New Lower Bounds for the Vehicle Routing Problem with Simultaneous Pickup and Delivery

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Abstract

This work deals with the Vehicle Routing Problem with Simultaneous Pickup and Delivery (VRPSPD). We propose undirected and directed two-commodity flow formulations, which are based on the one developed by Baldacci, Hadjiconstantinou and Mingozzi for the Capacitated Vehicle Routing Problem. These new formulations are theoretically compared with the one-commodity flow formulation proposed by Dell'Amico, Righini and Salani. The three formulations were tested within a branch-and-cut scheme and their practical performance was measured in well-known benchmark problems. The undirected two-commodity flow formulation obtained consistently better results. We also ran the three formulations in a particular case of the VRPSPD, namely the Vehicle Routing Problem with Mixed Pickup and Delivery (VRPMPD). Several optimal solutions to open problems with up to 100 customers and new improved lower bounds for instances with up to 200 customers were found.

1 Introduction

The Vehicle Routing Problem with Simultaneous Pickup and Delivery (VRPSPD) is a variant of the Capacitated Vehicle Routing Problem (CVRP), in which clients require both pickup and delivery services. This problem was first proposed two decades ago by Min [12]. The VRPSPD is clearly $\mathcal{NP}$-hard since it can be reduced to the CVRP when all the pickup demands are equal to zero.

The VRPSPD can be defined as follows. Let $G = (V, E)$ be a complete graph with a set of vertices $V = \{0, ..., n\}$, where the vertex 0 represents the depot and the remaining ones the customers. Each edge $\{i, j\} \in E$ has a non-negative cost c_{ij} and each client $i \in V' = V - \{0\}$ has non-negative demands d_i for delivery and p_i for pickup. Let $C = \{1, ..., m\}$ be a set of homogeneous vehicles with capacity Q. The VRPSPD consists in constructing a set up to m routes in such a way that: (i) every route starts and ends at the depot; (ii) all the pickup and delivery demands are accomplished; (iii) the vehicle's capacity is not exceeded in any part of a route; (iv) a customer is visited by only a single vehicle; (v) the sum of costs is minimized.

A particular case of the VRPSPD, known as the Vehicle Routing Problem with Mixed Pickup and Delivery (VRPMPD), arises when customers either have a pickup or a delivery demand. More precisely, if $d_i > 0$, then $p_i = 0$ and vice-versa. In principle, any VRPSPD solution method can be directly applied to solve the VRPMPD.

Applications of the VRPSPD can be found especially within the Reverse Logistics context. Companies are increasingly faced with the task of managing the reverse flow of finished goods or raw-materials. Thus, one should consider not only the Distribution Logistics, but also the management of the reverse flow. Both the Distribution Logistic and Reverse Logistic should act together with an aim to guarantee the synchronization between the pickup and delivery operations, as well as their impact on the company's supply chain, resulting in the customer's satisfaction and minimization of the operational efforts.

Although heuristic strategies are by far the most employed to solve the VRPSPD, some exact algorithms were also explored in the literature. A branch-and-price algorithm was developed by Dell'Amico et al. [3], in which two different strategies were used to solve the subpricing problem: (i) exact dynamic programming and (ii) state space relaxation. The authors managed to find optimal solutions for instances with up to 40 clients.

Angelelli and Manisini [1] also developed a branch-and-price approach based on the set covering formulation, but for the VRPSPD with time-windows constraints. The subproblem was formulated as a shortest-path with resource constraints but without the elementary condition and it was solved by applying a permanent labeling algorithm. The authors were able find optimal solutions for instances with up to 20 clients.

Three-index formulations for the VRPSPD were proposed by Dethloff [4] and Montané and Galvão [13], however only the last authors had tested it in practice. They ran their formulation in CPLEX 9.0 within a time limit of 2 hours and had reported the lower bounds produced for benchmark instances with 50-400 customers.

In this work we propose an undirected and a directed two-commodity flow formulations for the VRPSPD. These formulations extend the one developed by Baldacci et al. [2] for the CVRP. They were compared with the one-commodity flow formulation presented by Dell'Amico et al. [3]. In addition, the three formulations were implemented within a branch-and-cut algorithm, including cuts from the CVRPSEP library [10], and they were tested in well-known VRPSPD and VRPMPD benchmark problems with up to 200 customers.

The remainder of this report is organized as follows. Section 2 describes the one-commodity flow formulation [3]. In Section 3 we present the undirected and the directed two-commodity flow formulations for the VRPSPD and we compare these formulations with the one developed in [3]. Section 4 describes the branch-and-cut approach. Section 5 contains the experimental results. Section 6 presents the concluding remarks.

2 One-commodity flow formulation

Reasonably simple and effective formulations for the CVRP can be defined only over the natural edge variables (arc variables in the asymmetric case), see [18]. Similar formulations are not available for the VRPSPD. This difference between these two problems can be explained as follows. In the CVRP, the feasibility of a route can be determined by only checking whether the sum of its client demands does not exceed the vehicle's capacities. In contrast, the feasibility of a VRPSPD route depends crucially on the sequence of visitation of the clients. In the example shown in Figure 1, the route $0 \to 2 \to 3 \to 1 \to 0$ is feasible, but the shorter routes $0 \to 1 \to 2 \to 3 \to 0$ or $0 \to 3 \to 2 \to 1 \to 0$ are not. This fact suggests the use of extended formulations, where auxiliary flow variables are used to enforce route feasibility.

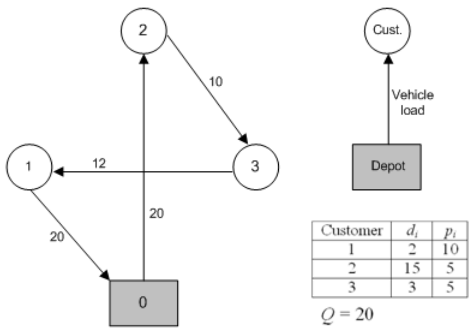


Figure 1: VRPSPD example

The following directed one-commodity flow formulation for the VRPSPD was proposed by Dell'Amico et al. [3]. Define A as the set of arcs consisting of a pair of opposite arcs (i, j) and (j, i) for each edge $\{i, j\} \in E$ and let D_{ij} and P_{ij} be the flow variables which indicate, respectively, the delivery and pickup loads carried along the arc $(i, j) \in A$. Let x_{ij} be 1 if the arc $(i, j) \in A$ is in the solution and 0 otherwise. The Mixed Integer Programming formulation F1C is described next.

$$\min \sum_{i \in V} \sum_{j \in V} c_{ij} x_{ij} \tag{1}$$

subject to

$$\sum_{j \in V} x_{ij} = 1 \qquad \forall i \in V' \tag{2}$$

$$\sum_{j \in V} x_{ji} = 1 \qquad \forall i \in V' \tag{3}$$

$$\sum_{j \in V'} x_{0j} \leq m \tag{4}$$

$$\sum_{j \in V} D_{ji} - \sum_{j \in V} D_{ij} = d_i \qquad \forall i \in V' \tag{5}$$

$$\sum_{j\in V} P_{ij} - \sum_{j\in V} P_{ji} = p_i \qquad \forall i \in V' \tag{6}$$

$$D_{ij} + P_{ij} \le Q x_{ij} \qquad \forall (i,j) \in A \tag{7}$$

$$D_{ij} \ge 0 \qquad \forall (i,j) \in A \tag{8}$$

$$P_{ij} \ge 0 \qquad \forall (i,j) \in A \tag{9}$$

$$x_{ij} \in \{0,1\} \qquad \forall (i,j) \in A \tag{10}$$

The objective function (1) minimizes the sum of the travel costs. Constraints (2)-(3) impose that each client should be visited exactly once. Constraints (4) refer to the number of vehicles available. Constraints (5)-(7) are the flow conservation equalities. Constraints (8)-(10) are related to the nature of the decision variables.

Dell'Amico et al. [3] basically extended the one-commodity flow formulation proposed by Gavish and Graves [7] for the CVRP by adding constraints (6) and (9), and the term P_{ij} in (7). Gouveia [8] showed that it is possible to obtain stronger inequalities for D_{ij} by using the tighter bounds (11) instead of (8) in the Gavish and Graves formulation. Accordingly, we can apply the same idea to develop stronger inequalities for P_{ij} by replacing (9) with (12) and for $D_{ij} + P_{ij}$ by replacing (7) with (13).

$$d_j x_{ij} \le D_{ij} \le (Q - d_i) x_{ij} \qquad \forall (i,j) \in A \tag{11}$$

$$p_i x_{ij} \le P_{ij} \le (Q - p_j) x_{ij} \qquad \forall (i,j) \in A \tag{12}$$

$$D_{ij} + P_{ij} \le (Q - \max\{0, p_j - d_j, d_i - p_i\}) x_{ij} \qquad \forall (i,j) \in A \tag{13}$$

It should be noticed that a lower bound for (13) is implicit in (11) and (12), i.e, $D_{ij} + P_{ij} \ge d_j x_{ij} + p_i x_{ij}$. Another valid inequality for F1C, given by (14), is due to the fact that each edge not adjacent to the depot is traversed at most once.

$$x_{ij} + x_{ji} \le 1 \qquad \forall i, j, i < j, \in V' \tag{14}$$

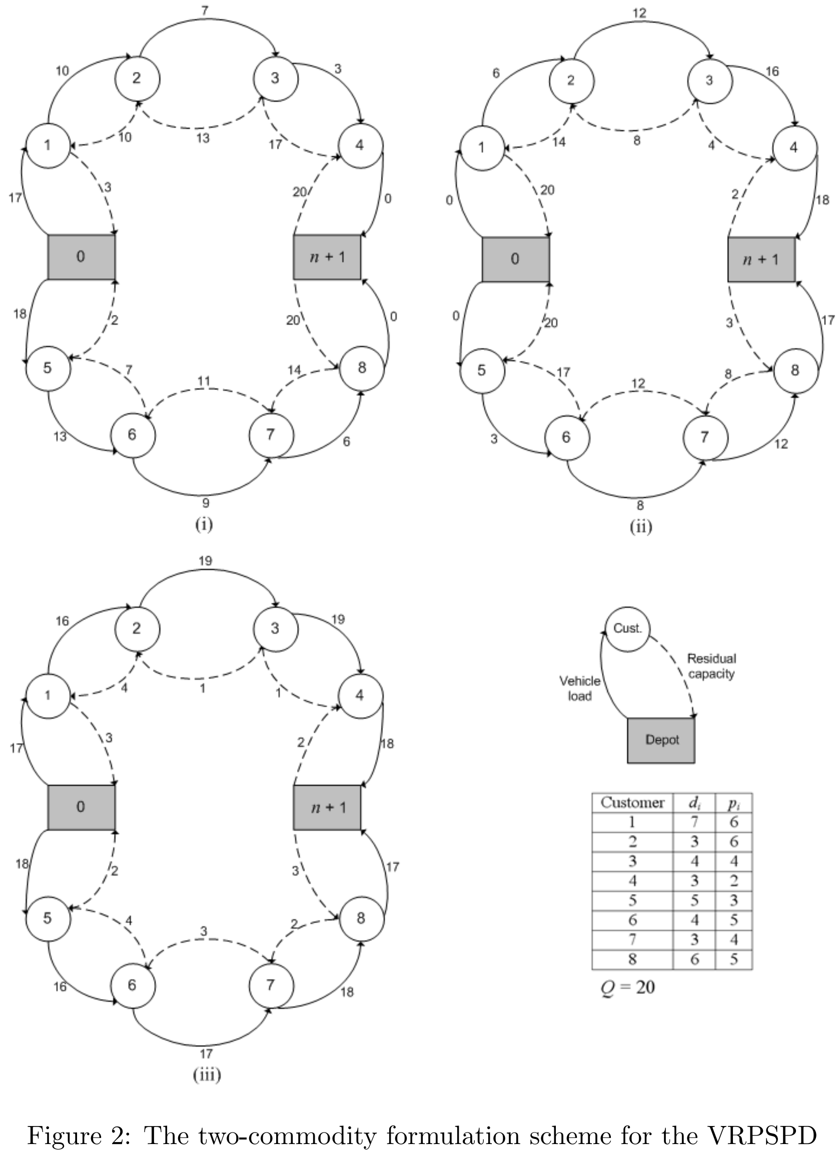
3 Two-commodity flow formulations

In this section we present both an undirected and a directed two-commodity flow formulations for the VRPSPD which are based on the one proposed by Baldacci et al. [2] for the CVRP.

3.1 Undirected two-commodity flow formulation

For the sake of convenience let vertex $n+1$ be a copy of the depot, $\bar{V} = V \cup \{n+1\}$ and $\bar{E}$ be the complete set of edges $\bar{E}$, excepting $\{0, n+1\}$. Let x'_{ij} be 1 if the edge $\{i,j\} \in \bar{E}$ is in the solution and 0 otherwise. Let the variables D'_{ij}, P'_{ij} and SPD_{ij} denote, respectively, the delivery, pickup and simultaneous pickup and delivery flows when a vehicle goes from $i \in \bar{V}$ to $j \in \bar{V}$ and let the same variables denote, respectively, the associated residual capacities when a vehicle goes from $j \in \bar{V}$ to $i \in \bar{V}$, in such a way that $D'_{ij} + D'_{ji} = Qx'_{ij}$, $P'_{ij} + P'_{ji} = Qx'_{ij}$ and $SPD_{ij} + SPD_{ji} = Qx'_{ij}$. Also, an integer variable v, which denotes the number of vehicles utilized, is included with an upper bound m. In the Baldacci et al. formulation [2] the precise number of vehicles m is assumed to be known in advance, since their formulation will produce feasible solutions with exact m vehicles.

Fig. 2 shows an example of the scheme used by the two-commodity flow formulation for the VRPSPD, where (i), (ii) and (iii) denote, respectively, the delivery, pickup and simultaneous pickup and delivery flows. In this case, $Q = 20$ and two routes are considered where r_1 is the one composed by $0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow n+1$ and r_2 is composed by $0 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow n{+}1$. Moreover, it can be observed that the flows when the vehicle is leaving the depot are equivalent in (i) and (iii), whereas the flows when the vehicle is returning to the depot are equivalent in (ii) and (iii). This fact can be generalized to any VRPSPD instance by means of the following relationships: $SPD_{0j} = D'_{0j}$, $SPD_{j0} = D'_{j0}$, $SPD_{jn+1} = P'_{jn+1}$ and $SPD_{n+1j} = D'_{n+1j}$, $\forall j \in V'$.



Customer	d_i	p_i
1	7	6
2	3	6
3	4	4
4	3	2
5	5	3
6	4	5
7	3	4
8	6	5

$Q = 20$

Figure 2: The two-commodity formulation scheme for the VRPSPD

The undirected formulation F2C-U is as follows.

$$\min \sum_{\{i,j\}\in \bar{E}} c_{ij}x'_{ij} \tag{15}$$

subject to

$$\sum_{i\in \bar{V}, i<k} x'_{ik} + \sum_{j\in \bar{V}, j>k} x'_{kj} = 2 \qquad \forall k \in V' \tag{16}$$

$$\sum_{j\in \bar{V}} (D'_{ji} - D'_{ij}) = 2d_i \qquad \forall i \in V' \tag{17}$$

$$\sum_{j\in V'} D'_{0j} = \sum_{i\in V'} d_i \tag{18}$$

$$\sum_{j\in V'} D'_{j0} = vQ - \sum_{i\in V'} d_i \tag{19}$$

$$\sum_{j\in \bar{V}} (P'_{ij} - P'_{ji}) = 2p_i \qquad \forall i \in V' \tag{20}$$

$$\sum_{j\in V'} P'_{jn+1} = \sum_{i\in V'} p_i \tag{21}$$

$$\sum_{j\in V'} P'_{n+1j} = vQ - \sum_{i\in V'} p_i \tag{22}$$

$$\sum_{j\in \bar{V}} (SPD_{ji} - SPD_{ij}) = 2(d_i - p_i) \qquad \forall i \in V' \tag{23}$$

$$SPD_{0j} = D'_{0j} \qquad \forall j \in V' \tag{24}$$

$$SPD_{j0} = D'_{j0} \qquad \forall j \in V' \tag{25}$$

$$SPD_{jn+1} = P'_{jn+1} \qquad \forall j \in V' \tag{26}$$

$$SPD_{n+1j} = D'_{n+1j} \qquad \forall j \in V' \tag{27}$$

$$D'_{ij} + D'_{ji} = Qx'_{ij} \qquad \forall \{i,j\} \in \bar{E} \tag{28}$$

$$P'_{ij} + P'_{ji} = Qx'_{ij} \qquad \forall \{i,j\} \in \bar{E} \tag{29}$$

$$SPD_{ij} + SPD_{ji} = Qx'_{ij} \qquad \forall \{i,j\} \in \bar{E} \tag{30}$$

$$D'_{jn+1} = P'_{0j} = 0 \qquad \forall j \in V' \tag{31}$$

$$\sum_{j\in V'} D'_{n+1j} = \sum_{j\in V'} P'_{j0} = vQ \tag{32}$$

$$\sum_{j\in V'} x'_{0j} = \sum_{j\in V'} x'_{jn+1} = v \tag{33}$$

$$0 \leq v \leq m \tag{34}$$

$$D'_{ij} \geq 0, D'_{ji} \geq 0 \qquad \forall \{i,j\} \in \bar{E} \tag{35}$$

$$P'_{ij} \geq 0, P'_{ji} \geq 0 \qquad \forall \{i,j\} \in \bar{E} \tag{36}$$

$$SPD_{ij} \geq 0, SPD_{ji} \geq 0 \qquad \forall \{i,j\} \in \bar{E} \tag{37}$$

$$x'_{ij} \in \{0,1\} \qquad \forall (i,j) \in \bar{E} \tag{38}$$

The objective function (15) minimizes the sum of the travel costs. Constraints (16) are the degree equations. Constraints (17) ensure that the delivery demands are satisfied. Constraints (18) state that the sum of the vehicle loads leaving the vertex 0 must be equal to the sum of the demand of all costumers. Constraints (19) enforce that the sum of the vehicle loads arriving at the vertex 0 must be equal to the sum of the residual capacity of all vehicles. Constraints (20)-(22) are related to the pickup flow and their meaning are, respectively, analogous to (17)-(19). Constraints (23) guarantee that the pickup and delivery demands are simultaneously satisfied. Constraints (24)-(27) are self-explanatory. Constraints (28)-(30) state, respectively, that the sum of the delivery, pickup and combined loads arriving and leaving each customer must be equal to the vehicle capacity. Constraints (31)-(32) are self-explanatory. Constraint (33) is related to the number of vehicles. Constraints (34)-(38) define the domain of the decision variables.

The formulation F2C-U was obtained by simply adding constraints (20)-(27), (29)-(34) and (36)-(37) to the formulation presented in [2]. As in F1C, stronger inequalities can be developed by tightening the bounds of the flow variables, i.e, replacing (35)-(36) with (39)-(40) and (37) with (41).

$$D'_{ij} \geq d_j x'_{ij} \qquad \forall (i,j) \in \bar{E} \qquad (39)$$

$$P'_{ij} \geq p_i x'_{ij} \qquad \forall (i,j) \in \bar{E} \qquad (40)$$

$$SPD_{ij} \geq \max\{0, d_j - p_j, p_i - d_i\} x'_{ij} \qquad \forall (i,j) \in \bar{E} \qquad (41)$$

Although the lower bounds of the flow variables are not explicit in (39)-(41) it can be easily verified that they become inherent to the formulation when these upper bound inequalities are combined with (28)-(30), resulting in $D'_{ij} \leq (Q - d_i)x'_{ij}$, $P'_{ij} \leq (Q - d_j)x'_{ij}$ and $SPD_{ij} \leq (Q - \max\{0, d_i - p_i, p_j - d_j\})x'_{ij}$.

3.2 Directed two-commodity flow formulation

Let $\bar{A}$ be the set of arcs (i,j), $\forall i,j \in \bar{V}$ and $\bar{x}_{ij}$ be 1 if the arc $(i,j) \in \bar{A}$ is in the solution and 0 otherwise. A directed version of the two-commodity flow formulation (F2C-D) is as follows.

$$\min \sum_{i \in \bar{V}} \sum_{j \in \bar{V}} c_{ij} \bar{x}_{ij} \qquad (42)$$

subject to

$$\sum_{j \in \bar{V}} \bar{x}_{ij} = 1 \qquad \forall i \in V' \qquad (43)$$

$$\sum_{j \in \bar{V}} \bar{x}_{ji} = 1 \qquad \forall i \in V' \qquad (44)$$

$$\bar{x}_{j0} = \bar{x}_{n+1j} = 0 \qquad \forall j \in V' \qquad (45)$$

$$D'_{ij} + D'_{ji} = Q(\bar{x}_{ij} + \bar{x}_{ji}) \qquad \forall (i,j), i < j, \in A \tag{46}$$

$$P'_{ij} + P'_{ji} = Q(\bar{x}_{ij} + \bar{x}_{ji}) \qquad \forall (i,j), i < j, i \neq 0 \in \bar{A} \tag{47}$$

$$SPD_{ij} + SPD_{ji} = Q(\bar{x}_{ij} + \bar{x}_{ji}) \qquad \forall i,j, i < j, \in V' \tag{48}$$

$$\sum_{j \in V'} \bar{x}_{0j} = \sum_{j \in V'} \bar{x}_{jn+1} = v \tag{49}$$

$$\bar{x}_{ij} \in \{0,1\} \qquad \forall (i,j) \in \bar{A} \tag{50}$$

(17)-(32) and (34)-(37)

Constraints (43)-(44) are the degree equations. Constraint (45) is self-explanatory. Constraints (46)-(48) are the capacity equalities. Constraints (49)-(50) have already been defined.

The stronger flow inequalities defined for F2C-U also hold for F2C-D as can be observed in (51)-(53). Also, the arc inequalities (14) used in F1C can be directly converted to F2C-D as shown in (54).

$$D'_{ij} \geq d_j(\bar{x}_{ij} + \bar{x}_{ji}) \qquad \forall (i,j) \in \bar{A} \tag{51}$$

$$P'_{ij} \geq p_i(\bar{x}_{ij} + \bar{x}_{ji}) \qquad \forall (i,j) \in \bar{A} \tag{52}$$

$$SPD_{ij} \geq (\max\{0, d_j - p_j, p_i - d_i\})(\bar{x}_{ij} + \bar{x}_{ji}) \qquad \forall (i,j) \in \bar{A} \tag{53}$$

$$\bar{x}_{ij} + \bar{x}_{ji} \leq 1 \qquad \forall i,j, i < j, \in V' \tag{54}$$

F2C-D is clearly at least as strong as F2C-U since the degree constraints (43)-(44) along with (54) dominate (16) and the linear relaxation of (38), whereas the remaining constraints are equivalent in both formulations.

Letchford and Salazar-Gonzalez [9] have shown that the one-commodity formulation and the directed two-commodity flow formulation with their respective stronger inequalities are equivalent for the CVRP. However, this fact is not verified for the VRPSPD as stated by Proposition 1.

Proposition 1. *The linear relaxation of F1C with (11)-(14) is stronger than the one obtained by F2C-D with (51)-(54).*

Proof. First we shall prove that given the solution vector (x^*, D^*, P^*) with cost z^* of the linear programming relaxation of the one-commodity flow formulation, it is possible to build a feasible solution of the linear program of F2C-D (in terms of $(\bar{x}, D', P', SPD, v)$) with the same cost.

The values of the variables of F2C-D can be directly obtained by means of (55)-(68).

For the sake of simplicity let $P_{jn+1} = P_{j0}$ and $P_{n+1j} = P_{0j}$, $\forall j \in V'$.

$$\bar{x}_{ij} = x_{ij} \qquad \forall i, j \in V' \qquad (55)$$

$$\bar{x}_{0j} = x_{0j} \qquad \forall j \in V' \qquad (56)$$

$$\bar{x}_{jn+1} = x_{j0} \qquad \forall j \in V' \qquad (57)$$

$$D'_{ij} = D_{ij} + (Q\bar{x}_{ji} - D_{ji}) \qquad \forall (i,j), i < j, \in A \qquad (58)$$

$$D'_{ji} = D_{ji} + (Q\bar{x}_{ij} - D_{ij}) \qquad \forall (i,j), i < j, \in A \qquad (59)$$

$$P'_{ij} = P_{ij} + (Q\bar{x}_{ji} - P_{ji}) \qquad \forall (i,j), i < j, i \neq 0 \in \bar{A} \qquad (60)$$

$$P'_{ji} = P_{ji} + (Q\bar{x}_{ij} - P_{ij}) \qquad \forall (i,j), i < j, i \neq 0 \in \bar{A} \qquad (61)$$

$$SPD_{ij} = D_{ij} + P_{ij} + (Q\bar{x}_{ji} - D_{ji} - P_{ji}) \qquad \forall i, j, i < j, \in V' \qquad (62)$$

$$SPD_{ji} = D_{ji} + P_{ji} + (Q\bar{x}_{ij} - D_{ij} - P_{ij}) \qquad \forall i, j, i < j, \in V' \qquad (63)$$

$$D'_{jn+1} = P'_{0j} = \bar{x}_{j0} = \bar{x}_{n+1j} = 0 \qquad \forall j \in V' \qquad (64)$$

$$D'_{n+1j} = Q\bar{x}_{n+1j}, P'_{j0} = Q\bar{x}_{j0} \qquad \forall j \in V' \qquad (65)$$

$$SPD_{0j} = D'_{0j}, SPD_{j0} = D'_{j0} \qquad \forall j \in V' \qquad (66)$$

$$SPD_{jn+1} = P'_{jn+1}, SPD_{n+1j} = P'_{n+1j} \qquad \forall j \in V' \qquad (67)$$

$$v = \sum_{j \in V'} \bar{x}_{0j} \qquad (68)$$

Note that constraints (46)-(48) are automatically satisfied since they can be easily obtained from (58)-(63). Constraints (51) are satisfied since, according to (11), $D_{ij} \geq d_j \bar{x}_{ij}$ and $Q\bar{x}_{ji} - D_{ji} \geq d_j \bar{x}_{ji}$, which implies in $D'_{ij} \geq d_j(\bar{x}_{ij} + \bar{x}_{ji})$. The same idea can be employed, using (12), to show that constraints (52) are also satisfied.

To verify if constraints (53) are not violated we need to prove the following statement: $D_{ij} + P_{ij} + (Q\bar{x}_{ij} - D_{ji} - P_{ji}) \geq (\max\{0, d_j - p_j, p_i - d_i\})(\bar{x}_{ij} + \bar{x}_{ji})$. Using the fact that $D_{ij} + P_{ij} \geq d_j \bar{x}_{ij} + p_i \bar{x}_{ij}$ (see (11)-(12)) and after some algebraic manipulation we obtain: $d_j \bar{x}_{ij} + p_i \bar{x}_{ij} + (Q - \max\{0, d_j - p_j, p_i - di\})\bar{x}_{ji} \geq D_{ji} + P_{ji} + (\max\{0, d_j - p_j, p_i - d_i\})\bar{x}_{ij}$. From (13) we observe that $(Q - \max\{0, d_j - p_j, p_i - di\})\bar{x}_{ji} \geq D_{ji} + P_{ji}$ and it is clear that $d_j \bar{x}_{ij} + p_i \bar{x}_{ij} \geq (\max\{0, d_j - p_j, p_i - d_i\})\bar{x}_{ij}$, which proves that (53) is satisfied.

Thus we conclude that the vector $(\bar{x}, D', P', SPD, v)$ is indeed a feasible solution of the linear program of F2C-D.

On the other hand, given the solution vector $(\bar{x}^*, D'^*, P'^*, SPD^*, v^*)$ with cost $\bar{z}^*$ of the linear programming relaxation of F2C-D it is not always possible to build a feasible solution in terms of (x, D, P) with the same cost. Tables 1 and 3; 4 and 6; and 7 and 9; all presented in Section 5, show that the value of the linear relaxation obtained by the F1C is always greater or equal than the one found by F2C-D. □

4 A branch-and-cut approach

A simple branch-and-cut (BC) algorithm was employed to evaluate the formulations presented in this work. Traditional CVRP inequalities were used, namely the rounded capacity, multistar and comb inequalities. They can be directly applied to the VRPSPD. The cuts were separated using the CVRPSEP package [10]. The reader is referred to [11] for details concerning the separation routines.

At first, we try to separate the cuts using the delivery demands. When no valid inequalities are found we then use the pickup demands. All of the three kinds of cuts are generated at the root node, but just the rounded capacity cuts are used throughout the tree up to the 5th level. Preliminary tests have shown that the overhead of separating comb and multistar inequalities outside the root node was not worthwhile. For each separation routine of the CVRPSEP package we have established a limit of 50 violated cuts per iteration.

In the case of the VRPSPD instances, the best upper bound (UB) solutions pointed out in the literature were given as initial primal bound for the BC, namely those reported in [17]. This definitely helps the algorithm to find optimal solutions in much less computational time. As for the VRPMPD instances, the UBs found by Gajpal and Abad [6] were provided as a cutoff value for the BC.

5 Computational Experiments

The BC procedures were implemented using the CPLEX 11.2 callable library and executed in an Intel Core 2 Quad with 2.4 GHz and 4 GB of RAM running under Linux 64 bits (kernel 2.6.27-16). Only a single thread was used in our experiments. Each BC is respectively associated with the formulations F1C, F2C-U and F2C-D. A time limit of 2 hours of execution was imposed for the BC algorithms. In some very particular cases, the CPLEX have slightly exceeded this time limit, namely on few instances involving more than 100 customers.

5.1 VRPSPD

Three set of test-problems are available in the VRPSPD literature. These benchmark instances were proposed by Dethloff [4], Salhi and Nagy [16] and Montané and Galvão [13]. The first group contains 40 instances with 50 customers, the second contains 14 instances with 50-199 customers, while the third contains 12 instances with 100-200 customers. The number of vehicles is not explicitly specified in these 66 instances. The barrier algorithm was used to solve the initial linear relaxation of the last two group of instances. It is noteworthy to mention that we have not considered the Montané and Galvão's instances involving 400 customers.

In the tables presented hereafter, $\#v$ represents the number of vehicles in the best known solution, **LP** is the linear relaxation, **Root LB** indicates the root lower bound, after CVRPSEP cuts are added, **Root Time** is the CPU time in seconds spent at the root node, **Tree size** corresponds to the the number of nodes opened, **Total time** is the total CPU time in seconds of the BC procedure, **Prev. LB** is the lower bound obtained in [13], **New LB** is the best lower bound determined among the three flow formulations, **F-LB** is the lower bound found by the respective formulation, **UB** is the upper bound reported in [17], and **Gap** corresponds to the gap between the LB and the UB. Proven optimal solutions are highlighted in boldface. If the F-LB is the one associated with the New LB (F-LB = New LB), then its value is underlined only if New LB is not an optimal solution.

Tables 1, 2 and 3 contain, respectively, the results obtained by F1C, F2C-U and F2C-D on the set of instances of Dethloff. It can be seen that the three formulations were able to prove the optimality of almost all instances of 4 vehicles. F2C-U appears to be the most effective under this aspect, being capable of proving the optimality of 17 instances. The performance of the three formulations on the instances of 9 vehicles were inferior in terms of optimality proof, but their LBs are significantly better than the previous values reported in [13]. F2C-U also seems to be the most effective in terms of LBs, with an average gap of 0.94%, against 1.34% and 1.26% of F1C and F2C-D, respectively.

In order to check if the values of the UB of the instances SCA3-0, SCA3-6, SCA8-3, SCA8-6, CON3-2, CON8-1, CON8-4 and CON8-7 are optimal we ran F2C-U with a time limit of 48 hours. The formulation was successful to prove the optimality of each of these instances within up to 36 hours of execution.

The results found by F1C, F2C-U and F2C-D on the set of instances of Salhi and Nagy are presented, respectively, in Tables 4, 5 and 6. The optimality of the instances CMT1X and CMT1Y has been proven by all the three formulations. Nevertheless, to our knowledge these are the first LBs presented for these set of instances. Montané and Galvão [13] had reported LBs for the case where the demands were rounded to the nearest integer. When comparing the LBs obtained by each of the three formulations it can be verified that F2C-U produced superior results, with an average gap of 4,27 %, against 4,57% and 4,31% of F2C-D and F1C, respectively.

The results obtained by the three formulations on the set of instances of Montané and Galvão are shown in Tables 7, 8 and 9. Three optimal solutions were proven by all formulations, namely in the instances r201, c201 and rc201. The main characteristic of these three instances is the fact of having relatively very few vehicles. When comparing the LBs of the different formulations, it can be verified that F2C-U found the best results, with an average gap of 2.94%, whereas for F2C-U and F1C the average gap was 3.57% and 3.62%, respectively.

Tables 10-12 present the statistics of the root node of each formulation over a set of representative instances. In these tables, **Sep. Rounds** represent the number of calls to the separation routines, **LP Time** is the time in seconds spent solving the linear

Table 1: Results obtained by F1C on Dethloff's instances

Instance/ Customers	#v	LP	Root LB	Root Time (s)	Tree size	Total Time (s)	Prev. LB	New LB	F-LB	UB	Gap (%)
SCA3-0/50	4	551.14	613.38	41	73473	7200	583.77	627.66	622.73	635.62	2.03
SCA3-1/50	4	645.95	682.40	107	1712	1230	655.63	697.84	**697.84**	**697.84**	0.00
SCA3-2/50	4	592.56	658.35	19	1	19	627.12	659.34	659.34	659.34	0.00
SCA3-3/50	4	586.30	667.37	70	1083	415	633.56	680.04	**680.04**	**680.04**	0.00
SCA3-4/50	4	627.29	672.92	80	8718	1599	642.89	690.50	**690.50**	**690.50**	0.00
SCA3-5/50	4	604.31	646.14	83	15560	1901	603.06	659.90	**659.90**	**659.90**	0.00
SCA3-6/50	4	587.97	624.92	47	37655	7200	607.53	645.56	639.97	651.09	1.71
SCA3-7/50	4	584.69	654.30	82	26	103	616.40	659.17	**659.17**	**659.17**	0.00
SCA3-8/50	4	638.75	688.77	94	72785	7200	668.04	719.48	703.12	719.48	2.27
SCA3-9/50	4	597.02	668.09	82	1674	417	619.03	681.00	**681.00**	**681.00**	0.00
SCA8-0/50	9	849.35	922.36	96	8854	7200	877.55	936.89	933.12	961.50	2.95
SCA8-1/50	9	937.71	998.04	75	9948	7200	954.29	1020.28	1015.05	1049.65	3.30
SCA8-2/50	9	931.93	1008.83	78	10334	7200	950.74	1024.24	1019.99	1039.64	1.89
SCA8-3/50	9	874.31	954.55	74	11375	7200	905.29	975.87	970.88	983.34	1.27
SCA8-4/50	9	958.58	1022.44	72	12054	7200	972.62	1041.65	1036.45	1065.49	2.73
SCA8-5/50	9	923.50	996.01	79	9207	7200	940.60	1015.19	1011.57	1027.08	1.51
SCA8-6/50	9	870.58	933.57	133	6219	7200	885.34	959.91	944.53	971.82	2.81
SCA8-7/50	9	937.30	1013.86	65	12533	7200	955.86	1031.56	1029.97	1051.28	2.03
SCA8-8/50	9	962.50	1023.86	102	8510	7200	986.52	1048.93	1036.90	1071.18	3.20
SCA8-9/50	9	953.36	1012.73	89	9031	7200	978.90	1034.28	1031.51	1060.50	2.73
CON3-0/50	4	577.74	606.00	91	40753	4836	592.38	616.52	616.46	616.52	0.01
CON3-1/50	4	506.41	543.71	73	52033	6498	532.55	554.47	**554.47**	**554.47**	0.00
CON3-2/50	4	468.40	503.14	61	13874	7200	491.04	517.26	514.11	518.00	0.75
CON3-3/50	4	541.46	581.45	55	20044	1941	557.99	591.19	**591.19**	**591.19**	0.00
CON3-4/50	4	537.90	577.61	63	78398	7200	558.26	588.79	588.47	588.79	0.06
CON3-5/50	4	511.88	553.87	107	32652	5975	531.33	563.70	**563.70**	**563.70**	0.00
CON3-6/50	4	468.90	486.59	128	14248	7200	475.33	499.05	493.01	499.05	1.21
CON3-7/50	4	533.86	562.10	38	53629	5522	550.73	576.48	**576.48**	**576.48**	0.00
CON3-8/50	4	477.81	513.90	87	15317	1923	492.69	523.05	**523.05**	**523.05**	0.00
CON3-9/50	4	528.34	564.87	63	15461	5602	547.31	578.25	**578.25**	**578.25**	0.00
CON8-0/50	9	774.69	829.80	47	16498	7200	795.45	845.19	842.62	857.17	1.70
CON8-1/50	9	680.24	719.03	80	7552	7200	693.22	734.71	732.44	740.85	1.14
CON8-2/50	9	636.18	682.76	128	9856	7200	650.81	695.70	693.07	712.89	2.78
CON8-3/50	10	732.55	784.93	71	7536	7200	754.41	797.57	796.31	811.07	1.82
CON8-4/50	9	710.36	749.83	122	6374	7200	729.09	767.63	759.11	772.25	1.70
CON8-5/50	9	696.85	728.10	78	7901	7200	709.76	741.51	736.79	754.88	2.40
CON8-6/50	9	611.16	647.04	61	10400	7200	631.41	662.14	662.14	678.92	2.47
CON8-7/50	9	729.28	787.89	64	11861	7200	762.03	810.08	800.22	811.96	1.44
CON8-8/50	9	689.23	741.02	74	10324	7200	705.08	757.45	753.42	767.53	1.84
CON8-9/50	9	716.21	770.66	101	5435	7200	729.10	786.40	778.65	809.00	3.75
									Avg. Gap (%)		1.34

relaxations, **Sep. Time** is the time in seconds spent separating the cuts, **Root Time** is the sum of the LP Time and Sep. Time, and **Gap** is the gap between the root relaxation and the UB.

From the results of Tables 10-12 it can be seen that in most cases the Sep. Rounds increases with the number of vehicles, given a fixed number of customers. Also, it is possible to verify that the LP Time is considerably higher than the Sep. Time and, as expected, this difference tends to increase with the size of the instance as well as the number of vehicles. It appears that all the three formulations became very "heavy" in the instances involving 200 customers, since in almost all cases, they took about 2 hours to solve less than 13 linear programs. An attempt has been made to use the barrier algorithm to solve all the linear programs, but unfortunately the results were not satisfactory.

Table 13 shows a summary of the results obtained by the three formulations in all set of instances. In this table, **G1** is the average gap between the linear relaxation and the UB, **G2** is the average gap with respect to the root LB, including the CVRPSEP cuts,

Table 2: Results obtained by the F2C-U on Dethloff's instances

Instance/ Customers	#v	LP	Root LB	Root Time (s)	Tree size	Total Time (s)	Prev. LB	New LB	F-LB	UB	Gap (%)
SCA3-0/50	4	550.85	613.36	28	211456	7200	583.77	627.66	625.00	635.62	1.67
SCA3-1/50	4	645.59	682.33	42	1467	142	655.63	697.84	**697.84**	**697.84**	0.00
SCA3-2/50	4	592.44	658.89	12	1	12	627.12	659.34	**659.34**	**659.34**	0.00
SCA3-3/50	4	586.02	667.37	25	847	81	633.56	680.04	**680.04**	**680.04**	0.00
SCA3-4/50	4	626.93	673.28	50	2866	252	642.89	690.50	**690.50**	**690.50**	0.00
SCA3-5/50	4	603.95	646.29	38	19724	731	603.06	659.90	**659.90**	**659.90**	0.00
SCA3-6/50	4	587.85	624.89	27	176561	7200	607.53	645.56	644.37	651.09	1.03
SCA3-7/50	4	584.59	653.76	34	30	43	616.40	659.17	**659.17**	**659.17**	0.00
SCA3-8/50	4	638.41	693.71	42	108017	4899	668.04	719.48	**719.48**	**719.48**	0.00
SCA3-9/50	4	596.79	668.09	36	1452	96	619.03	681.00	**681.00**	**681.00**	0.00
SCA8-0/50	9	847.39	922.58	79	17695	7200	877.55	936.89	936.89	961.50	2.56
SCA8-1/50	9	933.44	997.07	56	18086	7200	954.29	1020.28	1020.28	1049.65	2.80
SCA8-2/50	9	931.34	1008.27	75	12789	7200	950.74	1024.24	1024.24	1039.64	1.48
SCA8-3/50	9	872.37	953.67	49	18487	7200	905.29	975.87	975.87	983.34	0.76
SCA8-4/50	9	955.74	1021.35	44	25853	7200	972.62	1041.65	1041.65	1065.49	2.24
SCA8-5/50	9	922.25	995.93	58	19464	7200	940.60	1015.19	1013.87	1027.08	1.29
SCA8-6/50	9	868.00	933.76	74	10467	7200	885.34	959.91	959.91	971.82	1.23
SCA8-7/50	9	935.55	1015.11	69	15193	7200	955.86	1031.56	1031.56	1051.28	1.88
SCA8-8/50	9	960.17	1023.60	87	8262	7200	986.52	1048.93	1048.93	1071.18	2.08
SCA8-9/50	9	952.34	1014.89	64	16262	7200	978.90	1034.28	1034.28	1060.50	2.47
CON3-0/50	4	577.52	606.82	46	3048	247	592.38	616.52	**616.52**	**616.52**	0.00
CON3-1/50	4	506.23	545.53	54	16039	823	532.55	554.47	**554.47**	**554.47**	0.00
CON3-2/50	4	468.22	504.44	59	22107	7200	491.04	517.26	516.23	518.00	0.34
CON3-3/50	4	541.40	582.83	35	6608	330	557.99	591.19	**591.19**	**591.19**	0.00
CON3-4/50	4	537.73	577.57	42	50663	3198	558.26	588.79	**588.79**	**588.79**	0.00
CON3-5/50	4	511.59	554.35	65	10191	729	531.33	563.70	**563.70**	**563.70**	0.00
CON3-6/50	4	468.75	486.61	100	48466	5230	475.33	499.05	**499.05**	**499.05**	0.00
CON3-7/50	4	533.73	561.87	37	9822	1141	550.73	576.48	**576.48**	**576.48**	0.00
CON3-8/50	4	477.45	514.13	71	5541	450	492.69	523.05	**523.05**	**523.05**	0.00
CON3-9/50	4	527.94	564.78	53	6372	790	547.31	578.25	**578.25**	**578.25**	0.00
CON8-0/50	9	773.46	827.14	74	13038	7200	795.45	845.19	845.19	857.17	1.40
CON8-1/50	9	678.95	719.09	67	13302	7200	693.22	734.71	734.71	740.85	0.83
CON8-2/50	9	635.23	682.37	127	9409	7200	650.81	695.70	695.70	712.89	2.41
CON8-3/50	10	731.55	785.00	71	18680	7200	754.41	797.57	797.57	811.07	1.66
CON8-4/50	9	708.64	751.32	60	15700	7200	729.09	767.63	767.63	772.25	0.60
CON8-5/50	9	696.08	727.26	66	9765	7200	709.76	741.51	741.51	754.88	1.77
CON8-6/50	9	610.20	646.78	94	11947	7200	631.41	662.14	661.36	678.92	2.59
CON8-7/50	9	726.55	788.64	74	5520	7200	762.03	810.08	810.08	811.96	0.23
CON8-8/50	9	688.25	741.76	81	13325	7200	705.08	757.45	757.45	767.53	1.31
CON8-9/50	9	713.85	770.85	109	12833	7200	729.10	786.40	786.40	809.00	2.79
										Avg. Gap (%)	0.94

and **G3** is average gap for the LB, possibly after branching, found within the time limit established. Those results can be explained as follows. The linear relaxation of is F1C is indeed a little better than the linear relaxations of F2C-D and F2C-U. However, after the cuts, there is no significant difference in the LB quality. This can be clearly seen in the column **G2** under Dethloff instances. For those smaller instances, the cut separation in the root node could always be completed within the time limit. In those cases, the small gap differences (2.96%, 2.94% and 2.92%) are not significant and can be attributed to the heuristic nature of the routines in the CVRPSEP library. The consistent advantage of formulation F2C-U shown in columns **G3** is explained by the fact that CPLEX has a significantly better performance when reoptimizing its LPs. This means that more cuts can be separated and more nodes can be explored within the same time limit.

Table 3: Results obtained by F2C-D on Dethloff's instances

Instance/ Customers	#v	LP	Root LB	Root Time (s)	Tree size	Total Time (s)	Prev. LB	New LB	F-LB	UB	Gap (%)
SCA3-0/50	4	550.93	613.35	25	132649	7200	583.77	627.66	627.66	635.62	1.25
SCA3-1/50	4	645.60	682.23	29	1262	114	655.63	697.84	**697.84**	**697.84**	0.00
SCA3-2/50	4	592.47	659.11	11	1	11	627.12	659.34	**659.34**	**659.34**	0.00
SCA3-3/50	4	586.02	667.35	21	374	50	633.56	680.04	**680.04**	**680.04**	0.00
SCA3-4/50	4	626.93	673.22	31	6156	334	642.89	690.50	**690.50**	**690.50**	0.00
SCA3-5/50	4	603.96	646.41	31	12594	330	603.06	659.90	**659.90**	**659.90**	0.00
SCA3-6/50	4	587.85	624.92	26	167625	7200	607.53	645.56	645.56	651.09	0.85
SCA3-7/50	4	584.59	654.30	33	23	40	616.40	659.17	**659.17**	**659.17**	0.00
SCA3-8/50	4	638.41	694.13	42	196322	7200	668.04	719.48	714.19	719.48	0.73
SCA3-9/50	4	596.79	668.09	38	1567	116	619.03	681.00	**681.00**	**681.00**	0.00
SCA8-0/50	9	847.73	922.85	107	3758	7200	877.55	936.89	933.89	961.50	2.87
SCA8-1/50	9	933.47	997.57	103	3661	7200	954.29	1020.28	1013.38	1049.65	3.46
SCA8-2/50	9	931.42	1008.87	147	2435	7200	950.74	1024.24	1019.31	1039.64	1.96
SCA8-3/50	9	872.45	953.21	98	3914	7200	905.29	975.87	968.55	983.34	1.50
SCA8-4/50	9	955.96	1022.13	134	4773	7200	972.62	1041.65	1032.49	1065.49	3.10
SCA8-5/50	9	922.32	996.33	184	14246	7200	940.60	1015.19	1015.19	1027.08	1.16
SCA8-6/50	9	868.05	933.74	143	1593	7200	885.34	959.91	943.47	971.82	2.92
SCA8-7/50	9	935.82	1013.12	102	3334	7200	955.86	1031.56	1028.04	1051.28	2.21
SCA8-8/50	9	960.27	1023.53	159	1559	7200	986.52	1048.93	1036.29	1071.18	3.26
SCA8-9/50	9	952.41	1013.82	111	7476	7200	978.90	1034.28	1031.54	1060.50	2.73
CON3-0/50	4	577.52	605.97	34	21249	1141	592.38	616.52	**616.52**	**616.52**	0.00
CON3-1/50	4	506.24	543.70	41	19638	1199	532.55	554.47	**554.47**	**554.47**	0.00
CON3-2/50	4	468.22	504.31	71	97732	7200	491.04	517.26	517.26	518.00	0.14
CON3-3/50	4	541.40	582.89	37	4116	213	557.99	591.19	**591.19**	**591.19**	0.00
CON3-4/50	4	537.73	577.59	28	78652	2932	558.26	588.79	**588.79**	**588.79**	0.00
CON3-5/50	4	511.60	554.43	50	16215	772	531.33	563.70	**563.70**	**563.70**	0.00
CON3-6/50	4	468.75	486.76	96	65792	6979	475.33	499.05	**499.05**	**499.05**	0.00
CON3-7/50	4	533.75	561.89	31	15895	1134	550.73	576.48	**576.48**	**576.48**	0.00
CON3-8/50	4	477.45	513.99	51	3833	269	492.69	523.05	**523.05**	**523.05**	0.00
CON3-9/50	4	527.95	564.77	48	4637	585	547.31	578.25	**578.25**	**578.25**	0.00
CON8-0/50	9	773.51	826.63	122	2526	7200	795.45	845.19	840.60	857.17	1.93
CON8-1/50	9	679.00	719.00	132	2885	7200	693.22	734.71	729.26	740.85	1.56
CON8-2/50	9	635.25	682.12	200	2416	7200	650.81	695.70	692.34	712.89	2.88
CON8-3/50	10	731.55	785.01	151	3406	7200	754.41	797.57	794.94	811.07	1.99
CON8-4/50	9	708.64	751.40	121	5468	7200	729.09	767.63	766.37	772.25	0.76
CON8-5/50	9	696.08	726.88	133	3688	7200	709.76	741.51	734.84	754.88	2.66
CON8-6/50	9	610.20	646.22	125	2458	7200	631.41	662.14	658.43	678.92	3.02
CON8-7/50	9	726.57	787.53	142	1995	7200	762.03	810.08	801.59	811.96	1.28
CON8-8/50	9	688.33	741.06	166	4021	7200	705.08	757.45	749.66	767.53	2.33
CON8-9/50	9	713.94	770.74	234	2307	7200	729.10	786.40	778.72	809.00	3.74
										Avg. Gap (%)	1.26

Table 4: Results obtained by F1C on Salhi and Nagy's instances (VRPSPD)

Instance/ Customers	#v	LP	Root LB	Root Time (s)	Tree size	Total Time (s)	New LB	F-LB	UB	Gap (%)
CMT1X/50	3	449.00	459.94	63	1691	245	466.77	**466.77**	**466.77**	0.00
CMT1Y/50	3	449.00	460.06	71	3225	369	466.77	**466.77**	**466.77**	0.00
CMT2X/75	6	632.14	652.90	1025	2190	7200	655.98	655.39	684.21	4.21
CMT2Y/75	6	632.14	652.66	939	1071	7200	655.41	653.78	684.21	4.45
CMT3X/100	5	682.20	694.61	1382	1399	7200	705.54	695.55	721.27	3.57
CMT3Y/100	5	682.20	694.56	1649	1262	7200	705.62	696.05	721.27	3.50
CMT12X/100	5	566.09	628.64	3017	252	7200	629.39	629.19	662.22	4.99
CMT12Y/100	5	566.09	628.60	2279	618	7201	629.18	629.18	662.22	4.99
CMT11X/120	4	689.87	774.78	7200	1	7204	776.35	774.78	833.92	7.09
CMT11Y/120	4	689.87	775.01	7253	1	7256	775.74	775.01	833.92	7.06
CMT4X/150	7	796.52	816.39	7033	1	7201	817.11	816.39	852.46	4.23
CMT4Y/150	7	796.52	814.67	7013	1	7200	816.99	814.67	852.46	4.43
CMT5X/200	10	933.43	949.19	7315	1	7319	954.87	949.19	1029.25	7.78
CMT5Y/200	10	933.43	950.48	6844	1	7201	953.56	950.48	1029.25	7.65
									Avg. Gap (%)	4.57

5.2 VRPMPD

A set of 21 VRPMPD instances involving 50-199 customers was proposed by Salhi and Nagy [16]. As in the VRPSPD, the number of vehicles is not specified. Also, Gajpal and

Table 5: Results obtained by the F2C-U on Salhi and Nagy's instances (VRPSPD)

Instance/ Customers	#v	LP	Root LB	Root Time (s)	Tree size	Total Time (s)	New LB	F-LB	UB	Gap (%)
CMT1X/50	3	449.00	459.98	102	2282	300	466.77	**466.77**	**466.77**	0.00
CMT1Y/50	3	449.00	460.02	70	3205	213	466.77	**466.77**	**466.77**	0.00
CMT2X/75	6	632.11	652.85	346	2073	7200	655.88	655.21	684.21	4.24
CMT2Y/75	6	632.11	653.13	449	2610	7200	655.41	655.41	684.21	4.21
CMT3X/100	5	682.18	701.10	504	13820	7200	705.54	704.35	721.27	2.35
CMT3Y/100	5	682.18	701.12	612	19865	7200	705.62	705.28	721.27	2.22
CMT12X/100	5	564.08	628.59	813	991	7201	629.39	629.39	662.22	4.96
CMT12Y/100	5	564.08	628.58	923	118	7201	629.18	629.09	662.22	5.00
CMT11X/120	4	687.42	775.51	4835	42	7201	776.35	776.35	833.92	6.90
CMT11Y/120	4	687.42	775.40	6138	22	7200	775.74	775.74	833.92	6.98
CMT4X/150	7	796.48	817.11	7288	1	7292	817.11	817.11	852.46	4.15
CMT4Y/150	7	796.48	816.99	5747	1	7201	816.99	816.99	852.46	4.16
CMT5X/200	10	933.21	954.87	6939	1	7201	954.87	954.87	1029.25	7.23
CMT5Y/200	10	933.21	953.56	6600	1	7202	953.56	953.56	1029.25	7.35
								Avg. Gap (%)		4.27

Table 6: Results obtained by F2C-D on Salhi and Nagy's instances (VRPSPD)

Instance/ Customers	#v	LP	Root LB	Root Time (s)	Tree size	Total Time (s)	New LB	F-LB	UB	Gap (%)
CMT1X/50	3	449.00	459.89	56	1971	204	466.77	**466.77**	**466.77**	0.00
CMT1Y/50	3	449.00	460.02	71	3204	239	466.77	**466.77**	**466.77**	0.00
CMT2X/75	6	632.11	653.05	788	5719	7200	655.88	655.88	684.21	4.14
CMT2Y/75	6	632.11	652.95	625	1800	7200	655.41	654.96	684.21	4.28
CMT3X/100	5	682.18	701.77	610	14470	7200	705.54	705.54	721.27	2.18
CMT3Y/100	5	682.18	701.74	460	13533	7200	705.62	705.62	721.27	2.17
CMT12X/100	5	564.18	628.58	1804	88	7200	629.39	628.81	662.22	5.05
CMT12Y/100	5	564.18	628.53	1564	88	7200	629.18	629.02	662.22	5.01
CMT11X/120	4	687.42	774.36	7222	1	7224	776.35	774.36	833.92	7.14
CMT11Y/120	4	687.42	774.56	7237	1	7239	775.74	774.56	833.92	7.12
CMT4X/150	7	796.48	816.90	7154	1	7201	817.11	816.90	852.46	4.17
CMT4Y/150	7	796.48	816.90	7185	1	7201	816.99	816.91	852.46	4.17
CMT5X/200	10	933.21	952.38	7419	1	7422	954.87	952.38	1029.25	7.47
CMT5Y/200	10	933.21	952.62	6798	1	7201	953.56	952.62	1029.25	7.45
								Avg. Gap (%)		4.31

Table 7: Results obtained by the F1C on Montané and Galvão's instances

Instance/ Customers	#v	LP	Root LB	Root Time (s)	Tree size	Total Time (s)	Prev. LB	New LB	F-LB	UB	Gap (%)
r101/100	12	939.75	972.57	3084	104	7201	934.97	973.91	973.17	1009.95	3.64
r201/100	3	643.08	664.87	562	17	575	643.65	666.20	**666.20**	**666.20**	0.00
c101/100	16	1070.82	1195.47	1788	909	7200	1066.19	1196.70	1196.70	1220.18	1.92
c201/100	5	598.51	657.97	260	88	325	278.05	662.07	**662.07**	**662.07**	0.00
rc101/100	10	946.99	1028.72	4006	82	7201	937.41	1029.38	1029.08	1059.32	2.85
rc201/100	3	600.31	672.31	323	1	324	602.70	672.92	**672.92**	**672.92**	0.00
r1_2_1/200	23	3023.35	3078.76	7015	1	7202	2951.12	3084.97	3078.76	3360.02	8.37
r2_2_1/200	5	1549.88	1607.89	6824	1	7201	1501.82	1618.76	1607.89	1665.58	3.46
c1_2_1/200	28	3326.05	3396.32	5377	1	7201	3299.07	3475.03	3396.32	3629.89	6.43
c2_2_1/200	9	1560.54	1611.74	5601	1	7201	1542.96	1647.83	1611.74	1726.59	6.65
rc1_2_1/200	23	3020.18	3092.57	7124	1	7202	2939.98	3093.30	3092.57	3306.00	6.46
rc2_2_1/200	5	1439.13	1513.14	6757	1	7200	1396.95	1551.07	1513.14	1560.00	3.00
									Avg. Gap (%)		3.57

Abad [6] did not report the number of vehicles associated with their UBs. The barrier algorithm was employed to solve the initial linear relaxation.

Tables 14, 15 and 16 present the results obtained, respectively, by F1C, F2C-U and F2C-D. It can be observed that the optimality of the instances CMT1H, CMT1Q, CMT1T, CMT3Q and CMT12T was proven by all formulations. When comparing the LBs, one can

Table 8: Results obtained by F2C-U on Montané and Galvão's instances

Instance/ Customers	#v	LP	Root LB	Root Time (s)	Tree size	Total Time (s)	Prev. LB	New LB	F-LB	UB	Gap (%)
r101/100	12	939.19	972.88	2910	122	7201	934.97	973.91	973.10	1009.95	3.65
r201/100	3	643.07	664.80	292	21	307	643.65	666.20	**666.20**	**666.20**	0.00
c101/100	16	1070.40	1195.53	1396	302	7201	1066.19	1196.70	1195.89	1220.18	1.99
c201/100	5	598.47	657.97	197	17	241	596.85	662.07	**662.07**	**662.07**	0.00
rc101/100	10	944.21	1028.15	2940	138	7201	937.41	1029.38	1029.38	1059.32	2.83
rc201/100	3	600.24	671.84	134	4	134	602.70	672.92	**672.92**	**672.92**	0.00
r1_2_1/200	23	3013.16	3084.97	6971	1	7200	2951.12	3084.97	3084.97	3360.02	8.19
r2_2_1/200	5	1549.60	1618.76	7869	1	7874	1501.82	1618.76	1618.76	1665.58	2.81
c1_2_1/200	28	3325.20	3475.03	7041	1	7202	3299.07	3475.03	3475.03	3629.89	4.27
c2_2_1/200	9	1560.22	1647.83	7370	1	7374	1542.96	1647.83	1647.83	1726.59	4.56
rc1_2_1/200	23	3015.44	3093.30	7064	1	7201	2939.98	3093.30	3093.30	3306.00	6.43
rc2_2_1/200	5	1438.91	1551.07	7301	1	7308	1396.95	1551.07	1551.07	1560.00	0.57
									Avg.	Gap (%)	2.94

Table 9: Results obtained by F2C-D on Montané and Galvão's instances

Instance/ Customers	#v	LP	Root LB	Root Time (s)	Tree size	Total Time (s)	Prev. LB	New LB	F-LB	UB	Gap (%)
r101/100	12	939.26	972.85	5867	12	7200	934.97	973.91	973.91	1009.95	3.57
r201/100	3	643.07	665.05	490	14	524	643.65	666.20	**666.20**	**666.20**	0.00
c101/100	16	1070.40	1196.16	2752	666	7200	1066.19	1196.70	1196.65	1220.18	1.93
c201/100	5	598.47	657.97	317	61	404	596.85	662.07	**662.07**	**662.07**	0.00
rc101/100	10	944.39	1028.49	6570	6	7200	937.41	1029.38	1028.52	1059.32	2.91
rc201/100	3	600.26	671.84	200	6	201	602.70	672.92	**672.92**	**672.92**	0.00
r1_2_1/200	23	3013.21	3074.77	7088	1	7201	2951.12	3084.97	3074.77	3360.02	8.49
r2_2_1/200	5	1549.62	1615.14	7383	1	7386	1501.82	1618.76	1615.14	1665.58	3.03
c1_2_1/200	28	3325.20	3389.36	7517	1	7521	3299.07	3475.03	3389.36	3629.89	6.63
c2_2_1/200	9	1560.39	1596.24	6028	1	7201	1542.96	1647.83	1596.24	1726.59	7.55
rc1_2_1/200	23	3015.98	3067.65	7280	1	7283	2939.98	3093.30	3067.65	3306.00	7.21
rc2_2_1/200	5	1439.01	1526.68	6896	1	7202	1396.95	1551.07	1526.68	1560.00	2.14
									Avg.	Gap (%)	3.62

Table 10: Root node statistics of F1C over a set of VRPSPD representative instances

Instance/ Customers	#v	Sep. Rounds	LP Time (s)	Sep. Time (s)	Root Time (s)	Gap (%)
SCA3-1/50	4	19	106.1	0.8	106.9	2.21
SCA8-1/50	9	17	72.8	1.8	74.6	4.92
CON3-1/50	4	26	71.5	1.5	73.1	1.94
CON8-1/50	9	37	73.9	6.5	80.4	2.95
CMT1X/50	3	38	59.0	3.7	62.7	1.46
CMT2X/75	6	65	1005.0	19.9	1025.0	4.58
CMT3X/100	5	38	1369.0	13.4	1382.3	3.70
CMT12X/100	5	47	3006.4	10.2	3016.5	5.07
CMT11X/120	4	87	7152.3	48.1	7200.4	7.09
CMT4X/150	7	21	7030.6	2.6	7033.2	4.23
CMT5X/200	10	7	7312.7	2.2	7314.9	7.78
r101/100	12	82	2946.5	137.5	3084.0	3.70
r201/100	3	50	551.7	10.5	562.2	0.20
r1_2_1/200	23	12	7008.9	6.5	7015.4	8.37
r2_2_1/200	5	4	6823.0	0.5	6823.5	3.46

verify that they are very similar, but F1C slightly outperformed the other formulations, with an average gap of 2.37% against 2.42% of F2C-U and 2.40% of F2C-D. This little difference in favor of F1C is mostly due to the significant better LBs found in all the three instances involving 200 customers.

Since the Gaps of the instances CMT12H and CMT12Q were relatively small for all formulations, we decided to verify if the UBs found by Gajpal and Abad [6] for these instances are indeed optimal solutions by running F1C with a time limit of 48 hours. The

Table 11: Root node statistics of F2C-U over a set of VRPSPD representative instances

Instance/ Customers	#v	Sep. Rounds	LP Time (s)	Sep. Time (s)	Root Time (s)	Gap (%)
SCA3-1/50	4	22	40.8	1.2	42.0	2.22
SCA8-1/50	9	29	50.9	5.1	56.0	5.01
CON3-1/50	4	33	52.4	1.6	54.1	1.61
CON8-1/50	9	33	62.3	4.4	66.7	2.94
CMT1X/50	3	45	96.4	5.2	101.5	1.46
CMT2X/75	6	44	330.8	15.0	345.8	4.58
CMT3X/100	5	39	493.0	10.6	503.6	2.80
CMT12X/100	5	41	805.5	7.1	812.6	5.08
CMT11X/120	4	88	4770.0	64.5	4834.5	7.00
CMT4X/150	7	55	7173.3	114.6	7287.9	4.15
CMT5X/200	10	8	6936.7	2.6	6939.3	7.23
r101/100	12	76	2802.9	106.8	2909.7	3.67
r201/100	3	28	288.7	3.6	292.4	0.21
r1_2_1/200	23	13	6966.9	3.6	6970.5	8.19
r2_2_1/200	5	8	7868.2	1.0	7869.2	2.81

Table 12: Root node statistics of F2C-D over a set of VRPSPD representative instances

Instance/ Customers	#v	Sep. Rounds	LP Time (s)	Sep. Time (s)	Root Time (s)	Gap (%)
SCA3-1/50	4	18	40.0	0.7	40.8	2.29
SCA8-1/50	9	26	78.2	2.8	81.0	5.35
CON3-1/50	4	25	39.8	1.4	41.2	1.98
CON8-1/50	9	55	126.0	6.1	132.1	3.04
CMT1X/50	3	33	53.1	2.7	55.8	1.50
CMT2X/75	6	49	778.3	9.9	788.2	4.77
CMT3X/100	5	45	586.9	22.7	609.6	2.78
CMT12X/100	5	44	1798.3	5.8	1804.1	5.35
CMT11X/120	4	80	7173.9	47.7	7221.6	7.69
CMT4X/150	7	29	7135.6	18.1	7153.8	4.35
CMT5X/200	10	6	7417.3	1.5	7418.7	8.07
r101/100	12	80	5776.4	90.8	5867.3	3.81
r201/100	3	56	472.4	17.2	489.5	0.17
r1_2_1/200	23	11	7084.8	2.9	7087.6	9.28
r2_2_1/200	5	8	7381.9	0.8	7382.7	3.12

Table 13: Summary of the results obtained by the three formulations

Formulation	Dethloff			Salhi and Nagy			Montané and Galvão		
	G1 (%)	G2 (%)	G3 (%)	G1 (%)	G2 (%)	G3 (%)	G1 (%)	G2 (%)	G3 (%)
F1C	9.74	2.96	1.34	9.21	4.85	4.57	8.75	3.66	3.57
F2C-U	9.85	2.92	0.94	9.30	4.62	4.27	8.82	3.04	2.94
F2C-D	9.85	2.94	1.26	9.30	4.66	4.31	8.82	3.57	3.62

BC algorithm was capable of proving the optimality of the instance CMT12H after 7.1 hours of execution. On the other hand, the optimal solution found by the BC algorithm (**729.25**) for the instance CMT12Q, after 22.7 hours of execution, was better than the UB found by Gajpal and Abad (729.46). Moreover, the number of vehicles associated with optimal solutions of the instances CMT12H and CMT12Q are, respectively, 5 and 7.

The statistics of the root node of each formulation over a set of representative instances is presented in Tables 17-19. The interpretation of the results contained in these tables are quite similar to those reported for the VRPSPD instances (see Tables 10-12). The amount of time spent by all formulations to solve the LPs considerably increases with the size of the instances. Since we are not aware of the estimated number of vehicles of most instances, a further analysis regarding their influence in the statistics of the root node

Table 14: Results obtained by F1C on Salhi and Nagy's instances (VRPMPD)

Instance/ Customers	#v	LP	Root LB	Root Time (s)	Tree size	Total Time (s)	New LB	F-LB	UB	Gap (%)
CMT1H/50	3	442.77	460.11	39	794	87	465.02	**465.02**	**465.02**	0.00
CMT1Q/50	4	468.98	488.10	36	15	40	489.74	**489.74**	**489.74**	0.00
CMT1T/50	5	488.08	513.07	47	908	193	520.06	**520.06**	**520.06**	0.00
CMT2H/75	-	622.00	643.52	292	6174	7200	647.84	647.84	662.63	2.23
CMT2Q/75	-	682.68	707.50	698	5827	7200	711.30	710.83	732.76	2.99
CMT2T/75	-	733.20	760.18	574	4996	7200	764.99	764.19	782.77	2.37
CMT3H/100	-	675.40	691.44	621	4735	7200	694.92	694.40	701.31	0.98
CMT3Q/100	6	719.22	744.42	901	309	1112	747.15	**747.15**	**747.15**	0.00
CMT3T/100	-	757.25	784.23	3602	1016	7200	787.12	786.14	798.07	1.50
CMT12H/100	-	542.24	623.03	526	20280	7200	627.32	627.32	629.37	0.33
CMT12Q/100	-	641.08	721.71	863	2880	7200	726.71	724.63	729.46	0.66
CMT12T/100	9	706.28	787.52	457	1	457	787.52	**787.52**	**787.52**	0.00
CMT11H/120	-	671.59	800.99	7200	1	7204	801.05	800.99	820.35	2.36
CMT11Q/120	-	816.16	926.87	7182	1	7200	928.74	926.87	939.36	1.33
CMT11T/120	-	904.02	984.35	7224	1	7228	985.03	984.35	998.80	1.45
CMT4H/150	-	778.93	798.15	7240	1	7242	798.38	798.15	831.39	4.00
CMT4Q/150	-	857.79	889.58	7133	1	7201	890.12	889.58	913.93	2.66
CMT4T/150	-	920.63	949.50	7101	1	7200	950.59	949.50	990.39	4.13
CMT5H/200	-	905.32	922.88	6913	1	7201	922.88	922.88	992.37	7.00
CMT5Q/200	-	1023.95	1040.25	6541	1	7201	1040.25	1040.25	1134.72	8.33
CMT5T/200	-	1118.60	1139.93	7448	1	7449	1139.93	1139.93	1232.08	7.48
								Avg. Gap (%)		2.37

Table 15: Results obtained by F2C-U on Salhi and Nagy's instances (VRPMPD)

Instance/ Customers	#v	LP	Root LB	Root Time (s)	Tree size	Total Time (s)	New LB	F-LB	UB	Gap (%)
CMT1H/50	3	442.09	460.12	52	2893	128	465.02	**465.02**	**465.02**	0.00
CMT1Q/50	4	468.58	488.13	49	15	54	489.74	**489.74**	**489.74**	0.00
CMT1T/50	5	488.08	512.83	71	1050	193	520.06	**520.06**	**520.06**	0.00
CMT2H/75	-	620.06	643.20	365	1642	7200	647.84	646.11	662.63	2.49
CMT2Q/75	-	681.52	707.91	765	1596	7200	711.30	709.96	732.76	3.11
CMT2T/75	-	733.01	760.81	511	2763	7200	764.99	763.47	782.77	2.47
CMT3H/100	-	674.46	691.53	1499	10735	7200	694.92	694.15	701.31	1.02
CMT3Q/100	6	718.88	744.50	1583	271	1788	747.15	**747.15**	**747.15**	0.00
CMT3T/100	-	757.23	784.43	4414	1267	7200	787.12	785.86	798.07	1.53
CMT12H/100	-	537.95	623.32	674	57192	7200	627.32	626.70	629.37	0.42
CMT12Q/100	-	639.80	721.95	1160	5675	7200	726.71	726.49	729.46	0.41
CMT12T/100	9	706.08	787.52	508	7	509	787.52	**787.52**	**787.52**	0.00
CMT11H/120	-	665.43	800.91	7057	4	7214	801.05	800.91	820.35	2.37
CMT11Q/120	-	811.84	928.74	7272	1	7275	928.74	928.74	939.36	1.13
CMT11T/120	-	903.15	984.67	5670	12	7201	985.03	985.03	998.80	1.38
CMT4H/150	-	778.19	798.38	7199	1	7202	798.38	798.38	831.39	3.97
CMT4Q/150	-	857.54	890.12	7280	1	7286	890.12	890.12	913.93	2.61
CMT4T/150	-	920.63	950.59	7255	1	7259	950.59	950.59	990.39	4.02
CMT5H/200	-	902.33	917.99	6756	1	7201	922.88	917.99	992.37	7.50
CMT5Q/200	-	1022.01	1039.20	7837	1	7840	1040.25	1039.20	1134.72	8.42
CMT5T/200	-	1118.44	1133.52	7538	1	7540	1139.93	1133.52	1232.08	8.00
								Avg. Gap (%)		2.42

could not be performed.

6 Concluding Remarks

This work dealt with Mixed Integer Programming formulations for the the Vehicle Routing Problem with Simultaneous Pickup and Delivery (VRPSPD). An undirected and a directed two-commodity flow formulations were proposed. They were tested within a branch-and-cut scheme and their results were compared with the one-commodity flow formulation of Dell'Amico et al. [3]. The optimal solutions of 30 VRPSPD open prob-

Table 16: Results obtained by F2C-D on Salhi and Nagy's instances (VRPMPD)

Instance/ Customers	#v	LP	Root LB	Root Time (s)	Tree size	Total Time (s)	New LB	F-LB	UB	Gap (%)
CMT1H/50	3	442.09	460.07	71	1748	148	465.02	**465.02**	**465.02**	0.00
CMT1Q/50	4	468.58	488.21	61	11	64	489.74	**489.74**	**489.74**	0.00
CMT1T/50	5	488.08	512.92	85	813	208	520.06	**520.06**	**520.06**	0.00
CMT2H/75	-	620.06	643.42	730	5735	7200	647.84	647.45	662.63	2.29
CMT2Q/75	-	681.52	707.13	953	6287	7200	711.30	711.30	732.76	2.93
CMT2T/75	-	733.01	760.91	918	6183	7200	764.99	764.99	782.77	2.27
CMT3H/100	-	674.46	691.53	1158	19031	7200	694.92	694.92	701.31	0.91
CMT3Q/100	6	718.88	744.47	1366	411	1608	747.15	**747.15**	**747.15**	0.00
CMT3T/100	-	757.23	784.13	3957	1904	7200	787.12	787.12	798.07	1.37
CMT12H/100	-	538.07	623.32	773	42656	7200	627.32	626.45	629.37	0.46
CMT12Q/100	-	639.80	721.90	1309	3676	7200	726.71	726.71	729.46	0.38
CMT12T/100	9	706.08	787.25	750	6	752	787.52	**787.52**	**787.52**	0.00
CMT11H/120	-	665.46	801.05	7273	1	7277	801.05	801.05	820.35	2.35
CMT11Q/120	-	811.92	926.91	7197	1	7200	928.74	926.91	939.36	1.33
CMT11T/120	-	903.19	984.48	7196	1	7200	985.03	984.48	998.80	1.43
CMT4H/150	-	778.19	798.15	7295	1	7298	798.38	798.15	831.39	4.00
CMT4Q/150	-	857.54	890.11	7168	1	7200	890.12	890.11	913.93	2.61
CMT4T/150	-	920.63	950.12	7152	1	7200	950.59	950.12	990.39	4.07
CMT5H/200	-	902.34	922.85	7262	1	7265	922.88	922.85	992.37	7.01
CMT5Q/200	-	1022.08	1037.29	7249	1	7251	1040.25	1037.29	1134.72	8.59
CMT5T/200	-	1118.44	1129.15	6383	1	7201	1139.93	1129.15	1232.08	8.35
								Avg.	Gap (%)	2.40

Table 17: Root node statistics of the F1C over a set of VRPMPD representative instances

Instance/ Customers	#v	Sep. Rounds	LP Time (s)	Sep. Time (s)	Root Time (s)	Gap (%)
CMT1H/50	3	27	37.0	1.8	38.8	1.06
CMT1T/50	5	34	44.5	2.1	46.6	1.34
CMT2H/75	-	45	280.8	11.1	291.9	2.88
CMT2T/75	-	62	553.4	20.5	573.8	2.89
CMT3H/100	-	33	604.0	16.8	620.8	1.41
CMT3T/100	-	91	3520.0	81.9	3601.9	1.73
CMT12H/100	-	67	7168.0	32.2	7200.2	1.01
CMT12T/100	9	72	7206.0	17.5	7223.5	0.00
CMT11H/120	-	32	522.4	4.0	526.4	2.36
CMT11T/120	-	39	455.0	1.7	456.7	1.45
CMT4H/150	-	22	7236.6	3.4	7240.1	4.00
CMT4T/150	-	30	7096.0	5.3	7101.3	4.13
CMT5H/200	-	13	6908.8	4.1	6912.9	7.00
CMT5T/200	-	8	7445.6	2.1	7447.8	7.48

Table 18: Root node statistics of F2C-U over a set of VRPMPD representative instances

Instance/ Customers	#v	Sep. Rounds	LP Time (s)	Sep. Time (s)	Root Time (s)	Gap (%)
CMT1H/50	3	30	49.6	2.0	51.7	1.05
CMT1T/50	5	41	68.3	3.2	71.4	1.39
CMT2H/75	-	53	355.2	9.4	364.7	2.93
CMT2T/75	-	58	492.1	19.2	511.3	2.81
CMT3H/100	-	52	1477.7	21.5	1499.2	1.39
CMT3T/100	-	110	4340.3	73.6	4413.9	1.71
CMT12H/100	-	45	671.4	2.9	674.3	0.96
CMT12T/100	9	39	505.1	2.9	508.0	0.00
CMT11H/120	-	83	6994.5	62.1	7056.6	2.37
CMT11T/120	-	118	5589.4	80.9	5670.3	1.41
CMT4H/150	-	31	7194.2	4.3	7198.5	3.97
CMT4T/150	-	30	7249.4	5.9	7255.2	4.02
CMT5H/200	-	4	6755.2	1.1	6756.4	7.50
CMT5T/200	-	4	7537.2	0.8	7538.0	8.00

lems were proved, as can be seen in Table 20. The three formulations were also tested in benchmark instances of the Vehicle Routing Problem with Mixed Pickup and Deliv-

Table 19: Root node statistics of F2C-D over a set of VRPMPD representative instances

Instance/ Customers	#v	Sep. Rounds	LP Time (s)	Sep. Time (s)	Root Time (s)	Gap (%)
CMT1H/50	3	32	68.2	2.9	71.1	1.06
CMT1T/50	5	39	82.8	2.4	85.3	1.37
CMT2H/75	-	75	704.4	25.3	729.7	2.90
CMT2T/75	-	82	883.9	33.9	917.8	2.79
CMT3H/100	-	44	1142.7	15.5	1158.2	1.39
CMT3T/100	-	90	3882.4	74.2	3956.7	1.75
CMT12H/100	-	49	769.0	3.8	772.8	0.96
CMT12T/100	9	37	748.0	2.2	750.2	0.03
CMT11H/120	-	65	7246.7	26.2	7272.9	2.35
CMT11T/120	-	74	7178.3	17.6	7195.9	1.43
CMT4H/150	-	30	7274.2	21.1	7295.2	4.00
CMT4T/150	-	21	7148.7	3.0	7151.7	4.07
CMT5H/200	-	8	7259.8	2.2	7261.9	7.01
CMT5T/200	-	3	6382.6	0.6	6383.2	8.35

ery (VRPMPD), which is a particular case of the VRPSPD, and were able to prove the optimality of 7 open problems (see Table 21). Furthermore, new lower bounds were produced for both VRPSPD and VRPMPD instances with up to 200 customers. In addition, although we have shown that the one-commodity flow formulation produces a stronger linear relaxation, the two-commodity flow formulations have found, on average, better lower bounds in the VRPSPD instances. As for the VRPMPD, the lower bounds were, on average, quite similar, but with a slight superiority of F1C.

Table 20: VRPSPD Optimal Solutions

Instance/ Customers	#v	OPT
SCA3-0/50	4	**635.62**
SCA3-1/50	4	**697.84**
SCA3-2/50	4	**659.34**
SCA3-3/50	4	**680.04**
SCA3-4/50	4	**690.50**
SCA3-5/50	4	**659.90**
SCA3-6/50	4	**651.09**
SCA3-7/50	4	**659.17**
SCA3-8/50	4	**719.48**
SCA3-9/50	4	**681.00**
SCA8-3/50	9	**983.34**
SCA8-6/50	9	**971.82**
CON3-0/50	4	**616.52**
CON3-1/50	4	**554.47**
CON3-2/50	4	**518.00**
CON3-3/50	4	**591.19**
CON3-4/50	4	**588.79**
CON3-5/50	4	**563.70**
CON3-6/50	4	**499.05**
CON3-7/50	4	**576.48**
CON3-8/50	4	**523.05**
CON3-9/50	4	**578.25**
CON8-1/50	9	**740.85**
CON8-4/50	9	**772.25**
CON8-7/50	9	**811.96**
CMT1X/50	3	**466.77**
CMT1Y/50	3	**466.77**
r201/100	3	**666.20**
c201/100	5	**662.07**
rc201/100	3	**672.92**

In order to improve the results of the instances that have a relatively large number of vehicles, one can combine the well-known CVRP cuts used in our branch-and-cut approach (rounded capacity, multistar and comb) with column generation towards a branch-and-cut-and-price (BCP) algorithm. This will probably lead to considerably better results. Recent works have shown that the BCP had turned out to be one of the best effective methods for solving VRPs ([5], [14]. [15]).

Table 21: VRPMPD Optimal Solutions

Instance/ Customers	#v	OPT
CMT1H/50	3	**465.02**
CMT1Q/50	4	**489.74**
CMT1T/50	5	**520.06**
CMT3Q/100	6	**747.15**
CMT12H/100	5	**629.37**
CMT12Q/100	7	**729.25**
CMT12T/100	9	**787.52**

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