Local path-following property of inexact interior methods in nonlinear programming

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Abstract. We study the local behavior of a primal-dual inexact interior point methods for solving nonlinear systems arising from the solution of nonlinear optimization problems or more generally from nonlinear complementarity problems. The algorithm is based on the Newton method applied to a sequence of perturbed systems that follows by perturbation of the complementarity equations of the original system. In case of an exact solution of the Newton system, it has been shown that the sequence of iterates is asymptotically tangent to the central path (Armand and Benoist, Math Program 115:199–222, 2008). The purpose of the present paper is to extend this result to an inexact solution of the Newton system. We give quite general conditions on the different parameters of the algorithm, so that this asymptotic property is satisfied.

Key words. constrained optimization, interior point methods, nonlinear programming, primal-dual methods, nonlinear complementarity problems, inexact Newton method

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1 Introduction

We consider nonlinear programming problems of the form

$$\Phi(x,y,z)=0 \quad \text{and} \quad 0\le x\perp z\ge 0$$

where $\Phi:\mathbb{R}^{n+m+n}\to\mathbb{R}^{n+m}$ is supposed to be smooth and the perp notation means that in addition to nonnegative bounds, the variables x and z are orthogonal. This problem is known as an *implicit mixed complementarity problem* (MCP for short) and naturally arises in numerous applications [8, 9]. Consider for example the nonlinear programming problem

$$\begin{array}{ll} \text{minimize} & f(x), \\ \text{subject to} & g(x)=0, \\ & x\ge 0, \end{array}$$

for differentiable functions $f:\mathbb{R}^n\to\mathbb{R}$ and $g:\mathbb{R}^n\to\mathbb{R}^m$. The first order necessary optimality conditions can be written under the above form with

$$\Phi(x,y,z)=\begin{pmatrix}\nabla f(x)+\nabla g(x)y+z\\ g(x)\end{pmatrix}.$$

For convenience, let us rewrite the MCP under the form

$$F(w)=0 \quad \text{and} \quad v\ge 0, \tag{1.1}$$

where $w=(x,y,z)$, $v=(x,z)$ and $F:\mathbb{R}^{n+m+n}\to\mathbb{R}^{n+m+n}$ is defined by

$$F(w)=\begin{pmatrix}\Phi(x,y,z)\\ XZe\end{pmatrix},$$

where $X=\operatorname{diag}(x_1,\dots,x_n)$, $Z=\operatorname{diag}(z_1,\dots,z_n)$ and e is the vector of all ones.

The algorithm that we consider belongs to the class of interior point methods, in which Newton-type iterations are applied to a perturbation of the complementarity equations, namely

$$F(w)=\mu\tilde{e}, \tag{1.2}$$

where $\mu>0$ and $\tilde{e}=\begin{pmatrix}0 & e^\top\end{pmatrix}^\top$. The perturbation parameter is progressively driven to zero during the iterations, while the components of v are kept sufficiently positive, so that at the limit we recover a solution of (1.1). Let us denote by w^* such a solution. Under standard nondegeneracy assumptions, the solution of (1.2) defines locally a smooth curve $\mu\mapsto w(\mu)$, called the *central path*, whose end point is $w^*=w(0)$.

In this context, we provide a local convergence analysis for a primal-dual algorithm of the form

$$F'(w_k)(w_k^+-w_k)+F(w_k)-\mu_{k+1}\tilde{e}=r_k, \tag{1.3}$$

for some suitable sequence $\{\mu_k\}$ of positive numbers converging to zero and some residual vector r_k. At the beginning, an initial starting guess w_0, with $v_0 > 0$, is supposed to be given. To maintain strict feasibility with respect to the nonnegative constraints, a safeguarding rule, the so-called *fraction to the boundary rule*, is applied at each iteration: let α_k be the greatest $\alpha \in (0,1]$ such that

$$v_k + \alpha(v_k^+ - v_k) \geq (1-\tau_k)v_k, \tag{1.4}$$

where $\tau_k \in (0,1)$ (the inequality is understood componentwise). The new iterate is then set according to

$$w_{k+1} = w_k + \alpha_k(w_k^+ - w_k). \tag{1.5}$$

Recently [2], it has been shown that the sequence of iterates generated by algorithm (1.3)–(1.5) with an exact solution of the Newton system, that is with $r_k = 0$, is asymptotically tangent to the central path in the sense that

$$w_k = w(\mu_k) + o(\mu_k),$$

provided that the rate of convergence of $\{\mu_k\}$ is subquadratic. This is a core property of interior point algorithms. It shows that the sequence of iterates *naturally* follows the central path asymptotically, regardless of any proximity condition. In particular it implies that the full Newton step ($\alpha_k = 1$) is asymptotically accepted by the fraction to the boundary rule and that the rate of convergence of the sequence $\{w_k\}$ is the same as those of $\{\mu_k\}$. Our purpose is to extend this result to an algorithm with inexact solutions of the Newton systems and by the way to provide conditions on the residual r_k, the perturbation parameter μ_k and the boundary parameter τ_k such that the same property holds.

The usual analysis of the truncated or inexact Newton algorithms [6, 7, 15, 14] for solving a nonlinear equation $G(x) = 0$, involves a *forcing sequence* $\{\eta_k\}$ such that the linear equation residual r_k is forced to zero at least as fast as η_k times the (non-linear) equation $G(x_k)$ to solve: $\|r_k\| \leq \eta_k\|G(x_k)\|$. Whenever $\eta_k < 1$ remains bounded away from one, linear convergence in ensured. If $\{\eta_k\}$ goes to zero, superlinear convergence is obtained, while if $\eta_k = O(\|G(x_k)\|^\xi)$ for $0 < \xi \leq 1$, we get superlinear convergence with order at least $1+\xi$. In our context, a natural extension should be $\|r_k\| \leq \eta_k\|F(w_k) - \mu_{k+1}\tilde{e}\|$. Note that this condition could be too restrictive in case of an iterate *on* the central path, in the sense that $F(w_k) = \mu_{k+1}\tilde{e}$, implying that $r_k = 0$. In our study, we therefore propose a residual of the form

$$\|r_k\| \leq \eta_k\|F(w_k) - \mu_{k+1}\tilde{e}\| + \zeta_k, \tag{1.6}$$

for a forcing sequence $\{\eta_k\}$, with $0 \leq \eta_k < 1$ for all k and a non-negative sequence $\{\zeta_k\}$ that converges to zero. The parameter ζ_k appears as a relaxation term, but we will see later that it also allows to encompass more general conditions on the residual. We will show in Section 3 that our inexact strategy yields a locally convergent algorithm even if $\{\eta_k\}$ remains only bounded away from one. However, since we strive to prove that $\{w_k\}$ asymptotically approaches in a tangential manner the central path, the extension of the linearly convergent variants (η_k not converging to 0) is hopeless. So, in our asymptotic analysis, developed in Section 5, we will only consider sequences $\{\eta_k\}$ converging to zero. The superlinear variants requires that $\eta_k\mu_k = o(\mu_{k+1})$ and $\zeta_k = o(\mu_{k+1})$. For the order $1+\xi$ variants, that is for a *dynamic* forcing term of the form $\eta_k = O(\|F(w_k) - \mu_{k+1}\tilde{e}\|^\xi)$ we will assume that $\mu_{k+1} \geq \beta\mu_k^{1+\sigma}$ for some $\beta > 0$ and $0 < \sigma < \xi$. This will be the most difficult case to analyze.

This study is motivated by practical considerations, indeed in practice, when solving large scale problems, iterative methods are often used to compute the Newton step. For instance, seven out of the ten papers in the special issue of *Computational Optimization and Applications* devoted to linear algebra in interior point methods were concerned with iteratively solving the linear systems [13]. For linear and nonlinear problems, in many papers including [4, 1, 5, 12], the stopping rule $\|r_k\| \leq \eta_k\mu_k$ is used. Note that our study encompasses this particular case by means of suitable choices of the forcing terms used in (1.6). The results in those references concerning inexact solutions of the linear systems vary from properties of the preconditioner or the iterative solver to asymptotic analysis, sometimes also addressing global convergence. In one paper [1] for linear optimization, a result about the iterates remaining in a neighborhood of the central path is given in a complexity analysis context. Our results complement the above cited contributions, and are mostly directly applicable to all those algorithms for non-linear problems including MCPs. By satisfying our conditions, the central path is asymptotically followed, and better numerical behavior is thus expected.

The remainder of the paper is organized as follows. After settling the notation and assumptions in Section 2, convergence in a suitable neighborhood is addressed in Section 3, while asymptotic behavior is explored in Section 4 and Section 5, then numerical experiments are presented in Section 6. In Section 3, we provide a general convergence analysis for any forcing sequence bounded away from one. These results can be seen as an extension of the local convergence analysis of inexact Newton methods, see e.g. [14, Chapter 6], when solving a parametrized system such as (1.2). Section 4 is devoted to technical lemmas on which the asymptotic results in Section 5 do rely; we herein prove that $\|w_k - w(\mu_k)\| = O(\mu_k)$ (Lemma 4.3). Note that this result is a nontrivial extension of [2, Lemma 7] and has required an entirely new proof. In Section 5, we provide analysis for the two inexact variants of forcing sequences converging to zero; we justify that the inexact Newton step w_k^+ satisfies $w_k^+ = w(\mu_{k+1}) + o(\mu_{k+1})$ (Lemma 5.4), and thus the unit stepsize asymptotically satisfies the fraction of the boundary rule, which allows to conclude that $w_k = w(\mu_k) + o(\mu_k)$ (Theorem 5.5). Therefore, asymptotically, our scheme naturally follows the primal-dual central path. We then provide in Section 6 some numerical illustrations which compare different choices of forcing sequences.

2 Notation, assumptions and basic results

For $i \in \{1, \ldots, n\}$, we denote by e_i the vectors of the canonical basis of $\mathbb{R}^n$ and the i-th component of a vector $x \in \mathbb{R}^n$ will be denoted by $[x]_i$. Moreover in this paper vector inequalities are understood componentwise. Given two vectors $x, y \in \mathbb{R}^n$, their Euclidean scalar product is denoted by $x^\top y$ and the associated ℓ_2 norm is $\|x\| = (x^\top x)^{1/2}$. The open Euclidean ball centered at x with radius $r > 0$ is denoted by $B(x, r)$, that is $B(x, r) := \{x' \in \mathbb{R}^n : \|x' - x\| < r\}$.

For a differentiable function $g : \mathbb{R}^n \to \mathbb{R}^m$ and for all $x \in \mathbb{R}^n$, $\nabla g(x)$ denotes the transpose of the Jacobian matrix of g at x, i.e., the $n \times m$ matrix whose i-th column is $\nabla g_i(x)$.

For two nonnegative scalar sequences $\{a_k\}$ and $\{b_k\}$ converging to zero, we use the Landau symbols: $a_k = o(b_k)$ if there exists a sequence $\{c_k\}$ converging to zero, such that $a_k = c_k b_k$ for large k; $a_k = O(b_k)$ if there exists a constant $c > 0$, such that $a_k \leq c b_k$ for large k. We use similar symbols with vector arguments, in which case they are understood normwise.

Throughout the paper, we assume that the original complementarity problem minimization problem has a local solution $w^* = (x^*, y^*, z^*) \in \mathbb{R}^{n+m+n}$. We also suppose that the following assumptions are satisfied. To simplify our presentation, the assumptions are stated according to the properties of the function F, instead to the function Φ of the original complementarity problem.

Assumption 2.1 The functions F is continuously differentiable and its Jacobian is Lipschitz continuous over an open neighbourhood of w^*, that is, there exist $\delta_1 > 0$ and $L > 0$ such that for all $w, w' \in B(w^*, \delta_1)$,

$$\|F'(w) - F'(w')\| \leq L\|w - w'\|.$$

Assumption 2.2 The Jacobian $F'(w^*)$ is nonsingular.

Note that with the particular form of the function F, the second assumption implies that strict complementarity holds at w^*. Indeed, we have

$$F'(w) = \begin{pmatrix} \Phi'_x & \Phi'_y & \Phi'_z \\ Z & 0 & X \end{pmatrix},$$

so that if $F'(w^*)$ is nonsingular, then

$$a := \min\{[x^*]_i + [z^*]_i : i = 1, \ldots, n\} > 0. \tag{2.1}$$

The two following lemmas are well known (see e.g. [11]). A proof of property (2.2) below is given in [3].

Lemma 2.3 *Let Assumptions 2.1-2.2 hold. There exist $\delta_2 > 0$, $K > 0$ and $K' > 0$ such that for all $w \in B(w^*, \delta_2)$,*

$$\|F'(w)^{-1}\| \leq K \quad \textit{and} \quad \|F'(w)\| \leq K'.$$

Lemma 2.4 *Let Assumptions 2.1-2.2 hold. There exist $\delta_3 > 0$, $\bar{\mu} > 0$ and a continuously differentiable function $w(\cdot) : (-\bar{\mu}, \bar{\mu}) \to \mathbb{R}^{n+m+n}$ such that for all $(w, \mu) \in B(w^*, \delta_3) \times (-\bar{\mu}, \bar{\mu})$,*

$$F(w) = \mu\tilde{e} \quad \textit{if and only if} \quad w(\mu) = w.$$

There exists $C > 0$, such that for all μ, $\mu' \in (-\bar{\mu}, \bar{\mu})$,

$$\|w(\mu) - w(\mu')\| \leq C|\mu - \mu'|.$$

In addition, let $w(\mu) := (x(\mu), y(\mu), z(\mu))$, then for all $i \in \{1, \ldots, n\}$,

$$[x^*]_i[z'(0)]_i + [x'(0)]_i[z^*]_i = 1. \tag{2.2}$$

We will first show in the next section that in a suitable neighborhood, the inexact path following method is indeed attracted by w^*, and further in section 5 that, using suitable forcing sequences, is attracted by the trajectory.

3 Convergence

We now address the convergence of the inexact interior point algorithm (1.3)–(1.5) in which the residual of the linear system satisfies (1.6). We consider the general case where the forcing sequence $\{\eta_k\}$ satisfies $0 \leq \eta_k \leq \bar{\eta} < 1$. The hardest case happens to be the case when this sequence is not assumed to converge to zero. For this case, the bounds given using an arbitrary norm would involve using the constants K and K' and are too crude to obtain convergence since in the linear convergence analysis, constants do matter. Therefore, we next use the norm induced by $F'(w(\mu))$ defined as

$$\|u\|_\mu := \|F'(w(\mu))u\|,$$

whenever the Jacobian matrix is nonsingular. Such method is classical for an analysis of an inexact Newton method, see for example [14]. Note that with this notation we have that

$$\|u\|_0 := \|F'(w^*)u\|$$

since $w(0) = w^*$. Moreover, by virtue of Lemma 2.3, for all value of μ such that $w(\mu) \in B(w^*, \delta_2)$ and for all vector u, we have

$$\frac{1}{K}\|u\| \leq \|u\|_\mu \leq K'\|u\|. \tag{3.1}$$

For w close to w^* and $\mu^+ > 0$, we define the inexact Newton iterate at (w, μ^+) by

$$F'(w)(w^+ - w) + F(w) - \mu^+\tilde{e} = r, \tag{3.2}$$

where r is the residual of the linear system. It is assumed that this residual satisfies

$$\|r\| \leq \eta \|F(w) - \mu^+ \tilde{e}\| + \zeta, \tag{3.3}$$

for given forcing terms $\eta \in [0,1)$ and $\zeta \geq 0$.

We hereafter assume that Assumption 2.1 holds for given positive constants δ_1 and L. We then assume that the positive constants δ_2, K, K', δ_3, $\bar{\mu}$ and C are chosen such that the conclusions of Lemma 2.3 and 2.4 hold. Let us define the constant

$$\bar{\delta} = \min\{\delta_1, \delta_2, \delta_3\}. \tag{3.4}$$

We also assume that $\bar{\mu}$ is such that $w(\mu) \in B(w^*, \bar{\delta})$ for all $\mu \in (0, \bar{\mu})$, otherwise $\bar{\mu}$ is reduced accordingly.

Lemma 3.1 *There exist $K_1 > 0$ and $K_2 > 0$ such that for all $w \in B(w^*, \bar{\delta})$, $\mu^+ \in [0, \bar{\mu})$, $\eta \in [0,1)$ and $\zeta \geq 0$, if the residual defined by (3.2) satisfies (3.3), then*

$$\|w^+ - w(\mu^+)\|_{\mu^+} \leq \eta \|w - w(\mu^+)\|_{\mu^+} + K_1 \|w - w(\mu^+)\|^2 + K_2 \, \zeta.$$

Proof. Let $w \in B(w^*, \bar{\delta})$, $\mu^+ \in [0, \bar{\mu})$, $\eta \in [0,1)$ and $\zeta \geq 0$. We have

$$\begin{aligned} F'(w(\mu^+))(w^+ - w(\mu^+)) &= (F'(w(\mu^+)) - F'(w))(w^+ - w(\mu^+)) \\ &\quad + F'(w)(w^+ - w) + F'(w)(w - w(\mu^+)). \end{aligned}$$

Using (3.2) and $F(w(\mu^+)) = \mu^+ \tilde{e}$, we also have

$$\begin{aligned} F'(w)(w^+ - w) &= r + \mu^+ \tilde{e} - F(w) \\ &= r + \int_0^1 F'(w + t(w(\mu^+) - w))(w(\mu^+) - w) \, \mathrm{d}t. \end{aligned}$$

We then deduce that

$$\begin{aligned} F'(w(\mu^+))(w^+ - w(\mu^+)) &= r + (F'(w(\mu^+)) - F'(w))(w^+ - w(\mu^+)) \\ &\quad + \int_0^1 (F'(w + t(w(\mu^+) - w)) - F'(w))(w(\mu^+) - w) \, \mathrm{d}t. \end{aligned}$$

Taking the norm, then applying Cauchy-Schwarz inequality and using Assumption 2.1, we get

$$\|w^+ - w(\mu^+)\|_{\mu^+} \leq \|r\| + L \|w^+ - w(\mu^+)\| \|w - w(\mu^+)\| + \frac{L}{2} \|w - w(\mu^+)\|^2. \tag{3.5}$$

On one hand, using the equality

$$F(w) - \mu^+ \tilde{e} = F'(w(\mu^+))(w - w(\mu^+)) + F(w) - F(w(\mu^+)) - F'(w(\mu^+))(w - w(\mu^+)),$$

then taking the norm, using (3.3) and again the fact that F' is Lipschitzian, we obtain

$$\|r\| \leq \eta\|w - w(\mu^+)\|_{\mu^+} + \eta\frac{L}{2}\|w - w(\mu^+)\|^2 + \zeta. \tag{3.6}$$

On the other hand, by using (3.2), (3.3) and Lemma 2.3, we have

$$\begin{aligned}
\|w^+ - w(\mu^+)\| &\leq \|w^+ - w\| + \|w - w(\mu^+)\| \\
&\leq \|F'(w)^{-1}(r + \mu^+\tilde{e} - F(w))\| + \|w - w(\mu^+)\| \\
&\leq K(\|r\| + \|F(w) - F(w(\mu^+))\|) + \|w - w(\mu^+)\| \\
&\leq K(1+\eta)\|F(w) - F(w(\mu^+))\| + K\zeta + \|w - w(\mu^+)\| \\
&\leq KK'(1+\eta)\|w - w(\mu^+)\| + K\zeta + \|w - w(\mu^+)\| \\
&= K_1'\|w - w(\mu^+)\| + K\zeta
\end{aligned} \tag{3.7}$$

where $K_1' = 1 + KK'(1+\eta)$. Finally, using (3.7) and (3.6) in (3.5), the conclusion follows with the constants $K_1 = \frac{1}{2}L(1 + \eta + 2K_1')$ and $K_2 = 1 + LK\bar{\delta}$. □

Next, we need to express the evolution of $w^+ - w^*$ with respect to $w - w^*$. We will use the norm $\|\cdot\|_0$ and the above lemma, bounding the conversion between norms.

Lemma 3.2 *There exist three positive constants K_3, K_4 and K_5 such that for all $w \in B(w^*, \bar{\delta})$, $\mu^+ \in [0, \bar{\mu})$, $\eta \in [0,1)$ and $\zeta \geq 0$, if the residual defined by (3.2) satisfies (3.3), then*

$$\|w^+ - w^*\|_0 \leq \eta\|w - w^*\|_0 + K_3\|w - w^*\|^2 + K_4\mu^+ + K_5\,\zeta. \tag{3.8}$$

Proof. Let $w \in B(w^*, \bar{\delta})$, $\mu^+ \in [0, \bar{\mu})$, $\eta \in [0,1)$ and $\zeta \geq 0$. From the definition of $\|\cdot\|_0$, Lemmas 2.3 and 2.4, we first have

$$\begin{aligned}
\|w^+ - w^*\|_0 &\leq \|w^+ - w(\mu^+)\|_0 + \|w(\mu^+) - w^*\|_0 \\
&\leq \|w^+ - w(\mu^+)\|_0 + K'C\mu^+.
\end{aligned} \tag{3.9}$$

Next, by using Assumption 2.1, Lemma 2.4, Lemma 3.1, inequality (3.7) and the fact that $ab \leq \frac{1}{2}(a^2 + b^2)$ for real numbers a and b, we obtain

$$\begin{aligned}
&\|w^+ - w(\mu^+)\|_0 \\
&= \|F'(w^*)(w^+ - w(\mu^+))\| \\
&\leq \|F'(w(\mu^+))(w^+ - w(\mu^+))\| + \|F'(w^*) - F'(w(\mu^+))\|\|w^+ - w(\mu^+)\| \\
&\leq \|w^+ - w(\mu^+)\|_{\mu^+} + LC\mu^+\|w^+ - w(\mu^+)\| \\
&\leq \eta\|w - w(\mu^+)\|_{\mu^+} + K_1\|w - w(\mu^+)\|^2 + LC\mu^+\|w^+ - w(\mu^+)\| + K_2\,\zeta \\
&\leq \eta\|w - w(\mu^+)\|_{\mu^+} + K_1\|w - w(\mu^+)\|^2 + LCK_1'\mu^+\|w - w(\mu^+)\| + K_2'\zeta \\
&\leq \eta\|w - w(\mu^+)\|_{\mu^+} + (K_1 + K_3')\|w - w(\mu^+)\|^2 + K_3'(\mu^+)^2 + K_2'\zeta
\end{aligned}$$

where $K_2' = KLC\bar{\mu} + K_2$ and $K_3' = \frac{1}{2}LCK_1'$.

Using $\|w - w(\mu^+)\| \leq \|w - w^*\| + \|w^* - w(\mu^+)\| \leq \|w - w^*\| + C\mu^+$ and the inequality $(a+b)^2 \leq 2(a^2+b^2)$ for real numbers a and b, we obtain

$$\|w^+ - w(\mu^+)\|_0 \leq \eta\|w - w(\mu^+)\|_{\mu^+} + 2(K_1 + K_3')\|w - w^*\|^2 + K_4'\mu^+ + K_2'\zeta \quad (3.10)$$

with $K_4' = (2(K_1 + K_3')C^2 + K_3')\bar{\mu}$.

At last, we have

$$\begin{aligned}
&\|w - w(\mu^+)\|_{\mu^+} \\
&\leq \|w - w^*\|_{\mu^+} + \|w^* - w(\mu^+)\|_{\mu^+} \\
&\leq \|w - w^*\|_{\mu^+} + K'C\mu^+ \\
&\leq \|F'(w^*)(w - w^*)\| + \|(F'(w(\mu^+)) - F'(w^*))(w - w^*)\| + K'C\mu^+ \\
&\leq \|w - w^*\|_0 + LC\mu^+\|w - w^*\| + K'C\mu^+ \\
&\leq \|w - w^*\|_0 + C(L\bar{\delta} + K')\mu^+. \qquad (3.11)
\end{aligned}$$

The conclusion of the lemma follows by using (3.11) and (3.10) in (3.9), defining the constants $K_3 = 2(K_1 + K_3')$, $K_4 = C(L\bar{\delta} + K') + K_4'$ and $K_5 = K_2'$. □

Corollary 3.3 *Let $\bar{\eta} \in [0,1)$. There exist two constants $0 < \delta \leq \bar{\delta}$ and $M > 0$ such that for all $w \in B(w^*, \delta)$, for all $\mu^+ \in [0, \bar{\mu})$ and for all $\eta \in [0, \bar{\eta}]$, if the residual defined by (3.2) satisfies (3.3), then the inexact Newton iterate w^+ at (w, μ^+) exists and*

$$\|w^+ - w^*\|_0 \leq \frac{1+\bar{\eta}}{2}\|w - w^*\|_0 + M(\mu^+ + \zeta). \quad (3.12)$$

Proof. Set $\delta = \min\{\frac{1-\bar{\eta}}{2KK_3}, \bar{\delta}\}$ and $M = \max\{K_4, K_5\}$, where K, K_3 and K_4 are constants chosen so that conclusions of Lemmas 2.3 and 3.2 hold. Let $w \in B(w^*, \delta)$, $\mu^+ \in [0, \bar{\mu})$ and $\eta \in [0, \bar{\eta}]$. Applying Lemma 2.3, we have

$$\begin{aligned}
\|w - w^*\| &= \|F'(w^*)^{-1}F'(w^*)(w - w^*)\| \\
&\leq K\|w - w^*\|_0.
\end{aligned}$$

By virtue of Lemma 3.2 we have

$$\|w^+ - w^*\|_0 \leq (\eta + KK_3\delta)\|w - w^*\|_0 + M(\mu^+ + \zeta),$$

from which the result follows by the choice of δ. □

The result that follows will be used to show that near the solution, the steplength computed by means of the fraction to the boundary rule (1.4) remains bounded away from zero. This property will be satisfied if the components of the residual corresponding to the complementarity conditions are smaller than μ^+, in the sense that

$$\|r^c\|_\infty < \mu^+, \tag{3.13}$$

where r^c is defined by

$$r^c := Z(x^+ - x) + X(z^+ - z) + XZe - \mu^+ e. \tag{3.14}$$

In practice, this condition can be easily satisfied by reducing the system thanks to the substitution

$$z^+ - z = X^{-1}(\mu^+ e - Z(x^+ - x)) \tag{3.15}$$

in (3.2). In that case, because $r^c = 0$, (3.13) is trivially satisfied.

Recalling the definition (2.1) of the constant a and according to the fact that $F'(w)$ is nonsingular in the ball $B(w^*, \bar{\delta})$, we have $a > \bar{\delta}$. We then define the positive constant

$$b := \frac{1}{a - \bar{\delta}}.$$

Lemma 3.4 *For all $w \in B(w^*, \bar{\delta})$ and $\mu^+ \in (0, \bar{\mu})$, if $v > 0$ and if the residual defined by (3.14) satisfies (3.13), then*

$$v + \frac{v^+ - v}{1 + b\|w^+ - w\|} \geq 0.$$

Proof. Let $w \in B(w^*, \bar{\delta})$ such that $v > 0$. Let us define

$$t := \frac{1}{1 + b\|w^+ - w\|}.$$

It is enough to prove that for all $i \in \{1, \ldots, n\}$,

$$[z]_i + t([z^+]_i - [z]_i) \geq 0. \tag{3.16}$$

By a symmetric argument the same property will hold for the components $[x]_i$.

Let $i \in \{1, \ldots, n\}$. If $[z^+]_i \geq 0$, then (3.16) is satisfied because $[z]_i > 0$ and $0 < t \leq 1$. Suppose then that $[z^+]_i < 0$. Using (3.14), we have

$$[z]_i([x^+]_i - [x]_i) + [x]_i([z^+]_i - [z]_i) = \mu^+ - [x]_i[z]_i + [r^c]_i,$$

and thus

$$[x^+]_i - [x]_i = \frac{\mu^+ + [r^c]_i}{[z]_i} - \frac{[x]_i[z^+]_i}{[z]_i}.$$

By assumptions we have $\mu^+ + [r^c]_i > 0$, we then deduce that

$$0 < -\frac{[x]_i[z^+]_i}{[z]_i} < [x^+]_i - [x]_i. \tag{3.17}$$

The remainder of the proof is completely similar to the proof of [2, Lemma 4], but we reproduce it here for completeness.

Let us define $t_i = \frac{[z]_i}{[z]_i - [z^+]_i}$. It follows that $0 < t_i < 1$ and by using the notation $v_i = ([x]_i, [z]_i)$, we obtain

$$\frac{1 - t_i}{t_i} = \frac{-[z^+]_i}{[z]_i} = \frac{(([x]_i [z^+]_i / [z]_i)^2 + ([z^+]_i)^2)^{1/2}}{\|v_i\|}.$$

Now using (3.17) and $0 < -[z^+]_i < [z]_i - [z^+]_i$, we obtain

$$\begin{aligned} \frac{1 - t_i}{t_i} &\leq \frac{(([x^+]_i - [x]_i)^2 + ([z^+]_i - [z]_i)^2)^{1/2}}{\|v_i\|} \\ &\leq \frac{\|w - w^+\|}{\|v_i\|}. \end{aligned} \tag{3.18}$$

The definition (2.1) of a and Assumption 2.2 imply

$$a - \bar{\delta} \leq \|v_i^*\| - \|v_i - v_i^*\| \leq \|v_i\|, \tag{3.19}$$

where $v_i^* = ([x^*]_i, [z^*]_i)$. From inequalities (3.18) and (3.19), we deduce that

$$\frac{1 - t_i}{t_i} \leq b\|w - w^+\|,$$

and thus

$$t \leq t_i.$$

Since $[z^+]_i - [z]_i < 0$, the last inequality implies that

$$[z]_i + t([z^+]_i - [z]_i) \geq [z]_i + t_i([z^+]_i - [z]_i) = 0,$$

from which (3.16) follows. □

Corollary 3.5 *[2, Corollary 1] Let $\mu^+ \in (0, \bar{\mu})$ and $w \in B(w^*, \bar{\delta})$ such that $v > 0$. Let w^+ be the Newton iterate defined by (3.2). Assume that the residual satisfies (3.13). Let α be the greatest value in $(0, 1]$ such that*

$$v + \alpha(v^+ - v) \geq (1 - \tau_v)v,$$

where $\tau_v \in (0, 1)$. Then the following inequalities hold:

$$\alpha \geq \tau_v \frac{a - \bar{\delta}}{a + \bar{\delta}} \tag{3.20}$$

and

$$1 - \alpha \leq 1 - \tau_v + b\|w - w^+\|. \tag{3.21}$$

This result allows to prove convergence in a suitable neighborhood.

Theorem 3.6 *Let Assumptions 2.1–2.2 hold. Let* $\bar\eta \in (0,1)$ *and* $\bar\tau \in (0,1)$*. There exists* $\delta^* \in]0,\bar\delta[$*,* $\zeta^* > 0$*,* $\mu^* \in]0,\bar\mu[$*,* $\kappa \in (0,1)$ *and* $M > 0$ *such that for all sequences* $\{\eta_k\} \subset [0,\bar\eta]$*,* $\{\zeta_k\} \subset [0,\zeta^*]$ *converging to zero,* $\{\tau_k\} \subset (\bar\tau,1)$*,* $\{\mu_k\} \subset (0,\mu^*)$ *converging to zero and for an initial point* $w_0 \in B(w^*,\delta^*)$ *with* $v_0 > 0$*, the inexact primal-dual algorithm, defined for all* k *by (1.3)–(1.5) with a residual* r_k *satisfying (1.6) and* $\|r_k^c\|_\infty < \mu_{k+1}$*, generates a sequence* $\{w_k\}$ *converging to* w^* *such that, for all integer* k*,* $v_k > 0$ *and*

$$\|F'(w^*)(w_{k+1}-w^*)\| \le \kappa\|F'(w^*)(w_k-w^*)\| + M(\mu_{k+1}+\zeta_k).$$

Proof. Let us choose

$$\delta^* = \frac{1}{KK'}\delta, \quad 0 < \zeta^* < \frac{(1-\bar\eta)\delta}{2KM} \quad \text{and} \quad \mu^* = \min\{\frac{(1-\bar\eta)\delta}{2KM} - \zeta^*, \bar\mu\},$$

where δ, K, K' and M are the constants defined in Lemma 2.3 and Corollary 3.3. We have $\delta^* \le \delta \le \bar\delta$ and $\mu^* \le \bar\mu$. Let $\{\eta_k\} \subset [0,\bar\eta]$ be a forcing sequence , $\{\tau_k\} \subset (\bar\tau,1)$ be a sequence and $\{\mu_k\} \subset (0,\mu^*)$ be a sequence converging to zero. Let $w_0 \in B(w^*,\delta^*)$, with $v_0 > 0$, be the starting point of the primal-dual algorithm.

Firstly we shall show by induction that for all integer k, the vector w_k exists and satisfies the inequality

$$\|w_k - w^*\|_0 \le \frac{\delta}{K}.$$

Our claim holds for $k=0$ by using the second inequality in (3.1). Assume it is true for a given $k \ge 0$. The first inequality in (3.1) implies that $\|w_k - w^*\| \le \delta$. By virtue of Corollary 3.3, the vector w_k^+ exists and we have

$$\begin{aligned}\|w_k^+ - w^*\|_0 &\le \frac{1+\bar\eta}{2}\|w_k - w^*\|_0 + M(\mu_{k+1}+\zeta_k) \\ &\le \frac{1+\bar\eta}{2}\frac{\delta}{K} + M(\mu^*+\zeta^*) \\ &\le \frac{\delta}{K}.\end{aligned} \tag{3.22}$$

Using (1.5) and the induction hypothesis, the vector w_{k+1} exists and we have

$$\begin{aligned}\|w_{k+1} - w^*\|_0 &= \|\alpha_k w_k^+ + (1-\alpha_k)w_k - w^*\|_0 \\ &\le \alpha_k\|w_k^+ - w^*\|_0 + (1-\alpha_k)\|w_k - w^*\|_0 \\ &\le \frac{\delta}{K}\end{aligned} \tag{3.23}$$

which ends our induction reasoning. Now put

$$\bar\alpha = \bar\tau\frac{a-\bar\delta}{a+\bar\delta} \quad \text{and} \quad \kappa = 1 - \frac{1-\eta}{2}\bar\alpha < 1.$$

Let $k \in \mathbb{N}$. Using (3.20), we have $\bar{\alpha} \leq \alpha_k$ and, according to (3.22) and (3.23), it follows

$$\begin{aligned} \|w_{k+1} - w^*\|_0 &\leq \alpha_k(\frac{1+\bar{\eta}}{2}\|w_k - w^*\|_0 + M(\mu_{k+1} + \zeta_k)) + (1-\alpha_k)\|w_k - w^*\|_0 \\ &\leq \kappa\|w_k - w^*\|_0 + M(\mu_{k+1} + \zeta_k) \end{aligned}$$

which is exactly the desired inequality. Moreover, recalling that the sequence $\{w_k\}$ is bounded and taking the limit superior in the above inequality, we deduce that $\limsup \|w_k - w^*\|_0 = 0$. It follows that the whole sequence $\{w_k\}$ converges to w^*. □

4 Technical lemmas

The section is devoted to the proof of a fundamental property intended to the asymptotic analysis. As we said in the introduction, Lemma 4.3 is a far from obvious extension of [2, Lemma 7]. Note that the statements of this section are valid in a general context and that this part can be skipped at first reading. The proof is divided in three parts. Before proving Lemma 4.3 we will show two important preliminary lemmas.

Lemma 4.1 *Let $\{\mu_k\} \subset (0, +\infty)$ be a sequence such that for all integer k*

$$\beta\mu_k^{1+\sigma} \leq \mu_{k+1} \leq \gamma\mu_k \tag{4.1}$$

for some constants $\beta > 0$, $\gamma \in (0, 1)$ and $\sigma > 0$. Let $\{e_k\} \subset \mathbb{R}_+$ be a sequence converging to zero and satisfying for all integer k

$$e_{k+1} \leq q^k e_k^{1+\xi} + \frac{\mu_{k+1}}{\mu_k}e_k \tag{4.2}$$

for two constants $q \in (0, 1)$ and $\xi > \sigma$. Then we have $e_k = O(\mu_k)$.

Proof. Firstly let us choose two real numbers σ' and ξ' such that $\sigma < \sigma' < \xi' < \xi$ and set $r := 1 + \sigma'$ and $s := 1 + \xi'$. Let us also define the real number

$$Q := \prod_{\ell=0}^{\infty}(1 + q^\ell),$$

which belongs to $(1, +\infty)$. Without loss of generality we can assume that the sequence $\{e_k\} \subset (0, +\infty)$. Indeed if $e_k = 0$ for some k, the relation (4.2) implies that the terms of the sequence vanish for all indices greater than k and the conclusion trivially holds. We can also assume that the sequence $\{e_k\}_{k \geq k_0}$ is non-increasing for some $k_0 \in \mathbb{N}$. Indeed, for sufficiently large $k \in \mathbb{N}$ we have

$$e_{k+1} \leq (q^k e_k^\xi + \frac{\mu_{k+1}}{\mu_k})e_k \leq (q^k e_k^\xi + \gamma)e_k \leq e_k.$$

Let us now define the set of indices $\mathcal{K} := \{k \in \mathbb{N} : \mu_{k+1} \leq \mu_k e_k^{\xi}\}$. We shall show that $\mathcal{K}$ has a finite number of elements. Suppose on the contrary that $\mathcal{K} := \{k_i\}_{i\in\mathbb{N}}$ is a subsequence of the sequence of nonnegative integers. From the definition of $\mathcal{K}$ and from (4.2), we have on one hand for all $k \notin \mathcal{K}$

$$\frac{e_{k+1}}{\mu_{k+1}} \leq (1+q^k)\frac{e_k}{\mu_k} \tag{4.3}$$

and on the other hand for all $i \in \mathbb{N}$

$$e_{k_i+1} \leq (1+q^{k_i})e_{k_i}^{1+\xi}. \tag{4.4}$$

Since $q \in (0,1)$ and the sequence $\{e_k\}$ tends to zero, we deduce from (4.4) that there exists $i_0 \in \mathbb{N}$ such that $k_{i_0} \geq k_0$ and for all $i \geq i_0$

$$e_{k_i+1} \leq e_{k_i}^{s}. \tag{4.5}$$

We now claim that there exists $i_1 \geq i_0$ such that for all $i \geq i_1$, there exists $t_i \in [0,1]$ such that

$$e_{k_{i+1}} \leq t_i e_{k_i}^{s} \quad \text{and} \quad \mu_{k_{i+1}} \geq t_i \mu_{k_i}^{r}.$$

Indeed, since $r > 1+\sigma$ and the sequence $\{\mu_k\}$ tends to zero, there exists $i_1 \geq i_0$ such that for all $i \geq i_1$, we have

$$\mu_{k_i+1} \geq \beta\mu_{k_i}^{1+\sigma} \geq \frac{\beta}{Q}\mu_{k_i}^{1+\sigma} \geq \mu_{k_i}^{r}. \tag{4.6}$$

Let $i \geq i_1$. If $k_i + 1 = k_{i+1}$, according to (4.5) and(4.6), we can take $t_i = 1$. Suppose now that $k_i + 1 < k_{i+1}$. Using that (4.3) holds for all index k such that $k_i < k < k_{i+1}$ and that the finite product $\prod_{\ell=k_i+1}^{k_{i+1}-1}(1+q^{\ell})$ is bounded above by A, we obtain

$$\frac{e_{k_{i+1}}}{\mu_{k_{i+1}}} \leq Q\frac{e_{k_i+1}}{\mu_{k_i+1}}.$$

Let us take $t_i := \frac{e_{k_{i+1}}}{e_{k_i+1}}$. Since the sequence $\{e_k\}_{k\geq k_0}$ is non-increasing, we have $t_i \leq 1$. By (4.5) we have $e_{k_{i+1}} = t_i e_{k_i+1} \leq t_i e_{k_i}^{s}$. We also have

$$\mu_{k_{i+1}} \geq \frac{1}{Q}\frac{e_{k_{i+1}}}{e_{k_i+1}}\mu_{k_i+1} \geq \frac{\beta}{Q}t_i\mu_{k_i}^{1+\sigma} \geq t_i\mu_{k_i}^{r},$$

so that the claim is proved.

In others words, for all $i \geq i_1$, we have

$$U_{i+1} \leq t_i U_i^{s} \quad \text{and} \quad V_{i+1} \geq t_i V_i^{r} \tag{4.7}$$

where $U_i = e_{k_i}$ and $V_i := \mu_{k_i}$. Moreover the definition of $\mathcal{K}$ combined with the first inequality in (4.1) yields

$$\beta\mu_{k_i}^{1+\sigma} \leq \mu_{k_i+1} \leq e_{k_i}^{\xi}\mu_{k_i}.$$

Since $\sigma < \xi$, we deduce that there exists $i_2 \geq i_1$ such that for all $i \geq i_2$.

$$V_i = \mu_{k_i} \leq \frac{1}{\beta^{1/\sigma}} e_{k_i}^{\xi/\sigma} \leq e_{k_i} = U_i.$$

Repeating the use of (4.7), we deduce that for all $i \geq i_2$ and for all $j \geq 0$ we have the chain of inequalities

$$\left(\prod_{\ell=0}^{j-1} t_{i+j-1-\ell}^{r^\ell}\right) V_i^{r^j} \leq V_{i+j} \leq U_{i+j} \leq \left(\prod_{\ell=0}^{j-1} t_{i+j-1-\ell}^{s^\ell}\right) U_i^{s^j} \leq \left(\prod_{\ell=0}^{j-1} t_{i+j-1-\ell}^{r^\ell}\right) U_i^{s^j},$$

the last inequality following from $r < s$ and $t_i \in (0,1]$ for all $i \geq i_2$. We then obtain for all $i \geq i_2$ and for all $j \geq 0$

$$V_i^{r^j} \leq U_i^{s^j},$$

which contradicts the fact that $r < s$.

Consequently $\mathcal{K}$ has a finite number of elements. Then there exists $k_1 \in \mathbb{N}$ such that (4.3) holds for all $k \geq k_1$. We deduce that for all $k \geq k_1$

$$\frac{e_k}{\mu_k} \leq Q \frac{e_{k_1}}{\mu_{k_1}}$$

and thus we conclude that $e_k = O(\mu_k)$. □

Lemma 4.2 *Let $\{\mu_k\} \subset (0, +\infty)$ be a sequence such that for all integer k*

$$\beta \mu_k^{1+\sigma} \leq \mu_{k+1} \leq \gamma \mu_k \tag{4.8}$$

for some constants $\beta > 0$, $\gamma \in (0,1)$ and $\sigma > 0$. Let $\{e_k\} \subset \mathbb{R}_+$ be a sequence converging to zero and satisfying for all integer k

$$e_{k+1} \leq e_k^{1+\xi} + \frac{\mu_{k+1}}{\mu_k} e_k \tag{4.9}$$

for some constant $\xi > \sigma$. Then we have $e_k = O(\mu_k)$.

Proof. By eventually renumbering the sequences, we can assume that $\mu_0 \leq 1$. Let us choose γ' and θ in $(0,1)$ such that $\gamma < \gamma' < \gamma^{1-\frac{\sigma}{\xi}} < \theta$. Then we can also choose $\bar{\sigma} \in (\sigma, \xi)$ sufficiently near ξ such that

$$\gamma^{1-\frac{\sigma}{\bar{\sigma}}} \leq \theta. \tag{4.10}$$

Since the sequence $\{e_k\}$ tends to zero, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$e_{k+1} \leq (e_k^{\xi} + \gamma) e_k \leq \gamma' e_k.$$

Let us define $a := \theta^{\xi}$, $\bar{e}_k := \frac{1}{\theta^{k+\frac{1}{\xi}}} e_k$ and $\bar{\mu}_k := \frac{1}{\theta^k}\mu_k$, such that (4.9) is rewritten as

$$\bar{e}_{k+1} \le a^k \bar{e}_k^{1+\xi} + \frac{\bar{\mu}_{k+1}}{\bar{\mu}_k}\bar{e}_k.$$

Let us show that the two sequences $\{\bar{\mu}_k\}$ and $\{\bar{e}_k\}$ satisfy all the assumptions of Lemma 4.1, so that we will have $\bar{e}_k = O(\bar{\mu}_k)$ and thus $e_k = O(\mu_k)$.

On one hand, assumption (4.2) is clearly satisfied and for all $k \ge k_0$ we have

$$\bar{e}_{k+1} = \frac{1}{\theta^{k+1+\frac{1}{\xi}}} e_{k+1} \le \frac{1}{\theta^{k+1+\frac{1}{\xi}}}\gamma' e_k = \frac{\gamma'}{\theta}\bar{e}_k$$

from which we deduce that the sequence $\{\bar{e}_k\}$ tends to zero. On the other hand, from (4.8) we have that for all $k \in \mathbb{N}$

$$\bar{\beta}\theta^{k\sigma}\bar{\mu}_k^{1+\sigma} \le \bar{\mu}_{k+1} \le \bar{\gamma}\bar{\mu}_k, \tag{4.11}$$

with $\bar{\beta} = \frac{\beta}{\theta}$ and $\bar{\gamma} = \frac{\gamma}{\theta} < 1$. In particular since $\bar{\mu}_0 = \mu_0 \le 1$, we have $\bar{\mu}_k \le \bar{\gamma}^k$ for all integer k. Recalling (4.10) which can be rewritten $\bar{\gamma} \le \theta^{\frac{\sigma}{\bar{\sigma}-\sigma}}$, we deduce that for all integer k, $\bar{\mu}_k \le \theta^{\frac{k\sigma}{\bar{\sigma}-\sigma}}$, and thus

$$\begin{aligned} \bar{\mu}_k^{1+\bar{\sigma}} &= \mu_k^{\bar{\sigma}-\sigma}\bar{\mu}_k^{1+\sigma} \\ &\le \theta^{k\sigma}\bar{\mu}_k^{1+\sigma}. \end{aligned}$$

Hence, relation (4.11) implies that for all integer k,

$$\bar{\beta}\bar{\mu}_k^{1+\bar{\sigma}} \le \bar{\mu}_{k+1} \le \bar{\gamma}\bar{\mu}_k.$$

It follows that assumption (4.1) is satisfied, which completes the proof. □

Lemma 4.3 *Let $\{\mu_k\} \subset (0, +\infty)$ be a sequence converging to zero and satisfying for all integer k*

$$\beta\mu_k^{1+\sigma} \le \mu_{k+1} \le \gamma\mu_k \tag{4.12}$$

for some constants β, γ and σ in $(0, +\infty)$. Let $\{e_k\} \subset \mathbb{R}_+$ be a sequence converging to zero and satisfying for all integer k

$$e_{k+1} \le e_k^{1+\xi} + \rho\frac{\mu_{k+1}}{\mu_k}e_k + c\mu_{k+1} \tag{4.13}$$

for some constants $\xi > \sigma$, $\rho \in (0,1)$ and $c \ge 0$. If $\rho\gamma < 1$, then we have

$$\limsup_{k\to\infty}\frac{e_k}{\mu_k} \le s := c\max\{(1-\rho)^{-1}, (1-\rho\gamma)^{-1}\}. \tag{4.14}$$

Proof. In order to prove the lemma, it suffices to show that for every $t > s$ there exists an integer k_t such that for all $k \geq k_t$, $e_k \leq t\mu_k$.

Let $t > s$. We first claim that there exists an integer $k_0 > \frac{1}{\sigma}$ such that, for all $k \geq k_0$

$$t^{\xi}\frac{\mu_k^{1+\xi}}{\mu_{k+1}} + \rho + \frac{c}{t} \leq 1.$$

Indeed the definition of s in (4.14) and $t > s$ imply that

$$t > \frac{c}{1-\rho} \quad \text{and} \quad t > \frac{c}{1-\rho\gamma}. \tag{4.15}$$

The first inequality of (4.12) implies that $\frac{\mu_k^{1+\xi}}{\mu_{k+1}} \leq \frac{\mu_k^{\xi-\sigma}}{\beta}$ and consequently the sequence $\{t^{\xi}\frac{\mu_k^{1+\xi}}{\mu_{k+1}}\}$ converges to zero. Moreover, the first inequality of (4.15) implies that $\rho + \frac{c}{t} < 1$, so that the existence of such k_0 is ensured.

If $e_k \leq t\mu_k$ for some $k \geq k_0$, then by applying inequality (4.13), we obtain

$$e_{k+1} \leq \left(t^{\xi}\frac{\mu_k^{1+\xi}}{\mu_{k+1}} + \rho + \frac{c}{t}\right) t\mu_{k+1} \leq t\mu_{k+1}.$$

Hence, if we can prove that $e_{k_t} \leq t\mu_{k_t}$ for some $k_t \geq k_0$, then by induction we have $e_k \leq t\mu_k$ for all $k \geq k_t$ and we are done. Therefore, to conclude this proof, it is enough to show that $e_k \leq t\mu_k$ for some $k \geq k_0$. We will prove this fact by contradiction.

Suppose that

$$e_k > t\mu_k \quad \text{for all } k \geq k_0. \tag{4.16}$$

Let $k \geq k_0$. Notice that (4.16) and (4.13) yield $\mu_{k+1} < \frac{e_{k+1}}{t}$ and

$$\left(1 - \frac{c}{t}\right) e_{k+1} \leq \left(e_k^{\xi} + \rho\frac{\mu_{k+1}}{\mu_k}\right) e_k.$$

Multiplying both sides of this last inequality by $\left(\frac{t}{t-c}\right)^{1+\frac{1}{\xi}}$ we obtain

$$\tilde{e}_{k+1} \leq \tilde{e}_k^{1+\xi} + \frac{\tilde{\mu}_{k+1}}{\tilde{\mu}_k}\tilde{e}_k \tag{4.17}$$

where $\tilde{e}_k := \left(\frac{t}{t-c}\right)^{\frac{1}{\xi}} e_k$ and $\tilde{\mu}_k := \varphi^k\mu_k$ with $\varphi := \frac{\rho t}{t-c}$. The first inequality of (4.15) implies that $\varphi < 1$. Multiplying inequalities (4.12) by φ^{k+1} we have

$$\varphi^{k+1}\beta\frac{(\varphi^k\mu_k)^{1+\sigma}}{\varphi^{k(1+\sigma)}} \leq \varphi^{k+1}\mu_{k+1} \leq \varphi\gamma\varphi^k\mu_k,$$

which is equivalent to

$$\varphi^{1-k\sigma}\beta\tilde{\mu}_k^{1+\sigma} \le \tilde{\mu}_{k+1} \le \varphi\gamma\tilde{\mu}_k.$$

Since $\varphi < 1$ and $k \ge k_0 > \frac{1}{\sigma}$, we have $\varphi^{1-k\sigma} > 1$ which implies

$$\tilde{\beta}\tilde{\mu}_k^{1+\tilde{\sigma}} \le \tilde{\mu}_{k+1} \le \tilde{\gamma}\tilde{\mu}_k \tag{4.18}$$

where $\tilde{\beta} := \beta$, $\tilde{\sigma} = \sigma$ and $\tilde{\gamma} := \varphi\gamma$. Moreover from the second inequality of (4.15) and since $\rho\gamma < 1$, we deduce that $\varphi\gamma = \frac{\rho t\gamma}{t-c} < 1$ or equivalently $\tilde{\gamma} < 1$.

Relations (4.17) and (4.18) imply that all assumptions of Lemma 4.2 are satisfied with the sequence $\{\tilde{\mu}_{k_0+k}\}_{k\ge 0}$ and the sequence $\{\tilde{e}_{k_0+k}\}_{k\ge 0}$ which clearly converge to zero. Applying Lemma 4.2 we conclude that $e_k = O(\varphi^k\mu_k)$, which contradicts (4.16) and completes the proof. □

5 Asymptotic behavior

In this section, we place our study in the context of Theorem 3.6. More precisely, we assume that Assumptions 2.1–2.2 holds. We fix two constants $\bar{\eta} \in (0,1)$ and $\bar{\tau} \in (0,1)$. We also fix sequences $\{\eta_k\} \subset [0,\bar{\eta}]$, $\{\zeta_k\} \subset [0,\zeta^*]$ converging to zero, $\{\tau_k\} \subset (\bar{\tau},1)$, $\{\mu_k\} \subset (0,\mu^*)$ converging to zero and an initial point $w_0 \in B(w^*,\delta^*)$ with $v_0 > 0$. Then, according to Theorem 3.6, the inexact primal-dual algorithm generates a sequence $\{w_k\}$ converging to w^*. To lighten the notation, we define the following sequences

$$\epsilon_k = \|w_k - w(\mu_k)\|, \quad \hat{\epsilon}_k = \|w_k - w(\mu_{k+1})\| \quad \text{and} \quad \epsilon_k^+ = \|w_k^+ - w(\mu_{k+1})\|.$$

We now address the asymptotic behavior of the inexact primal-dual algorithm. In this context, since we deal with superlinear convergence, the analysis may well be performed using any norm, the refined constants bounds not being required. We develop simultaneously the results for two cases depending on the choice of the forcing sequence:

- $\{\eta_k\}$ converges to 0,
- $\eta_k = O(\|F(w_k) - \mu_{k+1}\tilde{e}\|^{\xi})$, for some parameter $\xi > 0$.

Intuitively, the rate of convergence of the forcing sequence has to be faster than that of the barrier parameter to let the sequence of iterates becoming asymptotically tangent to the central path. For the first case we will assume that $\eta_k\mu_k = o(\mu_{k+1})$. For the second case it will be assumed that the rate of convergence of $\{\mu_k\}$ is at most superlinear with an order $1+\sigma$ for a given $0 < \sigma < \xi$. More precisely, in the sequel we add on the sequences $\{\eta_k\}$, $\{\zeta_k\}$ $\{\tau_k\}$ and $\{\mu_k\}$ the following assumption.

Assumption 5.1 *The sequences* $\{\mu_k\}$ *and* $\{\tau_k\}$ *satisfy for all integer* k

$$\beta\mu_k^{1+\sigma} \leq \mu_{k+1} \leq \gamma\mu_k \tag{5.1}$$

and

$$1 - \tau\frac{\mu_{k+1}}{\mu_k} \leq \tau_k < 1 \tag{5.2}$$

for constants $\beta > 0$, $\gamma > 0$, $\sigma \in (0,1)$ *and* $\tau < \min\{\frac{1}{\gamma}, \frac{1}{2}\}$. *The forcing sequence* $\{\zeta_k\}$ *satisfies*

$$\zeta_k = o(\mu_{k+1}). \tag{5.3}$$

The forcing sequence $\{\eta_k\}$ *satisfies either*

$$\eta_k = o\Big(\frac{\mu_{k+1}}{\mu_k}\Big) \tag{5.4}$$

or

$$\eta_k = O(\|F(w_k) - \mu_{k+1}\tilde{e}\|^\xi) \tag{5.5}$$

for some constant $\xi \in (\sigma, 1]$.

The left inequality in (5.1) implies in particular that

$$\mu_k^2 = o(\mu_{k+1}). \tag{5.6}$$

Note also that, for all integer k, since $\mu_{k+1} \leq \gamma\mu_k$ we have

$$|\mu_{k+1} - \mu_k| \leq \gamma'\mu_k \tag{5.7}$$

with $\gamma' = \max\{1, \gamma - 1\}$.

Now we can evaluate the progress made by the inexact Newton step w_k^+.

Lemma 5.2 *Let Assumption 5.1 holds. We have*

$$\epsilon_k^+ = O(\epsilon_k^2 + \eta_k\epsilon_k + \eta_k\mu_k + \mu_k^2 + \zeta_k). \tag{5.8}$$

Moreover, in the case where (5.5) holds, we get

$$\epsilon_k^+ = O(\epsilon_k^{1+\xi} + \mu_k^{1+\xi} + \zeta_k). \tag{5.9}$$

Proof. By virtue of Lemma 3.1 and by applying the inequalities (3.1), we have that

$$\epsilon_k^+ = O(\hat{\epsilon}_k^2 + \eta_k\hat{\epsilon}_k + \zeta_k). \tag{5.10}$$

Using Lemma 2.4 and (5.7), we also have for all integer k

$$\begin{aligned} \hat{\epsilon}_k &\leq \epsilon_k + \|w(\mu_k) - w(\mu_{k+1})\| \\ &\leq \epsilon_k + C|\mu_k - \mu_{k+1}| \\ &\leq \epsilon_k + C\gamma'\mu_k. \end{aligned} \tag{5.11}$$

Combining (5.10) and (5.11), we deduce the relation (5.8).

Assume now that (5.5) holds. In this case we have $\eta_k = O(\hat{\epsilon}_k^{\xi})$. The bound (5.10) implies that $\epsilon_k^+ = O(\hat{\epsilon}_k^{1+\xi} + \zeta_k)$. From (5.11) we get $\epsilon_k^+ = O((\epsilon_k + \mu_k)^{1+\xi} + \zeta_k)$. Finally applying for all integer k the following inequality of convexity

$$(\epsilon_k + \mu_k)^{1+\xi} \leq 2^{\xi}(\epsilon_k^{1+\xi} + \mu_k^{1+\xi}) \tag{5.12}$$

we get the result. □

In order to pursue, we need to take into account the fraction to the boundary steplength. We will prove that the sequence $\{\epsilon_k\}$ goes to zero at a speed at least as fast as μ_k. This in turn will allow to formally justify that the unit steplength will be taken, and thus that the above lemma remains valid when we replace w_k^+ by w_{k+1}.

Recall that in Assumption 5.1 we have $\tau < 1$ and $\tau\gamma < 1$. Therefore, we can choose a constant ρ such that

$$\tau < \rho < 1 \quad \text{and} \quad \rho\gamma < 1.$$

Under this notation we have the following properties.

Lemma 5.3 *Let Assumption 5.1 holds. In case where (5.4) holds, there exist constants $c > 0$ and $e > 0$ such that for all integer k,*

$$e_{k+1} \leq e_k^2 + \rho \frac{\mu_{k+1}}{\mu_k} e_k + c\,\mu_{k+1}. \tag{5.13}$$

In case where (5.5) holds, there exist constants $c > 0$ and $e > 0$ such that for all integer k,

$$e_{k+1} \leq e_k^{1+\xi} + \rho \frac{\mu_{k+1}}{\mu_k} e_k + c\,\mu_{k+1}. \tag{5.14}$$

Proof. Using (1.5), inequality (3.21) and the assumption (5.2), we have for all integer k

$$\begin{aligned}
\epsilon_{k+1} &= \|\alpha_k w_k^+ + (1-\alpha_k)w_k - w(\mu_{k+1})\| \\
&\leq \alpha_k \epsilon_k^+ + (1-\alpha_k)\hat{\epsilon}_k \\
&\leq \epsilon_k^+ + (1 - \tau_k + b\|w_k - w_k^+\|)\hat{\epsilon}_k \\
&\leq \epsilon_k^+ + \tau \frac{\mu_{k+1}}{\mu_k}\hat{\epsilon}_k + b(\hat{\epsilon}_k + \epsilon_k^+)\hat{\epsilon}_k
\end{aligned}$$

Then, using (5.11), we obtain for all integer k

$$\epsilon_{k+1} \leq \tau \frac{\mu_{k+1}}{\mu_k}\epsilon_k + R_k \tag{5.15}$$

with $R_k = \tau C\gamma'\mu_{k+1} + b(\epsilon_k + C\gamma'\mu_k)^2 + (1 + b\hat{\epsilon}_k)\epsilon_k^+$. Recalling that the sequence $\{\hat{\epsilon}_k\}$ converges to 0 and according to relation (5.6), we have

$$R_k = O(\mu_{k+1} + \epsilon_k^2 + \epsilon_k^+).$$

Then, using relation (5.8) and the assumption (5.3), we obtain

$$R_k = O(\epsilon_k^2 + \eta_k\epsilon_k + \eta_k\mu_k + \mu_{k+1}). \tag{5.16}$$

Assume that (5.4) holds. We then have $\eta_k\epsilon_k = o(\frac{\mu_{k+1}}{\mu_k}\epsilon_k)$. Then, according to relations (5.15) and (5.16), there exist $e > 0$ and $c > 0$ such that for k large enough

$$\epsilon_{k+1} \leq e\epsilon_k^2 + \rho\frac{\mu_{k+1}}{\mu_k}\epsilon_k + \frac{c}{e}\mu_{k+1}.$$

By multiplying both sides by e and by eventually increasing the value of c, inequality (5.13) is satisfied for all $k \in \mathbb{N}$.

Assume now that (5.5) holds. In this case we have $\eta_k = O(\hat{\epsilon}_k^\xi)$ which implies, $\eta_k(\epsilon_k + \mu_k) = O((\epsilon_k + \mu_k)^{1+\xi})$ from inequality (5.11). By applying (5.12), we get $\eta_k(\epsilon_k + \mu_k) = O(\epsilon_k^{1+\xi} + \mu_k^{1+\xi})$, from which with the assumption (5.1) we deduce that

$$\eta_k(\epsilon_k + \mu_k) = O(\epsilon_k^{1+\xi} + \mu_{k+1}).$$

Then, according to relations (5.15) and (5.16), the fact that $\epsilon_k^2 \leq \epsilon_k^{1+\xi}$ for k large enough, there exist $e > 0$ and $c > 0$ such that for sufficiently large k we get

$$\epsilon_{k+1} \leq e^\xi\epsilon_k^{1+\xi} + \rho\frac{\mu_{k+1}}{\mu_k}\epsilon_k + \frac{c}{e}\mu_{k+1}.$$

By multiplying both sides by e and by eventually increasing the value of c, inequality (5.14) is then satisfied for all $k \in \mathbb{N}$. □

Lemma 5.4 *Let Assumption 5.1 holds. We have* $\epsilon_k^+ = o(\mu_{k+1})$.

Proof. According to Lemma 4.3, we have $\epsilon_k = O(\mu_k)$. Assume that (5.4) holds. By using (5.8), we obtain

$$\epsilon_k^+ = O(\mu_k^2 + \eta_k\mu_k + \zeta_k).$$

Assume now that (5.5) holds. By using (5.9), we obtain

$$\epsilon_k^+ = O(\mu_k^{1+\xi} + \zeta_k).$$

In both cases, by Assumption 5.1, we can conclude that $\epsilon_k^+ = o(\mu_{k+1})$. □

Theorem 5.5 *Assume that all the assumptions of Theorem 3.6 are satisfied. Assume also that Assumption 5.1 holds. Then the inexact primal-dual algorithm, defined for all k by (1.3)–(1.5), generates a sequence $\{w_k\}$ converging to w^* such that for sufficiently large index k the unit step length is accepted by the fraction to the boundary rule (1.4), meaning that $w_{k+1} = w_k^+$ and we have*

$$w_k = w(\mu_k) + o(\mu_k).$$

Proof. Since $\|w(\mu_k) - w^*\| = O(\mu_k)$, the full step to w_k^+ should satisfy the fraction to the boundary rule, so we would have $\epsilon_k = o(\mu_k)$. This is proved in [2, section 8], and the proof only relies on Lemmas 2.4 and 5.4, so it still applies in our more general context. □

6 Numerical illustration

We choose to illustrate the behavior of an inexact interior point method by means of the numerical solution of a set of bound constrained quadratic convex minimization problems. We explore different choices for the forcing sequences and observe their impact on the efficiency of the algorithm. The experiments were done with Matlab. The problems come from [10] and are of the form

$$\begin{array}{ll} \min & \frac{1}{2}x^\top Hx + q^\top x \\ \text{s.t.} & x \geq 0, \end{array}$$

where $H \in \mathbb{R}^{n\times n}$ is positive definite and $q \in \mathbb{R}^n$. To simplify the presentation the bounds are assumed to be only nonnegative, but general bounds are handled in similar manner as described below.

There are two problems suites: 760 problems named *Ball* and 997 problems named *Topple*. The problems are of medium size, ranging from a minimum of 33 variables to a maximum of 288. Under our notation we have $w = (x, z)$ and

$$F(w) = \begin{pmatrix} Hx + q - z \\ XZe \end{pmatrix}.$$

The corresponding Newton system is first reduced thanks to the substitution (3.15), so that the linear equation to solve is of the form

$$(H + X^{-1}Z)(x^+ - x) + Hx + q - \mu^+ X^{-1}e = 0,$$

This linear system is solved by means of a preconditioned conjugate gradient method (`pcg.m`). Two preconditioners are suggested in [10]. The first one is the simple diagonal matrix $\mathrm{diag}(H + X^{-1}Z)$. The second one is obtained via an incomplete Cholesky factorization (`cholinc.m`) of the matrix $H + X^{-1}Z$ with a drop tolerance of 10^{-4}. Though such a factorization cannot be done in a real application where only products matrix-vector are available, we do it only to compare different choices of the parameters.

The general settings are done according to those described in [10], which correspond to the local algorithm (1.3)–(1.5) described here, except that two steplengths are used, one for the primal variable x and one for the dual variables z, each one been computed by means of the fraction to the boundary rule of the form (1.4). The choice of the barrier parameter was slightly modified from that of [10], in order to fulfill our theoretical assumptions. At the beginning, the barrier parameter is set to the duality gap, that is $\frac{1}{n}x^\top z$. Then the choice of μ^+ is done as follows. The value $\bar{\mu} = \frac{\theta}{n}x^\top z$ is at first computed, where θ is a centering parameter initially set to 0.1. Whenever one of the fraction to the boundary steplengths becomes too small (less than 0.1), θ is set to 0.3, otherwise it is set to $(0.1 + \theta)/2$. The new value of the barrier parameter is then set according to

$$\mu^+ = \begin{cases} \min\{\bar{\mu}, \gamma\mu\} & \text{if } \mu \geq 0.1, \\ \max\{\mu^{1+\sigma}, \min\{\bar{\mu}, \gamma\mu\}\} & \text{otherwise,} \end{cases}$$

with $\gamma = 2$ and $\sigma = 0.4$. At last, the choice of the fraction to the boundary parameter was done according to the formula $\max\{0.9995, 1 - \mu^+/(2\gamma\mu)\}$.

The stopping test of the conjugate gradient iterations is of the form

$$\|(H + X^{-1}Z)(x^+ - x) + Hx + q - \mu^+ X^{-1}e\| \leq \eta\|F(w) - \mu^+\tilde{e}\| + \zeta,$$

where the function F is defined above, except in one experiment for which the right hand side of the inequality is replaced by

$$\min\left\{\|F(w)\|^{1.5}, \tfrac{1}{k+1}\|F(w)\|\right\}, \tag{6.1}$$

where k is the iteration number. The results for different choices for the parameters η and ζ are given in Tables 1 and 2, which report the mean/max number of interior point iterations (IP Iter) and conjugate gradient iterations (CG Iter) on each problems class.

The first row of each table corresponds to a nearly exact solution of the linear system and was the choice made in [10]. We note that the number of IP iterations is quite stable, while the efficiency in terms of CG iterations is very sensitive regarding the choices of η and ζ. We also experimented with a dynamic formula of the form $\eta = \|F(w) - \mu^+\tilde{e}\|^\xi$, but have not reported the results because they are similar to those with $\eta = \mu^\xi$. The last experiments with formula (6.1) are interesting because they give a real dynamic choice of the stopping tolerance for the CG iterations and is supported by our convergence theory. Indeed, by using the triangle inequality and a convex inequality like (5.12), we have

$$\|F(w)\|^{1.5} \leq \sqrt{2}(\|F(w) - \mu^+\tilde{e}\|^{1.5} + (\|\tilde{e}\|\mu^+)^{1.5})$$

and thus assumptions (5.3) and (5.5) are satisfied. This choice gives the best numerical results for both classes of problems and both preconditioners.

		BALL		TOPPLE	
η	ζ	IP Iter	CG Iter	IP Iter	CG Iter
10^{-6}	0	13.1/21	1375/2335	13.2/22	1160/4928
μ	0	13.6/39	863/1867	17.0/41	807/3407
$\mu^{0.5}$	0	14.0/47	648/1404	18.8/52	553/2691
$\frac{\mu^+}{(k+1)\mu}$	0	13.3/29	681/1338	14.0/26	491/2452
0	μ^2	13.7/29	1054/2699	13.7/23	962/3934
0	$\mu^{1.5}$	13.5/27	863/1574	13.3/22	735/3774
0	$\frac{\mu^+}{k+1}$	13.0/21	857/1528	13.2/22	720/4012
$\mu^{0.5}$	$\frac{\mu^+}{k+1}$	14.0/47	628/1404	18.8/52	541/2691
(6.1)		13.7/28	612/1129	16.8/29	468/2269

Table 1: Diagonal preconditioner

		BALL		TOPPLE	
η	ζ	IP Iter	CG Iter	IP Iter	CG Iter
10^{-6}	0	13.1/21	53/98	13.2/22	72/170
μ	0	13.1/36	31/91	16.3/41	58/135
$\mu^{0.5}$	0	13.1/36	19/66	16.4/41	42/103
$\frac{\mu^+}{(k+1)\mu}$	0	13.1/26	19/55	13.2/23	38/99
0	μ^2	13.7/29	38/105	13.5/23	64/162
0	$\mu^{1.5}$	13.3/27	26/66	13.2/22	49/128
0	$\frac{\mu^+}{k+1}$	13.1/21	26/61	13.2/22	48/124
$\mu^{0.5}$	$\frac{\mu^+}{k+1}$	13.1/36	18/66	16.4/41	41/103
(6.1)		13.6/26	19/52	15.0/26	37/91

Table 2: Incomplete Cholesky factorization preconditioner

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