# The Chvátal-Gomory Closure of a Strictly Convex Body

Daniel Dadush
H. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology,
765 Ferst Drive NW, Atlanta, GA 30332-0205
email: dndadush@gatech.edu

Santanu S. Dey
H. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology,
765 Ferst Drive NW, Atlanta, GA 30332-0205
email: santanu.dey@isye.gatech.edu

Juan Pablo Vielma
Business Analytics and Mathematical Sciences Department, IBM T. J. Watson Research Center,
P.O. Box 218, Yorktown Heights, NY 10598
and
Department of Industrial Engineering, University of Pittsburgh
1048 Benedum Hall, Pittsburgh, PA 15261
email: jvielma@pitt.edu

In this paper, we prove that the Chv́atal-Gomory closure of a set obtained as an intersection of a strictly convex body and a rational polyhedron is a polyhedron. Thus, we generalize a result of Schrijver[30] which shows that the Chv́atal-Gomory closure of a rational polyhedron is a polyhedron.

*Key words*: nonlinear integer programming; cutting planes; Chv́atal-Gomory closure

*MSC2000 Subject Classification*: Primary: 90C10, 90C30 ; Secondary: 90C57

*OR/MS subject classification*: Primary: Integer - Nonlinear

---

**1. Introduction** A cutting plane, also known as a cut, is typically a linear inequality that separates fractional points from the convex hull of integer feasible solutions of an Integer Programming (IP) problem. Cutting planes have proven to be crucial in the development of successful IP solver technology. See [21, 26, 27, 28] for general expositions on cutting plane methods.

Chv́atal-Gomory (CG) cuts are one of the first classes of cutting planes presented in the literature [14]. They have been at the heart of various fundamental theoretical and computational breakthroughs in IP. For example, Gomory [14] introduced CG cuts to present the first finite cutting plane algorithm for bounded IP problems. CG cuts can be used to obtain the convex hull of integer feasible solutions of some sets such as the Matching Polytope, as shown by Edmonds [12], which is a pioneering result in the area of polyhedral combinatorics.

For a rational polyhedron $P \subseteq \mathbb{R}^n$, the CG cutting plane procedure [9, 14, 15] can be described as follows. For $a \in \mathbb{Z}^n$, let $d \in \mathbb{R}$ be such that $\{x \in \mathbb{R}^n : \langle a, x\rangle \leq d\} \supset P$ where $\langle u, v\rangle$ is the inner product between $u$ and $v$. We then have that $P_I := P \cap \mathbb{Z}^n \subseteq \{x \in \mathbb{R}^n : \langle a, x\rangle \leq \lfloor d\rfloor\}$ and hence the CG cut $\langle a, x\rangle \leq \lfloor d\rfloor$ is a valid inequality for $\text{conv}(P_I)$. The first CG closure of $P$ is defined as the convex set obtained by adding all CG cuts. Because the number of CG cuts is infinite we have that the CG closure is not automatically a polyhedron. The first proof of the polyhedrality of the CG closure was introduced by Schrijver in 1979.

Theorem 1.1 ([30]) *The CG closure of a rational polyhedron is a rational polyhedron.*

Convex Nonlinear Integer Programming, i.e. problems where the continuous relaxation of the feasible set is non-polyhedral convex set, has received considerable attention from the IP community recently. There has been significant progress in the development of practical algorithms that can be effective for many important applications (e.g. [1, 5, 6, 19, 24]). Building on work for linear IP, practical algorithms for convex nonlinear IP have benefited from the development of several classes of cutting planes or valid inequalities (e.g. [2, 3, 4, 7, 8, 13, 17, 18, 23, 16, 29, 32]). Many of these inequalities are based on the generalization of ideas used in linear IP.

One particular idea for generating cutting planes for convex nonlinear IP that has been motivated by the linear case is that of CG cuts. CG cuts for general convex IP were discussed implicitly in [30] and described explicitly in [8] for convex IP problems where the continuous relaxation of the feasible region is conic representable. CG cuts can be extended to the case of a general convex set $C \subseteq \mathbb{R}^n$ using its support function $\sigma_C(a) := \max_{x \in C} \langle a, x \rangle$. A valid cut for $\operatorname{conv}(C \cap \mathbb{Z}^n)$ is $\langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor$ where $a$ is an integral vector. Similar to the case of rational polyhedra, CG closure for $C$ is the convex set obtained by adding all CG cuts.

To the best of our knowledge, the only version of Theorem 1.1 for the case where $C$ is not a rational polyhedron was shown in [11]. It was shown in [11] that when $C$ is a full-dimensional bounded ellipsoid described by rational data the CG closure is a polytope. A set $C$ is called strictly convex when the strict convex combination of any two points belonging to $C$ lies in the relative interior of $C$. In this paper, we will verify Theorem 1.1 for the case where $C$ is either a strictly convex body (full dimensional compact strictly convex set) or an intersection of a strictly convex body and a rational polyhedron. The first result generalizes the result in [11] and the second result effectively generalizes Theorem 1.1.

We observe here that while various proofs of Theorem 1.1 have been presented, unfortunately none of them seem to extend to the case of non-polyhedral convex sets. For example, it not clear how to extend the proofs in [10, 30, 31] beyond rational polyhedra because they rely on properties that are characteristic of these sets such as totally dual integral systems and finite integral generating sets. Cut domination arguments, commonly used in polyhedrality proofs of closures, also do not seem to adapt well to the proof for non-polyhedral convex sets. Note that one key property of CG closure for rational polyhedron is that the CG closure of a facet of a rational polyhedron is equivalent to the CG closure of the rational polyhedron intersected with the facet. This property together with an induction on the dimension of the polyhedron can be used to prove polyhedrality of the CG closure of a rational polyhedron. While we will prove a similar statement regarding the zero dimensional facets of strictly convex sets, we will need to develop completely new techniques to do so. Moreover in the case of strictly convex sets there are an infinite number of facets, thus requiring a different approach than used in the case of rational polyhedron. Finally we note that our proof of the polyhedrality of a set obtained as the intersection of a bounded strictly convex set and a rational polyhedron will however use in parts some of the ideas used for the case of rational polyhedra.

Instead of attempting to use the proof techniques for rational polyhedra, another possibility for proving the polyhedrality of CG closures of bounded non-polyhedral convex set $C$ is to directly use the polyhedrality of the first CG closure of rational polyhedral approximations of $C$. One natural scheme could be to attempt constructing a sequence of rational polytope pairs $\{P_i, Q_i\}_{i \in \mathbb{N}}$ such that (i) $P_i \cap \mathbb{Z}^n = Q_i \cap \mathbb{Z}^n = C \cap \mathbb{Z}^n$, (ii) $P_i \subseteq C \subseteq Q_i$ and (iii) $\operatorname{vol}(Q_i \setminus P_i) \leq 1/i$. We then would have that that CG closure of $P_i$ is a subset of CG closure of $C$ which in turn is a CG closure of $Q_i$ for any $i$. Unfortunately, it is not clear how to show that there exists $i$ such that CG closure of $P_i$ is equal to the CG closure of $C$ which in turn equals the CG closure of $Q_i$.

We note that strictly convex sets are completely 'rounded' without any flat faces of dimension greater than 0, which is completely in contrast with polyhedra. Thus, in a sense the polyhedrality of CG closure of strictly convex sets represents a result which is on the other end of the spectrum with respect to the polyhedrality result for CG closure of rational polytopes. While the result on intersection of strictly convex sets and rational polyhedra does fill the void slightly, we believe that a whole set of new methodologies and insights need to be developed to understand the structure of CG closures of general convex sets.

The rest of the paper is organized as follows. In Section 2 we give some background material, formally state our main results and give an overview of their proofs. Then in Section 3 we study the separation of points in the boundary of strictly convex sets using CG cuts. The separation results in this section are key to the proofs of our main results. Section 4 contains the proof of the polyhedrality of the CG closure for strictly convex bodies and Section 5 does the same for the intersection of a strictly convex body and a rational polyhedron.

We follow most of the notational conventions of [20], which we will use as a reference for all convex analysis results. However, for completeness, the appendix includes proofs of any result that does not explicitly appear in [20].

**2. Background and Proof Outline** To formally define the CG closure of a closed convex set it is useful to use the following characterization.

PROPOSITION 2.1 *Let $C$ be a compact convex set and $\sigma_C(\cdot)$ be its support function. Then*

$$C = \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n \,:\, \langle a, x\rangle \leq \sigma_C(a)\} \tag{1}$$

This is the standard outer description of a closed convex set (e.g. Theorem C.2.2.2 in [20]) with the exception that we take an intersection over $a \in \mathbb{Z}^n$ instead to the usual $a \in \mathbb{R}^n$ or $a \in S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ where $\|\cdot\|$ is the Euclidean norm. Validity of this alternative representation is straightforward, but for completeness Proposition 2.1 is proven in the Appendix.

DEFINITION 2.1 *Let $C$ be a compact convex set. For any set $S \subseteq \mathbb{Z}^n$ let*

$$\mathrm{CGC}_S(C) = \bigcap_{a \in S} \{x \in \mathbb{R}^n \,:\, \langle a, x\rangle \leq \lfloor \sigma_C(a)\rfloor\} \tag{2}$$

*Let $\mathrm{CGC}(C) = \mathrm{CGC}_{\mathbb{Z}^n}(C)$, we recursively define the k-th CG closure $C^k$ of $C$ as $C^0 := C$ and $C^k := \mathrm{CGC}(C^{k-1})$ for $k \geq 1$.*

$\mathrm{CGC}_S(C)$ is a closed convex set containing $C_I := C \cap \mathbb{Z}^n$ for any $S \subseteq \mathbb{Z}^n$ and by Proposition 2.1 we also have $C^1 \subseteq C$. Then $\mathrm{conv}(C_I) \subseteq C^l \subseteq C^k \subseteq C^0 = C$ for all $l > k > 0$. The last two containments are strict unless $C = \mathrm{conv}(C_I)$ or $C^k = \mathrm{conv}(C_I)$ and, as noted in [30], the following theorem follows from [9, 30].

THEOREM 2.1 ([9, 30]) *For every convex body $C$ there exist $r \in \mathbb{N}$ such that $C^r = \mathrm{conv}(C_I)$.*

Theorem 2.1 is also shown in [8] for Conic Quadratic Programming problems with bounded feasible regions. However, the result neither implies nor requires the polyhedrality of $C^1$. In fact, the original proof of Theorem 2.1 in [9] does not use the polyhedrality of either $C$ or $C^1$. Although surprising, it could be entirely possible for Theorem 2.1 to hold and for $C^r$ to be the only polyhedron in the hierarchy $\{C^k\}_{l=1}^r$. Before presenting our results we formally define strictly convex sets.

DEFINITION 2.2 *We say a set $C$ is strictly convex if for all $u, v \in C$, $u \neq v$ we have that $\lambda u + (1-\lambda)v \in \mathrm{rel.int}(C)$ for all $0 < \lambda < 1$. We say $C$ is a strictly convex body if $C$ is a full dimensional, strictly convex and a compact set.*

Our first main result is the following.

THEOREM 2.2 *Let $C$ be a strictly convex body. Then $\mathrm{CGC}(C)$ is a rational polytope.*

Together with Theorem 1.1 this characterizes the polyhedrality of the CG closure of convex sets at two extremes of the curvature spectrum: strictly convex bodies sets and rational polyhedra. Unfortunately, it is not clear how to use these results or adapt their proofs to generalize the result to every convex body. However, our second main result is to prove that the CG closure of the intersection of a strictly convex body with a rational polyhedron is a rational polyope.

THEOREM 2.3 *Let $C$ be a strictly convex body and $P$ a rational polyhedron. Then $\mathrm{CGC}(C \cap P)$ is a rational polytope.*

Theorem 2.3 allows us to relax the full dimensional requirement of Theorem 2.2, but only if $\mathrm{aff}(C)$ is a rational affine subspace.

Figure 1: A procedure to generate the first CG closure for a strictly convex body $C \subseteq \mathbb{R}^n$

Step 1 Construct a finite set $S \subset \mathbb{Z}^n$ such that:

(C1) $\mathrm{CGC}_S(C) \subseteq C$.

(C2) $\mathrm{CGC}_S(C) \cap \mathrm{bd}(C) \subseteq \mathbb{Z}^n$.

Step 2 Update $S$ with a vector $a \in \mathbb{Z}^n$ such that CG cut $\langle a, x\rangle \leq \lfloor \sigma_C(a) \rfloor$ separates a point of $\mathrm{CGC}_S(C) \setminus \mathrm{CGC}(C)$ until no such $a$ exists.

**2.1 Proof Outline of Theorem 2.2** The general proof strategy for Theorem 2.2 is the same one used in [11] to show the result for rational ellipsoids. The main difference is the generalization of some separation results from rational ellipsoids to arbitrary strictly convex bodies. One of these separation results essentially states that any non-integral point in the surface of a strictly convex body can be separated by a CG cut. Using this property we can show that the CG closure of a strictly convex body can be generated using the procedure described in Figure 1.

To show that Step 1 can be accomplished, we use the separation result to cover the boundary $\mathrm{bd}(C)$ of $C$ with a possibly infinite number of open sets that are associated to the CG cuts. Then, if there are no integral points in the boundary of $C$, we use compactness of the boundary of $C$ to obtain a finite sub-cover that yields a finite number of CG cuts that separate every point on the boundary of $C$. If there are integer points on the boundary, then for every $z \in \mathrm{bd}(C) \cap \mathbb{Z}^n$ we use CG cuts to build a polyhedral cone with vertex $z$ which cuts off all the boundary points in some neighborhood around $z$. In this way, we are able to find a finite set of CG cuts to separate all the non-integral points on the boundary. We do this formally in Proposition 4.1.

To show that Step 2 terminates finitely, we simply show that the set of CG cuts that separate at least one point in $\mathrm{CGC}_S(C) \setminus \mathrm{CGC}(C)$ is finite. We do this formally in Proposition 4.2.

We note that the separation of non-integral points using CG cuts on the boundary of $C$, required in Step 1 of Figure 1, is not straightforward. A natural first approach to separate a non-integral point $u$ on the boundary of $C$ is to use $a \in (N_C(u) \cap \mathbb{Z}^n) \setminus \{\mathbf{0}\}$ where $N_C(u) := \{a \in \mathbb{R}^n : \langle a, x-u\rangle \leq 0 \, \forall x \in C\}$ is the normal cone to $C$ at $u$. Then $\langle a, u\rangle = \sigma_C(a)$ and if $\sigma_C(a) \notin \mathbb{Z}$ then CG cut $\langle a, x\rangle \leq \lfloor \sigma_C(a) \rfloor$ separates $u$. Of course, this can fail either because $N_C(u) \subset (\mathbb{R}^n \setminus \mathbb{Z}^n) \cup \{\mathbf{0}\}$ or because $\sigma_C(a) \in \mathbb{Z}$ for every $a \in N_C(u) \cap \mathbb{Z}^n$. This is illustrated by the following two examples.

EXAMPLE 2.1 *Let* $C := \left\{x \in \mathbb{R}^2 : \|x\| \leq 1\right\}$ *and* $u = (1/2, \sqrt{3}/2)^T \in \mathrm{bd}(C)$. *Then* $N_C(u) = \mathrm{cone}(\{u\}) \subset (\mathbb{R}^n \setminus \mathbb{Z}^n) \cup \{\mathbf{0}\}$.

EXAMPLE 2.2 *Let* $C := \left\{x \in \mathbb{R}^2 : \|x\| \leq 5\right\}$ *and* $u = (25/13, 60/13)^T \in \mathrm{bd}(C)$. *Then* $N_C(u) = \mathrm{cone}(\{u\}) = \mathrm{cone}\left(\left\{(5,12)^T\right\}\right)$ *and* $\sigma_C(a) \in \mathbb{Z}$ *for every* $a \in N_C(u) \cap \mathbb{Z}^n$.

Fortunately, for both examples we can select alternative left hand sides $a'$ for which the associated CG cut will separate $u$. For instance, for Example 2.1 we can use $a' = (1,1)$ for which $\sigma_C(a') = \sqrt{2}$. In Section 3 we will show there exists a systematic method to obtain this alternative left hand side.

**2.2 Proof Outline of Theorem 2.3** To prove Theorem 2.3 we again use the procedure in Figure 1 with $C$ replaced by $C \cap P$. However, this time we cannot achieve $\mathrm{CGC}_S(C \cap P) \cap \mathrm{bd}(C \cap P) \subseteq \mathbb{Z}^n$ in Step 1 because not all non-integral points in the boundary of a rational polyhedron can be separated by a CG cut. For instance, fractional points in the relative interior of a facet with integral extreme points cannot be separated. For this reason we replace condition (C2) by $\mathrm{CGC}_S(C \cap P) \cap \mathrm{bd}(C \cap P) \subseteq \mathrm{CGC}(C \cap P)$. To achieve this new condition we show that every point in $\mathrm{bd}(C \cap P) \setminus \mathrm{CGC}(C \cap P)$ can be separated by a finite number of CG cuts. For this, we divide $\mathrm{bd}(C \cap P)$ into points in $\mathrm{bd}(C) \cap P$ and points in $C \cap F$ where $F$ is a facet of $P$. The separation argument for the first case is the same as that for Theorem 2.2 and for the second case we apply induction on the dimension of $C \cap P$ by noting that $C \cap F$ is also the intersection of a strictly convex body and a rational polyhedron. The arguments for Step 2 are identical to those of Theorem 2.2.

**3. Separation** As mentioned in Section 2.1, a key step in the proof of Theorem 2.2 is to show that if $C \subseteq \mathbb{R}^n$ is a strictly convex body then every $u \in \operatorname{bd}(C) \setminus \mathbb{Z}^n$ can be separated by a CG cut. An initial strategy to achieve this is to take $s \in (N_C(u) \cap \mathbb{Z}^n) \setminus \{\mathbf{0}\}$ such that $\sigma_C(s) \notin \mathbb{Z}$ to obtain the CG cut $\langle s, x\rangle \le \lfloor \sigma_C(s) \rfloor$ which separates $u$. However, as illustrated in Examples 2.1 and 2.2 this can fail either because because

(i) $N_C(u) \subset (\mathbb{R}^n \setminus \mathbb{Z}^n) \cup \{\mathbf{0}\}$,
(ii) or $\sigma_C(s) \in \mathbb{Z}$ for every $s \in N_C(u) \cap \mathbb{Z}^n$.

A natural solution for case (i) is to approximate some $s \in N_C(u)$ by a sequence $\{s^i\}_{i\in\mathbb{N}} \subseteq \mathbb{Z}^n$ such that $\bar{s}^i \xrightarrow{i\to\infty} \bar{s}$, where $\bar{a} = a/\|a\|$, and hope that $\langle s^i, u\rangle > \lfloor \sigma_C(s^i)\rfloor$ for some $i$. This solution will in fact work for both cases but we will need sequence $\{s^i\}_{i\in\mathbb{N}}$ to additionally comply with the following two properties

(P1) $\lim_{i\to+\infty} \langle s^i, u\rangle - \sigma_C(s^i) = 0$
(P2) $\lim_{i\to+\infty} F(\sigma_C(s^i)) = \delta > 0$. (A weaker condition like $\limsup_{i\to+\infty} F(\sigma_C(s^i)) > 0$ is sufficient, but we will verify the stronger condition),

where $F(r) = r - \lfloor r \rfloor$.

A sequence with these properties will directly yield a separating CG cut because $\langle s^i, u\rangle - \lfloor\sigma_C(s^i)\rfloor = \langle s^i, u\rangle - \sigma_C(s^i) + F(\sigma_C(s^i))$. However, the existence of such a sequence requires a proof, as the conditions do not hold for every sequence such that $\bar{s}^i \xrightarrow{i\to\infty} \bar{s}$. For instance, let $D(H_{s^i,\sigma_C(s^i)}, u) := \frac{|\langle s^i,u\rangle - \sigma(s^i)|}{\|s^i\|}$ be the distance between hyperplane $H_{s^i,\sigma_C(s^i)} := \{x \in \mathbb{R}^n : \langle s^i, x\rangle = \sigma(s^i)\}$ and $u$. Then $\bar{s}^i \xrightarrow{i\to\infty} \bar{s}$ implies $\lim_{i\to+\infty} D(H_{s^i,\sigma_C(s^i)}, u) = 0$. However, this last condition is weaker than (P1) when $\|s^i\| \to +\infty$, which is necessary for complying with $s^i \in \mathbb{Z}^n$ when $\lambda s \notin \mathbb{Z}^n$ for every $\lambda > 0$. In fact $\|s^i\| \to +\infty$ will be useful condition to have even when $s \in \mathbb{Z}^n$.

The next example illustrates the need for conditions (P1)–(P2) and the fact that they are not automatically satisfied by every sequence such that $\bar{s}^i \xrightarrow{i\to\infty} \bar{s}$.

Example 3.1 (Continuation of Example 2.2) *Let* $C := \{x \in \mathbb{R}^2 : \|x\| \le 5\}$ *and* $u = (25/13, 60/13)^T \in \operatorname{bd}(C)$. *Then* $N_C(u) = \operatorname{cone}(\{u\}) = \operatorname{cone}\left(\{(5,12)^T\}\right)$ *and* $\sigma_C(a) \in \mathbb{Z}$ *for every* $a \in N_C(u) \cap \mathbb{Z}^n$.

*We can select* $s = (5,12)^T$ *and approximate* $s$ *with sequence* $\{s^i\}_{i\in\mathbb{N}}$ *given by* $s^i = (65i^2, 26i + 156i^2)^T$[1]. *This sequence complies with* $\bar{s}^i \to \bar{s}$ *and* $D(H_{s^i,\sigma_C(s^i)}, u) \to 0$. *However,* $\langle s^i, u\rangle - \sigma_C(s^i) < -1$ *for all* $i$ *so* $\langle s^i, x\rangle \le \lfloor\sigma(s^i)\rfloor$ *will never separate* $u$. *This illustrates the need for condition (P1) and the fact that (P1) is not necessarily satisfied by any sequences such that* $\bar{s}^i \xrightarrow{i\to\infty} \bar{s}$.

*A sufficient condition for (P1) for Euclidean balls, such as* $C$ *in this example, is* $\left\|\bar{s}^i - \bar{s}\right\|^2 \in o(1/\|s^i\|)$. *We hence need a sequence for which* $\bar{s}^i$ *converges to* $\bar{s}$ *faster than the growth of* $\|s^i\|$. *One such sequence is given by* $s^i = (65i, 26 + 156i)^T$. *Unfortunately, although it complies with (P1), we have that for this new sequence* $\langle s^i, u\rangle - \sigma_C(s^i) \le -F(\sigma(s^i))$ *and hence* $\langle s^i, u\rangle \le \lfloor\sigma(s^i)\rfloor$ *for all* $i$. *However, we can comply with condition (P2) without loosing (P1) by slightly perturbing the last sequence to obtain* $s^i = (65i, 25 + 156i)^T$. *For this last sequence we finally have* $\langle s^3, u\rangle > \lfloor\sigma(s^3)\rfloor$.

A general way to obtain a sequence $\{s^i\}_{i\in\mathbb{N}}$ complying with (P1)–(P2) is to pick any $s \in N_C(u)$ and construct a simple perturbation of the modified simultaneous Diophantine approximation of $s$ given by the following theorem.

Theorem 3.1 (Dirichlet) *Let* $s \in \mathbb{R}^n$. *There exists* $\left\{p^i, q_i\right\}_{i\in\mathbb{N}} \subseteq \mathbb{Z}^n \times \mathbb{Z}$ *such that* $1 \le q_i \le i^n$ *for all* $i \in \mathbb{N}$, $\max_{1\le j\le n}\left|p^i_j - q_i s_j\right| \le \frac{1}{i}$ *and* $\lim_{i\to\infty} q_i = +\infty$.

Note that Theorem 3.1 is usually written without condition $\lim_{i\to\infty} q_i = +\infty$. However, this additional condition is always satisfied for $s \in \mathbb{R}^n \setminus \mathbb{Q}^n$ and can be easily enforced for $s \in \mathbb{Q}^n$ (If $s = \frac{1}{q}p$ where

[1] Note that $s^i = (13 \times 5i^2, 26i + 13 \times 15i^2)^T$. We could have alternatively used $s^i = (i^2, 26i + 15i^2)^T$, but the first option will yield something closer to the construction in our general propositions.

$(p,q)\in\mathbb{Z}^n\times\mathbb{Z}$, then for sufficiently large $i$ set $p^i = ip$ and $q_i = iq$). To show compliance with (P1)–(P2) we will additionally need the following known property of strictly convex bodies, which we prove in the appendix for completeness.

LEMMA 3.1 *Let $C\subseteq\mathbb{R}^n$ be a strictly convex body. Let $m_K : S^{n-1}\to \mathrm{bd}(C)$ be such that*

$$m_K(v) = \arg\max_{x\in K}\langle v, x\rangle$$

*Then $m_K$ is a well-defined and continuous function on $S^{n-1}$.*

LEMMA 3.2 *Let $C\subseteq\mathbb{R}^n$ be a strictly convex body, $u\in\mathrm{bd}(C)$ and $s\in N_C(u)\setminus\{0\}$. Let $s^i = p^i + w$, where $\{p^i, q_i\}_{i\in\mathbb{N}}$ is the simultaneous Diophantine approximation of $s$ given by Theorem 3.1 and $w\in\mathbb{R}^n\setminus\{0\}$. Then for any $\varepsilon>0$, there exists $N_\varepsilon\geq 0$ such that for all $i\geq N_\varepsilon$ we have*

*(i)* $\left\|p^i - q_i s\right\|\leq\varepsilon$.

*(ii)* $\left\|\bar{s}^i-\bar{s}\right\|\leq(1+\varepsilon)\frac{\|w\|}{\|s^i\|}$.

*(iii)* $\sigma_C(s^i)\geq\left\langle s^i, u\right\rangle\geq\sigma_C(s^i)-\varepsilon$.

*(iv)* $\left|\sigma_C(s^i)-\langle q_i s + w, u\rangle\right|\leq\varepsilon$ *and if* $\langle q_i s+w,u\rangle\notin\mathbb{Z}$ *then* $\left|F\left(\sigma_C(s^i)\right)-F\left(\langle q_i s+w,u\rangle\right)\right|\leq\varepsilon$.

PROOF.

(i) Follows directly from $\left\|p^i-q_is\right\|\leq\sqrt{n}\max_{1\leq j\leq n}\left|p^i_j-q_is_j\right|$ and Theorem 3.1.

(ii) To prove this point we will use the fact that for $a,b\in\mathbb{R}^n\setminus\{0\}$ the geometric-arithmetic mean inequality inequality is equivalent to $\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\|\leq\frac{\|a-b\|}{\sqrt{\|a\|\|b\|}}$. In effect we have that

$$\begin{aligned}\|a\|\|b\|\leq\frac{1}{2}\langle a,a\rangle+\frac{1}{2}\langle b,b\rangle &\Leftrightarrow \|a\|\|b\|-\langle a,b\rangle\leq\frac{1}{2}(\langle a,a\rangle-2\langle a,b\rangle+\langle b,b\rangle)\\ &\Leftrightarrow \|a\|\|b\|-\langle a,b\rangle\leq\frac{1}{2}\|a-b\|^2\\ &\Leftrightarrow 2\left(1-\frac{\langle a,b\rangle}{\|a\|\|b\|}\right)\leq\frac{\|a-b\|^2}{\|a\|\|b\|}\\ &\Leftrightarrow\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\|\leq\frac{\|a-b\|}{\sqrt{\|a\|\|b\|}}.\end{aligned}\tag{3}$$

Now, since $\|p^i\|\geq\|q_is\|-\left\|p^i-q_is\right\|$, $\|s\|>0$ and $\left\|p^i-q_is\right\|\xrightarrow{i\to\infty}0$ we get that $\|p^i\|\xrightarrow{i\to\infty}\infty$. Similarly, since $\left\|s^i\right\|=\left\|p^i+w\right\|\geq\left\|p^i\right\|-\|w\|$ we get that $\left\|s^i\right\|\xrightarrow{i\to\infty}\infty$. Then for any $\delta>0$ there exists $N$ such that for all $i\geq N$, $\delta\left\|s^i\right\|\geq\|w\|+\left\|p^i-q_is\right\|$ and $\delta\|w\|\geq\left\|p^i-q_is\right\|$. Hence for $i\geq N$, we deduce that

$$\|q_is\|\geq\left\|s^i\right\|-\left\|s^i-q_is\right\|\geq\left\|s^i\right\|-\left\|p^i-q_is\right\|-\|w\|\geq(1-\delta)\left\|s^i\right\|$$

By additionally using (3) with $a=s^i$ and $b=q_is$ we finally obtain that for $i\geq N$ we have

$$\left\|\bar{s}^i-\bar{s}\right\|=\left\|\frac{s^i}{\|s^i\|}-\frac{q_is}{\|q_is\|}\right\|\leq\frac{\left\|s^i-q_is\right\|}{\sqrt{\|s^i\|\,\|q_is\|}}\leq\frac{\|w\|+\left\|p^i-q_is\right\|}{\sqrt{(1-\delta)}\,\|s^i\|}\leq\left(\frac{1+\delta}{1-\delta}\right)\frac{\|w\|}{\|s^i\|}$$

and the result follows by taking $\delta=\frac{\varepsilon}{2+\varepsilon}$.

(iii) By part (ii) and continuity of $m_C$ we have that for any $\varepsilon>0$ there exist $N$ such that for all $i\geq N$ we have $\left\|\bar{s}^i-\bar{s}\right\|\leq 2\frac{\|w\|}{\|s^i\|}$ and $\left\|m_C(\bar{s}^i)-m_C(\bar{s})\right\|\leq\frac{\varepsilon}{2\|w\|}$. Also, by the definition of $m_C$, we have that $\sigma_C\left(s^i\right)=\left\langle s^i,m_C\left(\bar{s}^i\right)\right\rangle\geq\left\langle s^i,m_C(\bar{s})\right\rangle=\left\langle s^i,u\right\rangle$ and $\sigma_C(\bar{s})=\langle\bar{s},m_C(\bar{s})\rangle\geq\left\langle\bar{s},m_C\left(\bar{s}^i\right)\right\rangle$.

Now we see that

$$\begin{aligned}\sigma_C\left(s^i\right) - \left\langle s^i, u\right\rangle &= \left\langle s^i, m_C\left(\bar{s}^i\right)\right\rangle - \left\langle s^i, m_C(\bar{s})\right\rangle \\ &= \left\|s^i\right\| \left(\left\langle \bar{s}^i, m_C\left(\bar{s}^i\right)\right\rangle - \left\langle \bar{s}^i, m_C(\bar{s})\right\rangle\right) \\ &\leq \left\|s^i\right\| \left(\left\langle \bar{s}^i, m_C\left(\bar{s}^i\right)\right\rangle - \left\langle \bar{s}^i, m_C(\bar{s})\right\rangle + \left\langle \bar{s}, m_C(\bar{s})\right\rangle - \left\langle \bar{s}, m_C\left(\bar{s}^i\right)\right\rangle\right) \\ &= \left\|s^i\right\| \left\langle \bar{s}^i - \bar{s}, m_C\left(\bar{s}^i\right) - m_C(\bar{s})\right\rangle \\ &\leq \left\|s^i\right\| \left\|\bar{s}^i - \bar{s}\right\| \left\|m_C(\bar{s}^i) - m_C(\bar{s})\right\| \\ &\leq \left\|s^i\right\| \left(2\frac{\|w\|}{\|s^i\|}\right)\left(\frac{\varepsilon}{2\|w\|}\right) = \varepsilon\end{aligned}$$

as needed.

(iv) For the first part we simply note that by parts (i) and (iii) we have that for every $\varepsilon > 0$ there exists $N$ such that for all $i \geq N$ we have

$$\begin{aligned}\sigma_C\left(s^i\right) \geq \left\langle s^i, u\right\rangle = \left\langle p^i, u\right\rangle + \langle w, u\rangle &= \left\langle p^i - q_i s, u\right\rangle + \langle q_i s, u\rangle + \langle w, u\rangle \\ \geq -\left\|p^i - q_i s\right\| \|u\| + \langle q_i s, u\rangle + \langle w, u\rangle &\geq \langle q_i s + w, u\rangle - \varepsilon\end{aligned}$$

and

$$\begin{aligned}\sigma_C\left(s^i\right) \leq \left\langle s^i, u\right\rangle + \varepsilon = \left\langle p^i, u\right\rangle + \langle w, u\rangle + \varepsilon &= \left\langle p^i - q_i s, u\right\rangle + \langle q_i s, u\rangle + \langle w, u\rangle + \varepsilon \\ \leq \left\|p^i - q_i s\right\| \|u\| + \langle q_i s, u\rangle + \langle w, u\rangle &\leq \langle q_i s + w, u\rangle + 2\varepsilon\end{aligned}$$

The second part follows from $F(s)$ being an affine function in a neighborhood of $s \notin \mathbb{Z}$.

□

By suitably scaling $s$, the sequence $\{s^i\}_{i\in\mathbb{N}}$ described in Lemma 3.2 will satisfy (P1) and (P2). In particular part (iii) of Lemma 3.2 implies that $\{s^i\}_{i\in\mathbb{N}}$ satisfies (P1). Part(iv) of Lemma 3.2 will be used to verify (P2). Using this construction we can prove the desired separation results for non-integral points in the boundary of $C$ and a slightly stronger result for points that are additionally close to integral points.

Proposition 3.1 *Let $C \subseteq \mathbb{R}^n$ be a strictly convex body. Take $u \in \operatorname{bd}(C)$.*

*S1 If $u \notin \mathbb{Z}^n$, there exists $a \in \mathbb{Z}^n$ such that $\langle a, u\rangle > \lfloor\sigma_C(a)\rfloor$*

*S2 If $u \in \mathbb{Z}^n$, then for every $v \in \mathbb{R}^n \setminus \operatorname{int}(T_C(u))$, $v \neq 0$, there exists $a \in \mathbb{Z}^n$ such that*

$$\langle a, u\rangle = \lfloor\sigma_C(a)\rfloor \quad \text{and} \quad \langle a, v\rangle > 0$$

Proof.

(i) Let $s \in N_C(u) \setminus \{0\}$. By possibly scaling $s$ by a positive scalar, we may assume that $\langle s, u\rangle \in \mathbb{Z}$. Since $u \notin \mathbb{Z}^n$, there exists $l$, $1 \leq l \leq n$, such that $u_l \notin \mathbb{Z}$. Let $s^i = p^i + e^l$, for $i \geq 0$, where $\{p^i, q_i\}_{i\in\mathbb{N}}$ is the simultaneous Diophantine approximation of $s$ given by Theorem and $e^l$ is the $l^{th}$ unit vector. Because $q_i \langle s, u\rangle \in \mathbb{Z}$ and $\left\langle e^l, u\right\rangle = u_l \notin \mathbb{Z}$ we have that $\langle q_i s + w, u\rangle \notin \mathbb{Z}$ and $\delta := F\left(\langle q_i s + w, u\rangle\right) > 0$. Then using Lemma 3.2 for $w = e^l$ and $\varepsilon < \delta/3$ we have that there exists $i$ such that $\left\langle s^i, u\right\rangle - \sigma_C(s^i) > -\delta/3$ and $F\left(\sigma_C(s^i)\right) > (2/3)\delta$. The result follows by setting $a = s^i$ and noting that $\left\langle s^i, u\right\rangle - \left\lfloor\sigma_C(s^i)\right\rfloor = \left\langle s^i, u\right\rangle - \sigma_C(s^i) + F\left(\sigma_C(s^i)\right) > (1/3)\delta > 0$.

(ii) Since $u \in \operatorname{bd}(C)$ and $v \in \mathbb{R}^n \setminus \operatorname{int}(T_C(u))$, $v \neq 0$, there exists $s \in N_C(u) \setminus \{0\}$ such that $\langle s, v\rangle \geq 0$. Again by possibly scaling $s$, we may assume that $\langle s, u\rangle \in \mathbb{Z}$. Let $s^i = p^i + w$, for $i \geq 0$, where $\{p^i, q_i\}_{i\in\mathbb{N}}$ is the simultaneous Diophantine approximation of $s$ given by Theorem and $w \in \mathbb{Z}^n$ is any integer vector such that $\langle w, v\rangle \geq \frac{2}{3}$. Then using Lemma 3.2 for $\varepsilon < \min\left\{\frac{1}{3\|v\|}, \frac{1}{2}\right\}$ we have that there exists $i$ such that

$$\sigma_C(s^i) \geq \left\langle s^i, u\right\rangle \geq \sigma_C(s^i) - \frac{1}{2} \tag{4}$$

and

$$\left\|p^i - q_i s\right\| \leq \frac{1}{3\|v\|}. \tag{5}$$

Because $\langle s^i, u\rangle \in \mathbb{Z}$ we have that (4) implies $\lfloor \sigma_C(s^i)\rfloor = \langle s^i, u\rangle$. Furthermore, together with $\langle s^i, v\rangle = \langle p^i - q_i s, v\rangle + q_i \langle s, v\rangle + \langle w, v\rangle$, we have that (5), $\langle s, v\rangle \geq 0$ and $\langle w, v\rangle \geq \frac{2}{3}$ imply that $\langle s^i, v\rangle \geq -\frac{1}{3} + 0 + \frac{2}{3} > 0$. The result then follows by setting $a = s^i$.

□

REMARK 3.1 *In the proof of part (iii) of Lemma 3.2 we obtained as a partial result that*

$$\sigma_C\left(s^i\right) - \langle s^i, u\rangle \leq \left\|s^i\right\| \left\|\bar{s}^i - \bar{s}\right\| \left\|m_C(\bar{s}^i) - m_C(\bar{s})\right\|$$

*If we could prove that* $\left\|s^i\right\| \left\|\bar{s}^i - \bar{s}\right\| \xrightarrow{i\to\infty} 0$ *instead of the weaker result in part (ii) of Lemma 3.2, we would obtain condition S1 for any convex body. Of course this cannot hold as it would imply that CG cuts can separate fractional points in the relative interior of the facets of an integral polytope, which clearly cannot happen. However, we should note that this impossibility is only due to the use of nonzero perturbation* $w$ *as we do have that* $\left\|p^i\right\| \left\|\bar{p}^i - \bar{s}\right\| \xrightarrow{i\to\infty} 0$ *and hence Diophantine approximation does give us condition (P1) with* $s^i = p^i$ *for any convex set. In contrast, to obtain condition (P1) and condition (P2) we need continuity of* $m_C$*.*

We end this section by proving the following corollary of Proposition 3.1 that we will need for the proof of Proposition 4.1.

COROLLARY 3.1 *Let* $C \subseteq \mathbb{R}^n$ *be a convex body for which condition S2 of Proposition 3.1 holds for every* $u \in \operatorname{bd}(C) \cap \mathbb{Z}^n$*. Then*

*(i) There exists a polyhedral cone* $T'(u) = \{x \in \mathbb{R}^n : \langle c_i, x\rangle \leq 0, 1 \leq i \leq k\}$ *such that*

$$c_i \in \mathbb{Z}^n \quad \text{and} \quad \lfloor \sigma_C(c_i)\rfloor = \langle c_i, u\rangle \quad \forall\, i, 1 \leq i \leq k \tag{6}$$

*and*

$$T'(u) \subseteq \operatorname{int}(T_C(u)) \cup \{0\} \tag{7}$$

*(ii) There exists an open neighborhood* $\mathcal{N}$ *of* $u$ *such that*

$$\mathcal{N} \cap \operatorname{bd}(C) \cap (u + T'(u)) = \{u\}$$

PROOF.

(i) Examine $K = S^{n-1} \setminus \operatorname{int}(T_C(u))$. Since $\operatorname{int}(T_C(u))$ is an open subset of $\mathbb{R}^n$ (as $\operatorname{int}(C) \neq \emptyset$) we have that $K$ is a closed subset of the sphere, and hence $K$ is compact. For each $v \in K$, we note that $v \in \mathbb{R}^n \setminus \operatorname{int}(T_C(u))$, $v \neq 0$, therefore by condition S2 of Proposition 3.1, there exists $c^v \in \mathbb{Z}^n$ such that

$$\lfloor \sigma(c^v)\rfloor = \langle c^v, u\rangle \quad \text{and} \quad \langle c^v, v\rangle > 0$$

Let $U_v = \{w \in K : \langle c^v, w\rangle > 0\}$. Clearly $v \in U_v$ and $U_v$ is an open subset of $K$ (relative to the subspace topology). Therefore the collection $\{U_v\}_{v\in K}$ is an open cover of $K$. Hence by compactness of $K$ there exists $\{U_i\}_{i=1}^k \subseteq \{U_v\}_{v\in C}$ a finite open subcover of $K$. Let $\{c^i\}_{i=1}^k \subseteq \mathbb{Z}^n$ denote the vectors corresponding to $\{U_i\}_{i=1}^k$ and let

$$T' = \{x \in \mathbb{R}^n : \langle c^i, x\rangle \leq 0, 1 \leq i \leq k\}$$

By construction $\{c^i\}_{i=1}^k$ satisfy (6), hence we need only verify that $T'$ satisfies (7) for which we prove its contrapositive. Assume that $v \in \mathbb{R}^n \setminus \operatorname{int}(T_K(u))$, $v \neq 0$. Then note that $\bar{v} \in K$, and hence there exists $i \in \{1, \ldots, k\}$, such that $\bar{v} \in U_i$. Therefore

$$\langle c^j, \bar{v}\rangle > 0 \Rightarrow \langle c^i, v\rangle > 0 \Rightarrow v \notin T'(u)$$

as needed.

(ii) By translating $C$ to $C - u$, we may assume that $u = 0$, and so $T'(u) = T'(0)$, $T_C(u) = T_C(0)$ and $N_C(u) = N_C(0)$. For notational convenience, we now denote $T'(0)$ as $T'$, $T_C(0)$ as $T_C$ and $N_C(0)$ as $N_C$. Hence we need to show that there exists an open neighborhood $\mathcal{N}$ of 0 such that

$$\mathcal{N} \cap \mathrm{bd}(C) \cap T' = \{0\}.$$

Because $T'$ is a polyhedral cone there exists vectors $\{v^i\}_{i=1}^r \subseteq T' \setminus \{0\}$ such that

$$T' = \left\{ x \in \mathbb{R}^n \;:\; \exists \mu \in \mathbb{R}^r_+ \text{ s.t. } x = \sum_{i=1}^r \mu_i v^i \right\}.$$

By (7) we also have that $T' \setminus \{0\} \subseteq \mathrm{int}(T_C) = \bigcup_{\lambda > 0} \lambda \, \mathrm{int}(C)$ (For the last equality see for example section A.5.3 of [20]). Then, by scaling them appropriately, we may assume that $\{v^i\}_{i=1}^r \subseteq \mathrm{int}(C)$ to obtain

$$\mathrm{conv}\left(\{0, v^1, \ldots, v^r\}\right) \cap \mathrm{bd}(C) = \{0\} \tag{8}$$

by using the fact that $\mathrm{int}(C) \cup \{0\}$ is a convex set. Now, let $c \in N_C$, $c_0 = \max_{i=1}^r \langle c, v^i \rangle$, $\mathcal{N} = \{x \in \mathbb{R}^n : \langle c, x \rangle > c_0\}$ and for $x \in \mathcal{N} \cap T'$ let $\mu \in \mathbb{R}^r_+$ be such that $x = \sum_{i=1}^r \mu_i v^i$. Then $c_0 < \langle c, x \rangle = \sum_{i=1}^r \mu_i \langle c, v^i \rangle \leq c_0 \sum_{i=1}^r \mu_i$. Because $v^i \in \mathrm{int}(T_C)$ and $N_C$ is the polar of $T_C$ we have that $c_0 < 0$ and hence $\sum_{i=1}^r \mu_i \leq 1$. Then $x \in \mathrm{conv}\left(\{0, v^1, \ldots, v^r\}\right)$ and hence $\mathcal{N} \cap T' \subseteq \mathrm{conv}\left(\{0, v^1, \ldots, v^r\}\right)$. Together with (8) this gives the desired result.

□

**4. CG Closure of a Strictly Convex Body** To prove Theorem 2.2 we first show that Step 1 of Figure 1 can be achieved. We assume that $\mathrm{int}(C) \cap \mathrm{CGC}(C) \neq \emptyset$ as the alternative case is trivial for strictly convex bodies. However instead of requiring strict convexity of convex body $C$ we simply require boundary separation conditions (S1) and (S2). These conditions are satisfied by every strictly convex body by Proposition 3.1, but they can also be satisfied by some convex bodies that are not strictly convex.

PROPOSITION 4.1 *Let $C$ be a convex body such that* $\mathrm{int}(C) \cap \mathrm{CGC}(C) \neq \emptyset$. *If conditions (S1) and (S2) of Proposition 3.1 hold for $C$, then there exists a finite set $S \subseteq \mathbb{Z}^n$ such that conditions (C1) and (C2) in Step 1 of Figure 1 hold.*

PROOF. Since $C$ is a bounded set, let $\mathcal{I} := \mathrm{bd}(C) \cap \mathbb{Z}^n$ be the finite (and possibly empty) set of integer points on the boundary of $C$. For each $u \in \mathcal{I}$, let $\mathcal{N}_u$ be the neighborhood of $u$ and $T'(u)$ be the polyhedral cone from Corollary 3.1 and let $S_u$ be such that $T'(u) = \{x \in \mathbb{R}^n : \langle c, x \rangle \leq 0 \, \forall c \in S_u\}$. Let $\mathcal{D} := \mathrm{bd}(K) \setminus \bigcup_{u \in \mathcal{I}} \mathcal{N}_u$. Observe that $\mathcal{D} \cap \mathbb{Z}^n = \emptyset$ by construction and that $\mathcal{D}$ is compact since it is obtained from a compact set $\mathrm{bd}(K)$ by removing a finite number of open sets. Now, for any $a \in \mathbb{Z}^n$ let $O(a) := \{x \in \mathcal{D} : \langle a, x \rangle > \lfloor \sigma(a) \rfloor\}$ be the set of points of $\mathcal{D}$ that are separated by the CG cut $\langle a, x \rangle \leq \lfloor \sigma(a) \rfloor$. This set is open with respect to $\mathcal{D}$. Furthermore, by Conditions S1 and the construction of $\mathcal{D}$, we have that $\mathcal{D} \subseteq \bigcup_{a \in \mathcal{A}} O(a)$ for a possibly infinite set $\mathcal{A} \subseteq \mathbb{Z}^n$. However, since $\mathcal{D}$ is a compact set we have that there exists a finite subset $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $\mathcal{D} \subseteq \bigcup_{a \in \mathcal{A}_0} O(a)$. Let $S := \mathcal{A}_0 \cup \bigcup_{u \in \mathcal{I}} S_u$, then $\mathrm{CGC}(S) \cap \mathrm{bd}(K) \subseteq \mathbb{Z}^n$ and hence (C2) holds. To verify (C1) we show that $w \notin \mathrm{CGC}_S(C)$ for any $w \in \mathbb{R}^n \setminus C$. Let $v \in \mathrm{int}(C) \cap \mathrm{CGC}(C)$, then there exists $\lambda \in (0, 1)$, such that $u = \lambda w + (1 - \lambda) v \in \mathrm{bd}(C)$. If $u \notin \mathbb{Z}^n$, then there exists a CG cut corresponding to an integer vector in $\mathcal{A}_0$ that separates $u$. However, since this CG cut does not separate $v$ and $u$ is a convex combination of $w$ and $v$, it must separate $w$. Thus $w \notin \mathrm{CGC}(S)$. If $u \in \mathbb{Z}^n$, then by Corollary 3.1, there exists a CG cut corresponding to an integer vector in $S_u$ that separates $w$ and therefore $w \notin \mathrm{CGC}(S)$. □

Conditions (C1) and (C2) are sufficient for achieving Step 2 of Figure 1. However, we we show that Step 2 of Figure 1 can be achieved if slightly weaker conditions are satisfied as well. Furthermore, we also show that these weaker conditions are in fact sufficient for the polyhedrality of the CG closure of a convex body even if strict convexity is not satisfied.

PROPOSITION 4.2 *Let $C$ be a convex body. If there exists a finite set $S \subseteq \mathbb{Z}^n$ such that*

*(C1')* $\mathrm{CGC}_S(C) \subseteq C$.

*(C2')* $\mathrm{CGC}_S(C) \cap \mathrm{bd}(C) \subset \mathrm{CGC}(C)$.

*then* $\mathrm{CGC}(C)$ *is a rational polytope.*

PROOF. Let $\mathrm{ext}(\mathrm{CGC}_S(C))$ be the set of vertices of the polytope $\mathrm{CGC}_S(C)$. Because of (C2') we have that any CG cut that separates a point $u \in \mathrm{CGC}_S(C) \setminus \mathrm{CGC}(C)$ must also separate a point in $\mathrm{ext}(\mathrm{CGC}_S(C)) \setminus \mathrm{bd}(C)$. It is then sufficient to show that the set of CG cuts that separates some point in $\mathrm{ext}(\mathrm{CGC}_S(C)) \setminus \mathrm{bd}(C)$ is finite.

Because $\mathrm{ext}(\mathrm{CGC}_S(C)) \setminus \mathrm{bd}(C) \subseteq C \setminus \mathrm{bd}(C) = \mathrm{int}(C)$ and $|\mathrm{ext}(\mathrm{CGC}_S(C))| < \infty$ we have that there exists $\varepsilon > 0$ such that

$$\varepsilon B^n + v \subseteq C \quad \forall v \in \mathrm{ext}(\mathrm{CGC}_S(C)) \setminus \mathrm{bd}(C) \tag{9}$$

where $B^n := \{x \in \mathbb{R}^n : \|x\| \le 1\}$. Now take $v \in \mathrm{ext}(\mathrm{CGC}_S(C))$ and take $a \in \mathbb{Z}^n$ such that $\|a\| \ge \frac{1}{\varepsilon}$. Now note that

$$\lfloor \sigma_C(a) \rfloor \ge \sigma_C(a) - 1 \ge \sigma_{v+\varepsilon B^n}(a) - 1 = \langle v, a\rangle + \varepsilon\|a\| - 1 \ge \langle v, a\rangle$$

Hence the CG cut associated with $a$ does not cut off $v$. Then $\mathrm{CGC}(C) = \mathrm{CGC}_{S\cup S'}(C)$ for $S' := \mathbb{Z}^n \cap \frac{1}{\varepsilon}B^n$. Since $|S'| < \infty$, the claim follows. □

Note that condition (C1') is identical to (C1) and (C2') is equivalent to (C2) for strictly convex sets. The extra generality of (C2') will be useful when dealing with the CG closure of the intersection of a strictly convex body and a rational polyhedron.

With these two propositions the proof of Theorem 2.2 is as follows.

PROOF OF THEOREM 2.2. We divide the proof into the following cases

(i) $\mathrm{CGC}(C) = \emptyset$.

(ii) $\mathrm{CGC}(C) \neq \emptyset$ and $\mathrm{CGC}(C) \cap \mathrm{int}(C) = \emptyset$.

(iii) $\mathrm{CGC}(C) \cap \mathrm{int}(C) \neq \emptyset$.

For the first case, the result follows directly. For the second case, by Proposition 3.1 and the strict convexity of $C$, we have that $|\mathrm{bd}(C) \cap \mathbb{Z}^n| = 1$ and $\mathrm{CGC}(C) = \mathrm{bd}(C) \cap \mathbb{Z}^n$ so the result again follows directly. For the third case the result follows from Propositions 3.1, 4.1 and 4.2. □

## 5. CG Closure of the Intersection of a Strictly Convex Set and a Rational Polyhedron

Before considering intersections with general rational polyhedra we first concentrate on intersections with rational affine subspaces. To achieve this we will need the following well known theorem (e.g. see page 46 of [31]).

THEOREM 5.1 (INTEGER FARKAS'S LEMMA) *Let $A$ be a rational matrix and $b$ be a rational vector. Then the system $Ax = b$ has integral solutions if and only if $\langle y, b\rangle$ is integer whenever $y$ is a rational vector and $A^T y$ is an integer vector.*

Using this result we can characterize the CG closure of convex sets that are not full dimensional.

PROPOSITION 5.1 *Let $C \subseteq \mathbb{R}^n$ be a closed convex set such that* $\mathrm{aff}(C) = W + w$, *where $w \in \mathbb{Q}^n$ and $W$ is a rational subspace with* $\dim(W) = k$. *Then one of the following conditions is satisfied:*

(*i*) $\mathrm{aff}(C) \cap \mathbb{Z}^n = \emptyset$ *and* $\mathrm{CGC}(\mathrm{aff}(C)) = \emptyset$

(*ii*) *There exist an affine map $L : \mathbb{R}^n \to \mathbb{R}^k$ such that*

$$\mathrm{CGC}(C) = L^{-1}(\mathrm{CGC}(L(C))) \cap \mathrm{aff}(C)$$

PROOF. Suppose $\mathrm{aff}(C) \cap \mathbb{Z}^n = \emptyset$. By Theorem 5.1 we have that there exists $a \in \mathbb{Z}^n$ and $a_0 \in \mathbb{Q} \setminus \mathbb{Z}$ such that $\mathrm{aff}(C) \subset \{x \in \mathbb{R}^n : \langle a, x\rangle = a_0\}$. Then, $\langle a, x\rangle \le \lfloor a_0 \rfloor$ and $\langle a, x\rangle \ge \lceil a_0 \rceil$ are valid CG cuts for $C$ and we obtain $\mathrm{CGC}(C) = \emptyset$. Now, suppose $\mathrm{aff}(C) \cap \mathbb{Z}^n \neq \emptyset$. We then may assume that $w \in \mathbb{Z}^n$ and $W = \{Ax + w : x \in \mathbb{R}^k\}$, where $A \in \mathbb{Z}^{n\times k}$ is a rank $k$ matrix. Then there exists an unimodular matrix $U \in \mathbb{Z}^{n\times n}$ and a non-singular matrix $B \in \mathbb{Z}^{k\times k}$ such that $\begin{bmatrix} B \\ \mathbf{0}_{n-k\times k} \end{bmatrix} = UA$, where $\mathbf{0}_{n-k\times k} \in \mathbb{R}^{n-k\times k}$ is the all zeros matrix (See for example Corollary 4.3b in page 49 of [31]). We can then take $L(x) = P_k(U(x - w))$ where $P_k$ is the projection onto the first $k$ variables. This transformation gives the desired result because of the following useful properties of the CG closure.

(i) If $C \subseteq \mathbb{R}^k$ and $\mathbf{0}_{n-k}$ is the all zeros vector in $\mathbb{R}^{n-k}$ then $\mathrm{CGC}(C \times \mathbf{0}_{n-k}) = \mathrm{CGC}(C) \times \mathbf{0}_{n-k}$.

(ii) If $C \subseteq \mathbb{R}^n$ is a closed convex set and $w \in \mathbb{Z}^n$ then $\mathrm{CGC}(C - w) = \mathrm{CGC}(C) - w$.

(iii) If $C \subseteq \mathbb{R}^n$ is a closed convex set and $U \in \mathbb{Z}^{n\times n}$ is an unimodular matrix then $U\,\mathrm{CGC}(C) = \mathrm{CGC}(UC)$.

The first two properties are direct. For the third one we can use the fact that $U^{-T}\mathbb{Z}^n = \mathbb{Z}^n$ to see that

$$\begin{aligned} U\,\mathrm{CGC}(C) &= \bigcup_{a\in\mathbb{Z}^n} \left\{x \in \mathbb{R}^m \,:\, \left\langle a, U^{-1}x\right\rangle \leq \lfloor \sigma_C(a) \rfloor\right\} \\ &= \bigcup_{a\in\mathbb{Z}^n} \left\{x \in \mathbb{R}^m \,:\, \left\langle U^{-T}a, x\right\rangle \leq \lfloor \sigma_C(a) \rfloor\right\} \\ &= \bigcup_{a\in\mathbb{Z}^n} \left\{x \in \mathbb{R}^m \,:\, \langle a, x\rangle \leq \left\lfloor \sigma_C\left(U^T a\right)\right\rfloor = \lfloor \sigma_{UC}(a) \rfloor\right\} = \mathrm{CGC}(UC). \end{aligned}$$

□

Using Proposition 5.1 the polyhedrality of the CG closure of a non full dimensional convex set $C$ is equivalent to the polyhedrality of full dimensional convex set $L(C)$. For instance, if $L(C)$ is a strictly convex set we can use Theorem 2.2 to deduce the polyhedrality of $L(C)$ and $C$. In particular we obtain the following corollary.

Corollary 5.1 *Let $C$ be a strictly convex body and let $V$ be an affine rational subspace. Then the CG closure of $V \cap C$ is a rational polytope.*

For the intersection with a general rational polyhedron $P$ we need to understand the role of each face of $P$ in the construction of the CG closure of $P \cap C$. Specifically, we would like to be able to replace $P$ by $P \cap C$ in the following lemma which is proven in page 340 of [31].

Lemma 5.1 *If $F$ is a face of rational polyhedron $P$, then* $\mathrm{CGC}(F) = \mathrm{CGC}(P) \cap F$.

To generalize Lemma 5.1 we will need to understand the support function of $C \cap P$ in relation to the support functions of $C$ and $P$. In other words, we need a Farka's type result for constraints $x \in C$ and $x \in P$, which requires some type of constraint qualification. If $P$ and $C$ satisfy constraint qualification $P \cap \mathrm{int}(C) \neq \emptyset$, then it allows us to use the following known result. For completeness, we present a short proof of this result in the appendix.

Proposition 5.2 *Let $C$ be a closed convex set and $P$ be a polyhedron such that $P \cap \mathrm{int}(C) \neq \emptyset$. Then*

$$\sigma_{P\cap C}(a) \leq \sigma_P(a_P) + \sigma_C(a_C) \quad . \tag{10}$$

*for all $a_P, a_C \in \mathbb{R}^n$ such that $a = a_P + a_C$. Furthermore, for every $a \in \mathbb{R}^n$ there exists $a_P, a_C \in \mathbb{R}^n$ such that* (10) *holds at equality.*

Lemma 5.2 *Let $C \subseteq \mathbb{R}^n$ be a strictly convex body and let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. Then if* $\dim(C \cap P) \geq 1$ *(i.e. $C \cap P$ is not empty or a single point), we have that $P \cap \mathrm{int}(C) \neq \emptyset$.*

Proof. By assumption $\dim(P \cap C) \geq 1$ we have that there exists $x, y \in P \cap C$ such that $x \neq y$ and $z := \frac{1}{2}(x + y) \in P$. Since $C$ is a strictly convex body we have that $z \in \mathrm{int}(C)$. □

With these results we obtain the following generalization of Lemma 5.1 and a direct corollary that describes the intersection of the CG closure of $C \cap P$ with the boundary of $C \cap P$.

Proposition 5.3 *Let $C$ be a strictly convex body and $P$ be a rational polyhedron. Let $F$ be any nonempty face of $P$, then for any $S \subseteq \mathbb{Z}^n$ there exists $S' \subseteq \mathbb{Z}^n$ such that* $\mathrm{CGC}_{S'}(P \cap C) \cap F = \mathrm{CGC}_S(F \cap C)$.

Proof. We show that for any CG cut $\langle a, x\rangle \leq \lfloor \sigma_{F\cap C}(a) \rfloor$ for $F \cap C$, there exists a CG cut $\langle b, x\rangle \leq \lfloor \sigma_{P\cap C}(b) \rfloor$ for $P \cap C$ such that if $u \in F \cap C$ then $\langle a, u\rangle > \lfloor \sigma_{F\cap C}(a) \rfloor$ is equivalent to $\langle b, u\rangle > \lfloor \sigma_{P\cap C}(b) \rfloor$. The proof is divided into two cases.

First consider the case where $\dim(F \cap C) = 0$ or $F \cap C = \emptyset$. If $F \cap C = \emptyset$, then the result is evident. Otherwise $F \cap C = \{v\}$ fore some $v$ such that either $v \in \mathrm{bd}(C)$ or $F = \{v\}$. In both case if $v \notin \mathbb{Z}^n$, then there exists a CG cut valid for $C$ or $P$ that separates this point (For the first case by Proposition 3.1 and for the second case by Lemma 5.1 and the fact that if $v \notin \mathbb{Z}^n$ then $\mathrm{CGC}(\{v\}) = \emptyset$). Thus $\mathrm{CGC}(F \cap C) = \mathrm{CGC}(P \cap C) \cap F = \emptyset$. On the other hand if $v \in \mathbb{Z}^n$, then $\mathrm{CGC}(F \cap C) = F \cap C = \mathrm{CGC}(P \cap C) \cap F$.

Now assume that $\dim(F \cap C) \geq 1$. By Lemma 5.2, we have $F \cap \mathrm{int}(C) \neq \emptyset$. Therefore, by Proposition 5.2 there exists $a_F$ and $a_C$ such that $\sigma_{F\cap C}(a) = \sigma_F(a_F) + \sigma_C(a_C)$ and $a = a_F + a_C$. Since $P$ is a rational polyhedron, let $P = \{x \in \mathbb{R}^n : A^{\leq}x \leq b^{\leq}, A^{=}x \leq b^{=}\}$ and $F = \{x \in \mathbb{R}^n : A^{\leq}x \leq b^{\leq}, A^{=}x = b^{=}\}$ where $A^{\leq}, A^{=}, b^{\leq}, b^{=}$ are integral. By the nonemptyness of $F$ there exists $y^{\leq}, y^{=}$ such that

$$(y^{\leq})^T A^{\leq} + (y^{=})^T A^{=} = (a_F)^T \tag{11}$$
$$\langle y^{\leq}, b^{\leq}\rangle + \langle y^{=}, b^{=}\rangle = \sigma_F(a_F) \tag{12}$$
$$y^{\leq} \geq 0. \tag{13}$$

Consider $\hat{a}_F$ and $\hat{r}$ defined as $\hat{a}_F := (y^{\leq})^T A^{\leq} + (y^{=} - \lfloor y^{=} \rfloor)^T A^{=}$ and $\hat{r}_F := \langle y^{\leq}, b^{\leq}\rangle + \langle y^{=} - \lfloor y^{=}\rfloor, b^{=}\rangle$, where $\lfloor\cdot\rfloor$ is taken componentwise. Then observe that $\langle \hat{a}_F, x\rangle \leq \hat{r}_F$ is a valid inequality for $P$.

Let $\hat{a} = a - (\lfloor y^{=}\rfloor)^T A^{=}$ and $\hat{r} = \sigma_{F\cap C}(a) - \langle \lfloor y^{=}\rfloor, b^{=}\rangle$. Then observe that $\hat{a} \in \mathbb{Z}^n$ and $\hat{a} = \hat{a}_F + a_C$ and $\hat{r} = \hat{r}_F + \sigma_C(a_C)$. Therefore, by Proposition 5.2, $\langle \hat{a}, x\rangle \leq \hat{r}$ is a valid inequality for $C \cap P$. Finally, observe that if $u \in F$ then

$$\begin{aligned}
\langle a, u\rangle - \lfloor \sigma_{F\cap C}(a)\rfloor &= \langle \hat{a}, u\rangle + \langle \lfloor y^{=}\rfloor, A^{=}u\rangle - \lfloor \hat{r} + \langle \lfloor y^{=}\rfloor, b^{=}\rangle\rfloor \\
&= \langle \hat{a}, u\rangle - \lfloor \hat{r}\rfloor + \langle \lfloor y^{=}\rfloor, A^{=}u\rangle - \langle \lfloor y^{=}\rfloor, b^{=}\rangle \\
&= \langle \hat{a}, u\rangle - \lfloor \hat{r}\rfloor.
\end{aligned}$$

□

COROLLARY 5.2 *Let $C$ be a strictly convex set and $P$ a rational polyhedron such that $C \cap P$ is full dimensional. Let $\{F_i\}_{i=1}^m$, denote the facets of $P$, then*

$$\mathrm{CGC}(C \cap P) \cap \mathrm{bd}(C \cap P) = (\mathbb{Z}^n \cap \mathrm{bd}(C) \cap P) \cup \cup_{i=1}^m \mathrm{CGC}(F_i \cap C)$$

PROOF. We first note that if $C \cap P$ is full dimensional then $\mathrm{bd}(C \cap P) = (\mathrm{bd}(C) \cap P) \cup (\mathrm{bd}(P) \cap C)$. By Proposition 3.1 we have that $\mathrm{CGC}(C) \cap \mathrm{bd}(C) = \mathbb{Z}^n \cap \mathrm{bd}(C)$ and hence $\mathrm{CGC}(C \cap P) \cap \mathrm{bd}(C) \cap P = \mathbb{Z}^n \cap \mathrm{bd}(C) \cap P$. Now, because $P$ is full dimensional we have that $\mathrm{bd}(P) = \bigcup_{i=1}^m F_i$ and hence $\mathrm{CGC}(C \cap P) \cap \mathrm{bd}(P) = \bigcup_{i=1}^m F_i \cap \mathrm{CGC}(C \cap P) = \bigcup_{i=1}^m \mathrm{CGC}(C \cap F_i)$, where the last equality is obtained using Proposition 5.3. The result then follows from $\mathrm{CGC}(C \cap F_i) \cap C = \mathrm{CGC}(C \cap F_i)$ for all $i$. □

Using these results the proof of Theorem 2.3 is as follows.

PROOF OF THEOREM 2.3. The proof is by induction on the dimension of $C \cap P$. The base case is when the dimension is 0 and in this case $\mathrm{CGC}(C \cap P)$ is trivially a polyhedron ($\mathrm{CGC}(C \cap P) = C \cap P$ if $C \cap P \in \mathbb{Z}^n$ and $\mathrm{CGC}(C \cap P) = \emptyset$ otherwise).

For the induction argument we have that, by Proposition 5.1, we may assume that $C \cap P$ is full-dimensional. By Theorem 2.2 , $\mathrm{CGC}(C)$ is a polyhedron. Let $S_0 \subseteq \mathbb{Z}^n$ be a finite set such that $\mathrm{CGC}_{S_0}(C) = \mathrm{CGC}(C)$. Now, let $\{F_i\}_{i=1}^m$, denote the facets of P. Then by the induction hypothesis, we have that $\mathrm{CGC}(C \cap F_i)$ is a polyhedron for each $i$. For each $S_i \subseteq \mathbb{Z}^n$ such that $\mathrm{CGC}_{S_i}(C \cap F_i) = \mathrm{CGC}(C \cap F_i)$ let $S_i' \subseteq \mathbb{Z}^n$ be the set given by Proposition 5.3 such that $\mathrm{CGC}_{S_i'}(C \cap P) \cap F_i = \mathrm{CGC}_{S_i}(C \cap F_i) = \mathrm{CGC}(C \cap F_i)$. Finally, let $S = S_0 \cup S_P \cup \bigcup_{i=1}^m S_i'$ where $S_P$ is the finite set such that $P = \{x \in \mathbb{R}^n : \langle a, x\rangle \leq \sigma_P(a)\, \forall a \in S_P\}$ (Observe that $S_P$ exists because $P$ is a rational polyhedron).

Noting than $\mathrm{CGC}_{S_0}(C) \subseteq C$ we obtain that $\mathrm{CGC}_S(C\cap P) \subseteq C \cap P$. Moreover, by Proposition 3.1 we have that $\mathrm{CGC}_{S_0}(C) \cap \mathrm{bd}(C) \subseteq \mathbb{Z}^n$ so

$$\begin{aligned}
\mathrm{CGC}_S(C\cap P)\cap \mathrm{bd}(C\cap P) &= (\mathrm{CGC}_S(C\cap P)\cap \mathrm{bd}(C)\cap P)\cup(\mathrm{CGC}_S(C\cap P)\cap C\cap \mathrm{bd}(P)) && (14)\\
&= (\mathrm{CGC}_S(C\cap P)\cap \mathrm{bd}(C)\cap P)\cup \bigcup_{i=1}^m (\mathrm{CGC}_S(C\cap P)\cap C\cap F_i) && (15)\\
&= (\mathbb{Z}^n\cap \mathrm{bd}(C)\cap P)\cup \bigcup_{i=1}^m \mathrm{CGC}(C\cap F_i) && (16)\\
&= \mathrm{CGC}(C\cap P)\cap \mathrm{bd}(C\cap P) && (17)
\end{aligned}$$

where the last two containments follow from the definition of $S$, Proposition 5.3 and Lemma 5.2. Then $S$ complies with the hypothesis of Proposition 4.2 and hence $\mathrm{CGC}(C\cap P)$ is a polyhedron. □

**Appendix A. Omitted Proofs** Proof of Proposition 2.1. By Corollary C.3.1.2 in [20] we have that

$$C = \bigcap_{a\in\mathbb{R}^n} \{x \in \mathbb{R}^n \,:\, \langle a, x\rangle \le \sigma(a)\}\,.$$

It then suffices to show that

$$\bigcap_{a\in\mathbb{Z}^n} \{x \in \mathbb{R}^n \,:\, \langle a, x\rangle \le \sigma(a)\} \subseteq \bigcap_{a\in\mathbb{Q}^n} \{x \in \mathbb{R}^n \,:\, \langle a, x\rangle \le \sigma(a)\} \subseteq \bigcap_{a\in\mathbb{R}^n} \{x \in \mathbb{R}^n \,:\, \langle a, x\rangle \le \sigma(a)\}\,.$$

The first containment follows from positive homogeneity of $\sigma_C(\cdot)$ and the second follows from density of $\mathbb{Q}^n$ in $\mathbb{R}^n$ and continuity of $\sigma_C(\cdot)$ and $\langle\cdot, x\rangle$. □

Proof of Lemma 3.1. Take $v \in S^{n-1}$. Since $C$ is compact, the linear form $\langle v, .\rangle$ achieves its maximum over $C$. By linearity of $\langle v, .\rangle$ and since $v \neq 0$, any maximizer must be contained in $\mathrm{bd}(C)$. To show that $m_C$ is well-defined, we need only guarantee that this maximum is unique. Assume then that $x, y \in \mathrm{bd}(K)$, $x \neq y$, such that

$$\langle v, x\rangle = \langle v, y\rangle = \sigma_C(v)$$

Then note that the line $[x, y] \subseteq \mathrm{bd}(C)$, since $C$ is convex and $\langle v, z\rangle \le \sigma_C(v)$ for all $z \in C$ implies $\langle v, z\rangle = \sigma_C(v)$ for all $z \in [x, y]$. But by the strict convexity of $C$, we have that $(x, y) \subseteq \mathrm{int}(C)$, a clear contradiction. Hence the form $\langle v, .\rangle$ has a unique maximizer as needed.

Now let $\{v^i\}_{i\in\mathbb{N}}$ be a sequence of vectors in $S^{n-1}$ such that $v^i \xrightarrow{i\to\infty} v$. Let $x = m_C(v)$. To show that $m_C$ is continuous it suffices to show that $m_K(v^i) \xrightarrow{i\to\infty} m_K(v) = x$. Assume not, then for some open neighborhood $\mathcal{N}$ of $x$, there exists a subsequence $\{v^{\alpha_i}\}_{i\in\mathbb{N}}$ such that $m_C(v^{\alpha_i}) \notin \mathcal{N}$ for all $i \in \mathbb{N}$. Note that the sequence $\{m_C(v^{\alpha_i})\}_{i\in\mathbb{N}}$ is an infinite sequence on a compact set $\mathrm{bd}(C)$. Hence there exists a convergent subsequence $\{m_C(v^{\beta_i})\}_{i\in\mathbb{N}}$ with limit $y \in \mathrm{bd}(C)$. Since by construction, $m_C(v^{\beta_i}) \notin \mathcal{N}$, we have that $\lim_{i\to\infty} m_C(v^{\beta_i}) = y \notin \mathcal{N}$. Since $x \in \mathcal{N}$, we have that $x \neq y$. Now we see that

$$\begin{aligned}
\langle v, y\rangle &= \left\langle \lim_{i\to\infty} v^{\beta_i}, \lim_{i\to\infty} m_C(v^{\beta_i})\right\rangle = \lim_{i\to\infty} \left\langle v^{\beta_i}, m_C(v^{\beta_i})\right\rangle\\
&\ge \lim_{i\to\infty} \left\langle v^{\beta_i}, x\right\rangle = \langle v, x\rangle \ge \langle v, y\rangle
\end{aligned}$$

Now since $x = m_C(v)$, we have that $y \in \mathrm{bd}(C)$ is a maximizer of the form $\langle v, .\rangle$. But by strict convexity this maximizer is unique, and since $x \neq y$, we get a contradiction. Hence $m_C$ is a continuous function as claimed. □

Proof of Proposition 5.2. For a convex set $C$ let

$$\mathrm{i}_C(x) := \begin{cases} 0 & x \in C\\ +\infty & o.w. \end{cases}$$

be its indicator and for convex function $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ let $f^*(a) := \sup\{\langle a, x\rangle - f(x) \,:\, x \in \mathrm{dom}(f)\}$ be its conjugate or Legendre-Fenchel transform. Now, let $g_1 = \mathrm{i}_C$ and $g_2 = \mathrm{i}_P$ so that $(g_1)^* = \sigma_C$, $(g_2)^* = \sigma_P$ and $\mathrm{i}_{C\cap P} = g_1 + g_2$. Because $P \cap \mathrm{int}(C) \neq \emptyset$ we have that $\mathrm{int}(\mathrm{dom}(g1)) \cap \mathrm{dom}(g_2) \neq \emptyset$ and we hence have qualification assumption (2.3.Q.jj') in page 228 of [20]. We can then use Theorem E.2.3.2 in [20] to obtain that $\sigma_{P\cap C}(a) = (\mathrm{i}_{C\cap P})^*(a) = \inf\{\sigma_P(a_P) + \sigma_C(a_C) \,:\, a = a_P + a_C\}$ and that for every $a \in \mathbb{R}^n$ there exists $a_P, a_C \in \mathbb{R}^n$ such that $\sigma_{P\cap C}(a) = \sigma_P(a_P) + \sigma_C(a_C)$. □

## References


[1] K. Abhishek, S. Leyffer, and J. T. Linderoth, *FilMINT: An outer-approximation-based solver for nonlinear mixed integer programs*, Preprint ANL/MCS-P1374-0906, Argonne National Laboratory, Mathematics and Computer Science Division, Argonne, IL, September 2006.

[2] A. Atamtürk and V. Narayanan, *Lifting for conic mixed-integer programming*, Research Report BCOL.07.04, IEOR, University of California-Berkeley, October 2007, Forthcoming in *Mathematical Programming*.

[3] ______, *The submodular 0-1 knapsack polytope*, Discrete Optimization **6** (2009), 333–344.

[4] ______, *Conic mixed-integer rounding cuts*, Mathematical Programming **122** (2010), 1–20.

[5] P. Bonami, L. T. Biegler, A. R. Conn, G. Cornuéjols, I. E. Grossmann, C. D. Laird, J. Lee, A. Lodi, F. Margot, N. Sawaya, and A. Waechter, *An algorithmic framework for convex mixed integer nonlinear programs*, Discrete Optimization **5** (2008), 186–204.

[6] P. Bonami, M. Kilinç, and J. Linderoth, *Algorithms and software for convex mixed integer nonlinear programs*, Technical Report 1664, Computer Sciences Department, University of Wisconsin-Madison, October 2009.

[7] S Ceria and J Soares, *Convex programming for disjunctive convex optimization*, Mathematical Programming **86** (1999), 595–614.

[8] M. T. Çezik and G. Iyengar, *Cuts for mixed 0-1 conic programming*, Mathematical Programming **104** (2005), 179–202.

[9] V. Chvátal, *Edmonds polytopes and a hierarchy of combinatorial problems*, Discrete Mathematics **4** (1973), 305–337.

[10] W. J. Cook, W. H. Cunningham, W. R. Pulleyblank, and A. Schrijver, *Combinatorial optimization*, John Wiley and Sons, Inc., 1998.

[11] S. S. Dey and J. P. Vielma, *The Chvátal-Gomory Closure of an Ellipsoid is a Polyhedron*, To appear in Integer Programming and Combinatorial Optimization, 14th International IPCO Conference, Lausanne, Switzerland, June 9-11, 2010, Proceedings, Lecture Notes in Computer Science, Springer.

[12] J. Edmonds, *Paths, trees, and flowers*, Canadian Journal of mathematics **17** (1965), 449–467.

[13] A. Frangioni and C. Gentile, *Perspective cuts for a class of convex 0-1 mixed integer programs*, Mathematical Programming **106** (2006), 225–236.

[14] R. E. Gomory, *Outline of an algorithm for integer solutions to linear programs*, Bulletin of the American Mathematical Society **64** (1958), 275–278.

[15] ______, *An algorithm for integer solutions to linear programs*, Recent advances in mathematical programming, McGraw-Hill, New York, 1963, pp. 269–302.

[16] I. Grossmann and S. Lee, *Generalized convex disjunctive programming: Nonlinear convex hull relaxation*, Computational Optimization and Applications **26** (2003), 83–100.

[17] O. Günlük, J. Lee, and R. Weismantel, *MINLP strengthening for separable convex quadratic transportation-cost UFL*, IBM Research Report RC24213, IBM, Yorktown Heights, NY, March 2007.

[18] O. Günlük and J. Linderoth, *Perspective relaxation of mixed integer nonlinear programs with indicator variables*, Mathematical Programming, Series B **To Appear** (2009).

[19] R. Hemmecke, M. Köppe, J. Lee, and R. Weismantel, *Nonlinear integer programming*, [22], pp. 561–618.

[20] J.-B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of convex analysis*, Springer Verlag, Heidelberg, 2001.

[21] E. L. Johnson, G. L. Nemhauser, and M. W. P. Savelsbergh, *Progress in linear programming-based algorithms for integer programming: an exposition*, INFORMS Journal on Computing **12** (2000), 2–23.

[22] M. Jünger, T. Liebling, D. Naddef, G. Nemhauser, W. Pulleyblank, G. Reinelt, G. Rinaldi, and L. Wolsey (eds.), *50 years of integer programming 1958-2008: From the early years to the state-of-the-art*, Springer-Verlag, New York, 2010.

[23] A. N. Letchford and M. M. Sørensen, *Binary positive semidefinite matrices and associated integer polytopes*, in Lodi et al. [25], pp. 125–139.

[24] S. Leyffer, J. T. Linderoth, J. Luedtke, A. Miller, and T. Munson, *Applications and algorithms for mixed integer nonlinear programming*, Journal of Physics: Conference Series **180** (2009).

[25] A. Lodi, A. Panconesi, and G. Rinaldi (eds.), *Integer programming and combinatorial optimization, 13th international conference, ipco 2008, bertinoro, italy, may 26-28, 2008, proceedings*, Lecture Notes in Computer Science, vol. 5035, Springer, 2008.

[26] G. Cornuejols M. Conforti and G. Zambelli, *Polyhedral approaches to mixed integer linear programming*, [22], pp. 343–386.

[27] H. Marchand, A. Martin, R. Weismantel, and L. A. Wolsey, *Cutting planes in integer and mixed integer programming*, **123** (2002), 397–446.

[28] G. L. Nemhauser and L. A. Wolsey, *Integer and combinatorial optimization*, Wiley-Interscience, 1988.

[29] J.-P. P. Richard and M. Tawarmalani, *Lifting inequalities: a framework for generating strong cuts for nonlinear programs*, Mathematical Programming **121** (2010), 61–104.

[30] A. Schrijver, *On cutting planes*, Annals of Discrete Mathematics **9** (1980), 291–296, Combinatorics 79 (Proc. Colloq., Univ. Montréal, Montreal, Que., 1979), Part II.

[31] A. Schrijver, *Theory of linear and integer programming*, John Wiley & Sons, Inc., New York, NY, 1986.

[32] R. A. Stubbs and S. Mehrotra, *A branch-and-cut method for 0-1 mixed convex programming*, Mathematical Programming **86** (1999), 515–532.