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Truss topology design with integer variables made easy*

Dedicated to Herbert Hörnlein on the occasion of his 65th birthday

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Abstract. We propose a new look at the problem of truss topology optimization with integer or binary variables. We show that the problem can be equivalently formulated as an integer *linear* semidefinite optimization problem. This makes its numerical solution much easier, compared to existing approaches. We demonstrate that one can use an off-the-shelf solver with default settings and solve problems considered in the current literature too hard or even impossible to be solved.

1. Introduction

In truss topology optimization we try to find an optimum structural design of the truss by finding optimal thickness or cross-sectional areas of structural elements. Often, from the manufacturing point of view it is highly desirable that variables attain only few given discrete values, for instance 0,1,2,3. Then the problem becomes an optimization problem with integer variables.

The structural optimization problem as such is most often formulated as a nonlinear and nonconvex optimization problem, where the nonlinearity is due to equilibrium conditions depending both on the state variable (displacement) and the design variable (thickness). When searching for binary or integer design, the resulting problem is then a *nonconvex mixed integer nonlinear program* (MINLP). These problems are, typically, extremely difficult to solve, both due to nonconvexity and the integer nature of some of the variables. There have been many attempts to solve these problems, most of them based on heuristic optimization methods that cannot give any guarantees about the solution. A few articles have recently appeared in the literature that are based on mathematical programming approach to the problem and that deliver a guaranteed global minimum of the problem; see [2–4].

In this article, we do not propose any new method for the solution of MINLP problems, neither a brand new reformulation of the optimization problem. What we propose is a new look at the problem. Using well known facts, we will show that the nonconvex MINLP problem is fully equivalent to a *linear conic problem with integer variables*, with the cone of positive semidefinite matrices. As such, it is much easier to solve than the original formulation. We will show that, to solve the new formulation, one can use

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an off-the-shelf solver with default settings (in our case YALMIP [8]) and solve problems considered in the current literature.

Not only we can solve problems from recent articles, we can also solve problems that are considered too hard or even impossible to be solved. In particular, we can add more linear (or convex) conic constraints to our formulation, such as constraints on free vibration or constraints related to the multiple load case. Unlike in other existing approaches, these constraints will not change the character of our formulation, only the size of the problem.

The goal of this paper is to point out a reformulation of a nonconvex mixed integer nonlinear problems as an integer linear conic problem. It is not our goal to design an efficient solver for these problems; however, in the last section we mention several points that could lead to substantial improvements in solution efficiency.

The paper is organized as follows. In Section 2 we introduce the problem of truss topology optimization with integer variables, describe current approaches to its solution and introduce the new approach. Still in this section, we consider extensions of the problem to multiple load case and problems with constraints on free vibrations. In Section 3 we present results of numerical experiments. Section 4 contains concluding remarks.

We use standard notation: $\mathbb{S}^m$ is the space of real symmetric matrices of dimension $m \times m$. Notation $A \succcurlyeq 0$ for $A \in \mathbb{S}^m$ means that the matrix A is positive semidefinite.

2. Truss topology design with integer variables

2.1. The problem

In *truss optimization* we want to design a pin-jointed framework called *truss*. The truss consists of m slender *bars* of constant mechanical properties characterized by their Young's modulus E. We will consider trusses in a dim-dimensional space, where $dim = 2$ or $dim = 3$. The bars are jointed at $\tilde{n}$ nodes. The system is under load, i.e., forces $f_j \in \mathbb{R}^{dim}$ are acting at some nodes j. They are aggregated in a vector f, where we put $f_j = 0$ for nodes that are not under load. This external load is transmitted along the bars causing displacements of the nodes that make up the displacement vector u. Let p be the number of fixed nodal coordinates, i.e., the number of components with prescribed discrete homogeneous Dirichlet boundary condition. We omit these fixed components from the problem formulation reducing thus the dimension of u to

$$n = dim \cdot \tilde{n} - p.$$

Analogously, the external load f is considered as a vector in $\mathbb{R}^n$.

The design variables in the system are the bar cross-sectional areas ("thicknesses") $t_1, \dots, t_m$. Typically, we want to minimize the weight of the truss. We assume to have a unique material (and thus density) for all bars, so this is equivalent to minimizing the volume of the truss, i.e., $\sum_{i=1}^m \ell_i t_i$, where ℓ_i is the length of the ith bar. The optimal truss should satisfy mechanical equilibrium conditions:

$$K(t)u = f\,; \tag{1}$$

here

$$K(t) := \sum_{i=1}^{m} t_i K_i, \quad K_i = \frac{E_i}{\ell_i} \gamma_i \gamma_i^\top \tag{2}$$

is the so-called stiffness matrix, E_i the Young modulus of the ith bar, and γ_i the $n-$vector of direction cosines.

We further introduce the compliance of the truss $f^T u$ that indirectly measures the stiffness of the structure under the force f. The *minimum volume problem* truss topology optimization problem reads as

$$\min_{t\in\mathbb{R}^m,\, u\in\mathbb{R}^n} \sum_{i=1}^{m} \ell_i t_i \tag{3}$$

subject to

$$K(t)u = f$$
$$f^T u \leq \overline{\gamma}$$
$$0 \leq t_i \leq \bar{t}, \quad i = 1, 2, \ldots, m\,.$$

Alternatively, one may seek a truss with minimum compliance subject to volume constraints. This leads to the following *minimum compliance problem*:

$$\min_{t\in\mathbb{R}^m,\, u\in\mathbb{R}^n} f^T u \tag{4}$$

subject to

$$K(t)u = f$$
$$\sum_{i=1}^{m} \ell_i t_i \leq \overline{V}$$
$$0 \leq t_i \leq \bar{t}, \quad i = 1, 2, \ldots, m\,.$$

In the following, we will only concentrate on the minimum volume problem and its reformulations. All formulations can be analogously developed for the minimum compliance problem.

The optimal truss, as a solution of one of the above problems, includes bars that may vary from tiny ones to those on the upper bound $\bar{t}$. Such a result is highly impractical. Hence a natural requirement arises: the bar thicknesses should only be chosen from a small set of discrete values. For the sake of simplicity, assume that these values are integers. So we want to add another constraint to our problem:

$$t_i \in \{0, 1, \ldots, T\}, \quad i = 1, 2, \ldots, m$$

where T is either equal to 1 (giving a binary variable) or to some small integer, say 3,4,or 5.

The minimum volume problem (3) thus becomes a mixed-integer nonlinear (and nonconvex) optimization problem

$$\min_{t\in\mathbb{R}^m,\ u\in\mathbb{R}^n} \sum_{i=1}^m \ell_i t_i \tag{5}$$

subject to

$$K(t)u = f$$
$$f^T u \le \overline{\gamma}$$
$$t_i \in \{0,1,\dots,T\}, \quad i=1,2,\dots,m\,.$$

Clearly, the original minimum volume problem (3) with $\bar{t} = T$ is a continuous relaxation of (5).

2.2. *Existing approaches*

A number of authors have proposed solution techniques for problem (5). Most of these techniques are based on heuristic algorithms, such as pattern search or genetic algorithm; see [7] and the references therein. Only a few authors attempted to solve these problems by techniques of global optimization, in particular, mixed-integer nonlinear programming (MINLP); see, for instance, recent articles by Achtziger and Stolpe [2–4]. The authors, in particular, use the fact that the dual to the relaxed minimum compliance problem problem (4) can be written as the following convex quadratically constrained quadratic problem (QCQP)

$$\min_{u\in\mathbb{R}^n, r\in\mathbb{R}^m, \lambda\in\mathbb{R}} -f^T u + \lambda V - \sum_{i=1}^m r_j \tag{6}$$

subject to

$$u^T K_i u - \lambda + r_j \le 0, \quad i=1,2,\dots,m$$
$$\lambda \ge 0\,.$$

This allows one to compute the global minimum of the nonconvex problem (4) and decide whether the problem is feasible or not. Furthermore, using the fact that $K_i = b_i b_i^T$, $b_i \in \mathbb{R}^n$, Achtziger and Stolpe show that the QCQP (6) can be further formulated as a convex quadratic optimization problem and thus solved very efficiently. Using a branch-and-bound algorithm for MINLP (5), they were able to solve problems with up to 750 binary variables and 90 continuous variables.

The success of Achtziger and Stolpe's approach is given by their ability to compute global maxima to the nonconvex relaxations of the MINLP and thus tight and reliable lower bounds in the branch-and-bound algorithm. This, in turn, relies on the fact that the exact dual to (4) is a convex problem. Once we change the original problem, either by adding more constraints or by solving the so-called multiple-load problem, this property may be lost. This is, indeed, documented in [2], where the authors try to solve the multiple-load truss topology problem with integer design variables. In this problem, we assume that there are several independent loads $f^1, \dots, f^L$. The goal is to find a design that is optimal with respect to all these load cases. The problem (in the minimum weight form) is formulated as follows:

$$\min_{t \in \mathbb{R}^m,\, u \in \mathbb{R}^{n \times L}} \sum_{i=1}^{m} \ell_i t_i \tag{7}$$

subject to

$$K(t) u^\ell = f^\ell, \quad \ell = 1, \dots, L$$
$$f^T u^\ell \le \overline{\gamma}, \quad \ell = 1, \dots, L$$
$$t_i \in \{0, 1, \dots, T\}, \quad i = 1, 2, \dots, m\,.$$

The continuous relaxation to (7) is again obtained by replacing the last constraint by

$$0 \le t_i \le T, \quad i = 1, 2, \dots, m\,.$$

In this case, the quadratic program obtained from the dual problem gives only a lower bound to the global optimum of the continuous relaxation and thus a poor lower bound (according to the authors) to the MINLP (7). The authors then report a rather poor performance of the branch-and-bound method for the multiple load problems, as compared to the single-load case.

Another difficulty arises, when we want to add more constraints to the original truss topology problem, for instance, constraints on vibrations or stability of the optimal structure. Then, again, the above approach cannot be used and one is left with general purpose MINLP codes that cannot guarantee global optimality.

2.3. The new approach

The proposed new approach to the solution of MINLP (5) is rather simple. It is based on the fact that the original (continuous) topology optimization problem (3) can be written as linear (and hence convex) semidefinite programming (SDP) problems. The reformulation is based on the following result (for the proof, see, e.g., [1, Prop.3.1]):

Proposition 1. *Let* $t \in \mathbb{R}^m$, $t \ge 0$, *and* $\gamma \in \mathbb{R}$ *be fixed, and fix an index* $\ell \in \{1, \dots, n_\ell\}$. *Then there exists* $u_\ell \in \mathbb{R}^n$ *satisfying*

$$K(t) u_\ell = f_\ell \qquad \textit{and} \qquad f_\ell^T u_\ell \le \gamma$$

if and only if

$$\begin{pmatrix} \gamma & -f_\ell^T \\ -f_\ell & K(t) \end{pmatrix} \succeq 0\,.$$

We will now formulate the SDP version of the multiple-load minimum volume problem. The single load version is then just a special case when $L = 1$. The problem reads as follows:

$$\min_{t\in\mathbb{R}^m} \sum_{i=1}^m \ell_i t_i \tag{8}$$

subject to

$$\begin{pmatrix} \overline{\gamma} & -f_\ell^T \\ -f_\ell & K(t) \end{pmatrix} \succeq 0, \quad \ell = 1,\dots,L$$

$$0 \le t_i \le \overline{t}, \quad i = 1,2,\dots,m\,.$$

Hence the original MINLP (5) can be written as *linear semidefinite integer program*:

$$\min_{t\in\mathbb{R}^m} \sum_{i=1}^m \ell_i t_i \tag{9}$$

subject to

$$\begin{pmatrix} \overline{\gamma} & -f_\ell^T \\ -f_\ell & K(t) \end{pmatrix} \succeq 0, \quad \ell = 1,\dots,L$$

$$t_i \in \{0,1,\dots,T\}, \quad i = 1,2,\dots,m\,.$$

As before, problem (8) with $\overline{t} = T$ is a continuous relaxation of (9), respectively.

Notice that, unlike in the Achtziger-Stolpe approach, there is no difference between the single and multiple-load cases. In both cases, the continuous relaxation is a convex problem giving best possible lower bound for the integer problem. In our approach, the original nonconvex MINLP problem is equivalently reformulated as linear SDP problem with integer variables.

Furthermore, due to the fact that we remain in the primal formulation, we can "easily" add more linear or convex constraints, without changing the character of the problem. As an example, in the next section we will solve the integer truss topology problem with vibration constraints.

2.4. Vibration constraints

We want to set an additional constraint on free vibrations of the optimal structure. The free vibrations are the squares of the eigenvalues of the following generalized eigenvalue problem

$$K(t)w = \lambda(M(t) + M_0)w\,. \tag{10}$$

Here

$$M(t) = \sum_{i=1}^m t_i M_i \tag{11}$$

is the *mass matrix* that collects information about the mass distribution in the truss. The matrices M_i are positive semidefinite and have the same sparsity structure as K_i. The non-structural mass matrix M_0 is a constant, typically diagonal matrix with very few nonzero elements.

Low vibrations are dangerous and may lead to structural collapse. A typical condition requires the smallest free vibration to be bigger than some threshold, that is:

$$\lambda_{\min} \geq \overline{\lambda} \quad \text{for a given } \overline{\lambda} > 0 \tag{12}$$

where $\lambda_{\min}$ is the smallest eigenvalue of (10). This constraint can be equivalently written as a linear matrix inequality

$$K(t) - \overline{\lambda}(M(t) + M_0) \succcurlyeq 0 \tag{13}$$

which is to be added to (3). This linear inequality then now be easily added to the minimum volume problem with integer variables (9); the complexity of the problem will increase (two matrix constraints instead of one) but the character will remain the same and we can just solve the new problem by the same algorithm as (9). Here is the new problem:

$$\min_{t\in\mathbb{R}^m} \sum_{i=1}^{m} \ell_i t_i \tag{14}$$

subject to

$$\begin{pmatrix} \overline{\gamma} & -f_\ell^T \\ -f_\ell & K(t) \end{pmatrix} \succeq 0, \quad \ell = 1,\dots,L$$
$$t_i \in \{0,1,\dots,T\}, \quad i = 1,2,\dots,m$$
$$K(t) - \overline{\lambda}(M(t) + M_0) \succcurlyeq 0\,.$$

3. Numerical experiments

3.1. Solution procedure

The purpose of this article is to show that the truss topology optimization problem with integer variables can be easily solved by available software. To demonstrate it, instead of developing a special version of a branch-and-bound code that would be tailored to the problem, we will use an off-the-shelf general purpose branch-and-bound routine within the software YALMIP [8]. We cite from YALMIP website: “YALMIP is a modelling language for advanced modeling and solution of convex and nonconvex optimization problems. It is implemented as a free toolbox for MATLAB. The modelling language supports a large number of optimization classes, such as linear, quadratic, second order cone, semidefinite, mixed integer conic, geometric, local and global polynomial, multiparametric and robust programming…BNB is an implementation of a standard branch-and-bound algorithm for mixed integer linear/quadratic/second order cone and semidefinite programming solver. The solver relies on external solvers for solving the node problems.”

A YALMIP code for the solution of the binary minimum volume single load case truss problem is shown below. The reader can see that, having the problem data available, the coding is a question of five minutes.

```
% Single load truss topology minimum volume problem
% with binary variables
   % read problem data
m=par.m; n=par.n; BI=par.BI; length=par.length; ff=par.f;
compl=10;
   % define binary variables
t=binvar(m,1);
   % define the stiffness matrix K(t)
Kstiff=zeros(n,n);
for i=1:m, Kstiff=Kstiff+t(i)*BI(i,:)'*BI(i,:); end
   % define the constraints
Z = [compl -ff'; -ff Kstiff]; F = set(Z > 0);
F = F + set(0<t<1);
   % define the objective
obj = sum(t.*length);
   % solve the problem
solvesdp(F,obj);
   % plot the resulting figure
t = double(t); pic(par,t);
```

The SDP code used in all our experiments was SeDuMi [12], version 1.1, as it is reliable in detecting infeasible or unbounded problems. The experiments were performed on a notebook with Intel Core Duo 1.20 GHz CPU and 2GB memory, running Windows XP Professional. Default values of all parameters in YALMIP-BNB and SeDuMi have been used.

3.2. Example 1: cantilever

The first test example is the classic cantilever beam. The initial ground-structure, a 11×3 truss with neighbour nodes connected is depicted in Figure 1. The dimensions are $n =$

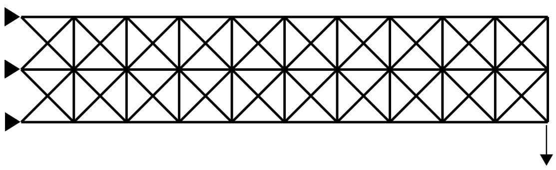

Fig. 1. Cantilever: initial design

60 and $m = 90$. All nodes on the left-hand side are fixed in both directions. The right-bottom node is subjected to a single force. The upper bound on compliance was chosen as $\overline{\gamma} = 1.0$. The number of nodes visited by the branch-and-bound code was 395. The optimal objective value of the relaxed problem was 14.5554 and of the binary problem 16.0711. The compliance of the binary solution was 0.9914. The optimal solutions for the relaxed and the binary problems are shown in Figure 2.

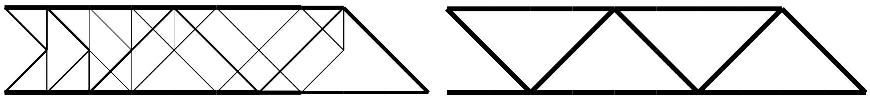

Fig. 2. Cantilever: relaxed and binary solution

3.3. *Example 2: bridge*

The goal is to find an optimal design of a bridge. The ground-structure, a 6×3 truss with all nodes connected, is shown in Figure 3. The dimensions are $m = 105$ an $n = 33$. We will consider three problems:

- single load case; all forces act simultaneously as one load
- two load cases; forces 1 and 2 form the first load case, forces 3 and 4 the second one
- four load case; all forces act as independent load cases

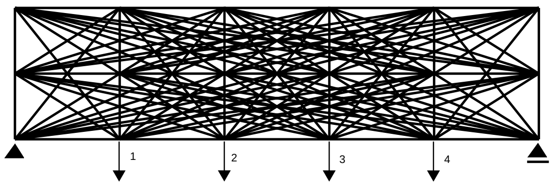

Fig. 3. Bridge: initial design

3.3.1. Single load case We considered this as a problem with integer variable $t_i \in \{0, 1, 2, 3\}$. The upper bound for compliance was 25.0. To find the integer solution, the branch-and-bound code needed to generate 14535 nodes. The optimal objective value of the relaxed problem was 93.5573, and of the integer problem 96.7684. The corresponding compliance of the integer problem was 24.9819. The optimal solutions for the relaxed and the integer problems are shown in Figure 4.

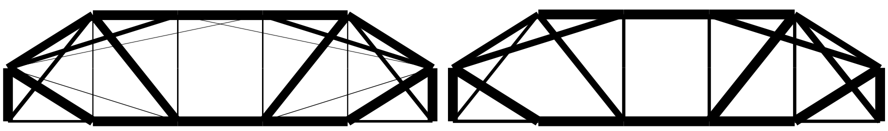

Fig. 4. Single load case: relaxed and integer solution

3.3.2. Two load cases We now set the upper bound for compliance $\overline{\gamma} = 20.0$. To find the binary solution, the code generated 14049 nodes. The optimal objective value of the relaxed problem was 43.0939, and of the binary problem 48.1394. The compliance of the binary solution was 19.3738 (in both load cases). The optimal solutions for the relaxed and the integer problems are shown in Figure 5.

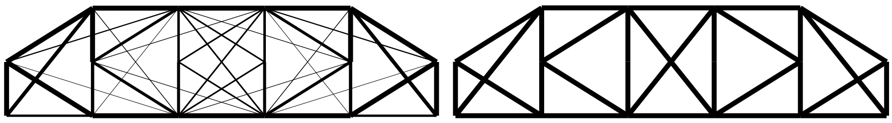

Fig. 5. Two load cases: relaxed and binary solution

3.3.3. Four load cases This time, the upper bound for compliance was 10.0. The code generated 4051 nodes to find the optimal binary solution. The optimal objective value of the relaxed problem was 28.4377, and of the binary problem 33.4309. The compliances corresponding to the optimal binary solution in the four load cases were (7.6668; 9.7606; 9.8046; 6.0568). The optimal solutions for the relaxed and the binary problems are shown in Figure 6.

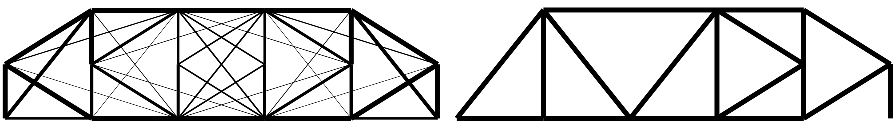

Fig. 6. Four load cases: relaxed and binary solution

3.4. Example 3: vibration constraints

We now consider the minimum volume problem with the additional vibration constraint (14) The ground-structure is shown in Figure 7. The dimensions are $m = 36$ an $n = 12$. The upper bound on compliance was chosen as $\overline{\gamma} = 2.0$ and the bound on the smallest eigenfrequency $\overline{\lambda} = 0.01$. To find the optimal binary solution we only needed to visit 203 nodes. The optimal objective value of the relaxed problem was 0.524, and of the binary problem 2.118. The corresponding compliance in the binary problem was 1.0. The optimal solutions for the relaxed and the binary problems are shown in Figure 8.

3.5. Example 4: vibration multiple mass

Here we propose an extension to the original problem proposed in [1]. Assume that we have two matrices $M_0^{(k)}$, $k = 1, 2$, corresponding to two different non-structural masses

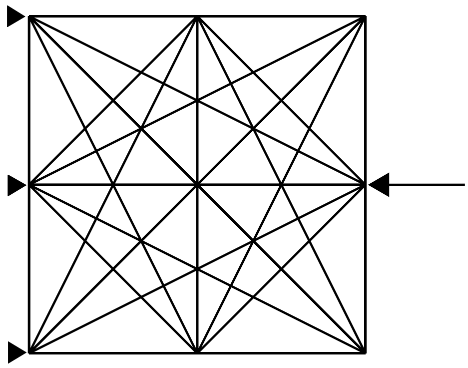

Fig. 7. Vibration constraints: initial design

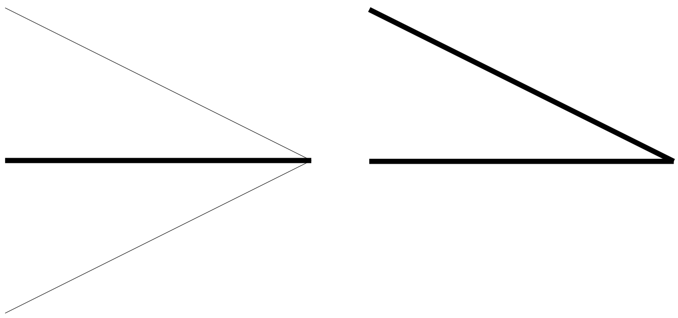

Fig. 8. Vibration constraints: relaxed and binary solution

that can be applied independently. The corresponding eigenvalue constraint extending the constraint "$\lambda_{\min}(t) \geq \overline{\lambda}$" in problem (14) is then be stated as

$$\lambda_{\min}(t, M_0^{(k)}) \geq \overline{\lambda} \qquad k = 1, 2$$

Generalizing the minimum volume problem (5) with vibration constraint (13), we arrive at the following formulation possessing the same problem structure:

$$\begin{aligned}
&\min_{t\in\mathbb{R}^m,\ u\in\mathbb{R}^n} \sum_{i=1}^{m} \ell_i t_i \\
&\text{subject to} \\
&\qquad K(t)u = f \\
&\qquad f^T u \leq \overline{\gamma} \\
&\qquad t_i \in \{0, 1, \ldots, T\}, \quad i = 1, 2, \ldots, m \\
&\qquad K(t) - \overline{\lambda}(M(t) + M_0^{(k)}) \succeq 0, \quad k = 1, 2\,.
\end{aligned}$$

In an analogous way we extend the SDP problems (14) to the case of multiple masses. (Obviously, in the same way, we can consider a problem with a general number of independent nonstructural masses.)

Consider a 4-by-4 truss with all nodes connected by potential bars. The nodes on the left-hand side are fixed in both directions, two balls (non-structural masses) are placed in the corners on the right-hand side; see Figure 9. The dimensions are $m = 120$ an $n = 24$. The bound on the smallest eigenfrequency was $\overline{\lambda} = 0.001$. In this example, we do not consider the compliance constraint, the design should only be stiff with respect to possible vibrations. We will consider two problems:

- single-mass problem;
- two-mass problem cases;

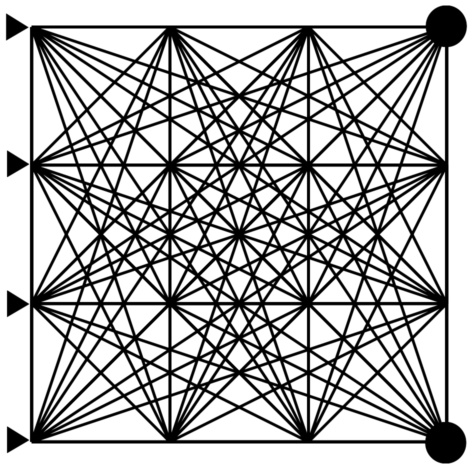

Fig. 9. Multiple-mass problem: initial design

3.5.1. Single mass problem We first consider the case when both non-structural masses act as a "single" non-structural mass. The mass of each ball was set to 300.0. To find the binary solution, the branch-and-bound code needed to generate 7799 nodes. (The optimal design was actually found in iteration 1683 already, the remaining iterations were needed to confirm global optimality.) The optimal objective value of the relaxed problem was 5.4966, and of the binary problem 6.3194. The optimal solutions for the relaxed and the binary problems are shown in Figure 10.

3.5.2. Multiple mass problem We now examine the case when the two non-structural masses are considered being independent from each other. The mass of each ball was set to 400.0. To find the binary solution, the branch-and-bound code needed to generate 8415 nodes. The optimal objective value of the relaxed problem was 4.7977, and of the binary problem 5.8284. The optimal solutions for the relaxed and the binary problems are shown in Figure 11.

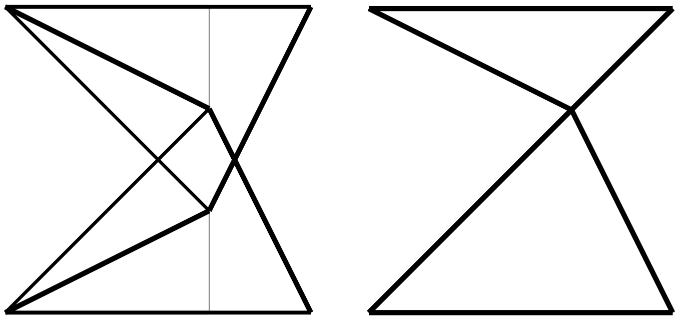

Fig. 10. Single mass problem: relaxed and binary solution

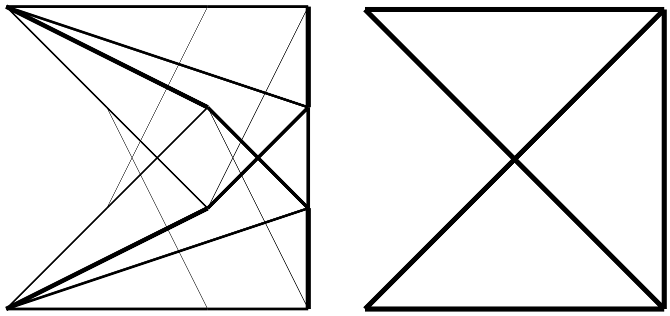

Fig. 11. Multiple mass problem: relaxed and binary solution

4. Final remarks

4.1. Room for improvement

4.1.1. The branch-and-bound code The author of YALMIP warns the users: "Note that the internal branch-and-bound algorithm is rudimentary and useful only for small problems." This is certainly an understatement. However, using a sophisticated branch-and-bound code tuned to the topology optimization problems would most probably reduce the number of visited nodes and subproblems solved significantly.

4.1.2. Solving the lower-bound relaxations The main idea of this paper is to formulate the integer truss topology optimization problems as linear semidefinite programs with integer variables. The continuous relaxation is then a standard linear SDP with guaranteed global optimum (when it exists). In our numerical experiments, we solve the problems as they are, i.e., as semidefinite programs, using SDP solvers. However, in order to compute the solution of the relaxed problems, we can still switch to the dual, just like in the Achtziger-Stolpe's articles. The dual problem must be, of course, easier to solve, if we want to benefit from it. This is the case of single load problems, when the dual is a convex QCQP. It is also the case of the problems with vibration constraints, when the dual can also be reformulated as a standard convex nonlinear program [9].

4.1.3. Warm starts It is well known that interior point methods, as well as penalty methods, are not particularly suitable for warm starting. These are the algorithms on which most SDP software is based. Still, warm starts can be used and can save some significant amount of CPU time. However, in this case the software needs not only good approximation of the prima solution but also of the dual variable. In addition, parameters of the codes may need to be tuned for such cases. YALMIP's branch-and-bound code only provides warm starts in the primal variable and hence does not fully use the potential of the codes.

4.2. Extensions

In topology optimization of continuum (solid) structures we try to find an optimum structural design by finding an optimal distribution of a given material [6]. It is well known (see, e.g., [5]) that this problem may not have a solution and, in order to find one, one has to relax it. The relaxation leads to "gray" designs, which are "mathematically correct" but often unacceptable from the engineering point of view—the user may require to have a 0-1 (material-no material) solution. The solution of binary topology optimization problems for continuum structures was one of the goals of the recently finished PLATO-N project (see project website www.plato-n.org for details).

After discretization by finite elements, the topology optimization problem has exactly the same structure as the truss design problem (4) or (3). The only difference is in the structure of the stiffness matrix K which is no longer a sum of dyadic products. We can just use the same trick as in Section 2, i.e., write this MINLP problem equivalently as a linear SDP problem with integer variables, the continuous relaxation to which is a standard linear SDP. The different structure of the stiffness matrix, however, makes the problem much harder to solve than the truss topology problems. From this very reason, the "SDP approach" proposed in this article has been initially excluded in the PLATO-N project [10, 11]. Nevertheless, we believe that by careful implementation of the branch-and-bound code and warm-started version of an SDP code, this approach could lead to significant improvements of existing methods.

4.3. Conclusion

In this article, we did not bring any new formulation of the topology optimization problem, rather a new look at the problems with integer variables. We formulate the problems as binary/integer linear (semidefinite) problems and utilize the consequences of this formulation. In particular, we can "easily" add more constraints in the form of linear matrix inequalities.

We believe that there are more applications that are being treated as mixed-integer nonlinear problems, yet can be formulated as integer linear conic problems, either with the semidefinite or the second-order cone.

Codes for mixed integer linear conic problems are already available, let us name YALMIP or BARON [13], though they may be underdeveloped, due to the lack of interest of potential users. The example of topology optimization presented in this article shows that integer conic programming deserves more attention.

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