On mixed integer reformulations of monotonic probabilistic programming problems with discrete distributions

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May 16, 2010

Abstract. The paper studies large scale mixed integer reformulation approach to stochastic programming problems containing probability and quantile functions, under assumption of discreteness of the probability distribution involved. Jointly with general sample approximation technique and contemporary mixed integer programming solvers the approach gives a regular framework to solution of practical probabilistic programming problems. In the literature this approach is applied basically to chance constrained problems with discrete probability distribution. In the present paper we extend it to more complicated problems, containing several probability/quantile functions both in the objective and constraints; in particular we consider probability and quantile functions optimization problems. First we show that the approach is applicable to probabilistic programming problems with objective and constraint functions monotonically depending on several probability functions. Next we remark that the general framework is applicable to optimization problems containing quantile functions, provided the last are uniformly bounded from below. As a solution technique beside general solvers we specialize branch and bound method for optimization of the probability function. To speed up the branch and bound method we derive dominance between integral variables basing on the dominance between probabilistic scenarios. Finally we note that some of probability functions may be defined on different probability spaces, this allows to treat within the considered framework some ambiguous probabilistic problems with uncertain probability distributions. As illustrations of the general framework we consider applications to (robust) safety first portfolio selection and quantile optimization.

***Keywords**: stochastic programming, probabilistic programming, probability function, quantile function, mixed integer programming, branch and bound method.*

1.Introduction

Probabilistic programming (P) problems include probability or quantile functions in their formulations, for instance, as objective (Roy 1952) or as chance constraints (Charnes and Cooper 1959; Kataoka 1963; Miller and Wagner 1965). The probability function expresses probability that some quantity depending on parameters falls below a given threshold. A chance constraint puts a bound on the probability function. Quantile function designates the minimal threshold for which the chance constraint is fulfilled. In decision making theory these concepts are used to model and to measure risks (potential losses). Probabilistic optimization problems constitute an important branch of stochastic programming theory (Ruszczyński and Shapiro 2003), dealing with decision making problems under uncertainty, and have variety of applications (Prékopa 1995, 2003; Kibzun and Kan 1996) (see also references there in). P-problems are difficult for solution, because they are generally nonconvex, nondifferentiable and even discontinuous, in particular all this complexities take place in case of discrete probability distributions. For decades P-problems are the subject of intensive investigation. Conditions of continuity and differentiability of probability functions are reviewed, for example, in (Raik 1972; Kibzun and Uryasev 1998; Kibzun and Kan 1996; Marti 2005; Shapiro, Dentcheva, Ruszczyński 2009). Their convexity conditions basically are due to Prékopa (1970, 1995), unfortunately they are not fulfilled in many practical cases, e.g. for general linear stochastic problems. Sample average approximations of P-problems are studied in (Calafiore and Campi 2005; Vogel 2005; Nemirovski and Shapiro 2005, 2006; Ludtke and Ahmed 2008). Solution techniques for P-problems are reviewed in (Prékopa 1995, 2003; Kibzun and Kan 1996; Shapiro, Dentcheva, Ruszczyński 2009). Recent advances in solution of P-problems with discrete probability distributions are connected with their reduction to mixed integer programming (MIP) problems and in the development of the improved solution techniques (Prékopa 1990; Sen 1992; Morgan, Eheart, Valocchi 1993; Dentcheva, Prékopa, Ruszczyński 2000; Vizvari 2002; Ruszczyński 2002; Beraldi and Ruszczyński 2002a, 2002b, 2005; Noyan, Rudolf and Ruszczyński 2006; Luedtke 2008; Noyan and Ruszczynski 2008; Beraldi and Bruni 2009; Luedtke, Ahmed, Nemhauser 2010; Saxena, Goyal, Lejeune 2010; Tanner and Ntaimo 2010; Küçükyavuz 2010).

Probabilistic programming has important applications to financial portfolio optimization. Quantitative theory of an optimal portfolio selection starts from Markowtz' (1952) idea to select financial portfolio by criteria "average return - risk", where variance (or a standard deviation) is considered as a measure of risk. Markowitz's approach consists in construction of an efficient portfolio frontier having minimal variance under given mean return and selection of an effective portfolio maximizing some utility function (Markowitz 1959). In the same 1952 year independently English economist Roy (1952) published a very close paper, where he suggested to select financial portfolio by minimization of a probability of default (safety first criterion). Since minimization of the probability was a difficult problem at that time, Roy (1952) suggested to minimize its upper bound obtained from Chebyshev's inequality. For an approximating problem he gave an elegant geometric solution, lying on the efficient frontier of the nondominated portfolios in the plane "mean return - variance of return". Although Roy's approach has not become much popular, his ideas continued to develop, for instance, in (Telser 1955/56; Kataoka 1963; Bawa 1978; Li et al. 1998; Jansen et al. 2000; Goodall 2002; Haque et al. 2004, 2007). Recent advances in mathematical portfolio selection theory are connected with a concept of a risk measure (see, e.g., (Pflug and Romisch 2007; Shapiro, Dentcheva, Ruszczyński 2009)).

The original safety first approach to portfolio optimization consists in minimization of the probability of default (Roy 1952) or in solution of a chance constrained problem (Kataoka 1963), which both are rather difficult tasks. A help comes from the side of stochastic programming theory, to be exact from its probabilistic programming branch basically developed by Prékopa (1970, 1995, 2003) and his followers. A new substantial contribution to safety first portfolio selection was made by A.Ruszczynski, who jointly with his collaborators developed a theory and solution techniques for optimal portfolio selection subject to stochastic dominance constraints (Ogryczak and Ruszczyński 1999, 2001; Ruszczyński 2002; Ruszczyński and Vanderbei 2003; Dentcheva and Ruszczyński 2003, 2004a, 2004b, 2006; Noyan, Rudolf and Ruszczyński 2006; Noyan and Ruszczyński 2008). First order stochastic dominance constraint is equivalent to a continuum of chance constraints, but in case of discrete reference distribution it is equivalent to a finite collection of probabilistic constraints (Noyan and Ruszczyński 2008), the last admits a mixed integer reformulation. Recent advances in mixed integer reformulations of chance constrained portfolio optimization problems are considered in Luedtke (2008). Important result is that the second order stochastic dominance constraints is a relaxations for the first order one (Dentcheva and Ruszczyński 2004b).

The present paper studies large scale mixed integer (in fact mixed Boolean) reformulation approach to stochastic programming problems containing probability and quantile functions, under assumption of discreteness of the probability distribution involved. Reformulation trick uses large bounds on the random problem functions, it is known in the integer programming as large bound trick (Korbut and Finkelstein 1969). In the literature this approach is applied basically to chance constrained problems with discrete probability distribution and fixed chance bounds. Jointly with sample approximation technique (Ludtke and Ahmed 2008) and contemporary large scale MIP solvers the approach gives a regular framework for solution of practical probabilistic programming problems.

In the present paper we extend this approach to more complicated problems, containing several probability/quantile functions both in the objective and constraints; in particular we consider probability and quantile functions optimization problems. First we show that the approach is applicable to probabilistic programming problems with objective and constraint functions monotonically depending on several probability functions. The number of Boolean variables in the transformed problem equals to the number of scenarios times the number of probability functions involved; the number of mixed integer constraints equals to the total number of constraints under probability signs times the number of scenarios. Next we remark that the general framework is applicable to optimization problems containing quantile functions, provided the last are uniformly bounded from below. As a solution technique beside available general solvers we consider and specialize branch and bound (B&B) method for optimization of the probability function. To speed up the B&B method we derive, following (Ruszczyński 2002), dominance between integral variables basing on dominance between probabilistic scenarios. Finally, we note that some of probability (or/and quantile) functions may be defined on different probability spaces; this allows to treat within the considered framework some ambiguous probabilistic problems with an uncertain probability distribution. As illustrations of the general framework we consider applications to (robust) safety first portfolio selection and quantile optimization.

The paper is organized as follows. In section 2.1 we transform general nonlinear probabilistic programming problems to mixed integer one, and in section 2.2 the same is done for problems containing quantiles. Section 3 discusses peculiarities of the branch and bound method when applied to arising MIP problems. Section 4 considered applications to (robust) safety first portfolio selection and optimization of a single quantile function. Section 5 describes computational experiment. Section 6 concludes.

2. Main results

In what follows we have to consider equivalent transformations of (inf or sup) optimization problems. To make considerations unified we follow the next framework (Mihalevich, Gupal, Norkin 1987, p. 131).

Definition 1. Two optimization problems are called equivalent if they both either do not have or have feasible solutions, and in the latter case there are simple (e.g. polynomial in the problem size number of elementary operations) algorithms, which allow to restore an optimal solution of one problem from an optimal solution of the other.

As elementary operations arithmetic operations and calculations of problem functions and of their components are considered.

The following statements are obvious (are proved by contradiction).

Statement 1 (sufficient condition for equivalence of two optimization problems). Two optimization problems are equivalent if there are simple (e.g. polynomial) algorithms, which allow for every feasible solution of one problem to point out a feasible solution of the other with not worse objective function value.

Statement 2. In assumptions of Statement 1 in case of feasibility optimal values (finite or infinite) of equivalent optimization problems coincide and both are either achieved or not.

2.1. Optimization problems with probability functions

In this subsection we reduce probabilistic programming problems with multiple probability functions in both objective and constraints to a mixed integer programming problem.

Let X be some set; J, K and $\{I_k, k \in K\}$ are finite ordered index sets; $(\Omega, \Sigma, \mathrm{P})$ be a probability space; $f_{ki}(x,\omega): X \times \Omega \to R^1$ be a measurable in $\omega \in \Omega$ function for each fixed $x \in X$, $i \in I_k$, $k \in K$; $G_j(x, y_K): X \times R_+^K \to R^1$ be monotonic (non decreasing componentwise) in $y_K = \{y_k, k \in K\} \in R_+^K$ function for each $x \in X$ and $j \in J$; R_+^K denotes Tihonov's product of K half spaces R_+^1 of nonnegative numbers. For $k \in K$ consider indicator functions

$$\mathbf{I}_k(x,\omega)=\begin{cases}1, & f_{ki}(x,\omega)\le 0 \;\; \forall i\in I_k,\\ 0, & \textit{otherwise},\end{cases}$$

and probability functions

$$g_k(x)=\Pr\left\{\omega\in\Omega:\; f_{ki}(x,\omega)\le 0 \;\forall i\in I_k\right\}=\int_{\Omega}\mathbf{I}_k(x,\omega)P(d\omega),$$

where the integral is reduced to summation for a discrete probability space. Remark that the set of simultaneous constraints $f_{ki}(x,\omega)\le 0 \;\forall i\in I_k$ can be equivalently converted into one inequality, $f_k(x,\omega)=\max_{i\in I_k} f_{ki}(x,\omega)\le 0$. Denote $g_K(x)=\left\{g_k(x),\, k\in K\right\}$.

Consider optimization problem

$$G_0\left(x,g_K(x)\right)\to \sup\nolimits_{x\in X}, \tag{1}$$

$$G_j\left(x,g_K(x)\right)\ge 0, \qquad j\in J. \tag{2}$$

As particular cases (1), (2) includes a probability optimization problem

$$g_0\left(x\right)\to \sup\nolimits_{x\in X}, \tag{3}$$

and a chance constrained one

$$G_0\left(x\right)\to \sup\nolimits_{x\in X}, \tag{4}$$

$$g_k(x)\ge \alpha_k, \qquad k\in K, \tag{5}$$

with fixed chance bounds α_k.

First we remark that general problem (1), (2) is reduced to a chance constrained problem (with variable chance bounds):

$$G_0\left(x,y_K\right)\to \sup\nolimits_{x\in X,\,y_K}, \tag{6}$$

$$G_j\left(x,y_K\right)\ge 0, \qquad j\in J; \tag{7}$$

$$g_k(x)\ge y_k, \qquad k\in K; \tag{8}$$

$$0\le y_k\le 1, \qquad k\in K. \tag{9}$$

Thus a probability optimization problem is reduced to a chance constrained one (with variable chance bound). We consider form (6) - (9) (with variable chance bounds) because it admits a relatively straightforward branch and bound solution techniques (see the next section).

Theorem 1. For finite J, K problems (1), (2) and (6) - (9) are equivalent in the sense of Definition 1; x-component of a solution of (6) - (9) is a solution of (1), (2).

Proof. Let x be a feasible point of (1), (2). Put $y_k(x) = g_k(x)$, $k \in K$. Consider mapping $x \to \left(x, y_K(x)\right)$. Obviously, $\left(x, y_K(x)\right)$ is a feasible point for (6)-(9) with the same value of the objective function. Conversely, let $\left(x, y_K\right)$ be a feasible point for (6) - (9), consider mapping $\left(x, y_K\right) \to x$. Due to condition (8) and by monotonicity of $G_j(x,\cdot)$ we conclude that x is feasible for (1), (2) with better value of the objective function. By statements 1, 2 problems (1), (2) and (6) - (9) are equivalent in the sense of Definition 1. The proof is complete.

Next we reduce problem (1), (2) to a mixed integer (in fact mixed Boolean) programming problem. For the case of chance constrained optimization problems (with fixed chance bounds) a similar transformation (large bound trick) was suggested in (Morgan et al. 1993) (see also, (Ruszczyński 2002; Noyan and Ruszczyński 2008)). Remark that a similar large bound trick (for a problem with multiple alternatives) is known in the integer programming for a long time (Korbut and Finkelstein 1969, Ch. 2, § 4, section 4.2). In the next theorem we extend the large bound trick to more complex probabilistic problems, in particular to chance constrained problems (6) - (9) with variable chance bounds y_k.

Assume that there exist measurable functions (large bounds) $N_{ki}(\omega)$ such that

$$\sup_{x\in X} f_{ki}(x,\omega) \le N_{ki}(\omega) < \infty, \qquad i \in I_k, \qquad k \in K. \tag{10}$$

Denote $z_k(\omega): \Omega \to \{0,1\}$ measurable in ω Boolean valued functions and linear functionals $p_k(x, z_k) = \int_\Omega z_k(\omega) P(d\omega)$; denote $z_K = \{z_k(\omega),\ \omega \in \Omega,\ k \in K\}$, $p_K(x, z_K) = \{\, p_k(x, z_k),\ k \in K\}$. Consider a problem

$$G_0\left(x, p_K(x, z_K)\right) \to \sup_{x\in X,\, z_K}, \tag{11}$$

$$G_j\left(x, p_K(x, z_K)\right) \ge 0, \ j \in J; \tag{12}$$

$$f_{ki}(x,\omega) \le N_{ki}(\omega)\left(1 - z_k(\omega)\right), \qquad i \in I_k,\ \omega \in \Omega,\ k \in K; \tag{13}$$

$$z_k(\omega) \in \{0,1\}, \qquad \omega \in \Omega, \qquad k \in K. \tag{14}$$

Theorem 2. Assume J, K, $\{I_k, k \in K\}$ and Ω are finite. Under made assumptions problems (1), (2) and (11) - (14) are equivalent in the sense that they both have or do not have feasible solutions; in case of feasibility their inf-optimal values (finite or infinite) coincide; in case of feasibility their solutions both are achieved or not achieved. If x^* is an optimal solution for (1), (2), then $\left(x^*, \left\{z_k^*(\omega) = \mathbf{I}_k(x^*, \omega),\ \omega \in \Omega,\ k \in K\right\}\right)$ is an optimal solution for (11) - (14);

conversely, if $\left(x^*, \left\{z_k^*(\omega),\ \omega \in \Omega,\ k \in K\right\}\right)$ is an optimal solution of problem (11) - (14), then x^* is an optimal solution for (1), (2).

Proof. Let us fix an arbitrary feasible solution x of (1), (2). Calculate $z_k(\omega) = \mathbf{I}_k(x,\omega)$ for all $\omega \in \Omega_k$ и $k \in K$. Consider mapping $x \to \left(x, \left\{z_k(\omega) = \mathbf{I}_k(x,\omega)\right\}\right)$. By measurability assumption on $f_{ki}(x,\omega)$ it follows measurability of $z_k(\omega)$. Take an arbitrary $k \in K$ and $\omega \in \Omega_k$. If $\max_{i \in I_k} f_k(x,\omega) \le 0$, then $z_k(\omega) = 1$ and inequality constraint in (13), obviously, is fulfilled. If $\max_{i \in I_k} f_k(x,\omega) > 0$, then $z_k(\omega) = 0$ and inequality constraint in (13) is fulfilled due to assumption (10). Thus in all cases pare $\left(x, \left\{z_k(\omega) = \mathbf{I}_k(x,\omega\right\}\right)$ satisfies (13). Integrating $\mathbf{I}_k(x,\omega) = z_k(\omega)$ over $\omega \in \Omega$ we obtain

$$g_k(x) = \int_\Omega \mathbf{I}_k(x,\omega)P\left(d\omega\right) = \int_\Omega z_k(\omega)P(d\omega) = p_k(x, z_k)$$

and

$$G_j\left(x, g_1(x), ..., g_K(x)\right) = G_j\left(x, p_1(x,z_1), ..., p_K(x,z_K)\right), \qquad j \in J \cup \{0\}.$$

Hence pare $\left(x, \left\{z_k(\omega) = \mathbf{I}_k(x,\omega)\right\}\right)$ satisfy constraints (12) by (2). Thus for each feasible solution x of problem (1), (2) we point out a feasible solution $\left(x, \left\{z_k(\omega)\right\}\right)$ of (11) - (14) with not less (in fact equal) objective function value.

Conversely, assume $\left(x, \left\{z_k(\omega)\right\}\right)$ be a feasible solution of (11) - (14) and show that x is a required feasible solution of (1), (2). Take an arbitrary $k \in K$ and $\omega \in \Omega$. If $z_k(\omega) = 0$, then $\mathbf{I}_k(x,\omega) \ge 0 = z_k(\omega)$ (and (10) holds). If $z_k(\omega) = 1$, then (13) implies that $\max_{i \in I_k} f_k(x,\omega) \le 0$, and, hence, $\mathbf{I}_k(x,\omega) = 1 = z_k(\omega)$. Thus in any case $\mathbf{I}_k(x,\omega) \ge z_k(\omega)$ и hence

$$g_k(x) = \int_\Omega \mathbf{I}_k(x,\omega)P\left(d\omega\right) \ge \int_\Omega z_k(\omega)P(d\omega) = p_k(x, z_k).$$

By monotonisity of functions $G_j(x, y_K)$ over arguments $y_K \ge 0$, holds

$$G_j\left(x, g_K(x)\right) \ge G_j\left(x, p_K(x, z_K)\right), \qquad j \in J \cup \{0\}.$$

Thus for each feasible solution $\left(x, \left\{z_k(\omega)\right\}\right)$ of (11) - (14) we have a feasible point x of problem (1), (2) with not less objective function value,

$$G_0\left(x, g_K(x)\right) \ge G_0\left(x, p_K(x, z_K)\right).$$

The proof is complete.

Next statement establishes relations between integral variables of (11) - (14). Similar statement for a chance constrained problem was given in (Ruszczynski 1992, Lemma 3).

Theorem 3 (on valid inequalities between integral variables). If for some $k' \in K$, some $\omega', \omega'' \in \Omega$ holds

$$f_{k'}(x,\omega') = \max_{i\in I_{k'}} f_{k'i}(x,\omega') \le \max_{i\in I_{k'}} f_{k'}(x,\omega'') = f_{k'}(x,\omega''), \forall x \in X \quad (15)$$

then problem (11) - (14) is equivalent to the same problem supplemented with additional inequality

$$z_{k'}(\omega') \ge z_{k'}(\omega''), \quad (16)$$

and any solution of (11) - (14), (17) is a solution of (11) - (14).

Proof. Obviously, any feasible solution of (11) - (14), (18) is feasible for (11) - (14). Conversely, let (x, z_K) be a feasible solution for (11) - (14). Define $\hat{z}_K = \{\hat{z}_k(\omega)\}$, where $\hat{z}_k(\omega) = z_k(\omega)$ for all $(k,\omega) \in K \times \Omega \setminus ((k',\omega'') \cup (k',\omega''))$ and

$$\hat{z}_{k'}(\omega') = \begin{cases} z_{k'}(\omega'), & f_{k'}(x,\omega') > 0, \\ z_{k'}(\omega'), & f_{k'}(x,\omega') \le 0,\ f_{k'}(x,\omega'') > 0, \\ \max\{z_{k'}(\omega'), z_{k'}(\omega'')\}, & f_{k'}(x,\omega') \le 0,\ f_{k'}(x,\omega'') \le 0; \end{cases}$$

$$\hat{z}_{k'}(\omega'') = \begin{cases} z_{k'}(\omega''), & f_{k'}(x,\omega') > 0, \\ z_{k'}(\omega''), & f_{k'}(x,\omega') \le 0,\ f_{k'}(x,\omega'') > 0, \\ \max\{z_{k'}(\omega'), z_{k'}(\omega'')\}, & f_{k'}(x,\omega') \le 0,\ f_{k'}(x,\omega'') \le 0; \end{cases}$$

Consider correspondence $(x, z_K) \to (x, \hat{z}_K)$, where $\hat{z}_K = \{\hat{z}_k(\omega), \omega \in \Omega, k \in K\}$. Let us show that $(x, \hat{z}_K)$ is feasible for (11) - (14), (16). Obviously, $\hat{z}_k(\omega) \in \{0,1\}$ and $\hat{z}_k(\omega) \ge z_k(\omega)$ for all ω, k, so by monotonisity of $G_0(x,\cdot)$, $G_j(x,\cdot)$ point $(x, \hat{z}_K)$ satisfy (12), (14) and $G_0(x,\hat{z}_K) \ge G_0(x, z_K)$. For $(k,\omega) \in K \times \Omega \setminus ((k',\omega'') \cup (k',\omega''))$ constraints (13) are fulfilled by feasibility assumption. It remains to check (16) and inequalities

$$f_{k'i}(x,\omega') \le N_{k'i}(\omega')(1 - \hat{z}_{k'}(\omega')), \qquad i \in I_{k'}; \quad (19)$$

$$f_{k'i}(x,\omega'') \le N_{k'i}(\omega'')(1 - \hat{z}_{k'}(\omega'')), \qquad i \in I_{k'}. \quad (20)$$

If $f_{k'}(x,\omega') > 0$, then by assumption $f_{k'}(x,\omega') \ge f_{k'}(x,\omega') > 0$ and hence $z_{k'}(\omega') = z_{k'}(\omega'') = \hat{z}_{k'}(\omega') = \hat{z}_{k'}(\omega'') = 0$, thus (16), (19), (20) are satisfied.

If $f_{k'}(x,\omega') \le 0,\ f_{k'}(x,\omega'') > 0$, then $z_{k'}(\omega'') = \hat{z}_{k'}(\omega'') = 0 \le z_{k'}(\omega') = \hat{z}_{k'}(\omega')$, so (16), (19), (20) are fulfilled.

If $f_{k'}(x,\omega') \le 0,\ f_{k'}(x,\omega'') \le 0$, then

$$f_{k'}(x,\omega') \le 0 \le N_{k'}(\omega')(1 - \hat{z}_{k'}(\omega')), \quad f_{k'}(x,\omega'') \le 0 \le N_{k'}(\omega'')(1 - \hat{z}_{k'}(\omega'')),$$

$\hat{z}_{k'}(\omega') = \hat{z}_{k'}(\omega'') = \max\{z_{k'}(\omega'), z_{k'}(\omega'')\}$ and hence (16), (19), (20) are satisfied. The proof is complete.

Remark 1. Theorem 3 implies that problem (11) - (14) can be safely supplemented by as much inequalities (16) as there are relations (15). Additional constraints (16) may be useful for speeding up solution methods for problem (11) - (14). If functions $\{ f_{k'i}(x,\omega), i \in I_k \}$ are monotonic (nondecreasing componentwise) in $\omega \in \Omega$ for any $x \in X$, then condition (15) is fulfilled for any $\omega' \le \omega''$ (componentwise domination). In general, for any ω', ω'' and k' condition (15) can be verified by checking inequality

$$\min_{x\in X} \left[f_{k'}(x,\omega'') - f_{k'}(x,\omega') \right] \ge 0 .$$

Remark 2. In this section probability space $\{ p(\omega), \omega \in \Omega \}$ is common for all probability functions involved, i.e. for all $k \in K$. But the space may be also different for different functions, i.e. $\{ p_k(\omega), \omega \in \Omega_k \}$. The latter case can be easily accommodated by putting $p_k(x, z_k) = \sum_{\omega \in \Omega_k} p_k(\omega) z_k(\omega)$ in (11), (12) and replacing Ω in (13), (14) by Ω_k . We encounter with this situation in Example 2.

2.2. Optimization problems with quantile functions

In this subsection we extend results of previous subsection to optimization problems with quantiles.

Let $\varphi_k : X \times \Omega \to R^1$ and $f_k : X \times \Omega \to R^1$ be finite valued functions; Ω be a support of probability measure P ; X be an arbitrary set; K be an ordered finite set. For $k \in K$ consider probability functions of the form

$$g_k(x, y_k) = \Pr\{ \omega \in \Omega : \ \varphi_k(x,\omega) \le y_k ,\ f_k(x,\omega) \le 0 \}$$

with two simultaneous constraints under probability sign, $\varphi_k(x,\omega) - y_k \le 0$ and $f_k(x,\omega) \le 0$. For any $x \in X$ function $g_k(x, y_k)$ is upper semicontinuous in y_k , and

$$g_k(x, y_k) \le \Pr\{ \omega \in \Omega : \ \varphi_k(x,\omega) \le y_k \} \to 0 \quad \text{as } y_k \to -\infty .$$

So a corresponding quantile function

$$q_k(x) = \min\{ y \in R^1 : g_k(x, y) \ge \alpha_k > 0 \}$$

is well defined.

Denote $y_K = \{ y_k ,\ k \in K \}$, $q_K(x) = \{ q_k(x),\ k \in K \}$. Consider optimization problems

$$G_0\left(x, q_K(x) \right) \to \inf_{x\in X} , \tag{21}$$

$$G_j\left(x, q_K(x) \right) \le 0 , \qquad j \in J , \tag{22}$$

where $G_j(x, y_K): X \times R^K \to R^1$ be monotonic in $y_K = \{y_k, k \in K\} \in R^K$ (non decreasing componentwise) function for each $x \in X$ and $j \in J$, J is some index set; R^K is Tihonov's product of K copies of R^1.

Next theorem reduces a complex optimization problem containing several quantile functions to a chance-constrained one and thus generalizes a similar result from (Kibzun and Kan 1996, Lemma 4.4), where single quantile minimization function case is considered.

Theorem 4. Problem (21), (22) is equivalent to

$$G_0(x, y_K) \to \inf_{x \in X, y_K \in R^K}, \tag{23}$$

$$G_j(x, y_K) \le 0, \qquad j \in J, \tag{24}$$

$$g_k(x, y_k) = \Pr\{\omega \in \Omega: \varphi_k(x,\omega) \le y_k, f_k(x,\omega) \le 0\} \ge \alpha_k, \quad k \in K. \tag{25}$$

Proof. To prove equivalence we again follow the scheme, outlined in Definition 1 and Statements 1, 2. Let $x \in X$ be feasible for (21), (22). This means that quantiles

$$q_k(x) = \min\{y \in R^1 : g_k(x, y) \ge \alpha_k\}$$

are known. Consider correspondence $x \to (x, q_K(x))$. By definition of the quantile,

$$g_k(x, q_k(x)) = \Pr\{\varphi_k(x,\omega) \le q_k(x), f_k(x,\omega) \le 0\} \ge \alpha_k.$$

Thus $(x, y_K = q_K(x))$ is feasible for (23)-(25) with the same value of the objective function.

Conversely, let (x, y_K) be feasible for (23)-(25). Consider correspondence $(x, y_K) \to x$. By the quantile definition $y_k \ge q_k(x)$ and by monotonicity of $G_j(x,\cdot)$, holds $G_j(x, y_K) \ge G_j(x, q_K(x))$. Hence $x \in X$ satisfies (22) and $G_0(x, q_K(x)) \le G_0(x, y_K)$. The proof is complete.

Next we reduce problems (21), (22) and (23)-(25) to a mixed integer (in fact mixed Boolean) programming problems. Assume that $p(\omega) > 0 \ \forall \omega \in \Omega$ and there exist constants m_k and measurable functions $M_k(\omega)$, $N_k(\omega)$ such that

$$\sup_{x \in X} f_k(x,\omega) \le N_k(\omega) < \infty, \qquad k \in K; \tag{26}$$

$$m_k \le \varphi_k(x,\omega) \le M_k(\omega) < \infty \quad \forall x \in X, \qquad \forall \omega \in \Omega, \qquad k \in K. \tag{27}$$

Assumptions (26), (27) differ from the analogues assumption (10) only in the left hand inequality (27), the latter means that quantile function $q_k(x)$ is assumed to be uniformly bounded from below. Denote $z_k(\omega): \Omega \to \{0,1\}$ measurable in ω

Boolean valued functions and linear functionals $p_k(x,z_k)=\int_\Omega z_k(\omega)P(d\omega)$, $k\in K$. Consider the problem

$$G_0\left(x,y_K\right)\to\inf_{x\in X,\{y_k\geq m_k\},\{z_k(\omega)\in\{0,1\}\}}, \quad (28)$$

$$G_j\left(x,y_K\right)\leq 0, \qquad j\in J; \quad (29)$$

$$\int_\Omega z_k(\omega)P(d\omega)\geq\alpha_k, \qquad k\in K; \quad (30)$$

$$\varphi_k(x,\omega)-y_k\leq\left(M_k(\omega)-m_k\right)\left(1-z_k(\omega)\right), \quad \omega\in\Omega, \qquad k\in K; \quad (31)$$

$$f_k(x,\omega)\leq N_k(\omega)\left(1-z_k(\omega)\right), \qquad \omega\in\Omega, \qquad k\in K; \quad (32)$$

$$z_k(\omega)\in\{0,1\}, \qquad \omega\in\Omega, \qquad k\in K. \quad (33)$$

Theorem 5. Under assumptions (26), (27) problems (21), (22) and (23)-(25) are equivalent to (28)-(33); x-component of a solution of problem (28)-(33) is a solution for problems (21), (22) and (23)-(25)

The statement follows from Theorems 4, 2.

Theorem 6. If for some $k'\in K$, some $\omega',\omega''\in\Omega$ holds

$$\varphi_{k'}(x,\omega')\leq\varphi_{k'}(x,\omega''), \qquad f_{k'}(x,\omega')\leq f_{k'}(x,\omega''), \qquad \forall x\in X,$$

then problem (28) - (33) is equivalent to the same problem supplemented with additional inequality

$$z_{k'}(\omega')\geq z_{k'}(\omega''), \quad (34)$$

and any solution of (28) - (33), (34) is a solution of (28) - (33).

The proof is similar to the one of Theorem 3.

Remark 3. In this subsection probability space is common for all quantile functions, but it can be different for some of them. This can be easily accommodated as indicated in Remark 2.

Remark 4. Probabilistic optimization problem may contain both probability and quantile functions. Obviously, it can be also reduced to mixed integer one combining formulations (11) - (14) and (28) - (33).

3. Remarks on solution techniques

If functions $G_j\left(x,y_K\right)$, $f_k(x,\omega)$, $\varphi_k(x,\omega)$ are linear in variables x,y_K, then problems (11) - (14) and (28) - (33) are mixed integer linear ones (MILP), so can be solved by general MILP solvers. Improvements of the general MILP algorithms and specialized ones are considered in (Prékopa 1990; Sen 1992; Morgan, Eheart, Valocchi 1993; Dentcheva, Prékopa, Ruszczyński 2000; Vizvari

2002; Ruszczyński 2002; Beraldi and Ruszczyński 2002a, 2002b, 2005; Noyan, Rudolf and Ruszczyński 2006; Luedtke 2008; Noyan and Ruszczynski 2008; Beraldi and Bruni 2009; Luedtke, Ahmed, Nemhauser 2010; Saxena, Goyal, Lejeune 2010; Tanner and Ntaimo 2010; Küçükyavuz 2010). In this section we consider some peculiarities of MIP problem (11)-(14), which can be exploited for acceleration of a general branch and bound method. With obvious changes the considerations are applicable to the quantiles optimization problems of subsection 2.2. Remark that certain B&B method can be applied also directly to the primal probabilistic problems (3) and (4), (5) (Norkin 1998).

3.1. Case *J*=Ø: optimization of probabilities

Suppose that in (1), (2) probability functions enter only the objective function, $J=\varnothing$. Let us discuss peculiarities of the B&B method in this case. The method subdivides integrality constraints (14) into subsets $S_0=\{(\omega,k)\in\Omega\times K : z_k(\omega)=0\}$, $S_1=\{(\omega,k)\in\Omega\times K : z_k(\omega)=1\}$ and $S=\{(\omega,k)\in\Omega\times K : z_k(\omega)\in\{0,1\}\}$. Remark that constraints (13) can be safely reduced to

$$f_k(x,\omega)\le N_k(\omega)\left(1-z_k(\omega)\right), \qquad (\omega,k)\in S\cup S_1; \tag{35}$$

Relax integrality constraints (14) to the following one:

$$\begin{aligned} &z_k(\omega)=0,\quad (\omega,k)\in S_0;\qquad z_k(\omega)=0,\quad (\omega,k)\in S_1;\\ &0\le z_k(\omega)\le 1,\quad (\omega,k)\in S. \end{aligned} \tag{36}$$

Assume that the relaxed problem (11), (12), (35), (36) is efficiently solvable, let $\left(x^*,z_K^*\right)$ be its solution. Then its optimal value $G_0\left(x^*,p_K(x^*,z_K^*)\right)$ gives an upper bound on the optimal value of (11)-(14). Round noninteger components $z_k^*(\omega)$ to zero and denote the resulting integral vector $\hat{z}_K^*$. Obviously, $\left(x^*,\hat{z}_K^*\right)$ is feasible for (11)-(14) and $G_0\left(x^*,p_K(x^*,\hat{z}_K^*)\right)$ is a lower bound for the optimal value of (11)-(14).

Exploiting dominance constraints (16) for the selection of branching variables (select that free variable for branching, which dominates maximal number of the other free variables) substantially speeds up the performance of the outlined branch and bound method.

3.2. General case *J*≠Ø: chance constraints

The general case $J\ne\varnothing$ in certain sense is reduced to the considered particular case $J=\varnothing$. For the B&B method upper bounds are the same, assuming problem (11), (12), (35), (36) is efficiently solvable. To reduce search in the B&B scheme, one needs to use lower bounds. Once found any feasible solution of (11)-(14) gives a lower bound and can be used for deletion of bad subproblems. Feasible solutions are either automatically obtained as singleton solutions of subproblems, or by solving problems

$$\max_{j\in J} G_j\left(x, p_K(x, z_K)\right) \to \min_{x\in X,\ z_K}$$

subject to constraints (35), (36) till feasibility condition, $\max_{j\in J} G_j\left(x, p_K(x, z_K)\right) \le 0$, for example, by the B&B method, outlined in subsection 3.1.

4. Application examples

In this section we give some examples, illustrating general results of section 2.

Example 1 (Safety first (SF) portfolio selection). Let vector $x' = \left(x_1, ..., x_n\right) \in R^n$ denotes financial portfolio weights; X is a set of admissible portfolio structures, for example, $X = \left\{x \in R^n : \sum_{i=1}^n x_i = 1,\ x_i \ge 0\right\}$; $\omega = \left(\omega_1, ..., \omega_n\right) \in \Omega$ denotes a random vector of portfolio returns with probability $p(\omega)$; $\Omega \subset R^n$ is a finite set of vectors. Denote $f(x,\omega) = \sum_{i=1}^n x_i\omega_i$ random portfolio return and $\mu(x) = \sum_{i=1}^n x_i \sum_{\omega\in\Omega} p(\omega)\omega_i$ its average return. Safety first portfolio selection according to (Roy 1952) consists finding such portfolio $x \in X$ that guarantees average return z and minimizes the probability that the return falls below a critical level u. SF-portfolio can be found as solution of the problem

$$\Pi(u,z) = \min_{x\in X,\ \mu(x)\ge z}\left[P(x) = \Pr\left\{ f(x,\omega) < u\right\} = 1 - \Pr\left\{ f(x,\omega) \ge u\right\}\right]. \quad (37)$$

Denote $N(\omega) = -\min_{1\le i\le n} \omega_i$. Then problem (37) is reduced to the following mixed integer one:

$$\sum_{\omega\in\Omega} p(\omega)z(\omega) \to \min_{x\in X,\ \mu(x)\ge z}, \quad (38)$$

$$-\sum_{i=1}^n x_i\omega_i + u \le \left(N(\omega) + u\right)z(\omega), \qquad \omega\in\Omega. \quad (39)$$

If portfolio return $f(x,\omega) = \sum_{i=1}^n x_i\omega_i$ is component-wise monotonic in ω for any $x \in X$, then by Theorem 3 we can add to the problem formulation valid relations

$$z(\omega') \ge z(\omega'') \qquad \forall\, \omega', \omega'' \in \Omega:\ \omega' \le \omega'' \text{ (component-wise)}. \quad (40)$$

All such relations can be found by just analyzing the set of vectors of returns Ω. In section 5 we give an example of a numerical solution of (38), (39).

Kataoka (1963) suggested to formalize SF portfolio selection through chance constrained programming:

$$\begin{aligned} &\mu(x) \to \max_{x\in X}, \\ &P(x) = \Pr\left\{ f(x,\omega) < u\right\} \le \pi. \end{aligned} \quad (41)$$

According to (Goodall 2002) SF portfolio selection problem consists in maximization of the probability of high returns (greater than v) subject to a bound α on the critical fall of the return (below level u):

$$\begin{aligned} &P_0(x) = \Pr\{ f(x,\omega) \ge v\} \to \max_{x\in X}, \\ &P_1(x) = \Pr\{ f(x,\omega) < u\} \le \alpha. \end{aligned} \tag{42}$$

Other SF portfolio selection settings are considered in (Telser 1955/56; Arzac and Bawa 1977; Bawa 1978; Kibzun and Kan 1996; Li et al. 1998; Haque et al. 2004, 2007). Roy's (1952) approximation approach to solution of problem (37) consists in replacement of the probability by its upper bound obtained from Chebyshev inequality. In (Norkin and Boyko 2010) Roy's approach is improved (more accurate bound is used) and beside in case of finite scenario set Ω optimization problems (37) - (42) are equivalently reduced to mixed binary ones.

Example 2 (robust safety-first portfolio selection, ambiguous probabilistic optimization). Scenarios $\omega \in \Omega$ may be fixed but not their probabilities $p(\omega)$, they may be known to belong to some ambiguous probability set D, for example,

$$D = \left\{ p(\omega) \ge 0 : \, p(\omega) \in \left[a(\omega), b(\omega) \right], \textstyle\sum_{\omega\in\Omega} p(\omega) = 1 \right\}.$$

Then robust safety-first portfolio selection consists in solution of the problem

$$f(x) = \sup_{P\in D} \Pr\{ f(x,\omega) < u\} \to \min_{x\in X}. \tag{43}$$

Since

$$f(x) = \sup_{P\in D} \Pr\{ f(x,\omega) < u\} = \sup_{\{p(\omega)\}\in D} \textstyle\sum_{\omega\in\Omega} p(\omega) I_{\{f(x,\omega)<u\}},$$

then $\sup_{\{p(\omega)\}\in D}$ is achieved at extreme points of D. If D is polyhedral, supremum can be replaced by maximum over finite number of vertices $P_k = \{ p_k(\omega),\, \omega \in \Omega\}$, $k \in K$, of D and the problem is replaced by

$$f(x) = \sup_{P\in D} \Pr\{ f(x,\omega) < u\} = \max_{P_k, k\in K} \Pr\{ f(x,\omega) < u\} \to \min_{x\in X}.$$

The last problem can be reduced to MIP one as shown in subsection 2.1 and Remark 2. Function $f(x) = \sup_{P\in D} \Pr\{ f(x,\omega) < u\}$ is called an ambiguous probability function. Such functions may enter constraints in a probabilistic optimization problem, in this a case the latter is call ambiguous chance constrained problem (Iyengar and Erdogan 2006). As indicated, in case of polyhedral ambiguous probability set an ambiguous chance constrained problem is reduced to probabilistic programming problem with several probability functions and thus admits an equivalent MIP formulation. A different sampling based approach to ambiguous chance constrained optimization is considered in (Nemirovski and Shapiro 2006; Iyengar and Erdogan 2006).

Example 3 (quantile optimization problem (Kataoka 1963; Kibzun and Kan 1996)). Let $(\Omega, \Sigma, \mathrm{P})$ be a probability space, X be some set, $\varphi : X \times \Omega \to R^1$ and

$f : X \times \Omega \to R^1$ be measurable in the second argument functions. Define a probability function

$$g(x, y) = \Pr\left\{\varphi(x,\omega) \le y,\ f(x,\omega) \le 0\right\}$$

and a quantile function

$$q_\alpha(x) = \inf\left\{y \in R^1 : g(x, y) \ge \alpha\right\} = \inf\left\{y \in R^1 : \Pr\left\{\varphi(x,\omega) \le y,\ f(x,\omega) \le 0\right\} \ge \alpha\right\}.$$

The so-called inverse probabilistic problem has the form

$$q_\alpha(x) \to \min_{x\in X}. \tag{44}$$

It is equivalent to the problem (see (Kibzun and Kan 1996, section 4.2)),

$$\begin{aligned} &y \to \min_{x\in X,\, y\in R^1}, \\ &P(x, y) = \Pr\left\{\varphi(x,\omega) - y \le 0,\ f(x,\omega) \le 0\right\} \ge \alpha. \end{aligned}$$

Assume that

$$\begin{aligned} &p(\omega) > 0 \quad \forall \omega \in \Omega, \\ &\inf_{x\in X,\omega\in\Omega} \varphi(x,\omega) \ge m > -\infty, \\ &\sup_{x\in X} \varphi(x,\omega) \le M(\omega) < \infty, \\ &\sup_{x\in X} f(x,\omega) \le N(\omega) < \infty. \end{aligned},$$

Then

$$q_\alpha(x) = \inf\left\{y \in R^1 : g(x, y) \ge \alpha > 0\right\} = \inf\left\{y \ge m : g(x, y) \ge \alpha\right\}$$

And problem (44) is equivalent to

$$\begin{aligned} &y \to \min_{x\in X,\, y\ge m}, \\ &P(x, y) = \Pr\left\{\varphi(x,\omega) - y \le 0,\ f(x,\omega) \le 0\right\} \ge \alpha. \end{aligned} \tag{45}$$

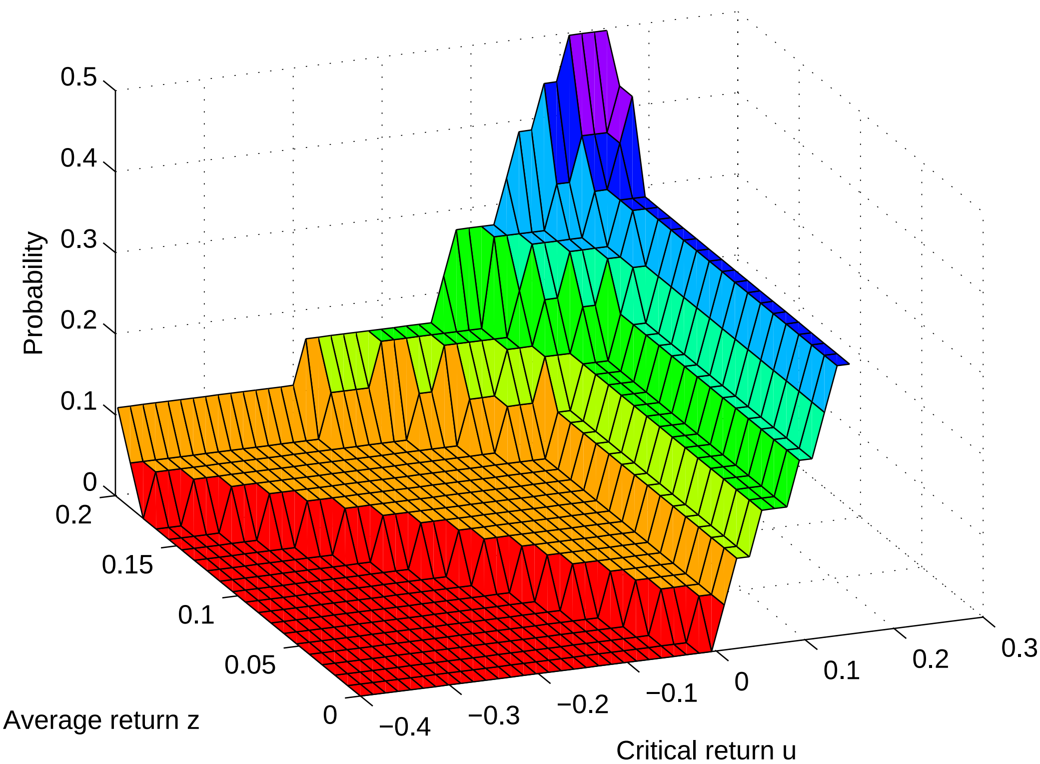
Let $z(\omega) : \Omega \to \{0,1\}$ be a measurable in ω Boolean valued function, define a linear functional $p(x, z) = \int_\Omega z(\omega) P(d\omega)$. Now (45) is equivalent by Theorem 5 to the following problem

$$\begin{aligned} &y \to \min_{x\in X,\, y\ge m,\ z(\omega)\in\{0,1\}}, \\ &\int_\Omega z(\omega) P(d\omega) \ge \alpha; \\ &\varphi(x,\omega) - y \le \left(M(\omega) - m\right)\left(1 - z(\omega)\right),\ \ \omega \in \Omega;, \\ &f(x,\omega) \le N(\omega)\left(1 - z(\omega)\right), \quad \omega \in \Omega; \\ &z(\omega) \in \{0,1\}, \quad \omega \in \Omega. \end{aligned}$$

5. Numerical experiment

We illustrate performance of the B&B method outlined in subsection 3.1 on the safety first portfolio selection problem (37) from Example 1. Portfolio consists of 9 securities, stocks of 9 large American firms, with recorded 18 annual returns since 1937 till 1954 as shown in (Markowitz 1959, Table1, page 13). Thus vectors x and ω have nine components, and the set of (equi-probable) scenarios Ω consists of 18 vectors. Figure 1 presents optimal value function $\Pi(u,z)$ (37) found by solving problem (38), (39) by the B&B method from subsection 3.1 implemented in MATLAB 7.9.0 (solution time is 601.5 seconds for 800 greed points (u,z) on AMD Athlon 2.6 GHz). Unlike (Norkin and Boyko 2010) in the present paper we exploit relations (40) for selection of branching variables and obtain a substantial acceleration (two times) of the B&B method. The constructed surface $\Pi(u,z)$ indicates that the more average profit the more downfall risk for the optimal SF portfolio.

Fig. 1. Probability surface Ï(u,z)



6. Concluding remarks

In the present paper we extended mixed integer reformulation approach for solution of the chance constrained problems (with discrete probability distribution) to more complicated problems, containing multiple probability/quantile functions both in the objective and constraint parts of the problem. The basic assumption is that problem functions monotonically depend on the probability/quantile functions involved. The resulting mixed integer problem is large scale. The number of Boolean variables in the transformed problem equals to the number of scenarios times the number of probability functions involved; the number of mixed integer constraints equals to the total number of constraints under probability signs times the number of scenarios. The resulting problem can be solved by available MIP solvers, however for the probability function optimization we implemented in MATLAB our own version of the B&B method. Using certain preference relations between scenarios we obtained valid inequalities between corresponding binary variables and used them to speed up B&B procedure.

The approach under examination is rather general, combined with scenario approximation technique it allows to solve real world probabilistic problems. The main difficulty in application of the approach is a possible extremely large scale of arising MIP problems, especially in dynamic applications. So quality of the probability function approximation, scenarios reduction technique, improvement and adaptation of existing MIP solvers, development of heuristic solution methods are the main topics in the development of the approach.

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