ON MAXIMAL S-FREE CONVEX SETS

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Abstract. Let $S \subseteq \mathbb{Z}^n$ satisfy the property that $\text{conv}(S) \cap \mathbb{Z}^n = S$. Then a convex set K is called an S-free convex set if $\text{int}(K) \cap S = \emptyset$. A maximal S-free convex set is an S-free convex set that is not properly contained in any S-free convex set. We show that maximal S-free convex sets are polyhedra. This result generalizes a result of Basu et al. [6] for the case where S is the set of integer points in a rational polyhedron and a result of Lovász [18] and Basu et al. [5] for the case where S is the set of integer points in some affine subspace of $\mathbb{R}^n$.

Key words. Integer nonlinear programming, Cutting planes, Maximal lattice-free convex sets

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1. Introduction. A convex set is called *lattice-free* if it contains no integer points in its interior. A *maximal lattice-free convex set* is a lattice-free convex set not strictly contained in any lattice-free convex set. Lovász [18] and Basu et al. [5] proved that maximal lattice-free convex sets are polyhedra. Given a convex set $\mathcal{P} \subseteq \mathbb{R}^n$, let S be the set of integer points contained in $\mathcal{P}$, that is $S = \mathcal{P} \cap \mathbb{Z}^n$. A set K is called *S-free convex set* if $\text{int}(K) \cap S = \emptyset$. Hence the concept of S-free convex sets represents a generalization of the concept of lattice-free convex sets. *Maximal S-free convex sets* are defined analogously to maximal lattice-free convex sets. Dey and Wolsey [12] show that maximal S-free convex sets are polyhedra under restrictive conditions. Fukasawa and Günlük [14] show that maximal S-free convex sets are polyhedra where $\mathcal{P}$ is a rational polyhedron in $\mathbb{R}^2$. Basu et al. [6] prove that if $\mathcal{P}$ is any general rational polyhedron, then maximal S-free convex sets are polyhedra. A natural question then is to ask whether the polyhedrality of maximal S-free convex sets carries through to the case where $\mathcal{P}$ is an arbitrary convex set, that is the case where $S \subseteq \mathbb{Z}^n$ is assumed only to satisfy the property that $\text{conv}(S) \cap \mathbb{Z}^n = S$.

In this paper we verify that maximal S-free convex sets are polyhedra, where S is the set of integer points in any convex set $\mathcal{P} \subseteq \mathbb{R}^n$. Since the proof in Basu et al. [6] explicitly uses that fact that $\text{conv}(S)$ is a rational polyhedron when S is the set of integer points contained in a rational polyhedron, the result in this paper also provides an alternative proof to the result in [6]. Finally, we establish the polyhedrality of maximal S-free convex sets in some cases where S does not satisfy $\text{conv}(S) \cap \mathbb{Z}^n = S$.

We now briefly describe one motivation for the study of maximal S-free convex sets. A key algorithmic technique in solving mixed integer optimization problems is to sequentially obtain tighter approximations of the convex hull of integer feasible solutions. This is achieved by the addition of cutting planes, that is inequalities that separate fractional point from the convex hull of integer feasible points. See for example Marchand et al. [19] and Johnson et al. [17] for description of cutting plane methods. A connection between S-free convex sets and cutting planes for mixed integer linear programming was first discovered by Balas [3]. The main idea is the following. Consider the mixed integer set

$$T := \{(x, y) \in \mathbb{Z}^p \times \mathbb{R}^q \; : \; (x, y) \in \mathcal{C}\}, \tag{1.1}$$

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where $\mathcal{C}$ is a convex set. Now suppose that we are able to identify a set $S \subseteq \mathbb{Z}^n$ such that $S \supseteq \text{Proj}_x(T)$. If $B \subseteq \mathbb{R}^p$ is an S-free convex set, then by letting $\hat{B} = B \times \mathbb{R}^q$ we can construct a valid relaxation of conv(T) as conv$(\mathcal{C} \setminus \text{int}(\hat{B}))$. Often by a good choice of B, conv$(\mathcal{C} \setminus \text{int}(\hat{B}))$ is a much better approximation of the convex hull of T than $\mathcal{C}$. A classical example of this procedure is that of split disjunctions; see Cook et al. [10]. Notice that if B^1 and B^2 are S-free convex sets such that $B^1 \supseteq B^2$, then conv$(\mathcal{C} \setminus \text{int}(\hat{B^1})) \subseteq$ conv$(\mathcal{C} \setminus \text{int}(\hat{B^2}))$. This motivates the search for maximal S-free convex sets. Various families of cutting planes based on maximal S-free convex sets (for different S) have been proposed. See for example Andersen et al. [1], Andersen et al. [2], Basu et al. [6], Borozan and Cornuéjols [9], Cornuéjols and Margot [11], Dey and Wolsey [12], Fukasawa and Günlük [14], Johnson [16], Zambelli [23]. Since we verify that maximal S-free convex sets are polyhedra, one possible way to compute conv$(\mathcal{C} \setminus \text{int}(\hat{B}))$ is by computing the convex hull of the following disjunction

$$\bigcup_{i=1}^{t} (\mathcal{C} \cap \{(x, y) \,:\, \langle a_i, x\rangle \geq b_i\}), \tag{1.2}$$

where $B = \{x \in \mathbb{R}^p \,:\, \langle a_i, x\rangle \leq b_i,\ i \in \{1, ..., t\}\}$. For many classes of convex sets $\mathcal{C}$, the convex hull of the union of sets in (1.2) can be described 'conveniently'; see for example the case of second order conic representable sets in Ben-Tal and Nemirovski [8].

The rest of the paper is organized as follows. In Section 2, we review some standard results from convex analysis and present a key result about maximal lattice-free convex sets in affine subspaces due to Basu et al. [5]. In Section 3, we present the characterization of maximal S-free convex sets. In Section 4, we discuss some differences between the properties of maximal S-free convex sets in the general case to the case where S is the set of integer points contained in a rational polyhedron and point out some generalizations of the results presented in Section 3.

2. Preliminaries. Let W be an affine subspace of $\mathbb{R}^n$. We use $\text{int}_W(A)$ to denote the interior of the set $A \subseteq W$ with respect to the topology induced by $\mathbb{R}^n$ on W. Therefore $\text{rel.int}(A) = \text{int}_{\text{aff.hull(A)}}(A)$. We will call a set H a half-space (resp. hyperplane) of W if H is the intersection of W with a half-space (resp. hyperplane) of $\mathbb{R}^n$ and $W \nsubseteq H$. For $u \in \mathbb{R}^n$ and $\varepsilon > 0$, $B(u, \varepsilon)$ is the open ball around u of radius $\varepsilon > 0$, that is $B(u, \varepsilon) := \{x \in \mathbb{R}^n \,:\, \|x - u\| < \varepsilon\}$.

We will frequently use the following basic result from convex analysis that we prove for completeness.

LEMMA 2.1. *Let $U \subseteq V \subseteq \mathbb{R}^n$ be affine subspaces and let $A \subseteq V$ be a convex set such that $\text{int}_V(A) \cap U \neq \emptyset$. Then $\text{int}_V(A) \cap U = \text{int}_U(A \cap U)$.*

Proof. The inclusion $\text{int}_V(A) \cap U \subseteq \text{int}_U(A \cap U)$ is straightforward. For the other inclusion, assume that $y \in \text{int}_U(A \cap U)$. Then there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \cap U \subseteq A \cap U$. Since $\text{int}_V(A) \cap U \neq \emptyset$, there exists $x \in \text{int}_V(A) \cap U$. If $x = y$, then the proof is complete. Otherwise, since U is a affine subspace, we obtain that $z = y + \frac{\varepsilon}{2\|x-y\|}(y - x) \in B(y, \varepsilon) \cap U$. It follows that y is a strict convex combination of $z \in A$ and $x \in \text{int}_V(A)$, so $y \in \text{int}_V(A)$, and then $y \in \text{int}_V(A) \cap U$. □

Using Lemma 2.1, we restate a version of separation theorem for convex sets that we use later.

THEOREM 2.2 (Separation Theorem). *Let $W \subseteq \mathbb{R}^n$ be an affine subspace. If $A, B \subseteq W$, A and B are convex sets, and $\text{rel.int}(A) \cap \text{rel.int}(B) = \emptyset$, then there exists half-space H of W such that $A \subseteq H$ and $\text{int}_W(H) \cap B = \emptyset$.*

Proof. By Theorem 4.14 in Hiriart-Urrut and Lemaréchal [15], there exists $s \in \mathbb{R}^n$ such that

$$\sup_{y \in A} \langle s, y \rangle \leq \inf_{y \in B} \langle s, y \rangle \tag{2.1}$$

$$\inf_{y \in A} \langle s, y \rangle < \sup_{y \in B} \langle s, y \rangle. \tag{2.2}$$

Let $\tilde{H} = \{x \in \mathbb{R}^n : \langle s, x \rangle \leq \sup_{y \in A} \langle s, y \rangle\}$ and $H = \tilde{H} \cap W$. By (2.2), s is not orthogonal to W. Therefore H is a halfspace of W. Finally, by Lemma 2.1 we obtain $\text{int}_{\mathbb{R}^n}(\tilde{H}) \cap W = \text{int}_W(H)$, which completes the proof. □

Listed below are some properties of interiors and relative interiors of convex sets that are also used frequently. See Hiriart-Urrut and Lemaréchal [15] and Rockafellar [20] for proofs.

PROPOSITION 2.3. *Let W be an affine subspace and let $A, B \subseteq W$ be convex sets. Then,*

1. $\text{int}_W(A) \cap \text{int}_W(B) = \text{int}_W(A \cap B)$.
2. $\text{rel.int}(A) \cap \text{rel.int}(B) \subseteq \text{rel.int}(A \cap B)$.
3. *If* $\text{rel.int}(A) \cap \text{rel.int}(B) \neq \emptyset$, *then* $\text{rel.int}(A) \cap \text{rel.int}(B) = \text{rel.int}(A \cap B)$.
4. $A \subseteq \overline{\text{rel.int}(A)}$, *where $\overline{C}$ represent the closure of C.*
5. $\text{rel.int}(A) + \text{rel.int}(B) = \text{rel.int}(A + B)$.

DEFINITION 2.4 (S-free and Lattice-free Convex sets). *Let $W \subseteq \mathbb{R}^n$ be an affine subspace of dimension* $\dim(W) \geq 1$, *$\mathcal{P} \subseteq W$ be a convex set and $S = \mathcal{P} \cap \mathbb{Z}^n$. A set K is called S-free (resp. lattice-free) convex set of W if $K \subseteq W$, K is convex and* $\text{int}_W(K) \cap S = \emptyset$ *(resp.* $\text{int}_W(K) \cap \mathbb{Z}^n = \emptyset$*). A convex set $K \subseteq W$ is called maximal S-free (resp. lattice-free) convex set of W if K is S-free (resp. lattice-free) convex set and there does not exists a set $K' \subseteq W$ such that $K' \neq K$, $K' \supseteq K$ and K' is an S-free (resp. lattice-free) convex set of W.*

Basu et al. [5] proved that maximal lattice-free convex sets of affine subspaces are polyhedra. This is a crucial ingredient in the proof presented in this paper. We present this result next.

THEOREM 2.5 (Basu et al. [5]). *Let $W \subseteq \mathbb{R}^n$ be an affine subspace containing an integral point and V be the affine hull of $W \cap \mathbb{Z}^n$. A set $K \subseteq W$ is a maximal lattice-free convex set of W if and only if one of the following holds:*

1. *K is a polyhedron in W whose dimension equals* $\dim(W)$, *$K \cap V$ is a maximal lattice-free convex set of V whose dimension equals* $\dim(V)$, *and for every facet F of K, $F \cap V$ is a facet of $K \cap V$,*
2. *K is an affine hyperplane of W such that $K \cap V$ is an irrational hyperplane of V,*
3. *K is a half-space of W that contains V on its boundary.*

The proof of Theorem 2.5 in Basu et al. [5] involves showing that if K is a lattice-free convex set of W, then K must be contained in maximal lattice-free convex set of W. We will therefore use the following simplified version of Theorem 2.5.

THEOREM 2.6 (Basu et al. [5]). *Let $W \subseteq \mathbb{R}^n$ be an affine subspace containing an integral point. If $K \subseteq W$ is a lattice-free convex set of W, then it is contained in a maximal lattice-free convex set B of W, where B is a polyhedron.*

3. Maximal S-free Convex Sets. As observed in Basu et al. [5], the existence of maximal S-free convex sets is a consequence of Zorn's Lemma.

We will verify the following results in this section.

THEOREM 3.1. *Let $W \subseteq \mathbb{R}^n$ be an affine subspace of dimension* $\dim(W) \geq 1$, *V the affine hull of $W \cap \mathbb{Z}^n$, $\mathcal{P} \subseteq W$ a convex set and $S = \mathcal{P} \cap \mathbb{Z}^n$. If $K \subseteq W$ is a maximal S-free convex set, then one of the following holds:*

1. $\dim(K) = \dim(W)$ *and* $\text{rel.int}(K) \cap \text{rel.int}(P) \neq \emptyset$*: Then* K *is a polyhedron in* W*,* $K \cap V$ *is a maximal* S*-free convex set of* V *whose dimension equals* $\dim(V)$*, and for every facet* F *of* K*,* $F \cap V$ *is a facet of* $K \cap V$*,*
2. $\dim(K) < \dim(W)$*: Then* K *is an hyperplane of* W*,*
3. $\dim(K) = \dim(W)$ *and* $\text{rel.int}(K) \cap \text{rel.int}(P) = \emptyset$*: Then* K *is a halfspace of* W*,*
4. $S = \emptyset$ *and* $K = W$*.*

If S is the set of integer points contained in an rational polyhedron, then every facet of maximal S-free polyhedron contains a point of S in its relative interior. A slightly weaker result holds in the general case. Before we present this result, we require some additional notation. Let K be a polyhedron of W, that is let $K = \{x \in W \ : \ \langle a_i, x\rangle \leq b_i \ \forall i \in \{1, ..., N\}\}$. Then we denote the i^{th} facet of K as $F_i(K)$. Also for all $\varepsilon > 0$, let $F_i^{\varepsilon}(K) = \{x \in W \ : \ \langle a_j, x\rangle < b_j \ \forall \ j \neq i \text{ and } b_i < \langle a_i, x\rangle < b_i + \varepsilon\}$.

THEOREM 3.2. *Let* $W \subseteq \mathbb{R}^n$ *be an affine subspace of dimension* $\dim(W) \geq 1$*,* $\mathcal{P} \subseteq W$ *a convex set and* $S = \mathcal{P} \cap \mathbb{Z}^n$*. Let* K *be a* S*-free convex set such that* $\dim(K) = \dim(W)$*. If* K *is a maximal* S*-free convex set, then* K *is a polyhedron with* N *facets such that*

1. $(\text{rel.int}(F_i) \cup F_i^{\varepsilon}(K)) \cap S \neq \emptyset \qquad \forall \ \varepsilon > 0 \ \forall \ i \in \{1, \ldots, N\}$ *and*
2. $N \leq 2^{\dim(\text{conv}(S))}$*.*

The rest of the section is organized as follows. In Section 3.1, we prove that maximal S-free convex sets are polyhedra. In Section 3.2, we verify properties of facets of maximal S-free convex sets. In particular, if $\dim(K) = \dim(W)$ and $\text{rel.int}(K) \cap \text{rel.int}(P) \neq \emptyset$, then we show that $K \cap V$ is a maximal S-free convex set of V whose dimension equals $\dim(V)$, and for every facet F of K, $F \cap V$ is a facet of $K \cap V$ (where $V = \text{aff.hull}(W \cap \mathbb{Z}^n)$). We also verify part (*1.*) of Theorem 3.2 in this section. Finally in Section 3.3, we obtain an upper bound on the number of facets of maximal S-free convex sets, completing the proof of Theorem 3.2. This upper bound is obtained using the upper bound result for maximal lattice-free sets (Doignon [13], Bell [7], Scarf [21]) but involves a little more work as facets of maximal S-free convex sets do not in general contain points of S in their relative interior.

3.1. Polyhedrality of Maximal S-free Convex Sets. To show that a maximal S-free convex set is a polyhedron, it is sufficient to show that every S-free convex set is contained in a polyhedral S-free convex set. This is verified next.

PROPOSITION 3.3. *Let* $W \subseteq \mathbb{R}^n$ *be an affine subspace of dimension* $\dim(W) \geq 1$*,* $\mathcal{P} \subseteq W$ *be a convex set and* $S = \mathcal{P} \cap \mathbb{Z}^n$*. Let* $K \subseteq W$ *be an* S*-free convex set of* W*. Then one of the following holds:*

1. $\dim(K) = \dim(W)$ *and* $\text{rel.int}(K) \cap \text{rel.int}(P) \neq \emptyset$*:* K *is contained in an* S*-free convex set* $B \subseteq W$ *such that* B *is a polyhedron,*
2. $\dim(K) < \dim(W)$*: Then* K *is contained in an* S*-free hyperplane of* W*,*
3. $\text{rel.int}(K) \cap \text{rel.int}(P) = \emptyset$*: Then* K *is contained in an* S*-free halfspace of* W*,*
4. $S = \emptyset$ *and* W *is an* S*-free convex set (*K *is contained in* W*).*

Proof. Consider first the case where $\dim(W) = 1$. Then since $K \subseteq W$ is a convex set and $\dim(K) \leq 1$, we conclude that $\overline{K}$ is an S-free polyhedron. Therefore the cases (*1.*), (*2.*), (*3.*) and (*4.*) are easily verifiable when $\dim(W) = 1$.

The proof is now by induction on the dimension of W.

If $S \cap W = \emptyset$, then W is an S-free convex set and this completes the proof. Hence, we can assume $S \cap W \neq \emptyset$. Denote $P = \text{conv}(S)$. Observe that since $\mathcal{P}$ is convex, $S = P \cap \mathbb{Z}^n$.

If $\dim(K) < \dim(W)$, then K is contained in a hyperplane H of W. Since $\text{int}_W(H) \cap S = \emptyset$, the proof is complete.

If $\text{rel.int}(K) \cap \text{rel.int}(P) = \emptyset$, then by separation theorem there exists a half-space $H^{\leq}$ of W such that $K \subseteq H^{\leq}$ and $P \cap \text{int}_W(H^{\leq}) = \emptyset$. Thus, K is contained in an S-free half-space. Therefore we can assume $\dim(K) = \dim(W)$ and $\text{rel.int}(K) \cap \text{rel.int}(P) \neq \emptyset$.

Since K is a S-free convex set, we obtain that $\text{int}_W(K \cap P) \cap \mathbb{Z}^n = \text{int}_W(K) \cap \text{int}_W(P) \cap \mathbb{Z}^n \subseteq \text{int}_W(K) \cap (P \cap \mathbb{Z}^n) = \text{int}_W(K) \cap S = \emptyset$. So we conclude $K \cap P$ is a lattice-free convex set of W. Also since $S \cap W \neq \emptyset$, we obtain that $W \cap \mathbb{Z}^n \neq \emptyset$. Hence, by Theorem 2.6, there exists a maximal lattice-free convex set of W, $Q \subseteq W$ such that $K \cap P \subseteq Q$ and Q is a polyhedron. Observe also that since $S \neq \emptyset$, $Q \subsetneq W$. We therefore obtain that

$$Q = \bigcap_{i=1}^{m} H_i^{\leq},$$

where each $H_i^{\leq}$, $i = 1, \ldots, m$ is a half-space of W defining a facet of Q. For $i \in I = \{1, \ldots, m\}$, denote $H_i^{\geq} = W \setminus (\text{int}_W(H_i^{\leq}))$ and $H_i^{=} = H_i^{\leq} \cap H_i^{\geq}$.

We partition the set I into four disjoints subsets.

1. $I_1 := \{i \in I \;:\; \text{rel.int}(P) \cap \text{rel.int}(H_i^{\geq}) \neq \emptyset\}$. If $i \in I_1$, then $\text{rel.int}(P \cap H_i^{\geq}) = \text{rel.int}(P) \cap \text{rel.int}(H_i^{\geq})$. Therefore

$$\begin{aligned} \text{rel.int}(K) \cap \text{rel.int}(P \cap H_i^{\geq}) &= \text{rel.int}(K) \cap \text{rel.int}(P) \cap \text{rel.int}(H_i^{\geq}) \\ &= \text{rel.int}(K \cap P) \cap \text{rel.int}(H_i^{\geq}) \\ &\subseteq Q \cap \text{rel.int}(H_i^{\geq}) = \emptyset. \end{aligned}$$

The second equality holds because $\text{rel.int}(K) \cap \text{rel.int}(P) \neq \emptyset$. Since $\text{rel.int}(K) \cap \text{rel.int}(P \cap H_i^{\geq}) = \emptyset$, we obtain that there exists G_i, a half-space of W, such that

$$K \subseteq G_i, \tag{3.1}$$
$$\text{int}_W(G_i) \cap P \cap H_i^{\geq} = \emptyset. \tag{3.2}$$

2. Consider $I_2 := \{i \in I \setminus I_1 \;:\; \text{int}_W(K) \cap H_i^{=} = \emptyset,\; S \cap H_i^{=} \neq \emptyset\}$.
In this case, we verify that $K \subseteq H_i^{\leq}$. Since $K \subseteq \overline{\text{rel.int}(K)} = \overline{\text{int}_W(K)}$ and $H_i^{\leq}$ is a closed convex set, it is sufficient to verify that $\text{int}_W(K) \subseteq H_i^{\leq}$. Assume by contradiction, that there exists $x \in \text{int}_W(K) \cap \text{int}_W(H_i^{\geq})$. Since $\emptyset \neq \text{rel.int}(K) \cap \text{rel.int}(P) \subseteq K \cap P \subseteq H_i^{\leq}$, we obtain that $\text{rel.int}(K) \cap H_i^{\leq} \neq \emptyset$. However, note that since the affine hull of $\text{rel.int}(K)$ and $H_i^{\leq}$ is W, $\text{rel.int}(K)$ is an open set (wrt W) and $H_i^{\leq}$ is a half space of W, we obtain that $\text{rel.int}(K) \cap \text{rel.int}(H_i^{\leq}) \neq \emptyset$. (Choose $\tilde{y} \in \text{rel.int}(K) \cap H_i^{\leq}$. If $\tilde{y} \in \text{rel.int}(H^{\leq})$, then we are done. Otherwise, since $\tilde{y} \in \text{rel.int}(K)$ and affine hull of $\text{rel.int}(K)$ is W, it is possible to choose a neighborhood $B(\tilde{y}, \varepsilon) \cap W$ of $\tilde{y}$ contained in $\text{rel.int}(K)$. However, since $\tilde{y} \notin \text{rel.int}(H_i^{\leq})$, we obtain that $\tilde{y} \in H_i^{=}$. Therefore, $B(\tilde{y}, \varepsilon) \cap W \cap \text{rel.int}(H_i^{\leq}) \neq \emptyset$ and thus there exists $y \in \text{rel.int}(K) \cap \text{rel.int}(H_i^{\leq})$). Let $y \in \text{rel.int}(K) \cap \text{rel.int}(H_i^{\leq})$. Hence every convex

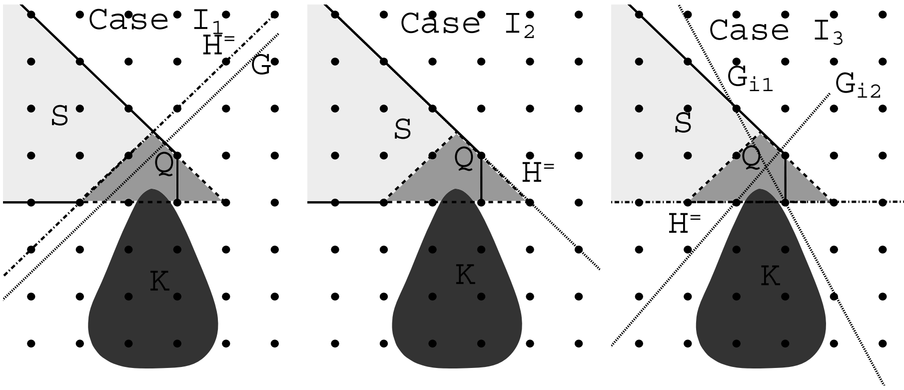

FIG. 3.1. *Illustration of case I_1, I_2, and I_3.*

combination of x and y belongs to $\text{int}_W(K)$ since $x, y \in \text{int}_W(K)$. Moreover, since $x \in \text{int}_W(H_i^{\geq})$ and $y \in \text{int}_W(H_i^{\leq})$, there exists a convex combination of x and y that belongs to $\text{int}_W(K) \cap H_i^{=}$. Therefore $\text{int}_W(K) \cap H_i^{=} \neq \emptyset$, which contradicts the fact that $i \in I_2$. If $i \in I_2$, then define $G_i = H_i^{\leq}$. Therefore from the above, we obtain

$$K \subseteq G_i, \tag{3.3}$$

$$H_i^{=} \cap \text{int}_W(G_i) = \emptyset. \tag{3.4}$$

3. Consider $I_3 := \{i \in I \setminus I_1 \ : \ \text{int}_W(K) \cap H_i^{=} \neq \emptyset, \ S \cap H_i^{=} \neq \emptyset\}$. If $i \in I_3$, then by Lemma 2.1 we obtain that $\text{int}_{H_i^{=}}(K \cap H_i^{=}) = \text{int}_W(K) \cap H_i^{=}$.
Since K is an S-free convex set and $\text{int}_{H_i^{=}}(K \cap H_i^{=}) = \text{int}_W(K) \cap H_i^{=}$ we obtain that $\text{int}_{H_i^{=}}(K \cap H_i^{=}) \cap S = \emptyset$. Hence, $K \cap H_i^{=}$ is a $(S \cap H_i^{=})$-free convex set of $H_i^{=}$. Note that since $H_i^{\leq} \subsetneq W$, we obtain that $\dim(H_i^{=}) = \dim(W) - 1$. By the induction hypothesis, there exists a polyhedron $T_i \subseteq H_i^{=}$ such that $K \cap H_i^{=} \subseteq T_i$, T_i is a $(S \cap H_i^{=})$-free convex set of $H_i^{=}$. Note that since $(S \cap H_i^{=}) \neq \emptyset$, we obtain that $T_i \neq H_i^{=}$. Therefore $T = \bigcap_{j=1}^{m_i} F_{ij}^{\leq}$, where $F_{ij}^{\leq} \subseteq H_i^{=}$, $j = 1, \ldots m_i$ are the half-spaces of $H_i^{=}$ defining the facets of T_i. Denote $F_{ij}^{\geq} = H_i^{=} \setminus (\text{int}_{H_i^{=}}(F_{ij}^{\leq}))$. We have:

$$\begin{aligned} \text{rel.int}(F_{ij}^{\geq}) \cap \text{rel.int}(K) &= \text{rel.int}(F_{ij}^{\geq}) \cap H_i^{=} \cap \text{int}_W(K) \\ &\subseteq F_{ij}^{\geq} \cap \text{int}_{H_i^{=}}(K \cap H_i^{=}) \\ &\subseteq F_{ij}^{\geq} \cap \text{int}_{H_i^{=}}(T_i) = \emptyset. \end{aligned}$$

Therefore there exists G_{ij}, half-spaces of W, such that:

$$K \subseteq G_{ij}, \tag{3.5}$$

$$F_{ij}^{\geq} \cap \text{int}_W(G_{ij}) = \emptyset. \tag{3.6}$$

In this case, define $G_i = \bigcap_{j=1}^{m_i} G_{ij}$.
We verify a property that we require later. Note that since $\text{int}_W(K) \subseteq \text{int}_W(G_{ij})$ and $\text{int}_W(K) \cap H_i^{=} \neq \emptyset$, we obtain that $\text{int}_W(G_{ij}) \cap H_i^{=} \neq \emptyset$. Therefore using Lemma 2.1 we obtain the equality $\text{int}_{H_i^{=}}(G_{ij} \cap H_i^{=}) = \text{int}_W(G_{ij}) \cap$

$H_i^=$. Therefore, (3.6) is equivalent to

$$F_{ij}^{\geq} \cap \text{int}_{H_i^=}(G_{ij} \cap H_i^=) = \emptyset. \tag{3.7}$$

4. Finally define $I_4 := \{i \in I \setminus I_1 \ : \ S \cap H_i^= = \emptyset\}$. Note that $I = I_1 \cup I_2 \cup I_3 \cup I_4$.

Before the final step in the proof, we verify that if $\text{rel.int}(P) \cap \text{rel.int}(H_i^{\geq}) = \emptyset$, then

$$P \cap \text{rel.int}(H_i^{\geq}) = \emptyset. \tag{3.8}$$

As $\text{rel.int}(P) \cap \text{rel.int}(H_i^{\geq}) = \emptyset$, we obtain $\text{rel.int}(P) \subseteq H_i^{\leq}$. Since $H_i^{\leq}$ is a closed set, we obtain $\overline{\text{rel.int}(P)} \subseteq H_i^{\leq}$. Finally, note that since P is a convex set, we obtain $P \subseteq \overline{\text{rel.int}(P)} \subseteq H_i^{\leq}$ (see Proposition 2.3). Thus $P \cap \text{rel.int}(H_i^{\geq}) = \emptyset$.

To complete the proof we will verify that the set $B = \bigcap_{i \in I \setminus I_4} G_i \subseteq W$ is an S-free convex set of W containing K. By (3.1), (3.3) and (3.5), we obtain that $K \subseteq B$. We need to prove that B is an S-free convex set to complete the proof. Assume by contradiction that there exists $y \in S \cap \text{int}_W(B)$, that is $y \in S$ and $y \in \text{int}_W(G_i)$ $\forall i \in I \setminus I_4$.

We have

$$\begin{aligned}
P &= W \cap P \\
&= \left[\bigcup_{i \in I} H_i^{\geq} \cup \text{int}_W(Q)\right] \cap P \\
&\subseteq \left[\bigcup_{i \in I} (H_i^{\geq} \cap P)\right] \cup \text{int}_W(Q) \\
&= \left[\bigcup_{i \in I_1} (H_i^{\geq} \cap P) \cup \bigcup_{i \in I_2} (H_i^= \cap P) \cup \bigcup_{i \in I_3} (H_i^= \cap P) \cup \bigcup_{i \in I_4} (H_i^= \cap P)\right] \\
&\quad \cup \text{int}_W(Q).
\end{aligned} \tag{3.9}$$

The last equality is a consequence of (3.8), since for $i \in I_2 \cup I_3 \cup I_4$ we have that $(H_i^{\geq} \cap P) = (\text{rel.int}(H_i^{\geq}) \cap P) \cup (H_i^= \cap P) = (H_i^= \cap P)$.

Since $y \in S = P \cap \mathbb{Z}^n$ and $S \cap H_i^= = \emptyset$ for $i \in I_4$, we obtain that $y \notin H_i^= \cap P$ for $i \in I_4$. Moreover Q is lattice-free. Therefore, using (3.9) there are three cases:

1. For $i \in I_1$, if $y \in H_i^{\geq} \cap P$ and $y \in \text{int}_W(G_i)$, then we obtain a contradiction to (3.2).
2. For $i \in I_2$, if $y \in H_i^=$ and $y \in \text{int}_W(G_i)$, then we obtain a contradiction to (3.4).
3. For $i \in I_3$, if $y \in H_i^=$, then since $y \in \bigcap_{j=1}^{m_i} \text{int}_W(G_{ij}) \cap H_i^= = \bigcap_{j=1}^{m_i} \text{int}_{H_i^=}(G_{ij} \cap H_i^=) \subseteq \bigcap_{j=1}^{m_i} (H_i^= \setminus F_{ij}{}^{\geq}) = \bigcap_{j=1}^{m_i} \text{int}_{H_i^=}(F_{ij}{}^{\leq})$ (the last inclusion is a consequence of (3.7)), we obtain that $y \in \text{int}_{H_i^=}(T_i)$ which is in contradiction with the fact that T_i is an $S \cap H_i^=$-free convex set of $H_i^=$.

Therefore B is S-free polyhedron containing K. □

Cases (*2.*), (*3.*), (*4.*) of Theorem 3.1 follow from maximality of K and cases (*2.*), (*3.*), (*4.*) of Proposition 3.3 respectively. Case (*1.*) of Proposition 3.3 shows that maximal S-free convex sets are polyhedra when $\dim(K) = \dim(W)$ and $\text{rel.int}(K) \cap \text{rel.int}(P) \neq \emptyset$. To complete the proof of Theorem 3.1, we need to show that $K \cap V$ is a maximal S-free convex set of V whose dimension equals $\dim(V)$, and for every facet F of K, $F \cap V$ is a facet of $K \cap V$ (where $V = \text{aff.hull}(W \cap \mathbb{Z}^n)$). This is verified in the next section.

3.2. Structure of Facets of Maximal S-free Convex Sets. For convenience we repeat the definition of some notation. Let K be the polyhedron of W, that is let $K = \{x \in W \ : \ \langle a_i, x\rangle \leq b_i \ \forall i \in \{1, ..., N\}\}$. Then we denote the i^{th} facet of K as $F_i(K)$. Also for all $\varepsilon > 0$, let $F_i^{\varepsilon}(K) = \{x \in W \ : \ \langle a_j, x\rangle < b_j \ \forall j \neq i \text{ and } \ b_i < \langle a_i, x\rangle < b_i + \varepsilon\}$.

Before completing the proof of Theorem 3.1, we prove part (*1.*) of Theorem 3.2.

Proposition 3.4. *Let* $K = \{x \in W \ : \ \langle a_i, x\rangle \leq b_i, \ \forall \ i \in \{1, \dots, N\}\} \subseteq \mathbb{R}^n$ *be a maximal S-free convex set, such that* $\dim(K) = \dim(W)$. *Then*

$$(\text{rel.int}(F_i(K)) \cup F_i^{\varepsilon}(K)) \cap S \neq \emptyset \qquad \forall \ \varepsilon > 0 \ \forall \ i \in \{1, \dots, N\}.$$

Proof. For $\varepsilon > 0$ and $i \in \{1, ..., N\}$ consider the set

$$K_i^{\varepsilon} = \{x \in W \ : \ \langle a_j, x\rangle \leq b_j \ \forall j \neq i \ \text{ and } \langle a_i, x\rangle \leq b_i + \varepsilon\}.$$

Since $K \subsetneq K_i^{\varepsilon}$ we obtain that K_i^{ε} is not an S-free convex set, that is, $\text{int}_W(K_i^{\varepsilon}) \cap S \neq \emptyset$. Hence, the set $\text{int}_W(K_i^{\varepsilon}) \setminus \text{int}_W(K)$ must contain a point of S. Observe that:

$$\begin{aligned}\text{int}_W(K_i^{\varepsilon}) \setminus \text{int}_W(K) &= \{x \in W \ : \ \langle a_j, x\rangle < b_j \ \forall \ j \neq i \text{ and } b_i \leq \langle a_i, x\rangle < b_i + \varepsilon\} \\ &= F_i^{\varepsilon}(K) \cup \text{rel.int}(F_i(K)).\end{aligned}$$

Therefore $\emptyset \neq \text{int}_W(K_i^{\varepsilon}) \setminus \text{int}_W(K) \cap S = (F_i^{\varepsilon}(K) \cup \text{rel.int}(F_i(K))) \cap S$. □

Note that Proposition 3.4 highlights an important difference between maximal S-free convex sets for general S and for the case where S is the set of integer points contained in a rational polyhedron. In the case of general S, there is no guarantee that every facet of a maximal S-free convex set contains points belonging to S in its relative interior. This is illustrated in the next example.

Example 3.5. *Let* $S = \{x \in \mathbb{Z}^2 \ : \ \sqrt{2}x_1 + x_2 \leq 0, x_1 \geq 1\}$. *Then the set* $K = \{x \in \mathbb{R}^2 \ : \ \sqrt{2}x_1 + x_2 \geq 0\}$ *is a maximal S-free convex set, but the facet of K defined by* $\{x \in \mathbb{R}^2 \ : \ \sqrt{2}x_1 + x_2 = 0\}$ *contains no point belonging to S.*

Now we complete the proof of Theorem 3.1. The proof of the first part of the next proposition is similar to the proof of a similar property for maximal lattice-free convex sets, appearing in Basu et al. [5].

Proposition 3.6. *Let* $K = \{x \in W \ : \ \langle a_i, x\rangle \leq b_i, \ \forall \ i \in \{1, \dots, N\}\} \subseteq W$ *be a maximal S-free convex set such that* $\dim(K) = \dim(W)$ *and* $\text{int}_W(K) \cap \text{rel.int}(\text{P}) \neq \emptyset$. *Let V be the affine hull of $W \cap \mathbb{Z}^n$. Then $K \cap V$ is a maximal S-free convex set of V such that* $\dim(K \cap V) = \dim(V)$ *and $F \subseteq V$ is a facet of $K \cap V$ if and only if $F = F_i(K) \cap V$ for some $i \in \{1, \dots, N\}$.*

Proof. Since $\text{int}_W(K) \cap \text{rel.int}(\text{P}) \neq \emptyset$, we obtain that $\text{int}_W(K) \cap V \neq \emptyset$. Therefore Lemma 2.1 implies that $\text{int}_W(K) \cap V = \text{int}_V(K \cap V)$.

We first verify that $K \cap V$ is a maximal S-free convex set of V. Since K is an S-free convex set and $\text{int}_W(K) \cap S = \text{int}_W(K) \cap S \cap V = \text{int}_V(K \cap V) \cap S$, we obtain that $K \cap V$ is an S-free convex set of V. If $K \cap V$ is not maximal, then there exist $B \subseteq V$, an S-free convex set, such that $B \supsetneq K \cap V$. Consider $K' = \text{conv}(K \cup B)$. Then $K' \cap V = B$. We have $\text{int}_W(K') \cap S = \text{int}_W(K') \cap S \cap V = \text{int}_V(B) \cap S = \emptyset$. Therefore K' is an S-free convex set of W. Since $K' \supseteq K$ and $B \supsetneq K \cap V$, we obtain $K' \supsetneq K$ which is in contradiction with the fact that K is maximal S-free convex set. Therefore, $K \cap V$ is a maximal S-free convex set of V.

Since $\text{int}_V(K \cap V) \neq \emptyset$, we obtain $\dim(K \cap V) = \dim(V)$. Finally, we verify that $F \subseteq V$ is a facet of $K \cap V$ if and only if $F = F_i(K) \cap V$ for some $i \in \{1, \ldots, N\}$.

($\Rightarrow$) If F is a facet of $K \cap V = \{x \in V \ : \ \langle a_i, x\rangle \leq b_i, \ \forall\ i \in \{1, \ldots, N\}\}$, then $F = F_i(K) \cap V$ for some $i \in \{1, \ldots, N\}$.

($\Leftarrow$) Given $\varepsilon > 0$, by the use of Proposition 3.4 we obtain that for all $i = 1, \ldots, N$, $\emptyset \neq (\text{relint}(F_i(K)) \cup F_i^{\varepsilon}(K)) \cap S = (\text{rel.int}(F_i(K)) \cup F_i^{\varepsilon}(K)) \cap V \cap S \subseteq \text{int}_W(\{x \in W \ : \ \langle a_k, x\rangle \leq b_k, \ \forall\ k \in \{1, \ldots, N\} \setminus \{i\}\}) \cap V \cap S = \text{int}_V(\{x \in V \ : \ \langle a_k, x\rangle \leq b_k, \ \forall\ k \in \{1, \ldots, N\} \setminus \{i\}\}) \cap S$. Note that the last equality is obtained as a consequence of Lemma 2.1, $\{x \in W \ : \ \langle a_k, x\rangle \leq b_k, \ \forall\, k \in \{1, \ldots, N\}\setminus\{i\}\} \supseteq K$ and $\text{int}_W(K) \cap V \neq \emptyset$. As $K \cap V$ is an S-free convex set of V and $\dim(K \cap V) = \dim(V)$, we conclude that $\langle a_i, x\rangle \leq b_i$ must define a facet of $K \cap V$, that is $F_i(K) \cap V$ is a facet of $K \cap V$ for all $i \in \{1, ..., N\}$. □

3.3. Upper Bound on the Number of Facets of Maximal S-free Convex Sets. When S is the set of integer points contained in a general convex set, maximal S-free convex sets do not need to have points of S in the relative interior of each facet. Therefore, proving upper bound on the number of facets is slightly more involved than the case of maximal lattice-free convex sets.

We first begin with a corollary of Theorem 3.1 and Proposition 3.4. Alternatively, this also follows from results in Basu et al. [6].

Corollary 3.7. *Let $K' \subseteq W$, $\mathcal{P}'$ a rational polytope, $S' = \mathcal{P}' \cap \mathbb{Z}^n$,* $\dim(K') = \dim(W)$*, then K' is a maximal S'-free convex set of W if and only if K' is a S-free polyhedron with a point of S' in the relative interior of each of its facets.*

The next lemma is a standard result. See Schrijver [22] for a proof.

Lemma 3.8. *If $L \subseteq \mathbb{R}^n$ is a rational affine subspace (i.e.* $\text{aff.hull}(L \cap \mathbb{Z}^n) = L$*), then there exists an affine transformation $T : \mathbb{R}^{\dim(L)} \longmapsto L$ such that T is invertible and $T(\mathbb{Z}^{\dim(L)}) = L \cap \mathbb{Z}^n$.*

Using Corollary 3.7 and Lemma 3.8 and a proof similar to that in Doignon [13], Bell [7], Scarf [21] we obtain the following result.

Lemma 3.9. *Let $K' \subseteq W$, $\mathcal{P}'$ a polytope, $S' = \mathcal{P}' \cap \mathbb{Z}^n$,* $\dim(K') = \dim(W)$*. If K' is a maximal S'-free convex set of W, then K' is a polyhedron with at most $2^{\dim(\mathcal{P}')}$ facets.*

We now have all the tools needed to verify the upper bound on the number of facets of maximal S-free convex sets.

Proposition 3.10. *If $K = \{x \in W \ : \ \langle a_i, x\rangle \leq b_i, \ \forall\ i \in \{1, \ldots, N\}\} \subseteq W$ is a maximal S-free convex set such that* $\dim(K) = \dim(W)$*, then* $N \leq 2^{\dim(\text{conv}(S))}$.

Proof. Let $\varepsilon > 0$. Consider the sets I_1 and I_2 defined as:

1. $\forall\ i \in I_1$, $S \cap \text{rel.int}(F_i(K)) \neq \emptyset$.
2. $\forall\ i \in I_2$, $S \cap \text{rel.int}(F_i(K)) = \emptyset$ and, $F_i^{\varepsilon}(K) \cap S \neq \emptyset$.

By Proposition 3.4, these sets are well defined and $I_1 \cup I_2 = \{1, ..., N\}$. If $I_1 \neq \emptyset$, then for each $i \in I_1$ take a point $x_i \in \text{rel.int}(F_i(K)) \cap S$ and if $I_2 \neq \emptyset$, then for each $i \in I_2$, take a point $x_i \in F_i^{\varepsilon}(K) \cap S$.

Define $\mathcal{P}' = \text{conv}(\{x_1, \ldots, x_N\})$ and $S' = \mathcal{P}' \cap \mathbb{Z}^n$. Since $\mathcal{P}' \subseteq \mathcal{P}$, we have that $S' \subseteq S$. This implies that $\text{int}_W(K) \cap S' \subseteq \text{int}_W(K) \cap S = \emptyset$, so K is an S'-free convex set.

Note that $\mathcal{P}'$ is a rational polytope, so by Corollary 3.7 and Lemma 3.9 every maximal S'-free convex set K' is a polyhedron that has at most $2^{\dim(\mathcal{P}')}$ facets and contains an integer point of S' in the relative interior of each of its facets. Observe that $\dim(\mathcal{P}') \leq \dim(\text{conv}(S))$.

If $I_2 = \emptyset$, then K is a maximal S'-free convex set. Therefore $N \leq 2^{\dim(\mathcal{P}')} \leq 2^{\dim(\text{conv}(S))}$.

If $I_2 \neq \emptyset$, then consider $i \in I_2$. We will construct a polyhedron K_1 with N facets such that:

1. K_1 is an S'-free convex set.
2. $F_j^\varepsilon(K) \subseteq F_j^\varepsilon(K_1)$, $\forall\, j \in I_2 \setminus \{i\}$.
3. K_1 has N facets.
4. K_1 has at least $|I_1| + 1$ facets with a point of S' in the relative interior.

We construct K_1 from K by changing the right hand side of the i^{th} inequality. Since $\mathcal{P}'$ is bounded and $S' \subseteq \mathbb{Z}^n$, we obtain that $|S'|$ is finite. So $S' \cap F_i^\varepsilon(K)$ is also finite. Moreover, since $x_i \in S' \cap F_i^\varepsilon(K)$, we have $S' \cap F_i^\varepsilon(K) \neq \emptyset$. Hence, there exists $z_i \in S' \cap F_i^\varepsilon(K)$ such that

$$\min\{\langle a_i, x\rangle \;:\; x \in S' \cap F_i^\varepsilon(K)\} = \langle a_i, z_i\rangle.$$

Denote $b_i' = \langle a_i, z_i\rangle$. Consider the polyhedron

$$K_1 = \{x \in W \;:\; \langle a_i, x\rangle \leq b_i,\ \forall\ i \in I_1 \cup I_2 \setminus \{i\},\ \langle a_i, x\rangle \leq b_i'\}$$

We verify that K_1 satisfies (1.), (2.), (3.), and (4.):

1. Since K is S'-free, we only need to show that $\text{int}(K_1)\setminus\text{int}(K)$ does not contain points of S'. Since $b_i' \leq b_i + \varepsilon$ we have:

$$\begin{aligned}\text{int}(K_1) \setminus \text{int}(K) &= \{x \in W \;:\; \langle a_i, x\rangle < b_i,\ \forall\ i \in I_1 \cup I_2 \setminus \{i\},\ b_i \leq \langle a_i, x\rangle < b_i',\}\\ &\subseteq \{x \in W \;:\; \langle a_j, x\rangle < b_j\ \forall\ j \neq i \text{ and } b_i \leq \langle a_i, x\rangle < b_i + \varepsilon\}\\ &= F_i^\varepsilon(K) \cup \text{rel.int}(F_i(K))\end{aligned}$$

   Since $i \in I_2$, $S \cap \text{rel.int}(F_i(K)) = \emptyset$ so $(\text{int}(K_1) \setminus \text{int}(K)) \cap S' \subseteq F_i^\varepsilon(K) \cap S'$. On the other hand, since $\forall\ x \in S' \cap F_i^\varepsilon(K)$, $\langle a, x\rangle \geq b_i'$, we conclude $(\text{int}(K_1) \setminus \text{int}(K)) \cap S' = \emptyset$. Therefore, K_1 is S'-free convex set.
2. Using the fact that $b_i \leq b_i'$ and $j \in I_2 \setminus \{i\}$ we have,

$$\begin{aligned}F_j^\varepsilon(K) &= \{x \in W \;:\; \langle a_k, x\rangle < b_k\ \forall k \neq j \text{ and } \ b_j < \langle a_j, x\rangle < b_j + \varepsilon\}\\ &\subseteq \{x \in W \;:\; \langle a_k, x\rangle < b_k\ \forall k \neq i, j,\ \langle a_i, x\rangle < b_i' \text{ and } b_j < \langle a_j, x\rangle < b_j + \varepsilon\}\\ &= F_j^\varepsilon(K_1).\end{aligned}$$

3. By definition, the point z_i satisfies $\langle a_i, z_i\rangle = b_i'$ and $\forall\ j \neq i$, $\langle a_j, z_i\rangle < b_j$. Therefore the inequality $\langle a_i, x\rangle \leq b_i'$ is a facet defining inequality for K_1. Observe also that the inequalities $\langle a_j, x\rangle \leq b_j$ for $j \neq i$ are facet-defining inequality for K_1. Therefore K_1 has N facets.
4. As verified earlier, z_i belongs to the relative interior of the facet of K_1 defined by $\{x \in K_1 \;:\; \langle a_i, x\rangle = b_i'\}$. Also for $j \in I_1$, since $b_i \leq b_i'$, we have:

$$\begin{aligned}\text{rel.int}(F_j(K)) &= \{x \in W \;:\; \langle a_k, x\rangle < b_k\ \forall k \neq j \text{ and } \langle a_j, x\rangle = b_j\}\\ &\subseteq \{x \in W \;:\; \langle a_k, x\rangle < b_k\ \forall k \neq i, j,\ \langle a_i, x\rangle < b_i' \text{ and } \langle a_j, x\rangle = b_j\}.\end{aligned}$$

   Therefore, since $x_j \in \text{rel.int}(F_j) \cap S'$, the facet of K_1 defined by $\{x \in K_1 \;:\; \langle a_j, x\rangle = b_j\}$ has a point of S' in its relative interior. In conclusion, K_1 has at least $|I_1| + 1$ facets with a point of S' in its relative interior.

So we have a procedure to construct, from K, an S'-free polyhedron K_1 with N facets that has at least one more facet that contains a point of S' in its relative interior than K has. If K_1 is not a maximal S'-free convex set, then we can choose a facet of K_1 without a point of S' in its relative interior and by properties (1.) and (2.), repeat the above procedure. We can repeat this a finite number of times, obtaining a sequence $K_1, K_2, \ldots, K_T$ of polyhedra with the same number of facets as K and such that K_T does not have any facet without a point of S' in its relative interior. So K_T is a maximal S'-free convex set. Thus, $N \leq 2^{\dim(\mathcal{P}')} \leq 2^{\dim(\text{conv}(S))}$. □

4. Notes.

4.1. Differences Between General Maximal S-free Convex Sets and the Case Where S is the Set of Integer Points Contained in a Rational Polyhedron. As discussed in Section 3.2, in the case of general S, maximal S-free convex sets do not necessarily have integer points in the relative interior of each facet.

There is another difference between maximal S-free convex sets in the general case and the case when S is the set of integer points contained in a rational polyhedron. The result of Lovász [18] states that maximal lattice-free convex sets are in the form of a polytope plus a cylinder, that is, if r is a recession direction of maximal lattice-free convex set, then so is $-r$. Similarly, Basu et al. [6] prove the following result.

Proposition 4.1 (Basu et al. [6]). *If S is a nonempty set of integer points contained in a rational polyhedron $\mathcal{P}$, K is a maximal S-free convex set such that $K \cap \text{conv}(S)$ has nonempty interior, and r belongs to the recession cone of $\mathcal{P}$ and K, then $-r$ belongs to the recession cone of K.*

Proposition 4.1 is an important property and is useful in verifying many results. See for example Basu et al. [4]. The following example shows that this result is not true in general.

Example 4.2. *Let $\mathcal{P} = \{u \in \mathbb{R}^3 : u_3 \geq u_2 - \sqrt{2}u_1, u_3 \geq \sqrt{2}u_1 - u_2\}$. Then $K = \{v \in \mathbb{R}^3 : v_3 \leq 1, v_2 \geq 0\}$ is a maximal S-free convex set. Also $r = (1, \sqrt{2}, 0) \in \text{rec}(K \cap \mathcal{P})$. However, $-r \notin \text{rec}(K)$.*

We present some sufficient conditions on $\mathcal{P}$ for a property like the one presented in Proposition 4.1 to hold. For simplicity we restrict the discussion to the case where $W = \mathbb{R}^n$.

Given a closed convex set $\mathcal{P} \subseteq \mathbb{Z}^n$, and a non-zero vector $r \in \text{rec}(\mathcal{P})$, let $\mathcal{Z}(\mathcal{P}, r) = \{x \in \mathbb{Z}^n : \exists \lambda \geq 0, \text{ s.t. } x + \lambda r \in \mathcal{P}\}$.

Definition 4.3 (Convex Set with Dirichlet Property). *We say a closed convex set $\mathcal{P} \subseteq \mathbb{R}^n$ satisfies Dirichlet Property if $\mathcal{P}$ satisfies the following conditions: For all $r \in \text{rec}(\mathcal{P}) \setminus \{0\}$ and for all $x \in \mathcal{Z}(P, r)$, given any $\epsilon > 0$ and $\gamma \geq 0$ there exists $\bar{x} \in \mathcal{P} \cap \mathbb{Z}^n$ such that the distance between the integer point $\bar{x}$ and the half line $\{x + \lambda r : \lambda \geq \gamma\}$ is less than ϵ.*

We first show that the Dirichlet property indeed implies a property similar to that presented in Proposition 4.1.

Proposition 4.4. *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a closed convex set satisfying Dirichlet Property, let $S = \mathcal{P} \cap \mathbb{Z}^n$, and let K be a full-dimensional maximal S-free convex set. If r belongs to the recession cone of $\mathcal{P}$ and K, then $-r$ belongs to the recession cone of K.*

Proof. By part (5.) of Proposition 2.3, we obtain $\text{int}(K + \{\lambda r \,|\, \lambda \in \mathbb{R}\}) = \text{int}(K) + \text{rel.int}(\{\lambda r \,|\, \lambda \in \mathbb{R}\}) = \text{int}(K) + \{\lambda r \,|\, \lambda \in \mathbb{R}\}$.

To prove the result of this proposition we need to verify that $\text{int}(K + \{\lambda r \,|\, \lambda \in \mathbb{R}\}) \cap S = \emptyset$. Assume by contradiction that $x \in \text{int}(K + \{\lambda r \,|\, \lambda \in \mathbb{R}\}) \cap S$. By the previous claim, there exists $\bar{\lambda} \in \mathbb{R}_+$ such that $x + \bar{\lambda} r \in \text{int}(K)$. Therefore $B(0, \delta) +$

$\{x+\lambda r\,:\,\lambda\geq\bar{\lambda}\}$ is contained in the interior of K for some suitable small and positive δ.

Since r is a recession direction of $\mathcal{P}$, we obtain that $x\in\mathcal{Z}(\mathcal{P},r)$. As $\mathcal{P}$ satisfies Dirichlet property, there exists an integer point z belonging to $\mathcal{P}$ in the interior of the set $B(0,\delta)+\{x+\lambda r\,:\,\lambda\geq\bar{\lambda}\}$. However, this set lies in the interior of K. Therefore $z\in\mathcal{P}$ and lies in interior of K, a contradiction. □

Our motivation for the name 'Dirichlet property' is due to the fact that we often use the following consequence of Dirichlet Theorem to prove the 'Dirichlet property'.

Lemma 4.5 (Basu et al. [5]). *If $x\in\mathbb{Z}^n$ and $r\in\mathbb{R}^n$, then for all $\epsilon>0$ and $\gamma\geq 0$, there exists a point of $\mathbb{Z}^n$ at a distance less than ϵ from the half line $\{x+\lambda r\,:\,\lambda\geq\gamma\}$.*

4.1.1. Some Convex Sets Satisfying Dirichlet Property. Next we give examples of convex sets with Dirichlet Property.

Proposition 4.6. *Every full dimensional closed convex set in $\mathbb{R}^2$ is a convex set satisfying Dirichlet Property.*

Proof. Let $\mathcal{P}$ be a full-dimensional, closed convex set in $\mathbb{R}^2$. Consider any $\epsilon>0$ and $\gamma\geq 0$. Let r be a non zero vector in the recession cone of $\mathcal{P}$ and $x\in\mathcal{Z}(\mathcal{P},r)$. Denote $\eta=\min\{\lambda\in\mathbb{R}_+\,:\,x+\lambda r\in\mathcal{P}\}$.

1. Since $\mathcal{P}$ is a closed set, we obtain that if r is a rational vector, then there exists $\bar{x}\in\mathcal{P}\cap\mathbb{Z}^n$ such that the distance between $\bar{x}$ and the half line $\{x+\lambda r\,:\,\lambda\geq\gamma\}$ is 0.
2. Suppose now that r is not a rational vector (i.e. $\lambda r\notin\mathbb{Z}^2\ \forall\lambda>0$). Assume by contradiction that there exists no $\bar{x}\in\mathcal{P}\cap\mathbb{Z}^n$ such that the distance between $\bar{x}$ and the half line $\{x+\lambda r\,:\,\lambda\geq\gamma\}$ is less than ϵ. Since $\mathcal{P}$ is full dimensional and $L:=\{x+\lambda r\,:\,\lambda\geq\eta\}\subseteq\mathcal{P}$, the set $M:=\{y\in\mathcal{P}\,:\,\text{distance}(y,L)\leq\epsilon\}$ is a full-dimensional convex subset of $\mathcal{P}$. Note now that M is lattice-free and r belongs to the recession cone of M. Therefore M is contained in a maximal lattice-free convex set. Since the only class of unbounded full-dimensional maximal lattice-free convex set in $\mathbb{R}^2$ is the split set (a set of form $\{y\in\mathbb{R}^2\,:\,b\leq\langle a,y\rangle\leq b+1\}$ where $a\in\mathbb{Z}^2$ and $b\in\mathbb{Z}$), we obtain that r is a rational vector which is a contradiction.

□

We next verify that all the examples which do not satisfy the property similar to that presented in Proposition 4.1, also satisfy the condition that $\text{conv}(S)$ is not closed. Note that Proposition 4.6 can be used to construct examples where $\text{conv}(S)$ is not closed and yet the property similar to that presented in Proposition 4.1 is satisfied.

Proposition 4.7. *If $S\subseteq\mathbb{Z}^n$ and $\text{conv}(S)$ is full-dimensional and closed, then $\text{conv}(S)$ satisfies the Dirichlet property.*

Proof. Let $\mathcal{P}:=\text{conv}(S)$. If $n=1$, then the result is straightforward to verify.

The proof is now by induction on n. Consider any $\epsilon>0$ and $\gamma\geq 0$. Let r be a non zero vector in the recession cone of $\mathcal{P}$ and $x\in\mathcal{Z}(\mathcal{P},r)$. Denote $\eta=\min\{\lambda\in\mathbb{R}_+\,:\,x+\lambda r\in\mathcal{P}\}$.

1. Since $\mathcal{P}$ is a closed set, we obtain that if r is a rational vector, then there exists $\bar{x}\in\mathcal{P}\cap\mathbb{Z}^n$ such that the distance between $\bar{x}$ and the half line $\{x+\lambda r\,:\,\lambda\geq\gamma\}$ is 0.
2. Suppose now that r is not a rational vector. There are two cases:
   (a) Suppose $\exists\ \delta>0$ and $\zeta>0$ such that $B(x+\zeta r,\delta)\subseteq\mathcal{P}$. Let $\epsilon'=\min\{\delta,\epsilon\}$. Now as a consequence of the Dirichlet Theorem there exists an integer point $\bar{x}$ at a distance less that ϵ' from the half line $\{x+\lambda r\,:\,\lambda\geq\max\{\zeta,\gamma\}\}$. Then $\bar{x}$ belongs to $\mathcal{P}$, completing the proof.

(b) The half-line $\{x + \lambda r \,:\, \lambda \geq \eta\}$ lies on the boundary of $\text{conv}(S)$. Let F be a proper face of $\text{conv}(S)$ containing this half-line. (Recall that a face F of a closed convex set $\mathcal{P}$ is a closed convex subset such that if $x \in F$ and x can be written as convex combination of $x^1, x^2 \in \mathcal{P}$, then $x^1, x^2 \in F$). Then we claim:

  i. $F = \text{conv}(S \cap F)$: Since $S \cap F \subseteq F$ and F is a convex set, we obtain that $\text{conv}(S \cap F) \subseteq F$. For the other inclusion, observe that if $x \in F$, then x can be written as a convex combination of a finite number of vectors in S (since $F \subseteq \text{conv}(S)$). However by definition of a face, each of these vectors belong to F. Thus $x \in \text{conv}(S \cap F)$, or equivalently $F \subseteq \text{conv}(S \cap F)$.

  ii. aff.hull(F) is a rational affine half-space containing an integer point: By the above claim $S \cap F \neq \emptyset$ and aff.hull(F) can be generated by vectors in S, which are integral vectors. Thus, aff.hull(F) is a rational affine half-space.

Now, we apply the invertible affine transformation $A : \text{aff.hull}(F) \longmapsto \mathbb{R}^{\dim(\text{aff.hull}(F))}$ to F, where A is the function described in Lemma 3.8. Observe now that $A(F)$ is a closed convex set, $A(F) = \text{conv}(A(S \cap F))$, Ar is a recession direction of $A(F)$, $Ax \in \mathcal{Z}(A(F), Ar)$ and $1 \leq \dim(\text{aff.hull}(F)) < n$. Therefore by the induction hypothesis, for all $\delta > 0$, there exists an integer point $\hat{x}$ belonging to $A(\text{conv}(S \cap F))$ that lies within a distance of δ from the half-line $\{u \,:\, u = Ax + \lambda Ar, \lambda \geq \max\{\eta, \gamma\}\}$. However, this implies that there exists a point $\bar{x} \in S$ such that $\bar{x}$ lies at a distance less than ϵ to the half line $\{u \,:\, u = x + \lambda r, \lambda \geq \max\{\eta, \gamma\}\}$.

□

We remark here that by the use of Lemma 3.8, the result of Proposition 4.7 can be extended to the case where aff.hull$(\text{conv}(S))$ is not full-dimensional since aff.hull$(\text{conv}(S))$ is a rational affine subspace.

Next we show an interesting class of non-polyhedral convex sets that satisfy Dirichlet Property.

Definition 4.8. *A set $\mathcal{P}$ is called a strictly convex set if it satisfies the following property: If $u \in \mathcal{P}$ such that u is not an extreme point, then u belongs to the relative interior of $\mathcal{P}$.*

Proposition 4.9. *Every full dimensional, closed, strictly convex set satisfies Dirichlet Property.*

Proof. Let $\mathcal{P}$ be a full dimensional, closed, strictly convex set. Consider any $\epsilon > 0$ and $\gamma \geq 0$. Let r be a non zero vector in the recession cone of $\mathcal{P}$ and $x \in \mathcal{Z}(\mathcal{P}, r)$. Denote $\eta = \min\{\lambda \in \mathbb{R}_+ \,:\, x + \lambda r \in \mathcal{P}\}$.

1. Since $\mathcal{P}$ is a closed set, we obtain that if r is a rational vector, then there exists $\bar{x} \in \mathcal{P} \cap \mathbb{Z}^n$ such that the distance between $\bar{x}$ and the half line $\{x + \lambda r \,:\, \lambda \geq \gamma\}$ is 0.
2. Suppose now that r is not a rational vector. Let $\bar{\gamma} := \max\{\gamma, \eta + 1\}$. The point $x + \bar{\gamma} r$ belongs to $\mathcal{P}$ and is not an extreme point of $\mathcal{P}$. Since $\mathcal{P}$ is strictly convex, the point $x + \bar{\gamma} r$ lies in the interior of $\mathcal{P}$. Therefore, there exists a ball of radius $\delta > 0$ around the point $x + \bar{\gamma} r$ that lies in $\mathcal{P}$. Set $\epsilon' := \min\{\epsilon, \frac{\delta}{2}\}$. Now as a consequence of the Dirichlet Theorem there exists an integer point $\bar{x}$ at a distance less that ϵ' from the half line $\{x + \lambda r \,:\, \lambda \geq \bar{\gamma}\}$. Since $\bar{x}$ lies at a distance less than δ from the half line $\{x + \lambda r \,:\, \lambda \geq \bar{\gamma}\}$ and the

set $B(0,\delta) + \{x + \lambda r : \lambda \geq \bar{\gamma}\}$ belongs to $\mathcal{P}$, we obtain that $\bar{x}$ belongs to $\mathcal{P}$. Moreover, the point $\bar{x}$ lies at a distance less that ϵ from the half line $\{x + \lambda r : \lambda \geq \gamma\}$.

□

4.2. Other Extensions.

1. Instead of defining $S \subseteq \mathbb{Z}^n$, we can define S as a subset of points belonging to a general lattice. All the results in Section 3 carry through.
2. The condition that $S \subseteq \mathbb{Z}^n$ such that $\text{conv}(S) \cap \mathbb{Z}^n = S$ is not necessary for the polyhedrality of maximal S-free convex sets. For example, the following corollary of Theorem 3.1 can be proven.

Corollary 4.10. *Let $S_i \subseteq \mathbb{Z}^n \cap W$, $i = 1, \ldots, N$ such that $S_i = conv(S_i) \cap \mathbb{Z}^n$. Denote $S = \bigcup_{i=1}^N S_i$. Let $K \subseteq W$ be an S-free convex set of W. Then there exists an S-free polyhedron $B \subseteq W$ such that $K \subseteq B$.*

Proof. For all $i = 1, \ldots, N$, $\text{int}_W(K) \cap S_i \subseteq \text{int}_W(K) \cap S = \emptyset$. Therefore K is an S_i-free convex set of W. By Theorem 3.1, there exists an S_i-free polyhedron $B_i \subseteq W$ such that $K \subseteq B_i$. The polyhedron $B = \bigcap_{i=1}^N B_i$ satisfies $K \subseteq B$ and $\text{int}_W(B) \cap S = \bigcap_{i=1}^N \text{int}_W(B_i) \cap \bigcup_{j=1}^N S_j = \bigcup_{j=1}^N \left[S_j \cap \bigcap_{i=1}^N \text{int}_W(B_i)\right] \subseteq \bigcup_{j=1}^N [S_j \cap \text{int}_W(B_j)] = \emptyset$. Thus B is an S-free polyhedron containing K. □

An analysis similar to that in Section 3.1 and 3.2 can be carried out for this more general class of S. The upper bound on the number of facets of maximal S-free convex sets presented in Section 3.3 is not valid for this class of more general S. For example, if $S = \{(0,0), (1,0), (-1,1), (2,1), (0,2), (1,2)\}$, then the hexagon with vertices $\{(0.5, -0.25), (2, 0.5), (2, 1.5), (0.5, 2.25), (-1, 1.5), (-1, 0.5)\}$ is a maximal S-free convex set.

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