The Globally Uniquely Solvable Property of Second-Order Cone Linear Complementarity Problems

Wei Hong Yang* and Xiaoming Yuan†

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Abstract. The globally uniquely solvable (GUS) property of the linear transformation of the linear complementarity problems over symmetric cones has been studied recently by Gowda *et al.* via the approach of Euclidean Jordan algebra. In this paper, we contribute a new approach to characterizing the GUS property of the linear transformation of the second-order cone linear complementarity problems (SOCLCP) via some basic linear algebra properties of the involved matrix of SOCLCP. Some more concrete and checkable sufficient and necessary conditions for the GUS property are thus derived.

Key words. Second-order cone; Linear complementarity problem; Globally uniquely solvable property

AMS subject classifications. 90C33, 65K05

1 Introduction

The linear complementarity problem (LCP) is to find a vector $x \in \mathbb{R}^n$ such that

$$LCP(M, q): \quad x \geq 0, \quad q + Mx \geq 0, \quad x^T(q + Mx) = 0, \tag{1.1}$$

where the vector $q \in \mathbb{R}^n$ and the matrix $M \in \mathbb{R}^{n \times n}$ are given. We refer to [6] for the comprehensive study of LCP. Theoretically, it is of obvious interest to characterize the uniqueness of the solution of $LCP(M, q)$, and a popular approach toward this objective is to investigate the linear algebra properties of the involved matrix M such that the solution of $LCP(M, q)$ is unique for all vectors $q \in \mathbb{R}^n$. In the literature, it is well known that the following statements are equivalent (see [6, Thm.3.3.4]):

*School of Mathematical Sciences, Fudan University, Shanghai 200433, China (whyang@fudan.edu.cn).
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†Department of Mathematics, Hong Kong Baptist University, Hong Kong, China (xmyuan@hkbu.edu.hk).
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(1) M is a P-matrix, i.e., all principal minors of M are positive.

(2) $z_i(Mz)_i \le 0$ for all $i \Rightarrow z = 0$.

(3) The $LCP(M, q)$ has a unique solution for all $q \in \mathbb{R}^n$.

(4) All real eigenvalues of M and its principal submatrices are positive.

Recent years have witnessed impressive developments on extending $LCP(M, q)$, which is essentially the LCP over the cone $\mathbb{R}^n_+$, to symmetric cones, especially to the positive semidefinite cone and the second-order cone. To mention a few, see e.g. [1, 3, 4, 7, 8, 9, 10, 11, 12, 14, 17]. However, the favorable characterization stated in (1)-(4) above is not true any more for the LCPs over generic symmetric cones, and this inspired the interesting work in [8, 9, 10] where the P-property, the cross commutative property and the globally uniquely solvable (GUS) property were intensively studied for the linear transformations of LCPs over symmetric cones. More specifically, via the approach of Euclidean Jordan algebra, the authors discovered that the linear transformation of the LCP over a symmetric cone has the GUS property if and only if it has both the P-property and the cross commutative property (see [10, Thm. 14]). Thus, this work contributes the novel approach to characterizing the GUS property of the involved linear transformations of LCPs over symmetric cones via the Euclidean-Jordan-Algebra-based approach. Note that there are difficulties to verify the P-property (which requires verifying the operator commutability of the corresponding Lyapunov transformations of x and Mx for an arbitrary x) and the cross commutative property (which requires knowing all solutions) of the LCPs over symmetric cones. On the other hand, for the linear transformation of a LCP over a symmetric cone, the Euclidean-Jordan-Algebra-based approach fails to reveal the relationship between the GUS property and the basic linear-algebra-related properties of its involved matrix explicitly.

In this paper, we focus on the LCP over the second-order cone (SOCLCP for short), which captures various applications in many fields, see e.g. [1, 3, 4, 7, 12] and references therein. More specifically, the second-order cone (SOC) in $\mathbb{R}^n$, which is also well known as the Lorentz cone, is defined by

$$\mathcal{K}^n = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} : \|x_2\| \le x_1\},$$

where $\|\cdot\|$ is the Euclidean norm. The SOCLCP is to find vectors $x, y \in \mathbb{R}^n$ satisfying

$$x^T y = 0, \quad x \in \mathcal{K}^n, \quad y \in \mathcal{K}^n, \quad y = Mx + q, \tag{1.2}$$

where $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. We denote it by $LCP(M, \mathcal{K}^n, q)$. Throughout, the set of all solutions of $LCP(M, \mathcal{K}^n, q)$ is denoted by $SOL(M, \mathcal{K}^n, q)$. Rather than the dominant Euclidean-Jordan-Algebra-based approach in the literature of LCPs over symmetric cones, in this paper, we contribute a completely new approach to characterizing the GUS property of M for $LCP(M, \mathcal{K}^n, q)$ directly via some basic linear-algebra-related properties of M. Some sufficient and necessary conditions for this characterization are thus derived. The impetus behind this new approach is that we find out the fact that when $q \notin -M\mathcal{K}^n \cup \mathcal{K}^n$, finding $x \in SOL(M, \mathcal{K}^n, q)$ is equivalent to solving the root $(s, \xi) \in \mathbb{R}_{++} \times bd(\mathcal{K}^n)$ of the equation:

$$(M - sJ_n)\xi = -q,$$

where $J_n = diag(1, -1, \cdots, -1)$. Therefore, it is possible to characterize the GUS property of M for $LCP(M, \mathcal{K}^n, q)$ via the properties of the derived matrix $M - sJ_n$.

The organization of the paper is as follows. In Section 2, we introduce some notations and definitions to be used; and delineate some preliminary results on the proper cones and second-order cone. In Section 3, for $LCP(M, \mathcal{K}^n, q)$, we derive some necessary conditions when M has the GUS property. In this section, we shall prove that all nonnegative eigenvalues of MJ_n equal to the same value $\tau > 0$. Moreover, we shall show that $(M - \tau J_n)\mathcal{K}^n$ is a hyperplane, and that the cone $(M - sJ_n)\mathcal{K}^n$ lies in one side of this hyperplane when $0 \leq s < \tau$ while it lies in the other side of the hyperplane when $s > \tau$. In Section 4, we present some sufficient and necessary conditions to characterize the GUS property of M for $LCP(M, \mathcal{K}^n, q)$. Finally, some conclusions are drawn in Section 5.

2 Notations and Preliminaries

In this section, we introduce the notations, definitions, and preliminary results which will be used throughout the paper. We use $\mathbb{R}^n_+$ ($\mathbb{R}^n_{++}$) to denote the set of vectors which have nonnegative (positive) components. For $a, b \in \mathbb{R}^n$, (a, b) denotes the set $\{c \in \mathbb{R}^n : c = \lambda a + (1 - \lambda)b, 0 < \lambda < 1\}$. For $x \in \mathbb{R}^n$, define $x^\perp := \{y \in \mathbb{R}^n : x^T y = 0\}$. We use the notation $B(x; r)$ to denote the closed ball with center x and radius r. The symbol $\mathbb{S}^n$ refers to the unit sphere in $\mathbb{R}^n$. For a set $C \subset \mathbb{R}^n$, the boundary (resp. interior, closure, cone hull) of C is denoted by $bd(C)$ (resp. $int(C)$, $\overline{C}$, $cone(C)$). The distance from $u \in \mathbb{R}^n$ to C is denoted by $d(u, C) := \min_{x \in C} \|u - x\|$. For $x \in C$, the normal cone of C at x is defined by

$$N_C(x) := \{z \in \mathbb{R}^n : z^T(y - x) \leq 0, \quad \forall y \in C\}.$$

By the definition above, it is easy to prove that if $y \in int(C)$, then

$$z^T(y - x) < 0 \;\text{ for all nonzero vector }\; z \in N_C(x). \tag{2.1}$$

For a matrix $M \in \mathbb{R}^{n \times n}$, we use $Ker(M) := \{x \in \mathbb{R}^n : Mx = 0\}$ to denote the kernel of M. We use $\mathcal{R}(M) := \{y \in \mathbb{R}^n : y = Mx \text{ for some } x \in \mathbb{R}^n\}$ to denote the range of M. The norm of M is defined by $\|M\| := \max_{\|x\|=1} \|Mx\|$. As usual, I_n denotes the identity matrix in $\mathbb{R}^{n \times n}$.

Let K be a cone. If $int(K) \neq \emptyset$, we say that K is a solid cone and if $K \cap -K = \{0\}$, we say that K is a pointed cone. By the definition, if K is pointed, then K contains no line. A closed solid pointed convex cone is called a proper cone [2, p. 43]. The dual cone of K is defined by

$$K^* = \{y \in \mathbb{R}^n : x^T y \geq 0 \;\text{ for all }\; x \in K\}.$$

We shall need the following elementary results on a proper cone.

Lemma 2.1. *Let A be a matrix and K a proper cone. The following statements hold:*

(1) *$int(AK) \neq \emptyset$ if and only if A is nonsingular.*

(2) *If A is nonsingular, then $bd(AK) = Abd(K)$, $int(AK) = Aint(K)$ and AK is a proper cone.*

(3) *For all $a \in int(K)$, $\mathbb{R}^n = \{K - ta : t \geq 0\}$.*

(4) *For all $q \notin int(K) \cup -int(K^*)$, there exist $x \in bd(K)$ and $y \in bd(K^*)$ such that $q = x - y$.*

(5) *For any $a, b \in int(K)$ such that $a \neq tb$ for any $t > 0$, there exist nonzero real numbers α and β such that $\alpha a + \beta b \in bd(K)$.*

Let K_1, K_2 be two convex cones. The Pompeiu-Hausdorff distance between K_1 and K_2 is defined by [16]

$$d(K_1, K_2) = \max\{\max_{u \in K_1 \cap \mathbb{S}^n} d(u, K_2), \max_{v \in K_2 \cap \mathbb{S}^n} d(v, K_1)\}. \tag{2.2}$$

In [13], Iusem and Seeger use the following metric between K_1 and K_2

$$\delta(K_1, K_2) = \sup_{\|x\| \leq 1} \left| d(x, K_1) - d(x, K_2) \right|, \tag{2.3}$$

which coincides with (2.2). It is easy to prove the following result and we omit the proof.

Lemma 2.2. *Let K_1 and K_2 be two closed convex cones such that $K_1 \cap K_2 = \{0\}$. Let K be a convex cone. There exists $\epsilon > 0$ such that if $d(K, K_1) < \epsilon$, then $K \cap K_2 = \{0\}$.*

The following three lemmas will be useful in the next sections of this paper.

Lemma 2.3. *Let K_1 be a convex cone such that $int(K_1) \neq \emptyset$ and K_2 a convex cone. For each $x \in int(K_1)$, there exists $\epsilon > 0$ such that if $d(K_1, K_2) < \epsilon$ then $x \in int(K_2)$.*

Proof. If the assertion is not true, then there exists a sequence of convex cones $\{C_i\}_{i=1}^{\infty}$ such that $d(K_1, C_i) < 1/i$ and $x \notin int(C_i)$. Without loss of generality, assume that $\|x\| < 1$. Then there exists $\sigma > 0$ such that $y \in int(K_1)$ and $\|y\| < 1$ for all $y \in B(x; \sigma)$. If $int(C_i) \neq \emptyset$, by the Convex Separation Theorem, there exist $0 \neq a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ such that

$$a_i^T x + b_i \geq 0 \text{ and } a_i^T z + b_i \leq 0 \;\; \forall z \in C_i.$$

Let $x_i = x + \sigma a_i / \|a_i\|$. Then $x_i \in B(x; \sigma) \subset K_1$, and so $\|x_i\| < 1$. It is obvious that $d(x_i, C_i) \geq \sigma$. From (2.3), it follows that $d(K_1, C_i) \geq \sigma$, a contradiction. If $int(C_i) = \emptyset$, there exists $0 \neq c_i \in \mathbb{R}^n$ such that $c_i^T x \geq 0 \geq c_i^T z$ for all $z \in C_i$. Similar to the proof above, we can derive a contradiction. □

Lemma 2.4. *Let A be a matrix and C a proper cone. If $Ker(A) \cap C = \{0\}$, then $bd(AC) \subset Abd(C)$.*

Proof. If A is nonsingular, by Lemma 2.1 (2), $bd(AC) = Abd(C)$. Assume that A is singular. Since C is proper and $Ker(A) \cap C = \{0\}$, by [15, Thm. 9.1], AC is a closed cone. Pick a nonzero vector $z \in Ker(A)$. Assume that $bd(AC) \neq \emptyset$. For any $y \in bd(AC)$, since AC is closed, there exists $x \in C$ such that $y = Ax$. If $x \in bd(C)$, then we are done. Assume that $x \in int(C)$. Since C contains no line, there exists $t > 0$ such that $x + tz \in bd(C)$ or $x - tz \in bd(C)$. Without loss of generality, assume that $x + tz \in bd(C)$. Then, $y = A(x + tz) \in Abd(C)$. Since y is arbitrary, we have $bd(AC) \subset Abd(C)$. □

Lemma 2.5. *Let $C_1, C_2 \subset \mathbb{R}^n$ be two proper cones. If $int(C_1) \cap int(C_2) \neq \emptyset$ and $bd(C_1) \cap C_2 = \{0\}$, then we have $C_2 \backslash \{0\} \subset int(C_1)$.*

Proof. Pick $x \in int(C_1) \cap int(C_2)$. Now we prove $C_2 \subset C_1$. If there exists $y \in C_2$ but $y \notin C_1$, there exists $z \in (x, y)$ such that $z \in bd(C_1)$. From $x, y \in C_2$, it follows that $z \in C_2$. Since C_2 is pointed, we have $z \neq 0$, which contradicts $bd(C_1) \cap C_2 = \{0\}$. Thus, we must have $C_2 \subset C_1$, which together with $bd(C_1) \cap C_2 = \{0\}$ implies that $C_2 \backslash \{0\} \subset int(C_1)$. □

In the following, we summarize some properties of $\mathcal{K}^n$ which will be used later on. The proof is straightforward based on the definitions.

Lemma 2.6. *The following statements hold:*

(1) *$\mathcal{K}^n$ is a self-dual proper cone, that is, $(\mathcal{K}^n)^* = \mathcal{K}^n$.*

(2) *For nonzero vectors $x, y \in \mathcal{K}^n$, $x^T y = 0$ if and only if $x \in bd(\mathcal{K}^n)$, $y \in bd(\mathcal{K}^n)$ and $y = \mu J_n x$ for some $\mu > 0$.*

(3) *Let $x \in \mathcal{K}^n$. Then $x^T y > 0$ for all nonzero vector $y \in \mathcal{K}^n$ if and only if $x \in int(\mathcal{K}^n)$.*

(4) *If $x \in bd(\mathcal{K}^n)$ and $x \neq 0$, then*

$$N_{\mathcal{K}^n}(x) = \{-tJ_n x : t \geq 0\}. \tag{2.4}$$

3 Necessary Conditions

Recall that $LCP(M, \mathcal{K}^n, q)$ is defined by (1.2). We say that M has the Globally Uniquely Solvable (GUS) property [10] if for all $q \in \mathbb{R}^n$, $LCP(M, \mathcal{K}^n, q)$ has a unique solution. In this section, we present some necessary conditions for M having the GUS property. For a matrix $M \in \mathbb{R}^{n \times n}$ and $s \geq 0$, we define $M_s = M - sJ_n$, where $J_n := diag(1, -1, -1, \cdots, -1)$. Let

$$K_s := M_s \mathcal{K}^n = \{M_s x : x \in \mathcal{K}^n\}.$$

With this notation, we have $K_0 = M\mathcal{K}^n$.

Lemma 3.1. *Assume that $x \in SOL(M, \mathcal{K}^n, q)$. Then the following statements hold:*

(1) *$x \in int(\mathcal{K}^n) \Longrightarrow q \in -M\big(int(\mathcal{K}^n)\big)$.*

(2) $x = 0 \Longrightarrow q \in \mathcal{K}^n$.

(3) *If $x \neq 0$ and $x \in bd(\mathcal{K}^n)$, then there exists $s \geq 0$ such that $q = -M_s x$.*

Proof. (1). Since $x \in int(\mathcal{K}^n)$, by (1.2) and Lemma 2.6 (3), we have $Mx + q = 0$, which implies that $q = -Mx \in -M(int(\mathcal{K}^n))$. The proof of (2) is obvious. (3). If $x \neq 0$ and $x \in bd(\mathcal{K}^n)$, by Lemma 2.6 (2), there exists $s \geq 0$ such that $Mx + q = sJ_n x$. Then

$$q = -Mx + sJ_n x = -M_s x, \tag{3.1}$$

which proves (3). □

The proof of the following lemma is straightforward and thus omitted, while it provides the equivalence of $LCP(M, \mathcal{K}^n, q)$ to an matrix equation and thus inspires the proposing new approach to characterizing the GUS-property of M for $LCP(M, \mathcal{K}^n, q)$.

Lemma 3.2. *If $q = -M_s(x)$ for some $s \geq 0$ and $0 \neq x \in bd(\mathcal{K}^n)$, then $x \in SOL(M, \mathcal{K}^n, q)$.*

For LCP over $\mathbb{R}^n_+$, the complementary cone [6, p. 17] plays an important role. Now, we extend this definition to $LCP(M, \mathcal{K}^n, q)$. For this purpose, we define the complementary cone for SOCLCP with parameters $\varrho \in bd(\mathcal{K}^n) \cap \mathbb{S}^n$ as:

$$C(\varrho) = -\overline{cone(\cup_{s \geq 0} M_s(\varrho))}.$$

Then, by Lemma 3.1 (3) and Lemma 3.2, it is easy to prove the following lemma.

Lemma 3.3. *Fix $\varrho \in bd(\mathcal{K}^n) \cap \mathbb{S}^n$. Let $D = \{-M\varrho, J_n\varrho\}$. We have $C(\varrho) = cone(D)$. There exist $\lambda, \mu \geq 0$ such that $q = -\lambda M\varrho + \mu J_n \varrho$, that is $q \in C(\varrho)$, if and only if $\lambda\varrho \in SOL(M, \mathcal{K}^n, q)$.*

The following lemma reveals the relationship between K_s and $-\mathcal{K}^n$.

Lemma 3.4. *The following statements hold:*

(1) $\lim_{s \to 0} d(K_s, M\mathcal{K}^n) = 0$.

(2) $\lim_{s \to \infty} d(K_s, -\mathcal{K}^n) = 0$.

(3) *For each $\eta \in -int(\mathcal{K}^n)$, there exists $r > 0$ such that if $s > r$ then $\eta \in int(K_s)$.*

Proof. The proof of (1) is elementary and we omit the proof. For $y \in -\mathcal{K}^n \cap \mathbb{S}^n$, let $x = -J_n y$. For $s > 0$, let $y_s = M_s x/s = (M - sJ_n)x/s$. Fix $\epsilon > 0$. If $s > 2\|M\|/\epsilon$, then $\|y_s - y\| = \|Mx\|/s < \epsilon/2$. Then we have

$$\begin{aligned} & \|\frac{M_s x}{\|M_s x\|} - y\| \leq \|\frac{y_s}{\|y_s\|} - y_s\| + \|y_s - y\| \\ = \; & |\|y_s\| - 1| + \|y_s - y\| \leq 2\|y_s - y\| < \epsilon, \end{aligned} \tag{3.2}$$

which implies $\max_{y \in (-\mathcal{K}^n) \cap \mathbb{S}^n} d(y, K_s) < \epsilon$. By the definition of M_s, there exists $r > 0$ such that M_s is nonsingular when $s > r$. Thus, if $s > r$, every $u \in K_s \cap \mathbb{S}^n$ can be represented as $u = M_s x/\|M_s x\|$ for some $x \in \mathcal{K}^n \cap \mathbb{S}^n$. By (3.2) again, if $s > \max\{2\|M\|/\epsilon, r\}$ then $\max_{u \in K_s \cap \mathbb{S}^n} d(u, -\mathcal{K}^n) < \epsilon$, and so $d(K_s, -\mathcal{K}^n) < \epsilon$. Then the conclusion of (2) holds. The assertion of (3) follows from (2) and Lemma 2.3. □

In the remaining of this section, we always assume that M **has the GUS property**, that is, for all $q \in \mathbb{R}^n$, $SOL(M, \mathcal{K}^n, q)$ is nonempty and has a unique element. By [10, Thm. 11 and 14], M must be nonsingular. In the following lemma, we present some properties on the relationship between $M\mathcal{K}^n$ and $\mathcal{K}^n$.

Lemma 3.5. *$M\mathcal{K}^n - \mathcal{K}^n = \mathbb{R}^n$, $M\mathcal{K}^n \cap -\mathcal{K}^n = \{0\}$ and $int(\mathcal{K}^n) \cap int(M\mathcal{K}^n) \neq \emptyset$.*

Proof. Fix any $q \notin -M\mathcal{K}^n \cup \mathcal{K}^n$. Assume that $SOL(M, \mathcal{K}^n, q) = \{x\}$. From Lemma 3.1, it follows that $x \neq 0$ and $x \in bd(\mathcal{K}^n)$. By (3.1), $q = -Mx + sJ_n x \in \mathcal{K}^n - M\mathcal{K}^n$ for some $s > 0$. Then we have $\mathcal{K}^n - M\mathcal{K}^n = \mathbb{R}^n$, and so $M\mathcal{K}^n - \mathcal{K}^n = \mathbb{R}^n$.

For the second assertion, suppose on the contrary, there exists $0 \neq x \in \mathcal{K}^n$ such that $Mx \in -\mathcal{K}^n$. Let $q = -Mx$. Then $q \in \mathcal{K}^n$, which implies that $SOL(M, \mathcal{K}^n, q)$ contains two elements x and 0. This contradicts the GUS property of M.

Since M is nonsingular, $int(M\mathcal{K}^n) \neq \emptyset$. If $int(\mathcal{K}^n) \cap int(M\mathcal{K}^n) = \emptyset$, by the Convex Separation Theorem, there exists $0 \neq a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$a^T x \geq b \quad \text{and} \quad a^T y \leq b \quad \forall x \in \mathcal{K}^n,\ \forall y \in M\mathcal{K}^n,$$

which implies that $a^T z \leq 0$ for all $z \in M\mathcal{K}^n - \mathcal{K}^n$. This contradicts $M\mathcal{K}^n - \mathcal{K}^n = \mathbb{R}^n$. □

Lemma 3.6.

$$\cup_{s>0} M_s\big(bd(\mathcal{K}^n)\backslash\{0\}\big) = \mathbb{R}^n \backslash (-\mathcal{K}^n \cup M\mathcal{K}^n).$$

Proof. Pick any $p \in \mathbb{R}^n \backslash (-\mathcal{K}^n \cup M\mathcal{K}^n)$. Let $q = -p$ and assume that $SOL(M, \mathcal{K}^n, q) = \{x\}$. Since $q \notin \mathcal{K}^n \cup -M\mathcal{K}^n$, by Lemma 3.1 (1) and (2), we have $x \neq 0$ and $x \in bd(\mathcal{K}^n)$. Then $q = -M_s x$ for some $s \geq 0$ by Lemma 3.1 (3). From (3.1) and $q \notin -M\mathcal{K}^n$, it follows that $s > 0$, and so $\cup_{s>0} M_s\big(bd(\mathcal{K}^n)\backslash\{0\}\big) \supseteq \mathbb{R}^n \backslash (-\mathcal{K}^n \cup M\mathcal{K}^n)$. Now we prove $\cup_{s>0} M_s\big(bd(\mathcal{K}^n)\backslash\{0\}\big) \subseteq \mathbb{R}^n \backslash (-\mathcal{K}^n \cup M\mathcal{K}^n)$. If it is not true, then there exist $0 \neq x \in bd(\mathcal{K}^n)$ and $s > 0$ such that $M_s x \in -\mathcal{K}^n$ or $M_s x \in M\mathcal{K}^n$. Let $z = -M_s x$. From Lemma 3.2, it follows that $x \in SOL(M, \mathcal{K}^n, z)$. If $M_s x \in -\mathcal{K}^n$, then $SOL(M, \mathcal{K}^n, z)$ contains at least two elements 0 and x, a contradiction; if $M_s x \in M\mathcal{K}^n$, then there exists $y \in \mathcal{K}^n$ such that $M_s x = My$. From $s > 0$ and $x \neq 0$, it follows that $x \neq y$. Thus $SOL(M, \mathcal{K}^n, z)$ contains at least two elements x and y, a contradiction. □

For a matrix $M \in \mathbb{R}^{n\times n}$, we define the following map associated with M by

$$\mathcal{M}(s, x) = M_s(x) \quad \forall s \geq 0, \forall x \in \mathbb{R}^n. \tag{3.3}$$

Lemma 3.7. *Let M be a nonsingular matrix such that $\mathcal{K}^n \cap -M\mathcal{K}^n = \{0\}$. Then M has the GUS property if and only if $\mathcal{M}$ is a one-to-one map from $\mathbb{R}_{++} \times \big(bd(\mathcal{K}^n)\backslash\{0\}\big)$ onto $\mathbb{R}^n \backslash (-\mathcal{K}^n \cup M\mathcal{K}^n)$.*

Proof. Necessity. If M has the GUS property, by Lemma 3.6, we only need to prove $\mathcal{M}$ is injective. Let $(s, x), (t, y) \in \mathbb{R}_{++} \times \big(bd(\mathcal{K}^n)\backslash\{0\}\big)$ be such that $M_s(x) = M_t(y)$. Let $q = -M_s x$. From Lemma 3.2, it follows that $x \in SOL(M, \mathcal{K}^n, q)$. By $q = -M_t y$, we also have $y \in SOL(M, \mathcal{K}^n, q)$. Since M has the GUS property, we must have $x = y$, which together with $M_s(x) = M_t(y)$ implies that $s = t$.

Sufficiency. Assume that $\mathcal{M}$ is a one-to-one map from $\mathbb{R}_{++} \times \big(bd(\mathcal{K}^n)\backslash\{0\}\big)$ onto $\mathbb{R}^n\backslash(-\mathcal{K}^n \cup M\mathcal{K}^n)$. Now we prove that M has the GUS property. Pick any $q \in \mathbb{R}^n\backslash(\mathcal{K}^n \cup -M\mathcal{K}^n)$. Then there exists a unique $(s, x) \in \mathbb{R}_{++} \times \big(bd(\mathcal{K}^n)\backslash\{0\}\big)$ such that $q = -\mathcal{M}(s, x) = -M_s x$. From Lemma 3.2, it follows that $x \in SOL(M, \mathcal{K}^n, q)$. Pick any $y \in SOL(M, \mathcal{K}^n, q)$. Since $q \notin \mathcal{K}^n \cup -M\mathcal{K}^n$, by Lemma 3.1, $q = -M_\mu(y)$ for some $\mu > 0$, and so $\mathcal{M}(\mu, y) = -q$. Since $\mathcal{M}$ is a one-to-one map, we have $y = x$ and $s = \mu$. Thus $SOL(M, \mathcal{K}^n, q)$ has a unique element for all $q \in \mathbb{R}^n\backslash(\mathcal{K}^n \cup -M\mathcal{K}^n)$.

If $q \in \mathcal{K}^n$, then $0 \in SOL(M, \mathcal{K}^n, q)$. Assume there exists $0 \neq y \in SOL(M, \mathcal{K}^n, q)$. We must have $My + q \neq 0$ (If $My + q = 0$, then $q \in -M\mathcal{K}^n \cap \mathcal{K}^n$. From Lemma 3.5, it follows that $q = 0$. Since M is nonsingular, we have $y = 0$, a contradiction). By Lemma 2.6 (2), we have $y \in bd(\mathcal{K}^n)$ and $My + q = \mu J_n y$ for some $\mu > 0$. That is, $q = -M_\mu(y)$ and so $\mathcal{M}(\mu, y) = -q \in -\mathcal{K}^n$, a contradiction.

If $q \in -M\mathcal{K}^n$, there exists $x \in \mathcal{K}^n$ such that $Mx + q = 0$, and so $x \in SOL(M, \mathcal{K}^n, q)$. Assume that there exists $y \neq x$ such that $y \in SOL(M, \mathcal{K}^n, q)$. We must have $y \neq 0$. (If $y = 0$, from Lemma 3.1 (2), it follows that $q \in \mathcal{K}^n$, and so $q \in -M\mathcal{K}^n \cap \mathcal{K}^n$. From 3.5, it follows that $q = 0$. Since M is nonsingular, we have $x = 0$, which contradicts the assumption $y \neq x$). Since $x = -M^{-1}q$ and $y \neq x$, we have $My + q \neq 0$. By Lemma 2.6 (2), we have $y \in bd(\mathcal{K}^n)$ and $My + q = \mu J_n y$ for some $\mu > 0$. Thus, one has that $\mathcal{M}(\mu, y) = -q \in M\mathcal{K}^n$, a contradiction. □

Corollary 3.1. *Assume that M has the GUS property. For any $s \geq 0$ and any nonzero vector $x \in bd(\mathcal{K}^n)$, $\mathcal{M}(s, x) = M_s(x) \neq 0$.*

For $s \geq 0$, we define

$$\mathcal{M}_s(t, x) = (M_s - tJ_n)(x) = \mathcal{M}(s + t, x) \quad \forall t \geq 0, \forall x \in \mathbb{R}^n. \tag{3.4}$$

where $\mathcal{M}$ is defined by (3.3). The following lemma will be used in the proof of the main theorem of this section.

Lemma 3.8. *For $s \geq 0$, let $\mathcal{M}_s$ be defined by* (3.4)*. If $M\mathcal{K}^n \subset K_s$, then $\mathcal{M}_s(t, \xi) = M_{s+t}\xi \notin K_s$ for all $0 \neq \xi \in bd(\mathcal{K}^n)$ and for all $t > 0$.*

Proof. If $s = 0$, then the assertion follows from Lemma 3.7. Fix $s > 0$ and assume that $M\mathcal{K}^n \subset K_s$. Since M has the GUS property, M is nonsingular and so $int(M\mathcal{K}^n) \neq \emptyset$, which together with $M\mathcal{K}^n \subset K_s$ implies $int(K_s) \neq \emptyset$. By Lemma 2.1 (1), M_s is nonsingular. If the assertion is not true, then there exists $t > 0$, $y \in \mathcal{K}^n$ and $\xi \in bd(\mathcal{K}^n)$ such that $M_{s+t}\xi = M_s y$. By Lemma 3.7, we must have $y \notin bd(\mathcal{K}^n)$ (otherwise we have $\mathcal{M}(s + t, \xi) = \mathcal{M}(s, y)$, contradicts the injectivity of $\mathcal{M}$ on $\mathbb{R}_{++} \times (bd(\mathcal{K}^n)\backslash\{0\})$). Thus, $y \in int(\mathcal{K}^n)$. Recall that M_s is nonsingular. Thus, we have $M_{s+t}\xi = M_s y \in int(K_s)$. It is obvious that $M_s\xi \in (M\xi, M_{s+t}\xi)$. Since $M_{s+t}\xi \in int(K_s)$ and $M\xi \in M\mathcal{K}^n \subset K_s$, by [15, Thm. 6.1], we have $M_s\xi \in int(K_s)$. But we also have $M_s\xi \in bd(K_s)$ because M_s is nonsingular and $\xi \in bd(\mathcal{K}^n)$, which is a contradiction. □

For proving the main results, we divide our proof into several steps. The following lemma tells us that M_s has the GUS property if $s > 0$ is sufficiently small.

Lemma 3.9. *There exists $r > 0$ such that for each $s \in (0, r)$, M_s is nonsingular,*

(i) $M\mathcal{K}^n \backslash \{0\} \subset int(K_s)$, *and*

(ii) M_s *has the GUS property.*

Proof. By Lemma 3.8, we have

$$M_s \xi \notin M\mathcal{K}^n \quad \forall s > 0, \forall\, 0 \neq \xi \in bd(\mathcal{K}^n). \tag{3.5}$$

Let $e = (1, 0, \cdots, 0)^T$. Then $Me \in int(M\mathcal{K}^n)$. Since M is nonsingular, there exists r such that M_s is nonsingular and $M_s e = Me - sJ_n e \in int(M\mathcal{K}^n)$ for all $s \in [0, r)$.

(i). Fix $s \in (0, r)$. Since M_s is nonsingular, by Lemma 2.1 (2), $bd(K_s) = M_s(bd(\mathcal{K}^n))$. From (3.5), it follows that $bd(K_s) \cap M\mathcal{K}^n = \{0\}$. By the non-singularity of M_s again, $M_s e \in int(K_s)$, and so $M_s e \in int(K_s) \cap int(M\mathcal{K}^n)$. Hence, by Lemma 2.5, we have $M\mathcal{K}^n \backslash \{0\} \subset int(K_s)$.

(ii). Recall that $-M\mathcal{K}^n \cap \mathcal{K}^n = \{0\}$. By Lemma 2.2 and Lemma 3.4 (1), there exists r_1 such that for all $s \in [0, r_1)$, $-K_s \cap \mathcal{K}^n = \{0\}$. Let $r' = min\{r_1, r\}$. Then for all $s \in (0, r')$, M_s is nonsingular and $-M_s\mathcal{K}^n \cap \mathcal{K}^n = \{0\}$. Fix $s \in (0, r')$. Let $\mathcal{M}_s$ be defined by (3.4). To prove that M_s has the GUS property, by Lemma 3.7, we only need to prove that $\mathcal{M}_s$ is a one-to-one map from $\mathbb{R}_{++} \times \big(bd(\mathcal{K}^n) \backslash \{0\}\big)$ onto $\mathbb{R}^n \backslash (-\mathcal{K}^n \cup M_s\mathcal{K}^n)$. Since M has the GUS property, by Lemma 3.7, $\mathcal{M}_s$ must be injective on $\mathbb{R}_{++} \times (bd(\mathcal{K}^n) \backslash \{0\})$, and $\mathcal{M}_s(t, \xi) \notin -\mathcal{K}^n$ for all $t > 0$ and $\xi \in bd(\mathcal{K}^n)$. From Lemma 3.8, it follows that $\mathcal{M}_s(t, \xi) \notin M_s\mathcal{K}^n$ for all $t > 0$ and $\xi \in bd(\mathcal{K}^n)$.

Now we prove that $\mathcal{M}_s$ maps $\mathbb{R}_{++} \times \big(bd(\mathcal{K}^n) \backslash \{0\}\big)$ onto $\mathbb{R}^n \backslash (-\mathcal{K}^n \cup M_s\mathcal{K}^n)$. Fix $q \in \mathbb{R}^n \backslash (-\mathcal{K}^n \cup M_s\mathcal{K}^n)$. By Lemma 3.7, there exists $\bar{s} > 0$ and $\xi \in bd(\mathcal{K}^n)$ such that $\mathcal{M}(\bar{s}, \xi) = M_{\bar{s}}\xi = q$. From $q \notin M_s\mathcal{K}^n$, it follows that $\bar{s} \neq s$. Assume that $\bar{s} < s$. By (i), $M\xi \in M_s\mathcal{K}^n$, which together with $M_{\bar{s}}\xi = \frac{s-\bar{s}}{s}M\xi + \frac{\bar{s}}{s}M_s\xi$ implies that $q = M_{\bar{s}}\xi \in M_s\mathcal{K}^n$, a contradiction. Thus, we must have $\bar{s} > s$ and so $\mathcal{M}_s(\bar{s} - s, \xi) = q$, which completes the proof. □

Define

$$\tau = \sup\{r : \text{for all } s \in (0, r), M\mathcal{K}^n \subseteq K_s \text{ and } M_s \text{ has the GUS property.}\} \tag{3.6}$$

From Lemma 3.9, it follows that $\tau > 0$.

Lemma 3.10. *Let τ be defined by (3.6). For all $0 < t < s < \tau$, we have $K_t \backslash \{0\} \subset int(K_s)$.*

Proof. Fix $s \in (0, \tau)$ and $t \in (0, s)$. We have $M_t y = \frac{s-t}{s}My + \frac{t}{s}M_s y$ for all $y \in \mathcal{K}^n$. From (3.6), it follows that $My \in M\mathcal{K}^n \subset K_s$. Thus we have $M_t y \in K_s$, and so $K_t \subset K_s$. By the definition of τ, M_s has the GUS property and so M_s is nonsingular. From Lemma 2.1 (2), $bd(K_s) = \cup_{\xi \in bd(\mathcal{K}^n)} M_s\xi$. For any $0 \neq \xi \in bd(\mathcal{K}^n)$, by Lemma 3.8, we have $M_s\xi \notin K_t$. Hence $bd(K_s) \cap K_t = \{0\}$, which together with $K_t \subset K_s$ implies $K_t \backslash \{0\} \subset int(K_s)$. □

Lemma 3.11. *Let τ be defined by* (3.6)*. Then $\tau < \infty$ and M_τ is singular.*

Proof. By Lemma 3.5, there exists $\eta \in int(\mathcal{K}^n) \cap int(M\mathcal{K}^n)$. From Lemma 2.1 (3), it follows that $\Theta := \{M\mathcal{K}^n - t\eta : t \geq 0\} = \mathbb{R}^n$. By Lemma 3.4 (3), there exists $r > 0$ such that $-\eta \in K_s$ for all $s > r$. Now we prove that $M\mathcal{K}^n \subseteq K_s$ does not hold for all $s > r$. Assume that $K_s \supseteq M\mathcal{K}^n$ for some $s > r$. Whether M_s is nonsingular or not, K_s is a proper subset of $\mathbb{R}^n$. But, since K_s is a convex cone, we have $K_s \supset \Theta = \mathbb{R}^n$, a contradiction. Thus, we have $\tau < \infty$.

To prove that M_τ is singular, we use a contrapositive argument. Assume that M_τ is nonsingular. First, we prove $M\mathcal{K}^n \subset K_\tau$. Since M_s is nonsingular for all $s \in [0, \tau)$, we have $\lim_{s\uparrow\tau} M_s^{-1} = M_\tau^{-1}$. Pick any $p \in M\mathcal{K}^n$. From (3.6), it follows that $M\mathcal{K}^n \subset K_s$ for all $s \in [0, \tau)$. Thus, there exists $\zeta_s \in \mathcal{K}^n$ such that $M_s\zeta_s = p$. Let $\zeta := M_\tau^{-1}p$. Then

$$\lim_{s\to\tau} \zeta_s = \lim_{s\to\tau} M_s^{-1}p = M_\tau^{-1}p = \zeta.$$

Since $\zeta_s \in \mathcal{K}^n$, we have $\zeta \in \mathcal{K}^n$. Since $p \in M\mathcal{K}^n$ is arbitrary, we have $M\mathcal{K}^n \subset K_\tau$.

Second, we prove that $-K_\tau \cap \mathcal{K}^n = \{0\}$. If not, pick any $0 \neq z \in -K_\tau \cap \mathcal{K}^n$. Then $z = -M_\tau y$ for some $0 \neq y \in \mathcal{K}^n$. By the definition of τ, for each $s \in (0, \tau)$, M_s has the GUS property. From Lemma 3.7, it follows that $K_s \cap -\mathcal{K}^n = M_s\mathcal{K}^n \cap -\mathcal{K}^n = \{0\}$. For $s \in (0, \tau)$, we have $0 \neq M_s y \in K_s$, and so $-M_s y \notin \mathcal{K}^n$. From $z = -M_\tau y = \lim_{s\uparrow\tau} -M_s y$, it follows that $z \in bd(\mathcal{K}^n)$. Thus, $-K_\tau \cap \mathcal{K}^n \subset bd(\mathcal{K}^n)$. Recall that we assume that M_τ is nonsingular, which implies $-int(K_\tau) \neq \emptyset$. We must have $-int(K_\tau) \cap \mathcal{K}^n = \emptyset$ (otherwise $-int(K_\tau) \cap int(\mathcal{K}^n) \neq \emptyset$, which contradicts $-K_\tau \cap \mathcal{K}^n \subset bd(\mathcal{K}^n)$) and so

$$-K_\tau \cap \mathcal{K}^n = -bd(K_\tau) \cap bd(\mathcal{K}^n).$$

Hence, $z = -M_\tau y \in -bd(K_\tau)$. Since M_τ is nonsingular, we have $y \in bd(\mathcal{K}^n)$. But from Lemma 3.7, it follows that $-z = M_\tau y \notin -\mathcal{K}^n$, which contradicts $z \in \mathcal{K}^n$. Thus, we must have $-K_\tau \cap \mathcal{K}^n = \{0\}$.

Let $\mathcal{M}_\tau$ be defined by (3.4). Since $M\mathcal{K}^n \subset K_\tau$, $-K_\tau \cap \mathcal{K}^n = \{0\}$ and M_τ is nonsingular, similar to the proof in Lemma 3.9 (ii), we can prove that $\mathcal{M}_\tau$ is a one-to-one map from $\mathbb{R}_{++} \times \big(bd(\mathcal{K}^n)\backslash\{0\}\big)$ onto $\mathbb{R}^n \backslash (-\mathcal{K}^n \cup M_\tau\mathcal{K}^n)$. Thus, by Lemma 3.7, $\mathcal{M}_\tau$ has the GUS property. By Lemma 3.9, there exists $r > 0$ such that for all $s \in (0, r)$, $(M_\tau)_s = M_{\tau+s}$ has the GUS property and $M\mathcal{K}^n \subset M_{\tau+s}\mathcal{K}^n = K_{\tau+s}$. Thus, for all $t \in [0, \tau + r)$, M_t has the GUS property and $M\mathcal{K}^n \subset K_t$, which contradicts (3.6). Hence, M_τ must be singular. □

Lemma 3.12. *Let τ be defined by* (3.6)*. Then $rank(M_\tau) = n - 1$ and there exists $\omega \in int(\mathcal{K}^n)$ such that*

$$Ker(M_\tau) = \{t\omega : t \in \mathbb{R}\}. \tag{3.7}$$

Proof. First, we prove that for all $0 \neq \omega \in Ker(M_\tau)$, we have $\omega \in int(\mathcal{K}^n)$ or $-\omega \in int(\mathcal{K}^n)$. If it is not true, there exists $\omega \neq 0$ such that $\omega \in Ker(M_\tau)$ and $\pm\omega \notin int(\mathcal{K}^n)$. By Lemma 2.1 (4), $\omega = \xi - \eta$ for some $\xi, \eta \in bd(\mathcal{K}^n)$. Since $\omega \in Ker(M_\tau)$, we have $M_\tau\xi = M_\tau\eta$. Let $\mathcal{M}$ be defined by (3.3). Then $\mathcal{M}(\tau, \xi) = \mathcal{M}(\tau, \eta)$. If $\xi \neq 0$ and $\eta \neq 0$, by Lemma 3.7, we have $\xi = \eta$ and so $\omega = 0$, a contradiction. If $\xi = 0$ and $\eta \neq 0$, then $\mathcal{M}(\tau, \eta) = 0$, which contradicts Corollary 3.1. For the same reason, $\eta = 0$ and $\xi \neq 0$ is impossible. Since $\omega \neq 0$, $\xi = \eta = 0$ is impossible also.

From the proof above, it is obvious that $rank(M_\tau) = n-1$ is equivalent to (3.7). Thus we only need to prove $rank(M_\tau) = n-1$. If it is not true, there exists $\omega_1 \in int(\mathcal{K}^n) \cap Ker(M_\tau)$ and $\omega_2 \in int(\mathcal{K}^n) \cap Ker(M_\tau)$ such that $\omega_1 \neq t\omega_2$ for any $t \in \mathbb{R}$. By Lemma 2.1 (5), there exist nonzero real numbers α and β such that $z := \alpha\omega_1 + \beta\omega_2 \in bd(\mathcal{K}^n)$. It is obvious $z \neq 0$. Since $\omega_1, \omega_2 \in Ker(M_\tau)$, we have $M_\tau z = 0$, which contradicts Corollary 3.1. The proof is complete. □

Lemma 3.13. *Let τ be defined by* (3.6)*. Then $\mathcal{R}(M_\tau) = \cup_{\xi \in bd(\mathcal{K}^n)} M_\tau \xi$.*

Proof. Let $G = \cup_{\xi \in bd(\mathcal{K}^n)} M_\tau \xi$. It is obvious $G \subseteq \mathcal{R}(M_\tau)$. By Lemma 3.12, there exists $\omega \in int(\mathcal{K}^n)$ such that $Ker(M_\tau) = \{t\omega : t \in \mathbb{R}\}$, which implies $\mathcal{R}(M_\tau) = \{M_\tau y : y \in \omega^\perp\}$. Pick any $0 \neq y \in \omega^\perp$. Then $y \notin \mathcal{K}^n$, and so here exists $\alpha > 0$ such that $\omega + \alpha y \in bd(\mathcal{K}^n)$. Let $\xi := \omega + \alpha y$. Then $M_\tau y = M_\tau \xi / \alpha \in G$. Since y is arbitrary, we have $G = \mathcal{R}(M_\tau)$. □

By Lemma 3.12, we have $rank(M_\tau) = n-1$, which is equivalent to $rank(M_\tau^T) = n-1$. Hence, there exists $0 \neq \theta \in \mathbb{R}^n$ such that

$$M_\tau^T \theta = (M^T - \tau J_n)\theta = 0, \tag{3.8}$$

which implies $\mathcal{R}(M_\tau) = \theta^\perp$.

Lemma 3.14. *Let θ be defined by* (3.8)*. Then $\theta \in int(\mathcal{K}^n)$ or $-\theta \in int(\mathcal{K}^n)$.*

Proof. Let $\tilde{K} = \cup_{0<s<\tau} K_s$. Recall that we have proved that $K_t \subset K_s$ and K_s is a proper cone if $0 < t < s < \tau$. It is easy to prove that $\tilde{K}$ is convex and has nonempty interior. By Lemma 3.8, $M_\tau \xi \notin \tilde{K}$ for all $0 \neq \xi \in bd(\mathcal{K}^n)$, which together with Lemma 3.13 implies that $\mathcal{R}(M_\tau) \cap \tilde{K} = \{0\}$. Pick any $\eta \in int(\mathcal{K}^n)$. It is obvious $\eta \in int(\tilde{K})$ and so $\eta \notin \mathcal{R}(M_\tau)$. Thus η must lies in one side of $\mathcal{R}(M_\tau)$. Let θ be defined by (3.8) such that $\theta^T \eta > 0$. Define $H := \{z \in \mathbb{R}^n : \theta^T z > 0\}$. Then $bd(H) = \mathcal{R}(M_\tau)$.

For each $0 \neq y \in \mathcal{R}(M_\tau)$, by Lemma 3.13, $y = M_\tau \xi$ for some $\xi \in bd(\mathcal{K}^n)$. Since $y = \lim_{s \uparrow \tau} M_s \xi$ and $y \notin \tilde{K}$, we have $y \in bd(\tilde{K})$. Thus $bd(H) = \mathcal{R}(M_\tau) \subset bd(\tilde{K})$. Fix $\rho \in H$. Let $\beta = \theta^T \rho / \theta^T \eta$. Then $h := \rho - \beta\eta \in bd(H)$. Note that $2h \in bd(\tilde{K})$ and $2\beta\eta \in int(\tilde{K})$. Since $\rho \in (2h, 2\beta\eta)$, by [15, Thm. 6.1], we have $\rho \in int(\tilde{K})$ and so $H \subseteq \tilde{K}$. Recall $bd(H) \cap \tilde{K} = \{0\}$. It is obvious $\tilde{K} = H \cup \{0\}$. By Lemma 3.10, for all $s \in (0, \tau)$ and $0 \neq \xi \in bd(\mathcal{K}^n)$, we have $M_s \xi \in K_s \subset int(\tilde{K})$, which together with $int(\tilde{K}) = H$ implies that

$$\theta^T M_s \xi = \theta^T (M\xi - sJ_n \xi) > 0 \quad \forall \xi \in bd(\mathcal{K}^n). \tag{3.9}$$

By (3.8), $\theta^T M_\tau \xi = 0$, which together with (3.9) implies $(\tau - s)\theta^T J_n \xi > 0$. Since $0 \neq \xi \in bd(\mathcal{K}^n)$ is arbitrary, we must have $\theta \in int(\mathcal{K}^n)$. □

By Lemma 3.14, there exists $\theta \in int(\mathcal{K}^n)$ such that (3.8) holds, and if $0 < s < \tau$, then K_s lies in one side of $\mathcal{R}(M_\tau)$. For the case $s > \tau$, we present some observations now. If $s > \tau$, for each $0 \neq x \in \mathcal{K}^n$, we have

$$\theta^T M_s x = \theta^T M_\tau x + (\tau - s)\theta^T x < 0, \tag{3.10}$$

which implies that K_s lies in another side of $\mathcal{R}(M_\tau)$. From (3.10), it follows that

$$\mathcal{R}(M_\tau) \cap K_s = \{0\} \tag{3.11}$$

and

$$Ker(M_s) \cap \mathcal{K}^n = \{0\}. \tag{3.12}$$

By Lemma 3.12, there exists $\omega \in int(\mathcal{K}^n)$ such that $Ker(M_\tau) = \{t\omega : t \in \mathbb{R}\}$. If $s > \tau$, then we have

$$M_s\omega = M_\tau\omega + (\tau - s)\omega = (\tau - s)\omega \in -int(\mathcal{K}^n). \tag{3.13}$$

Lemma 3.15. *If $s' > s > \tau$, then $M_s\xi \notin K_{s'}$ for all $0 \neq \xi \in bd(\mathcal{K}^n)$.*

Proof. Fix $s' > s > \tau$. Assume that $M_s\xi \in K_{s'}$ for some $0 \neq \xi \in bd(\mathcal{K}^n)$. From Corollary 3.1, it follows that $M_s\xi \neq 0$. If $M_s\xi \in bd(K_{s'})$, by (3.12) and Lemma 2.4, there exists $0 \neq \eta \in bd(\mathcal{K}^n)$ such that $M_s\xi = M_{s'}\eta$, which contradicts Lemma 3.7. Thus, $M_s\xi \in int(K_{s'})$, which implies $M_{s'}$ is nonsingular. By (3.11), $M_\tau\xi \notin K_{s'}$ and so there exists $\lambda \in (0, 1)$ such that $u := \lambda M_\tau\xi + (1-\lambda)M_s\xi \in bd(K_{s'})$. Let $\bar{s} = \lambda\tau + (1-\lambda)s$. Then $M_{\bar{s}}\xi = u \in bd(K_{s'})$. From Corollary 3.1, it follows that $u \neq 0$. Since $M_{s'}$ is nonsingular and $u \neq 0$, by Lemma 2.4 again, there exists $0 \neq \varsigma \in bd(\mathcal{K}^n)$ such that $M_{\bar{s}}\xi = M_{s'}\varsigma$, which contradicts Lemma 3.7. □

Now we can derive the following necessary conditions for M having the GUS property.

Lemma 3.16. *If $s > \tau$, then M_s is nonsingular and $-\mathcal{K}^n \backslash \{0\} \subset int(K_s)$.*

Proof. Let $\omega \in int(\mathcal{K}^n)$ be such that $Ker(M_\tau) = \{t\omega : t \in \mathbb{R}\}$. We must have $M_s\omega \in int(K_s)$ (If not, then $M_s\omega \in bd(K_s)$. By (3.12) and Lemma 2.4, $M_s\omega = M_s\xi$ for some $\xi \in bd(\mathcal{K}^n)$, which together with (3.13) implies $M_s\xi \in -\mathcal{K}^n$. This contradicts Lemma 3.7). By $int(K_s) \neq \emptyset$ and Lemma 2.1 (1), M_s is nonsingular. Thus $bd(K_s) = \cup_{\xi \in bd(\mathcal{K}^n)} M_s\xi$. From Lemma 3.7, it follows that $bd(K_s) \cap -\mathcal{K}^n = \{0\}$. Thus, to prove $-\mathcal{K}^n \backslash \{0\} \subset int(K_s)$, we only need to show that $-\mathcal{K}^n \subset K_s$. If not, there exists $z \in -\mathcal{K}^n$ but $z \notin K_s$. Then there exists $y \in (z, M_s\omega)$ such that $y \in bd(K_s)$. Recall that $M_s\omega \in -int(\mathcal{K}^n)$, and so $y \in -\mathcal{K}^n$. Since $-\mathcal{K}^n$ is a pointed cone, we have $y \neq 0$. Thus, $y = M_s\xi$ for some $0 \neq \xi \in bd(\mathcal{K}^n)$, which together with Lemma 3.7 implies $y \notin -\mathcal{K}^n$, a contradiction. □

Lemma 3.17. *If $s' > s > \tau$, then $K_{s'} \backslash \{0\} \subset int(K_s)$.*

Proof. Fix $s' > s > \tau$. Pick any $0 \neq x \in \mathcal{K}^n$. By Lemma 3.16, $-J_n x \in int(K_s)$. Let t be large enough such that $t > s'$ and $-J_n x + Mx/t \in int(K_s)$, which implies that $M_t x = Mx - tJ_n x \in int(K_s)$. Note that $M_{s'}x \in (M_s x, M_t x)$. Since $M_t x \in int(K_s)$ and $M_s x \in K_s$, by [15, Thm. 6.1], we have $M_{s'}x \in int(K_s)$. Since $x \neq 0$ is arbitrary, we have $K_{s'} \backslash \{0\} \subset int(K_s)$. □

Note that for all $s \in \mathbb{R}$,

$$(MJ_n - sI_n)v = (M - sJ_n)J_n v = M_s J_n v \quad \forall v \in \mathbb{R}^n, \tag{3.14}$$

which means that $J_n v \in Ker(M_s)$ if and only if v is the eigenvector of MJ_n associated with the eigenvalue s. By Lemma 3.12, Lemma 3.16 and (3.14), MJ_n has nonnegative eigenvalues, which equal to the same value τ, where τ is defined by (3.6). Moreover, $J_n\omega \in int(\mathcal{K}^n)$ is the eigenvector of MJ_n associated with τ, where ω is as in Lemma 3.12.

Analogously, $M^T J_n$ has nonnegative eigenvalues and they all equal to τ also. Let $\theta \in int(\mathcal{K}^n)$ be such that $Ker(M_\tau^T) = \{t\theta : t \in \mathbb{R}\}$. Then $J_n\theta \in int(\mathcal{K}^n)$ is the eigenvector of MJ_n associated with τ. Now we can summarize the above results as the following theorem.

Theorem 3.1. *For $LCP(M, \mathcal{K}^n, q)$, if M has the GUS property, then the following statements hold:*

(1) *MJ_n has nonnegative eigenvalues and there exists $\tau > 0$ such that all nonnegative eigenvalues of MJ_n equal to τ. Moreover, $rank(MJ_n - \tau I_n) = n - 1$.*

(2) *There exists $\omega \in int(\mathcal{K}^n)$ such that ω is the eigenvector of MJ_n associated with τ.*

(3) *The range of M_τ, $\mathcal{R}(M_\tau)$, is a hyperplane in $\mathbb{R}^n$. There exists $\theta \in int(\mathcal{K}^n)$ such that $\mathcal{R}(M_\tau) = \theta^\perp$. Moreover, $J_n\theta$ is the eigenvector of $M^T J_n$ associated with τ.*

(4) *For all $0 \leq t < s < \tau$, $K_t \backslash \{0\} \subset int(K_s)$ and K_s lies in one side of $\mathcal{R}(M_\tau)$; if $s > t > \tau$, then $-\mathcal{K}^n \backslash \{0\} \subset int(K_s)$, $K_s \backslash \{0\} \subset int(K_t)$ and K_t lies in another side of $\mathcal{R}(M_\tau)$.*

Remark 3.1. *By [10, Thm. 17], if M is positive definite (i.e., $x^T Mx > 0$ for all $x \neq 0$), then M has the GUS property. Thus, MJ_n satisfies (1)-(4) of the above theorem.*

4 Sufficient and Necessary Conditions

In this section, we will present sufficient and necessary conditions for M having the GUS property in $LCP(M, \mathcal{K}^n, q)$. Before presenting the main results, we need the following preliminary results.

Lemma 4.1. *Suppose that $A \in \mathbb{R}^{n \times n}$ is nonsingular. Let $C = A\mathcal{K}^n$ and let ξ be a nonzero vector in $bd(\mathcal{K}^n)$. We have $N_C(A\xi) = \{-tA^{-T}J_n\xi : t \geq 0\}$. Let $H = \{A\xi + y : y^T A^{-T} J_n \xi \leq 0\}$. Then $H \cap C = \{tA\xi : t \geq 0\}$.*

Proof. For any $\eta \in N_C(A\xi)$, we have $\eta^T(Ax - A\xi) \leq 0$ for all $x \in \mathcal{K}^n$, which implies that $A^T\eta \in N_{\mathcal{K}^n}(\xi)$. By (2.4), $A^T\eta = -tJ_n\xi$ for some $t \geq 0$, and therefore $\eta = -tA^{-T}J_n\xi$, which implies that $N_C(A\xi) = \{-tA^{-T}J_n\xi : t \geq 0\}$.

Since $\xi \in bd(\mathcal{K}^n)$, by Lemma 2.6 (2), we have $(A\xi)^T(A^{-T}J_n\xi) = 0$. Hence $tA\xi \in H$ for all $t \in \mathbb{R}$, and so $\{tA\xi : t \geq 0\} \subseteq H \cap C$. If $y \in H \cap C$, then $y = Ax$ for some $x \in \mathcal{K}^n$ and $(Ax - A\xi)^T A^{-T}J_n\xi \leq 0$, which implies that $x^TJ_n\xi \leq 0$. Since $\mathcal{K}^n$ is self-dual, $x^TJ_n\xi \geq 0$ and so $x^TJ_n\xi = 0$. By Lemma 2.6 (2), $x = t\xi$ for some $t \geq 0$, which implies that $y = tA\xi$. Thus $H \cap C \subseteq \{tA\xi : t \geq 0\}$ and so the equality holds. □

In the next two lemmas, we present some more necessary conditions for M having the GUS property. As we shall present in Theorem 4.1, these necessary conditions, together with the assumptions (1), (2) and (3) in Theorem 3.1, constitute the sufficient conditions to ensure the GUS property of M.

Lemma 4.2. *Assume that M has the GUS property and let τ be defined by* (3.6)*. Then $\xi^T M_s^{-1}\xi > 0$ for all $0 \neq \xi \in bd(\mathcal{K}^n)$ and $s \in (0, \tau)$.*

Proof. Fix $s \in (0, \tau)$ and $0 \neq \xi \in bd(\mathcal{K}^n)$. By Theorem 3.1, M_s is nonsingular and K_s is a proper cone. By Lemma 4.1, we have

$$N_{K_s}(M_sJ_n\xi) = \{-tM_s^{-T}\xi : t \geq 0\}. \tag{4.1}$$

From Theorem 3.1, it follows that $MJ_n\xi \in int(K_s)$, which together with (2.1) and $-M_s^{-T}\xi \in N_{K_s}(M_sJ_n\xi)$ implies that

$$0 < (M_s^{-T}\xi)^T(MJ_n\xi - M_sJ_n\xi) = s\xi^T M_s^{-1}\xi,$$

which implies the assertion. □

Corollary 4.1. *If M has the GUS property, then $\xi^T M^{-1}\xi \geq 0$ for all $\xi \in bd(\mathcal{K}^n)$.*

Lemma 4.3. *If M has the GUS property, then $\xi^T M\xi \geq 0$ for all $\xi \in bd(\mathcal{K}^n)$.*

Proof. Fix $0 \neq \xi \in bd(\mathcal{K}^n)$. Since M has the GUS property, by Lemma 3.5 and 3.6, $M_s\xi \notin -\mathcal{K}^n$ for all $s \geq 0$. By the definition of M_s, $\cup_{s\geq 0}M_s\xi$ is a ray, which has empty intersection with $-\mathcal{K}^n$. By the Convex Separation Theorem, there exists $a \neq 0$ such that

$$a^T(-x) \leq 0 \quad \forall x \in \mathcal{K}^n \tag{4.2}$$

and

$$a^T M_s\xi \geq 0 \quad \forall s \geq 0. \tag{4.3}$$

Since $\mathcal{K}^n$ is a self-dual cone, by (4.2), we have $a \in \mathcal{K}^n$, which implies that $a^TJ_n\xi \geq 0$. From $a^TM_s\xi = a^T(M\xi - sJ_n\xi) \geq 0$ for all $s > 0$, it follows that $a^TJ_n\xi \leq 0$. Thus we have $a^TJ_n\xi = 0$, which together with Lemma 2.6 (2) implies that $a = \lambda\xi$ for some $\lambda > 0$. Substituting it into (4.3) and letting $s = 0$, we obtain $\xi^TM\xi \geq 0$. □

Remark 4.1. *Note that for a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$,*

$$x \in SOL(q, A) \Longleftrightarrow Ax + q \in SOL(-A^{-1}q, A^{-1}). \tag{4.4}$$

Thus, by Corollary 4.1 and (4.4)*, we obtain Lemma 4.3 immediately. The current proof of Lemma 4.3 tells us that $\xi^T M\xi \geq 0,\ \forall \xi \in bd(\mathcal{K}^n)$, comes from the geometric phenomenon $(\cup_{s\geq 0} M_s\xi) \cap -\mathcal{K}^n = \emptyset$.*

Now, we are ready to state the main results of this section, i.e., some sufficient and necessary conditions of the GUS property of M for $LCP(M, \mathcal{K}^n, q)$.

Theorem 4.1. *For $LCP(M, \mathcal{K}^n, q)$, M has the GUS property if and only if it satisfies the following assumptions:*

(i) *MJ_n has nonnegative eigenvalues and there exists $\tau > 0$ such that all nonnegative eigenvalues of MJ_n equal to τ. Moreover, $rank(MJ_n - \tau I_n) = n - 1$. There exists $\omega \in int(\mathcal{K}^n)$ such that ω is the eigenvector of MJ_n associated with τ.*

(ii) *There exists $\theta \in int(\mathcal{K}^n)$ such that θ is the eigenvector of $M^T J_n$ associated with τ.*

(iii)

$$\xi^T M\xi \geq 0 \quad \forall \xi \in bd(\mathcal{K}^n). \tag{4.5}$$

(iv)

$$\xi^T M^{-1}\xi \geq 0 \quad \forall \xi \in bd(\mathcal{K}^n). \tag{4.6}$$

Proof. The necessity is obviously true based on Theorem 3.1, Corollary 4.1 and Lemma 4.3. Now we prove that the conditions (i)-(iv) are sufficient to ensure that M has the GUS property.

First of all, we have some observations. Let τ be as in assumption (i). By (3.14) and (i), $M_\tau\xi \neq 0$ for all $0 \neq \xi \in bd(\mathcal{K}^n)$. For any $s \geq 0$ satisfying $s \neq \tau$, by assumption (i) again, M_s is nonsingular. Thus, we have

$$M_s\xi \neq 0 \quad \forall s \geq 0,\ \forall 0 \neq \xi \in bd(\mathcal{K}^n). \tag{4.7}$$

Let θ be as in assumption (ii). Then $(M^T J_n - \tau I_n)\theta = 0$ and so $M_\tau^T J_n\theta = 0$, which implies

$$\mathcal{R}(M_\tau) = (J_n\theta)^\perp. \tag{4.8}$$

If $0 \leq s < \tau$, then

$$(J_n\theta)^T(M_s x) = (J_n\theta)^T(M_\tau x + (\tau - s)x) = (\tau - s)(J_n\theta)^T x > 0 \quad \forall 0 \neq x \in \mathcal{K}^n, \tag{4.9}$$

the last strict inequality follows from Lemma 2.6 (3) and $J_n\theta \in int(\mathcal{K}^n)$. By assumption (i) and (4.9), we obtain that

$$M \text{ is nonsingular and } \ M\mathcal{K}^n \cap -\mathcal{K}^n = \{0\}. \tag{4.10}$$

Similar to (4.9), if $s > \tau$, we can derive that

$$(J_n\theta)^T(M_s x) < 0 \quad \text{for all nonzero} \quad x \in \mathcal{K}^n. \tag{4.11}$$

To present the rest of the proof clearly, we summarize some results in the following lemmas and then complete the proof.

Lemma 4.4. *If $s > 0$, then $M_s\xi \notin M\mathcal{K}^n$ for all $0 \neq \xi \in bd(\mathcal{K}^n)$.*

Proof. Fix nonzero vector $\xi \in bd(\mathcal{K}^n)$. By (4.1), $N_{M\mathcal{K}^n}(M\xi) = \{-tM^{-T}J_n\xi : t \geq 0\}$. Assume that $M_s\xi \in M\mathcal{K}^n$ for some $s > 0$. Since $(-M^{-T}J_n\xi) \in N_{M\mathcal{K}^n}(M\xi)$, we have

$$(-M^{-T}J_n\xi)^T(M_s\xi - M\xi) = s\xi^T J_n M^{-1} J_n \xi \leq 0,$$

which together with (4.6) implies $\xi^T J_n M^{-1} J_n \xi = 0$. Thus, we have $M_s\xi \in \{M\xi + y : y^T M^{-T} J_n \xi \leq 0\}$. By Lemma 4.1, $M_s\xi = \kappa M\xi$ for some $\kappa \geq 0$. Since $\xi \neq 0$ and $s > 0$, it is impossible that $\kappa = 1$. If $\kappa < 1$, then $(M - \frac{s}{1-\kappa}J_n)\xi = 0$, which contradicts (4.7). If $\kappa > 1$, then $(\kappa - 1)M\xi = -sJ_n\xi$, which contradicts (4.10). Hence, we must have $M_s\xi \notin M\mathcal{K}^n$. □

Lemma 4.5. *If $s > 0$, then $M_s\xi \notin -\mathcal{K}^n$ for all $0 \neq \xi \in bd(\mathcal{K}^n)$.*

Proof. We use a contrapositive argument. Assume $M_s\xi \in -\mathcal{K}^n$ for some $0 \neq \xi \in bd(\mathcal{K}^n)$ and $s > 0$. From (4.10), it follows that $M\xi \notin -\mathcal{K}^n$. Thus, there exist $0 < s' \leq s$ such that $M_{s'}\xi \in -bd(\mathcal{K}^n)$. By (4.7), we have $M_{s'}\xi \neq 0$ and so

$$M_{s'}\xi = -\eta \quad \text{for some nonzero vector} \quad \eta \in bd(\mathcal{K}^n). \tag{4.12}$$

From (4.5), it follows that $0 \leq \xi^T M\xi = \xi^T(M - s'J_n)\xi = \xi^T M_{s'}\xi = -\xi^T\eta \leq 0$, and so $\xi^T\eta = 0$. By Lemma 2.6 (2), $\eta = tJ_n\xi$ for some $t > 0$, which together with (4.12) implies that $M\xi = (s' - t)J_n\xi$. If $s' \geq t$, then $M_{s'-t}\xi = 0$, which contradicts (4.7). If $s' < t$, then $M\xi \in -\mathcal{K}^n$, a contradiction. □

Lemma 4.6. *If $0 < s < \tau$, then $M\mathcal{K}^n \backslash \{0\} \subset int(K_s)$; if $s > \tau$, then $-\mathcal{K}^n \backslash \{0\} \subset int(K_s)$.*

Proof. Fix $s \in (0, \tau)$. From Lemma 4.4, it follows that $M_s\xi \notin M\mathcal{K}^n$ for all nonzero vector $\xi \in bd(\mathcal{K}^n)$. Since M_s is nonsingular, $bd(K_s) = M_s(bd(\mathcal{K}^n))$. Then we have $bd(K_s) \cap M\mathcal{K}^n = \{0\}$. Let ω be as in assumption (i). By (3.14), $M_s J_n\omega = (\tau - s)J_n\omega \in int(K_s) \cap int(\mathcal{K}^n)$. Thus, the assertion $M\mathcal{K}^n \backslash \{0\} \subset int(K_s)$ follows from Lemma 2.5.

Assume that $s > \tau$. Since M_s is nonsingular and $\omega \in int(\mathcal{K}^n)$, we have $M_s J_n \omega = (\tau - s)J_n\omega \in int(K_s) \cap -int(\mathcal{K}^n)$. From Lemma 4.5, it follows that $M_s\xi \notin -\mathcal{K}^n$ for all nonzero vector $\xi \in bd(\mathcal{K}^n)$. Note that $bd(K_s) = M_s(bd(\mathcal{K}^n))$, and so $bd(K_s) \cap -\mathcal{K}^n = \{0\}$. By Lemma 2.5, the assertion $-\mathcal{K}^n \backslash \{0\} \subset int(K_s)$ holds. □

Lemma 4.7. *If $0 < t < s < \tau$, then $K_t \backslash \{0\} \subset int(K_s)$; if $s' > s > \tau$, we have $K_{s'} \backslash \{0\} \subset int(K_s)$.*

Proof. Assume that $s \in (0, \tau)$ and $t \in (0, s)$. Pick any $0 \neq x \in \mathcal{K}^n$. Note that $M_t x \in (Mx, M_s x)$. By Lemma 4.6, we have $Mx \in int(K_s)$, which together with [15, Thm. 6.1] implies that $M_t x \in int(K_s)$. Since $x \neq 0$ is arbitrary, we have $K_t \backslash \{0\} \subset int(K_s)$.

Assume that $s' > \tau$ and $s \in (\tau, s')$. By Lemma 4.6, we have $-\mathcal{K}^n \backslash \{0\} \subset int(K_s)$. Then the proof of $K_{s'} \backslash \{0\} \subset int(K_s)$ is similar to that in Lemma 3.17, and we omit it. $\square$

With the lemmas 4.4–4.7, we can complete the proof of this theorem. Let $\mathcal{M}$ be defined by (3.3). Now we prove that $\mathcal{M}$ is a one-to-one map from $\mathbb{R}_{++} \times \big(bd(\mathcal{K}^n) \backslash \{0\}\big)$ onto $\mathbb{R}^n \backslash (-\mathcal{K}^n \cup M\mathcal{K}^n)$. If this is true, then by Lemma 3.7 and (4.10), M has the GUS property.

From Lemmas 4.4 and 4.5, it follows that

$$\cup_{s>0, 0 \neq \xi \in bd(\mathcal{K}^n)} \mathcal{M}(s, \xi) \subseteq \mathbb{R}^n \backslash (-\mathcal{K}^n \cup M\mathcal{K}^n).$$

Now we prove that $\mathcal{M}$ is injective. By (4.8), (4.9), (4.11) and Lemma 4.7, we obtain that $M_s \xi \neq M_t \varsigma$, where $s, t > 0$ with $s \neq t$, and $\xi, \varsigma \in bd(\mathcal{K}^n)$ are nonzero vectors. Thus, to prove the injectivity of $\mathcal{M}$, we only need to show that M_s is injective on $bd(\mathcal{K}^n)$ for all $s > 0$. If $s > 0$ and $s \neq \tau$, the injectivity of M_s on $bd(\mathcal{K}^n)$ follows from the non-singularity of M_s. If $s = \tau$, by assumption (i), $rank(M_\tau) = n - 1$ and so M_τ is injective on $bd(\mathcal{K}^n)$.

In the following, we show that $\mathcal{M}$ is onto, that is for each $q \notin (-\mathcal{K}^n \cup M\mathcal{K}^n)$, we need to prove that there exist $0 \neq \xi \in bd(\mathcal{K}^n)$ and $s > 0$ such that $M_s \xi = q$. Fix $q \in \mathbb{R}^n \backslash (-\mathcal{K}^n \cup M\mathcal{K}^n)$. Let θ be as in assumption (ii).

If $q^T J_n \theta = 0$, then by (4.8), there exist $y \neq 0$ such that $q = M_\tau y$. By assumption (i) and (3.14), $J_n \omega \in Ker(M_\tau)$, and so we can assume that $y \in (J_n \omega)^\perp$. Thus, there exists $\alpha > 0$ such that $J_n \omega + \alpha y \in bd(\mathcal{K}^n)$. Let $\xi := (J_n \omega)/\alpha + y$. Then $\xi \in bd(\mathcal{K}^n)$ and $q = M_\tau \xi$.

Assume that $q^T J_n \theta > 0$. Let $\tilde{K} = \cup_{0<s<\tau} K_s$. We can use the same argument as that in Lemma 3.14 to prove that $\tilde{K} = \{x : (J_n \theta)^T x > 0\} \cup \{0\}$, which implies that $q \in K_s$ for some $s \in (0, \tau)$. Let $\bar{s} = \inf\{s : q \in K_s\}$. Let $\delta = d(q/\|q\|, M\mathcal{K}^n)$. By Lemma 3.4 (1), there exists $s' > 0$ such that $d(M\mathcal{K}^n, K_{s'}) < \delta/2$, which implies that $q \notin K_{s'}$. Thus $0 < \bar{s} < \tau$. Let $x = M_{\bar{s}}^{-1} q$. Since $M_{\bar{s}}$ is nonsingular, we have $x \neq 0$. By the definition of $\bar{s}$, there exist $s_n \in (\bar{s}, \tau)$ and $x_n \in \mathcal{K}^n$ such that $\lim_{n\to\infty} s_n = \bar{s}$ and $M_{s_n} x_n = q$. We have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} M_{s_n}^{-1} q = M_{\bar{s}}^{-1} q$. From $x_n \in \mathcal{K}^n$, it follows that $x \in \mathcal{K}^n$. Let $y_k = M_{\bar{s}-\frac{1}{k}}^{-1} q$. By the definition of $\bar{s}$, we have $y_k \notin \mathcal{K}^n$. From $\lim_{k\to\infty} y_k = x$ and $x \in \mathcal{K}^n$, it follows that $x \in bd(\mathcal{K}^n)$.

Assume that $q^T J_n \theta < 0$. Let $\hat{K} = \cup_{s>\tau} K_s$. Similar to the proof of $\tilde{K} = \{x : (J_n \theta)^T x > 0\} \cup \{0\}$, we can prove that $\hat{K} = \{x : (J_n \theta)^T x < 0\} \cup \{0\}$, which implies that $q \in K_s$ for some $s > \tau$. Let $\hat{s} = \sup\{s : q \in K_s\}$. Then there exist $s_n > \hat{s}$ and $z_n \in \mathcal{K}^n$ such that $s_n \to \hat{s}$ and $M_{s_n} z_n = q$. Let $\sigma = d(q/\|q\|, -\mathcal{K}^n)$. By Lemma 3.4 (2), there exists $s' > 0$ such that $d(q/\|q\|, K_{s'}) < \delta/2$, which implies that $q \notin K_{s'}$. Thus $\hat{s} < \infty$, and so $\lim_{n\to\infty} z_n = \lim_{n\to\infty} M_{s_n}^{-1} q = M_{\hat{s}}^{-1} q$. Let $z = M_{\hat{s}}^{-1} q$. Since $\hat{s} > \tau$, $M_{\hat{s}}$ is nonsingular and so $z \neq 0$. From $z_n \in \mathcal{K}^n$ and $z_n \to z$, it follows that $z \in \mathcal{K}^n$. Let $w_k = M_{\bar{s}+\frac{1}{k}}^{-1} q$. Then $\lim_{k\to\infty} w_k = x$. By the definition of $\hat{s}$, we have $w_k \notin \mathcal{K}^n$ and so $z \in bd(\mathcal{K}^n)$, which completes the proof. $\square$

Remark 4.2. *Note that* $rank(MJ_n - \tau I_n) = rank(M_\tau) = rank(M_\tau^T) = rank(M^T J_n - \tau I_n)$ *always holds. By the condition* (i), $M^T J_n$ *has nonnegative eigenvalues and all nonnegative eigenvalues equal to* τ. *We also have* $rank(M^T J_n - \tau I_n) = n - 1$.

Remark 4.3. *The conditions* (iii) *and* (iv) *in Theorem 4.1 are easy to verify. Note that any nonzero* $\xi \in bd(\mathcal{K}^n)$ *can be written as* $\xi = \lambda(1, \bar{x}^T)^T$, *where* $\bar{x} \in \mathbb{S}^{n-1}$ *and* $\lambda > 0$. *Then, by partitioning the matrix* $M \in \mathbb{R}^{n \times n}$ *into*

$$M = \begin{bmatrix} a & b^T \\ c & \bar{M} \end{bmatrix},$$

we have that

$$\xi^T M \xi = \lambda^2(\bar{x}^T \bar{M} \bar{x} + (b + c)^T \bar{x} + a).$$

Let $\tilde{x} \in \mathbb{R}^{n-1}$ *be the optimum solution of the following problem:*

$$\begin{array}{ll} \min & f(\bar{x}) = \bar{x}^T \bar{M} \bar{x} + (b + c)^T \bar{x} + a \\ S.t. & \|\bar{x}\| = 1, \end{array} \qquad (4.13)$$

which can be solved easily by many existing state-of-the-art methods, e.g. [5, Sec. 7.3]. Then, if $f(\tilde{x}) \geq 0$, *the condition* (iii) *is satisfied. Analogously, the condition* (iv) *can be easily verified.*

Corollary 4.2. *M has the GUS property if and only if* M^T *has the GUS property.*

In [10, Thm. 17], Gowda *et al.* proved that if M is positive definite, then M has the GUS property. Now, we give a matrix which has the GUS property, while it is not even positive semidefinite.

Example 4.1. *Let*

$$M = \begin{pmatrix} 1/2 & 1/2 & 0 \\ -1/2 & -1/2 & 1 \\ 0 & -1 & -1/2 \end{pmatrix},$$

which is obviously not positive semidefinite.

It is easy to verify M *defined above satisfies all the conditions (*i)-(iv) in Theorem 4.1. Firstly, MJ_3 has the unique nonnegative eigenvalue $\tau = 1/2$. The associated eigenvector of MJ_3 with respect to τ is $(1, 0, -1/2)^T \in int(\mathcal{K}^3)$; and the associated eigenvector of $M^T J_3$ with respect to τ is $(1, 0, -1/2)^T \in int(\mathcal{K}^3)$. Thus, the conditions (i) and (ii) in Theorem 4.1 hold. Second, let

$$bd(\mathcal{K}^3) = \{t(1, x, y)^T : t \geq 0\},$$

where (x, y) satisfies $x^2 + y^2 = 1$. Then, we have

$$(1, x, y)M(1, x, y)^T = (1 - x^2 - y^2)/2 = 0$$

and

$$(1,x,y)M^{-1}(1,x,y)^T = 5/2 + 2y - x^2/2 \geq 0,$$

which imply the conditions (iii) and (iv) immediately. Therefore, by Theorem 4.1, M has the GUS property.

Note that this example also shows that the inequality in (4.5) is tight, i.e., the equality can be achieved.

Next, by some examples, we illustrate that none of the conditions (i)-(iv) in Theorem 4.1 can be removed in order to ensure that M has the GUS property.

Example 4.2. *Let*

$$M = \begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix}.$$

Then, MJ_2 has two eigenvalues: $\tau_1 = 3$ and $\tau_2 = -1$. The associated eigenvector of MJ_2 with respect to τ_1 is $(1, 1/3)^T \in int(\mathcal{K}^2)$, which indicates that the condition (i) *in Theorem 4.1 holds. In addition, the associated eigenvector of M^TJ_2 with respect to τ_1 is $(1,-1)^T \in bd(\mathcal{K}^2)$, which says that the* **condition** (ii) **does not hold**. *Note that $bd(\mathcal{K}^2) = \{t(1,1)^T : t \geq 0\} \cup \{t(1,-1)^T : t \geq 0\}$. Thus, it is easy to see that both the conditions* (iii) *and* (iv) *in Theorem 4.1 hold.*

On the other hand, let $q = (1,-1)^T$. For this case, it is trivial to verify that $SOL(M, \mathcal{K}^2, q)$ contains at least two elements: 0 and $(1,1)^T$.

*Therefore, all the conditions in Theorem 4.1 except for (*ii*) hold, while M does not have the GUS property.*

Example 4.3. *Let*

$$M = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}.$$

Analogous to the last example, for this M, it is easy to see that the **condition** (i) **does not hold**, *while all the other conditions in Theorem 4.1 hold.*

On the other hand, let $q = (-1,-1)^T$, then $SOL(M, \mathcal{K}^2, q) = \emptyset$ for this case. In fact, if $x = (u,v)^T \in \mathbb{R}^2 \in SOL(M^T, \mathcal{K}^2, q)$, then we have that $x \in \mathcal{K}^2$, which implies $u \geq |v|$ and

$$Mx + q = (2u + v - 1, -3u - 1)^T \in \mathcal{K}^2.$$

Thus, we get

$$2u + v - 1 \geq 3u + 1,$$

which is a contradiction. Hence, $SOL(M, \mathcal{K}^2, q) = \emptyset$.

Therefore, all the conditions in Theorem 4.1 except for (i) *hold, while M does not have the GUS property.*

Example 4.4. *Let*

$$M = \begin{pmatrix} 2/5 & 1/2 & 0 \\ -1/2 & -3/5 & 1 \\ 0 & -1 & -3/5 \end{pmatrix}. \tag{4.14}$$

Then, MJ_3 has the unique nonnegative eigenvalue $\tau = 0.3389$. The associated eigenvector of MJ_3 with respect to τ is

$$(0.9002, 0.1100, -0.4214)^T \in int(\mathcal{K}^3),$$

and the associated eigenvector of $M^T J_3$ with respect to τ is

$$(0.9002, -0.1100, -0.4214)^T \in int(\mathcal{K}^3),$$

Thus, the conditions (i) and (ii) in Theorem 4.1 hold. Moreover, let

$$bd(\mathcal{K}^3) = \{t(1, x, y)^T : t \geq 0\},$$

where (x, y) satisfies $x^2 + y^2 = 1$. Then, we have

$$(1, x, y)M(1, x, y)^T = 2/5 - 3(x^2 + y^2)/5 = -1/5$$

and

$$(1, x, y)M^{-1}(1, x, y)^T = (136 - 24x^2 + 100y + y^2)/394.$$

Thus, the **condition (iii) does not hold** *while the condition (*iv) does.

On the other hand, let $q = (2.6, -2.4, 1)^T$. For this case, $SOL(M, \mathcal{K}^3, q)$ contains at least two elements: 0 and $(1, 1, 0)^T$.

Therefore, all the conditions in Theorem 4.1 except for (iii) hold, while M does not have the GUS property.

Example 4.5. *Let $N = M^{-1}$, where M is defined by* (4.14)*. Then, NJ_3 has the unique nonnegative eigenvalue $\tau = 2.9508$. The associated eigenvector of NJ_3 with respect to τ is*

$$(0.9002, -0.1100, 0.4214)^T \in int(\mathcal{K}^3),$$

and the associated eigenvector of $N^T J_3$ with respect to τ is

$$(0.9002, 0.1100, 0.4214)^T \in int(\mathcal{K}^3).$$

Thus, the conditions (i) *and* (ii) *in Theorem 4.1 hold. Moreover, by the definition of N, it is obvious that the condition* (iii) *holds while* **the condition** (iv) **does not**.

On the other hand, let $q = (2.6, -2.4, 1)^T$ and let $q' = -Nq$. Then, $SOL(N, \mathcal{K}^3, q')$ contains at least two elements: q and $M(1, 1, 0)^T + q$.

*Therefore, all the conditions in Theorem 4.1 except for (*iv) hold, while M does not have the GUS property.

5 Conclusion

In this paper, we present a new approach to study the globally uniquely solvable (GUS) property of the linear transformation of the second-order cone linear complementarity problems (SOCLCP). Some sufficient and necessary conditions of the GUS property are thus derived. These sufficient and necessary conditions are characterized by some basic linear algebra properties of the involved matrix of SOCLCP, and they are easy to check. In [8, 9, 10], the authors have insightfully analyzed some important properties relevant to the linear transformation for LCP over generic symmetric cones, e.g. the Lipschitzian GUS property, the P, P_0, Q and R_0 properties. For the future research, it is interesting to apply the new approach proposed in this paper to characterize these important properties for SOCLCP, and thus derive some concrete and easily-checkable sufficient and necessary conditions in terms of basic linear algebra properties of the involved matrix of SOCLCP.

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