langle x\rangle$ for some $d', d \in \mathbb{N}$.

6.1.2. *Polynomial evaluations.* If $p \in \mathbb{R}^{d'\times d}\langle x\rangle$ is an NC polynomial and $X \in (\mathbb{R}^{n\times n})^g$, the evaluation $p(X) \in \mathbb{R}^{d'n\times dn}$ is defined by simply replacing x_i by X_i. For example, if $p(x) = Ax_1x_2$, where

$$A = \begin{bmatrix} -4 & 2 \\ 3 & 0 \end{bmatrix},$$

then

$$p\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = A \otimes \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 4 & 0 & -2 \\ -4 & 0 & 2 & 0 \\ 0 & -3 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, if $p(x) = A$ and $X \in (\mathbb{R}^{n\times n})^g$, then $p(X) = A \otimes I_n$.

Most of our evaluations will be on tuples of *symmetric* matrices $X \in (\mathbb{SR}^{n\times n})^g$; our involution fixes the variables x element-wise, so only these evaluations give rise to $*$-representations of NC polynomials.

6.2. **Archimedean quadratic modules and a Positivstellensatz.** In this subsection we use the linear Positivstellensatz (Corollary 3.7) to prove that linear pencils with bounded LMI sets give rise to archimedean quadratic modules. This is then used to prove a (nonlinear) Positivstellensatz for matrix NC polynomials positive (semi)definite on bounded matricial LMI sets.

Theorem 6.1. *Suppose $L \in \mathbb{SR}^{d\times d}\langle x\rangle$ is a monic linear pencil and $\mathcal{D}_L$ is bounded. Then for every symmetric polynomial $f \in \mathbb{R}^{\ell\times\ell}\langle x\rangle$ with $f|_{\mathcal{D}_L} \succ 0$, there are $A_j \in \mathbb{R}^{\ell\times\ell}\langle x\rangle$, and $B_k \in \mathbb{R}^{d\times\ell}\langle x\rangle$ satisfying*

$$f = \sum_j A_j^* A_j + \sum_k B_k^* L B_k. \tag{6.1}$$

Corollary 6.2. *Keep the assumptions of Theorem 6.1. Then for a symmetric polynomial $f \in \mathbb{R}^{\ell\times\ell}\langle x\rangle$ the following are equivalent:*

(i) $f|_{\mathcal{D}_L} \succeq 0$;
(ii) *for every $\varepsilon > 0$ there are $A_j \in \mathbb{R}^{\ell\times\ell}\langle x\rangle$, and $B_k \in \mathbb{R}^{d\times\ell}\langle x\rangle$ satisfying*

$$f + \varepsilon = \sum_j A_j^* A_j + \sum_k B_k^* L B_k. \tag{6.2}$$

Proof. Obviously, (ii) $\Rightarrow$ (i). Conversely, if (i) holds, then $f + \varepsilon|_{\mathcal{D}_L} \succ 0$ and we can apply Theorem 6.1. ■

We emphasize that convexity of $\mathcal{D}_L$ implies concrete bounds on the size of the matrices $X \in \mathcal{D}_L$ that need to be plugged into f to check whether $f|_{\mathcal{D}_L} \succ 0$:

Proposition 6.3 (cf. [HM04, Proposition 2.3]). *Let $L \in \mathbb{SR}^{d\times d}\langle x\rangle$ be a linear pencil with $\mathcal{D}_L$ bounded, and let $f = f^* \in \mathbb{R}^{n\times n}\langle x\rangle$ be of degree m. Set $s := n\sum_{j=0}^m g^j$. Then:*

(1) $f|_{\mathcal{D}_L} \succ 0$ *if and only if* $f|_{\mathcal{D}_L(s)} \succ 0$;
(2) $f|_{\mathcal{D}_L} \succeq 0$ *if and only if* $f|_{\mathcal{D}_L(s)} \succeq 0$.

Proof. In both statements the direction ($\Rightarrow$) is obvious. If $f|_{\mathcal{D}_L} \not\succ 0$, there is an ℓ, $X \in \mathcal{D}_L(\ell)$ and $v = \oplus_{j=1}^n v_j \in (\mathbb{R}^\ell)^n$ with

$$\langle f(X)v, v\rangle \leq 0.$$

Let

$$\mathcal{K} := \{w(X)v_j \mid w \in \langle x\rangle \text{ is of degree} \leq m,\ j = 1,\dots,n\}.$$

Clearly, $\dim \mathcal{K} \leq n\sum_{j=0}^m g^j = s$. Let P be the orthogonal projection of $\mathbb{R}^\ell$ onto $\mathcal{K}$. Then

$$\langle f(PXP)v, v\rangle = \langle f(X)v, v\rangle \leq 0.$$

Since $PXP \in \mathcal{D}_L(s)$, this proves (1). The proof of (2) is the same. ■

The crucial step in proving Theorem 6.1 is observing that the quadratic module generated by L in $\mathbb{R}^{\ell\times\ell}\langle x\rangle$ is archimedean. This is essentially a consequence of Corollary 3.7, i.e., of the linear Positivstellensatz as we now demonstrate.

Definition 6.4. Let $\mathcal{A}$ be a ring with involution $a \mapsto a^*$ and set $\operatorname{Sym}\mathcal{A} := \{a \in \mathcal{A} \mid a = a^*\}$. A subset $M \subseteq \operatorname{Sym}\mathcal{A}$ is called a **quadratic module** in $\mathcal{A}$ if

$$1 \in M, \quad M + M \subseteq M \quad \text{and} \quad a^*Ma \subseteq M \text{ for all } a \in \mathcal{A}.$$

We will be mostly interested in the case $\mathcal{A} = \mathbb{R}^{\ell\times\ell}\langle x\rangle$. In this case given a subset $S \subseteq \operatorname{Sym}\mathbb{R}^{d\times d}\langle x\rangle$, the quadratic module M_S^ℓ generated by S in $\mathbb{R}^{\ell\times\ell}\langle x\rangle$ is the smallest subset of $\operatorname{Sym}\mathbb{R}^{\ell\times\ell}\langle x\rangle$ containing all a^*sa for $s \in S\cup\{1\}$, $a \in \mathbb{R}^{d\times\ell}\langle x\rangle$, and closed under addition:

$$M_S^\ell = \Big\{ \sum_{i=1}^N a_i^* s_i a_i \mid N \in \mathbb{N},\ s_i \in S\cup\{1\},\ a_i \in \mathbb{R}^{d\times\ell}\langle x\rangle \Big\}.$$

This notion extends naturally to quadratic modules generated by $S \subseteq \bigcup_{d\in\mathbb{N}} \operatorname{Sym}\mathbb{R}^{d\times d}\langle x\rangle$.

Definition 6.5. A quadratic module M of a ring with involution $\mathcal{A}$ is **archimedean** if

$$\forall a \in \mathcal{A}\ \exists N \in \mathbb{N}:\ N - a^*a \in M. \tag{6.3}$$

To a quadratic module $M \subseteq \operatorname{Sym}\mathcal{A}$ we associate its **ring of bounded elements**

$$H_M(\mathcal{A}) := \{a \in \mathcal{A} \mid \exists N \in \mathbb{N}:\ N - a^*a \in M\}.$$

A quadratic module $M \subseteq \operatorname{Sym}\mathcal{A}$ is thus archimedean if and only if $H_M(\mathcal{A}) = \mathcal{A}$.

The name *ring* of bounded elements is justified by the following proposition:

Proposition 6.6 (Vidav [Vid59])**.** *Let $\mathcal{A}$ be an $\mathbb{R}$-algebra with involution, and $M \subseteq \operatorname{Sym}\mathcal{A}$ a quadratic module. Then $H_M(\mathcal{A})$ is a subalgebra of $\mathcal{A}$ and is closed under the involution.*

Hence it suffices to check the archimedean condition (6.3) on a set of algebra generators.

Lemma 6.7. *A quadratic module $M \subseteq \mathbb{R}^{\ell\times\ell}\langle x\rangle$ is archimedean if and only if there exists $N \in \mathbb{N}$ with $N - x^*x = N - \sum_i x_i^2 \in M$.*

Proof. The "only if" direction is obvious. For the converse, observe that $\mathbb{R}^{\ell\times\ell}\langle x\rangle$ is generated as an $\mathbb{R}$-algebra by x and the $\ell\times\ell$ matrix units E_{ij}, $i,j = 1,\dots,\ell$. By assumption,

$$N - x_i^2 = (N - \sum_i x_i^2) + \sum_{j\neq i} x_j^2 \in M,$$

so $x_j \in H_M(\mathbb{R}^{\ell\times\ell}\langle x\rangle)$ for every j. On the other hand, $E_{ij}^*E_{ij} = E_{jj}$ and thus

$$1 - E_{ij}^*E_{ij} = \sum_{k\neq j} E_{kk}^*E_{kk} \in M.$$

Hence by Proposition 6.6, $H_M(\mathbb{R}^{\ell\times\ell}\langle x\rangle) = \mathbb{R}^{\ell\times\ell}\langle x\rangle$ so M is archimedean. ∎

We are now in a position to give our crucial observation.

Proposition 6.8. *Suppose $L \in \mathbb{SR}^{d\times d}\langle x\rangle$ is a monic linear pencil and $\mathcal{D}_L$ is bounded. Then the quadratic module $M_{\{L\}}^\ell$ generated by L in $\mathbb{R}^{\ell\times\ell}\langle x\rangle$ is archimedean.*

To make the proof more streamlined we separate one easy argument into a lemma:

Lemma 6.9. *For $S \subseteq \bigcup_{d\in\mathbb{N}} \operatorname{Sym}\mathbb{R}^{d\times d}\langle x\rangle$ the following are equivalent:*

(i) *M_S^ℓ is archimedean for some $\ell \in \mathbb{N}$;*

(ii) M_S^ℓ *is archimedean for all* $\ell \in \mathbb{N}$.

Proof. (ii) $\Rightarrow$ (i) is obvious. For the converse assume (i) and let $p \in \mathbb{N}$ be arbitrary. By assumption, there is $N \in \mathbb{N}$ with $(N - x^*x)I_\ell \in M_S^\ell$. If $E_{ij}^{(s,q)}$ denote the $s \times q$ matrix units, then

$$(N - x^*x)E_{11}^{(p,p)} = (E_{11}^{(\ell,p)})^*(N - x^*x)I_\ell E_{11}^{(\ell,p)} \in M_S^p.$$

Now using permutation matrices we see $(N - x^*x)E_{jj}^{(p,p)} \in M_S^p$ for all j concluding the proof by the additivity of M_S^p. ∎

Proof of Proposition 6.8. Since $\mathcal{D}_L$ is bounded, there is $N \in \mathbb{N}$ with $N \geq \|X\|$ for all $X \in \mathcal{D}_L$. Consider the $(g+1) \times (g+1)$ monic linear pencil

$$\mathcal{J}_N(x) = \frac{1}{N}\begin{bmatrix} N & x^* \\ x & NI_g \end{bmatrix} \in \mathbb{S}\mathbb{R}^{(g+1)\times(g+1)}\langle x\rangle.$$

By taking Schur complements, we see $\mathcal{J}_N(X) \succeq 0$ if and only if $N - \frac{1}{N}\sum_j X_j^2 \geq 0$, i.e., if and only if $\|X\| \leq N$. This means $\mathcal{J}_N|_{\mathcal{D}_L} \succeq 0$ and so by Corollary 3.7 (in the new terminology), $\mathcal{J}_N \in M_{\{L\}}^{g+1}$. Since $M_{\{L\}}^{g+1}$ is closed under $*$-conjugation, we obtain

$$\begin{bmatrix} N & 0 \\ 0 & \left(N - \frac{1}{N}x^*x\right)I_g \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{N}x & 1 \end{bmatrix} N\mathcal{J}_N(x) \begin{bmatrix} 1 & 0 \\ -\frac{1}{N}x & 1 \end{bmatrix}^* \in M_{\{L\}}^{g+1}.$$

Again, using permutation matrices leads to $(N^2g - x^*x)I_{g+1} \in M_{\{L\}}^{g+1}$. By Proposition 6.6, $M_{\{L\}}^{g+1}$ is archimedean. Finally, Lemma 6.9 implies $M_{\{L\}}^\ell$ is archimedean. ∎

Corollary 6.10. *For a monic linear pencil L the following are equivalent:*

(i) $\mathcal{D}_L(1)$ *is bounded;*
(ii) *the quadratic module* $M_{\{L\}}^\ell$ *is archimedean for some* $\ell \in \mathbb{N}$*;*
(iii) *the quadratic module* $M_{\{L\}}^\ell$ *is archimedean for all* $\ell \in \mathbb{N}$.

Proof. Clearly, (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). On the other hand, (i) is equivalent to $\mathcal{D}_L$ being bounded by Proposition 2.4, so Proposition 6.8 applies and allows us to deduce (iii). ∎

Proof of Theorem 6.1; compare [HM04, Proposition 4.1]. The statement (6.1) holds if and only if $f \in M_{\{L\}}^\ell$. Now that the archimedeanity of the quadratic module $M_{\{L\}}^\ell$ has been established in Proposition 6.8, the proof is classical. We only list basic steps and refer the reader to [HM04] for detailed proofs.

The proof is by contradiction, so assume $f \notin M_{\{L\}}^\ell$. Archimedeanity of $M_{\{L\}}^\ell$ is equivalent to the existence of an order unit (also called algebraic interior point), namely 1, of the convex cone $M_{\{L\}}^\ell \subseteq \operatorname{Sym}\mathbb{R}^{\ell\times\ell}\langle x\rangle$. Thus the Eidelheit-Kakutani separation theorem yields a linear map $\varphi : \mathbb{R}^{\ell\times\ell}\langle x\rangle \to \mathbb{R}$ satisfying

$$\varphi(f) \leq 0 \quad \text{and} \quad \varphi(M_{\{L\}}^\ell) \subseteq \mathbb{R}_{\geq 0}.$$

Modding out $\mathcal{N} := \{f \in \mathbb{R}^{1\times\ell}\langle x\rangle \mid \varphi(p^*p) = 0\}$ out of $\mathbb{R}^{1\times\ell}\langle x\rangle$ leads to a vector space $\mathcal{H}_0$ and φ induces a scalar product

$$\langle \_,\_ \rangle : \mathcal{H}_0 \times \mathcal{H}_0 \to \mathbb{R}, \quad (\bar{p}, \bar{q}) \mapsto \varphi(q^*p).$$

Completing $\mathcal{H}_0$ with respect to this scalar product yields a Hilbert space $\mathcal{H}$. It is nonzero since $\sum_i \langle e_i, e_i\rangle = \varphi(1) = 1$, where e_i are the matrix units of $\mathbb{R}^{1\times\ell}$. Let $e = \oplus e_i \in \mathcal{H}^\ell$.

The induced left regular $*$-representation $\pi : \mathbb{R}\langle x\rangle \to B(\mathcal{H})$ is bounded (since $M^\ell_{\{L\}}$ is archimedean). Let $\hat{X}_i := \pi(X_i)$ and $\hat{X} := (\hat{X}_1, \dots, \hat{X}_g)$. The constructed scalar product extends naturally to $\mathcal{H}^\ell$. For every $\bar{p} \in \mathcal{H}_0^\ell$, we have

$$\langle L(\hat{X})\bar{p}, \bar{p}\rangle = \sum_{j,k} \langle L(\hat{X})_{j,k}\bar{p}_j, \bar{p}_k\rangle = \sum_{j,k} \varphi(p_k^* L(x)_{j,k} p_j) = \varphi(p^* L(x) p) \geq 0,$$

where p has been identified with a $\ell \times \ell$ matrix polynomial and the last inequality results from $p^*L(x)p \in M^\ell_{\{L\}}$. Hence $\hat{X} \in \mathcal{D}_L$. But now

$$0 \geq \varphi(f) = \langle f(\hat{X})e, e\rangle > 0,$$

a contradiction. ∎

The cautious reader will have noticed that the constructed $\hat{X}$ leading to the contradiction was (in general) not acting on a finite dimensional Hilbert space. However this is only a slight technical difficulty; we refer the reader to Proposition 6.3 or [HM04, Proposition 2.3] for a remedy.

6.3. **More constraints.** Additional constraints can be imposed on elements of a matricial LMI set. Given $S \subseteq \bigcup_{d\in\mathbb{N}} \operatorname{Sym}\mathbb{R}^{d\times d}\langle x\rangle$, define

$$\mathcal{D}_S(n) = \{X \in (\mathbb{S}\mathbb{R}^{n\times n})^g \mid \forall s \in S :\ s(X) \succeq 0\},$$

and let

$$\mathcal{D}_S = \bigcup_{n\in\mathbb{N}} \mathcal{D}_S(n)$$

denote the (matrix) **positivity domain**. Also of interest is the **operator positivity domain**

$$\mathcal{D}_S^\infty = \big\{X \in \operatorname{Sym}\mathcal{B}(\mathcal{H})^g \mid \forall s \in S :\ s(X) \succeq 0\big\}.$$

Here $\mathcal{H}$ is a separable Hilbert space, and $\operatorname{Sym}\mathcal{B}(\mathcal{H})$ is the set of all bounded symmetric operators on $\mathcal{H}$.

Theorem 6.11. *Suppose $L \in \mathbb{S}\mathbb{R}^{d\times d}\langle x\rangle$ is a monic linear pencil and $\mathcal{D}_L$ is bounded. Let $g_j \in \operatorname{Sym}\mathbb{R}^{d_j\times d_j}\langle x\rangle$ $(j \in \mathbb{N})$ be symmetric matrix polynomials. Then for every $f \in \operatorname{Sym}\mathbb{R}^{\ell\times\ell}\langle x\rangle$ with $f|_{\mathcal{D}^\infty_{\{L,\, g_j | j\in\mathbb{N}\}}} \succ 0$, we have $f \in M^\ell_{\{L,\, g_j | j\in\mathbb{N}\}}$.*

Proof. Since the quadratic module $M^\ell_{\{L,\, g_j | j\in\mathbb{N}\}} \supseteq M^\ell_{\{L\}}$ is archimedean, the same proof as for Theorem 6.1 applies. ∎

Remark 6.12. For a particularly appealing consequence (in commuting variables) of Theorem 6.11 see Section 7.1.

We conclude this section with a Nichtnegativstellensatz. It is a stronger form of the Nirgendsnegativsemidefinitheitsstellensatz [KS07] for matricial LMI sets.

Corollary 6.13. *Let $L \in \mathbb{SR}^{d\times d}\langle x\rangle$ be a monic linear pencil and suppose $\mathcal{D}_L$ is bounded. Let $g_j \in \operatorname{Sym}\mathbb{R}^{d_j\times d_j}\langle x\rangle$ $(j \in \mathbb{N})$ be symmetric matrix polynomials. Then for every $h \in \operatorname{Sym}\mathbb{R}^{\ell\times\ell}\langle x\rangle$ the following are equivalent:*

(i) $h|_{\mathcal{D}^\infty_{\{L,\, g_j|j\in\mathbb{N}\}}} \not\preceq 0$, *i.e., for every (nontrivial separable) Hilbert space $\mathcal{H}$ and tuple of symmetric bounded operators $X \in \mathcal{D}^\infty_{\{L,\, g_j|j\in\mathbb{N}\}}$ on $\mathcal{H}$, there is a $v \in \mathcal{H}$ with $\langle h(X)v, v\rangle > 0$;*

(ii) *there are $D_j \in \mathbb{R}^{\ell\times\ell}\langle x\rangle$ satisfying*

$$\sum D_j^* h D_j \in I_\ell + M^\ell_{\{L,\, g_j|j\in\mathbb{N}\}}. \tag{6.4}$$

Proof. (ii) ⇒ (i) is obvious. The converse is also easy. Just apply Theorem 6.11 with $f = -1$ and the positivity domain $\mathcal{D}^\infty_{\{L,\,-h,\, g_j|j\in\mathbb{N}\}} = \varnothing$. ∎

Remark 6.14. There does not seem to exist a clean linear Nichtnegativstellensatz. We found 4×4 monic linear pencils L_1, L_2 in nine variables with the following properties:

(1) $\mathcal{D}_{L_1}$ and $\mathcal{D}_{L_2}$ are bounded;

(2) $L_2|_{\mathcal{D}_{L_1}} \not\preceq 0$, or equivalently,

$$\left\{X \in \mathbb{R}^9 \;\middle|\; \begin{bmatrix} L_1(X) & 0 \\ 0 & -L_2(X)\end{bmatrix} \succeq 0\right\} = \varnothing;$$

(3) there do not exist real matrices U_j, V_k, W_ℓ with

$$\sum_j U_j^* L_2(x) U_j = I + \sum_\ell W_\ell^* W_\ell + \sum_k V_k^* L_1(x) V_k. \tag{6.5}$$

By Corollary (6.13), (1) and (2) imply that (6.5) holds with $U_j, V_k, W_\ell \in \mathbb{R}^{4\times4}\langle x\rangle$. A Mathematica notebook with all the calculations is available at `http://srag.fmf.uni-lj.si`.

7. More general Positivstellensätze

In this section we present two possible modifications of our theory. First, we apply our techniques to commuting variables and derive a "clean" classical Putinar Positivstellensatz on a bounded spectrahedron. This is done by adding symmetrized commutation relations to our list of constraints. In fact we can add any symmetric relation and get a clean Positivstellensatz on a subset of a bounded LMI set (this is Theorem 7.4). In Section 7.2 we also show how to deduce similar results for nonsymmetric noncommuting variables.

7.1. **Positivstellensätze on an LMI set in $\mathbb{R}^g$.** We adapt some of our previous definitions to commuting variables. Let $[y]$ be the monoid freely generated by $y=(y_1,\dots,y_g)$, i.e., $[y]$ consists of words in the g commuting letters $y_1,\dots,y_g$ (including the empty word $\varnothing$ which plays the role of the identity 1). Let $\mathbb{R}[y]$ denote the commutative $\mathbb{R}$-algebra freely generated by y, i.e., the elements of $\mathbb{R}[y]$ are polynomials in the commuting variables y with coefficients in $\mathbb{R}$.

More generally, for an abelian group R we use $R[y]$ to denote the abelian group of all R-linear combinations of words in $[y]$. Besides $R=\mathbb{R}$, the most important example is $R=\mathbb{R}^{d'\times d}$ giving rise to matrix polynomials. If $d'=d$, i.e., $R=\mathbb{R}^{d\times d}$, then $R[y]$ is an $\mathbb{R}$-algebra, and admits an involution fixing $\{y\}$ pointwise and being the usual transposition on $\mathbb{R}^{d\times d}$. We also use $*$ to denote the canonical mapping $\mathbb{R}^{d'\times d}[y]\to\mathbb{R}^{d\times d'}[y]$. If $p\in\mathbb{R}^{d'\times d}[y]$ is a polynomial and $Y\in\mathbb{R}^g$, the evaluation $p(Y)\in\mathbb{R}^{d'\times d}$ is defined by simply replacing y_i by Y_i.

The natural map $\langle x\rangle\to[y]$ is called the **commutative collapse**. It extends naturally to matrix polynomials.

For $A_0,A_1,\dots,A_g\in\mathbb{S}\mathbb{R}^{d\times d}$, a linear matrix polynomial

$$L(y)=A_0+\sum_{j=1}^{g}A_jy_j\in\mathbb{S}\mathbb{R}^{d\times d}[y], \tag{7.1}$$

is a linear pencil. If $A_0=I$, then L is monic. If $A_0=0$, then L is a truly linear pencil. Its spectrahedron is

$$\mathcal{D}_L(1)=\{Y\in\mathbb{R}^g\mid L(Y)\succeq 0\},$$

and for every $\ell\in\mathbb{N}$, L induces a quadratic module $Q^\ell_{\{L\}}$ in $\mathbb{R}^{\ell\times\ell}[y]$:

$$Q^\ell_{\{L\}}=\Big\{\sum_{i=1}^{N}a_i^*a_i+\sum_{j=1}^{N}b_j^*Lb_j\mid N\in\mathbb{N},\ a_i\in\mathbb{R}^{\ell\times\ell}[y],\ b_j\in\mathbb{R}^{d\times\ell}[y]\Big\}.$$

All the results on linear pencils and archimedeanity given in Section 6 carry over to the commutative setting. For instance, given a monic linear pencil $L\in\mathbb{S}\mathbb{R}^{d\times d}[y]$, we have:

(1) $Q^\ell_{\{L\}}$ is archimedean for some $\ell\in\mathbb{N}$ if and only if $Q^\ell_{\{L\}}$ is archimedean for all $\ell\in\mathbb{N}$;
(2) $Q^\ell_{\{L\}}$ is archimedean if and only if the spectrahedron $\mathcal{D}_L(1)$ is bounded.

Most importantly, we obtain the following clean version of Putinar's Positivstellensatz [Put93] on a bounded spectrahedron.

Theorem 7.1. *Suppose $L\in\mathbb{S}\mathbb{R}^{d\times d}[y]$ is a monic linear pencil and $\mathcal{D}_L(1)$ is bounded. Then for every symmetric polynomial $f\in\mathbb{R}^{\ell\times\ell}[y]$ with $f|_{\mathcal{D}_L(1)}\succ 0$, there are $A_j\in\mathbb{R}^{\ell\times\ell}[y]$, and $B_k\in\mathbb{R}^{d\times\ell}[y]$ satisfying*

$$f=\sum_j A_j^*A_j+\sum_k B_k^*LB_k. \tag{7.2}$$

Proof. Let $F \in \operatorname{Sym}\mathbb{R}^{\ell\times\ell}\langle x\rangle$ be an arbitrary symmetric matrix polynomial in noncommuting variables whose commutative collapse is f. By abuse of notation, let $L \in \mathbb{SR}^{d\times d}\langle x\rangle$ be the canonical lift of $L \in \mathbb{SR}^{d\times d}[y]$. Write

$$g_{ij} = -(x_ix_j - x_jx_i)^*(x_ix_j - x_jx_i) = (x_ix_j - x_jx_i)^2 \in \operatorname{Sym}\mathbb{R}\langle x\rangle$$

for $i, j = 1, \dots, g$. Note $g_{ij}(X) \succeq 0$ if and only if $X_iX_j = X_jX_i$. By the spectral theorem, $F|_{\mathcal{D}_{\{L,\, g_{ij}|i,j=1,\dots,g\}}} \succ 0$. So Theorem 6.11 implies and yields

$$F \in M^{\ell}_{\{L,\, g_{ij}|i,j=1,\dots,g\}}.$$

Applying the commutative collapse gives $f \in Q^{\ell}_{\{L\}}$, as desired. ∎

Corollary 7.2. *Suppose $L \in \mathbb{SR}^{d\times d}[y]$ is a monic linear pencil and $\mathcal{D}_L(1)$ is bounded. Let $g_1, \dots, g_s \in \mathbb{R}[y]$ and*

$$\mathcal{D}_L(g_1, \dots, g_s) := \{Y \in \mathbb{R}^g \mid L(Y) \succeq 0,\, g_1(Y) \geq 0, \dots, g_s(Y) \geq 0\}.$$

If $f \in \mathbb{R}[y]$ satisfies $f|_{\mathcal{D}_L(g_1,\dots,g_s)} > 0$, then there are $h_{ij} \in \mathbb{R}[y]$, and $B_k \in \mathbb{R}^{d\times 1}[y]$ satisfying

$$f = \sum_{j=0}^{s} g_j \sum_i h_{ij}^2 + \sum_k B_k^* L B_k, \tag{7.3}$$

where $g_0 := 1$.

Corollary 7.3. *Suppose $L \in \mathbb{SR}^{d\times d}[y]$ is a monic linear pencil and $\mathcal{D}_L(1)$ is bounded. Then for every polynomial $f \in \mathbb{R}[y]$ with $f|_{\mathcal{D}_L(1)} > 0$, there are $h_j \in \mathbb{R}[y]$, and $B_k \in \mathbb{R}^{d\times 1}[y]$ satisfying*

$$f = \sum_j h_j^2 + \sum_k B_k^* L B_k. \tag{7.4}$$

It is clear that a Nichtnegativstellensatz along the lines of Corollary 6.13 holds in this setting. We leave the details to the reader.

7.2. **Free (nonsymmetric) variables.** In this section we explain how our theory adapts to the free $*$-algebra. Let $\langle x, x^*\rangle$ be the monoid freely generated by $x = (x_1, \dots, x_g)$ and $x^* = (x_1^*, \dots, x_g^*)$, i.e., $\langle x, x^*\rangle$ consists of words in the $2g$ noncommuting letters $x_1, \dots, x_g, x_1^*, \dots, x_g^*$ (including the empty word $\varnothing$ which plays the role of the identity 1). Let $\mathbb{C}\langle x, x^*\rangle$ denote the $\mathbb{C}$-algebra freely generated by x, x^*, i.e., the elements of $\mathbb{C}\langle x, x^*\rangle$ are polynomials in the noncommuting variables x, x^* with coefficients in $\mathbb{C}$. As before, we introduce matrix polynomials $\mathbb{C}^{d'\times d}\langle x, x^*\rangle$. If $p \in \mathbb{C}^{d'\times d}\langle x, x^*\rangle$ is a polynomial and $X \in (\mathbb{C}^{n\times n})^g$, the evaluation $p(X, X^*) \in \mathbb{C}^{d'\times d}$ is defined by simply replacing x_i by X_i and x_i^* by X_i^*.

For $A_1, \dots, A_g \in \mathbb{C}^{d\times d}$, a linear matrix polynomial

$$L(x) = \sum_{j=1}^{g} A_j x_j \in \mathbb{C}^{d\times d}\langle x\rangle, \tag{7.5}$$

is a truly linear pencil. (Note: none of the variables x^* appears in such an L.) Its **monic symmetric pencil** is

$$\mathcal{L}(x,x^*) = I + L(x) + L(x)^* = I + \sum_{j=1}^{g} A_j x_j + \sum_{j=1}^{g} A_j^* x_j^* \in \operatorname{Sym}\mathbb{C}^{d\times d}\langle x,x^*\rangle.$$

The associated matricial LMI set is

$$\mathcal{D}_{\mathcal{L}} = \bigcup_{n\in\mathbb{N}} \{X \in (\mathbb{C}^{n\times n})^g \mid \mathcal{L}(X) \succeq 0\},$$

its operator-theoretic counterpart is

$$\mathcal{D}_{\mathcal{L}}^{\infty} = \{X \in \mathcal{B}(\mathcal{H})^g \mid \mathcal{L}(X) \succeq 0\},$$

and for every $\ell \in \mathbb{N}$, $\mathcal{L}$ induces a quadratic module $M^{\ell}_{\{\mathcal{L}\}}$ in $\mathbb{C}^{\ell\times\ell}\langle x,x^*\rangle$:

$$M^{\ell}_{\{\mathcal{L}\}} = \Big\{ \sum_{i=1}^{N} a_i^* a_i + \sum_{j=1}^{N} b_j^* \mathcal{L} b_j \mid N \in \mathbb{N},\, a_i \in \mathbb{C}^{\ell\times\ell}\langle x,x^*\rangle,\, b_j \in \mathbb{C}^{d\times\ell}\langle x,x^*\rangle \Big\}.$$

Like in the previous subsection, all our main results from Section 6 carry over to this free setting. As a sample, we give a Positivstellensatz:

Theorem 7.4. *Suppose $\mathcal{L} \in \operatorname{Sym}\mathbb{C}^{d\times d}\langle x,x^*\rangle$ is a monic symmetric linear pencil and $\mathcal{D}_{\mathcal{L}}$ is bounded. Let $g_j \in \operatorname{Sym}\mathbb{C}^{d_j\times d_j}\langle x,x^*\rangle$ $(j \in \mathbb{N})$ be symmetric matrix polynomials. Then for every $f \in \operatorname{Sym}\mathbb{C}^{\ell\times\ell}\langle x,x^*\rangle$ with $f|_{\mathcal{D}^{\infty}_{\{\mathcal{L},\, g_j | j\in\mathbb{N}\}}} \succ 0$, we have $f \in M^{\ell}_{\{\mathcal{L},\, g_j | j\in\mathbb{N}\}}$.*

As a special case we obtain a Positivstellensatz describing polynomials positive definite on *commuting* tuples X in a matricial LMI set. (Note: we are not assuming the entries X_i commute with the adjoints X_j^*.)

Corollary 7.5. *Suppose $\mathcal{L} \in \operatorname{Sym}\mathbb{C}^{d\times d}\langle x,x^*\rangle$ is a monic symmetric linear pencil and $\mathcal{D}_{\mathcal{L}}$ is bounded. Suppose $f \in \operatorname{Sym}\mathbb{C}^{\ell\times\ell}\langle x,x^*\rangle$ satisfies $f(X,X^*) \succ 0$ for all $X \in \mathcal{D}^{\infty}_{\mathcal{L}}$ with $X_iX_j = X_jX_i$ for all i,j.*

(1) *Let $c_{jk} = x_j x_k - x_k x_j$. Then*

$$f \in M^{\ell}_{\{\mathcal{L},\, c_{jk}+c_{jk}^*,\, i(c_{jk}-c_{kj}) | j,k=1,\dots,g\}}.$$

(2) *Let $d_{jk} = -c_{jk}^* c_{jk}$. Then*

$$f \in M^{\ell}_{\{\mathcal{L},\, d_{jk} | j,k=1,\dots,g\}}.$$

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