A probabilistic comparison of split and type 1 triangle cuts for two row mixed-integer programs

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Abstract

We provide a probabilistic comparison of split and type 1 triangle cuts for mixed-integer programs with two rows and two integer variables. Under a simple probabilistic model of the problem parameters, we show that a simple split cut, i.e. a Gomory cut, is more likely to be better than a type 1 triangle cut in terms of cut coefficients and volume cut off.

1 Introduction

This paper is concerned with valid inequalities for a two-row mixed-integer program (MIP) with two integer variables of the form

$$\begin{aligned} x &= f + \sum_{j=1}^{k} r^j y_j \\ x &\in \mathbb{Z}^2,\ y_j \geq 0, \end{aligned} \tag{1}$$

where $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and $r^j \in \mathbb{Q}^2 \setminus \{0\}$ for all j. Let X denote the set of solutions to (1). It has been shown (e.g Andersen et al. [1]) that any valid inequality for $\text{conv}(X)$ that cuts off the infeasible point $(x, y) = (f, 0)$ is an intersection cut (Balas [2]), corresponding to a convex set $L \in \mathbb{R}^2$ with $\text{int}(L) \cap \mathbb{Z}^2 = \emptyset$ (i.e. integer-free) and $f \in \text{int}(L)$. Such a cut is of the form

$$\sum_{j=1}^{k} \psi_L(r^j) y_j \geq 1\,, \tag{2}$$

where $\psi_L : \mathbb{Q}^2 \mapsto \mathbb{R}$ is given by

$$\psi_L(r) = \begin{cases} 0 & r \in \text{rec.cone}(L) \\ \frac{1}{\lambda} & \lambda > 0,\ f + \lambda r \in \text{boundary}(L). \end{cases} \tag{3}$$

Furthermore, minimal inequalities of the form (2) can be derived from maximal integer-free sets in $\mathbb{R}^2$ with non-empty interior. Such sets are of one of the following types (Lovasz [10]):

- A split S: $c \leq ax_1 + bx_2 \leq c + 1$, where $a, b, c \in \mathbb{Z}$ and $gcd(a, b) = 1$;
- A triangle with an integer point in the relative interior of each of the edges; these can be further classified in to one of the following three types (Dey and Wolsey [8]):

1. A type 1 triangle T_1: a triangle with integer vertices and exactly one integer point in the relative interior of each edge.
2. A type 2 triangle T_2: a triangle with more than one integer point on one edge and exactly one integer point in the relative interior of each of the other two edges.
3. A type 3 triangle T_3: a triangle with exactly one integer point in the relative interior of each edge and non-integral vertices.

- A quadrilateral Q with exactly one integer point in the relative interior of each edge such that the four integer points form a parallelogram of area one.

Inequalities of the form (2) corresponding to the above sets are called split, (type 1, 2 or 3) triangle, and quadrilateral cuts, respectively. From the maximality of the above integer-free sets, it follows that any non-trivial facet of $\text{conv}(X)$ is either a split, triangle or quadrilateral cut [1, 5].

Split cuts are the classical Gomory mixed integer (GMI) or mixed-integer rounding cuts [11]. Recently there has been a great deal of activity in comparing triangle and quadrilateral cuts to split cuts for two row MIPs. Basu et al. [4] compared the rank-1 closure (the convex set obtained by adding in a single round all possible cuts from the family) corresponding to the three cuts classes. They showed that the triangle closure (considering all three types of triangle cuts) and the quadrilateral closure are contained in the split closure, suggesting that triangle and quadrilateral cuts are in some sense stronger than split cuts. Dey [6] showed that type 2, type 3 triangle cuts and quadrilateral cuts have a finite split ranks (i.e. such a cut can be constructed via a finite sequence of split cuts) while only type 1 triangle cuts can have infinite split rank. However, empirical studies demonstrating the success of triangle and quadrilateral cuts in comparison to split (or GMI) cuts have been limited. Espinoza [9] reported some success with intersection cuts generated from some classes of integer-free triangles and quadrilaterals. Basu et al. [3] considered strengthened versions of a class of type 2 triangle cuts and showed that combining these cuts with GMI cuts give somewhat better performance than GMI cuts alone. Dey et al. [7] presented computational results on randomly generated multi-knapsack instances and showed that a subclass of type 2 triangle cuts can close more gap than GMI cuts.

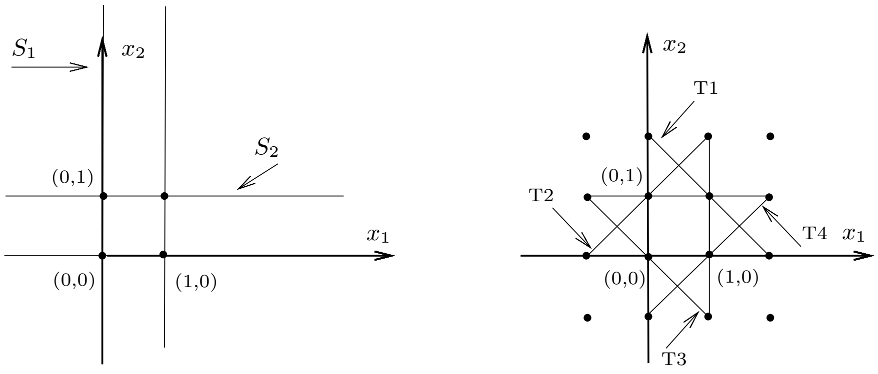
We present a probabilistic comparison of type 1 triangle cuts and split cuts. Specifically we address the question: what is the likelihood that a split cut will dominate with respect to cut coefficients or cut off more volume from the linear programming relaxation than a type 1 triangle cut for an arbitrary instance of the two-row MIP (1) given a specific probability distribution of the problem parameters? Our analysis reveals that, for the given distribution of the instances, such likelihood is high. The analysis also suggests some guidelines on when type 1 triangle cuts are likely to be more effective than split cuts and vice versa.

2 Setup

In this section, we discuss the distributional model for instances of the two-row MIP (1) and the two metrics used in our probabilistic comparison of type 1 triangle and split cuts.

Without loss of generality, (by translating x by $\lfloor f \rfloor$ and scaling y_j by $||r^j||_2$) we can assume that $0 \leq f_i < 1$ for $i = 1, 2$ and $||r^j||_2 = 1$ for all j in (1). Then $r_1^j = \cos\theta_j$ and $r_2^j = \sin\theta_j$ where θ_j is the angle between r^j and the positive x_1-axis.

The input model: We consider instances of (1) where f is a realization of a random vector $\mathbf{f}$ that is uniformly distributed with support $U := (0, 1)^2$, i.e., the open unit square in the plane, and θ_j is a realization of a random variable $\boldsymbol{\theta}_j$ that is uniformly distributed over $[0, 2\pi)$ for all j. (When



(a) Two simple splits (b) Four type 1 triangles

Figure 1: The integer-free bodies selected for comparison

f is on the boundary of $\text{cl}(U)$, the coefficients for some split and type 1 triangle cuts can be $+\infty$, causing technical issues in their comparison.) Moreover, $\mathbf{f}, \boldsymbol{\theta_1}, \ldots, \boldsymbol{\theta_k}$ are independent.

Under this probabilistic input model, the cut corresponding to the integer-free body L is of the form

$$\sum_{j=1}^{k} \psi_L(\mathbf{f}, \boldsymbol{\theta_j}) y_j \geq 1, \tag{4}$$

where the cut coefficient $\psi_L(\mathbf{f}, \boldsymbol{\theta_j})$ of variable y_j is a random variable depending on $\mathbf{f}$ and $\boldsymbol{\theta_j}$ and is given by (3). Our analysis compares the random cut (4) when the set L is a split or a type 1 triangle. To guarantee that $\mathbf{f} \in \text{int}(L)$ with probability one, we only consider integer-free splits and type 1 triangles that contain U. In particular, there are only two splits containing U (the valid inequality corresponds to the GMI cut for each row of system (1)) and there are only four type 1 triangles containing U, with one of the four vertices of U as its right-angle vertex (see Figure 1). There are various criteria for comparing cuts. We choose two criteria suitable for comparing two single cuts instead of cut families. The first one is based on cut dominance.

Definition 1. *Suppose $C_1 : \sum_{j=1}^{k} a_j y_j \geq 1$ and $C_2 : \sum_{j=1}^{k} b_j y_j \geq 1$ are two distinct valid inequalities for system* (1)*, then C_1* ***dominates*** *C_2 if $a_j \leq b_j$ for $j = 1, \cdots, k$ with at least one of the inequalities being strict. We use $C_1 \succ_D C_2$ to denote that C_1 dominates C_2.*

If $C_1 \succ_D C_2$, then C_2 is implied by C_1. The second criteria is based on the volume cut off by the cuts from the linear relaxation.

Definition 2. *Suppose $C_1 : \sum_{j=1}^{k} a_j y_j \geq 1$ and $C_2 : \sum_{j=1}^{k} b_j y_j \geq 1$ are two distinct valid inequalities for system* (1)*. Let X_{LP} be the linear relaxation of* (1)*. Then $C_1 \succ_V C_2$ if C_1* ***cuts off more volume*** *than C_2 from X_{LP}, i.e.*

$$vol(X_{LP} \cap \left\{ (x, y) : \sum_{j=1}^{k} a_j y_j \leq 1 \right\}) > vol(X_{LP} \cap \left\{ (x, y) : \sum_{j=1}^{k} b_j y_j \leq 1 \right\}).$$

We probabilistically compare split and type 1 triangle cuts with respect to these two metrics.

3 Conditional Probabilities with respect to f

We first analyze the conditional probabilities of split cuts dominating and cutting off more volume than triangle cuts with respect to the fractional point f. The analysis helps with computing the total probabilities in Section 4, and also provides some insight into values of f for which type 1 triangle cuts are likely to be better than split cuts and vice versa.

3.1 Cut coefficient comparison

Without loss of generality, we select one split from the two splits and one type 1 triangle from the four type 1 triangles in Figure 1. The analysis easily extends to the other splits and type 1 triangles by symmetry. The chosen split S_1 and type 1 triangle T_1 are shown in Figure 2. The split S_1 is defined by AD and BC and the type 1 triangle T_1 is defined by $\triangle AEF$. Suppose that C_{S_1} is the split cut for S_1 and C_{T_1} is the triangle cut for T_1, and recall that $\psi_{S_1}(\mathbf{f}, \boldsymbol{\theta_j})$ and $\psi_{T_1}(\mathbf{f}, \boldsymbol{\theta_j})$ are the corresponding (random) cut coefficients for variable y_j. We use $\Pr[\psi_{T_1}(\mathbf{f}, \boldsymbol{\theta_j}) < \psi_{S_1}(\mathbf{f}, \boldsymbol{\theta_j})|f]$ to denote the conditional probability of the event $\psi_{T_1}(\mathbf{f}, \boldsymbol{\theta_j}) < \psi_{S_1}(\mathbf{f}, \boldsymbol{\theta_j})$ when $\mathbf{f} = f$.

Lemma 1. *For each* $j = 1, \cdots, k$, $\Pr[\psi_{T_1}(\mathbf{f}, \boldsymbol{\theta_j}) < \psi_{S_1}(\mathbf{f}, \boldsymbol{\theta_j})|f] = \alpha(f)$, $\Pr[\psi_{S_1}(\mathbf{f}, \boldsymbol{\theta_j}) = \psi_{T_1}(\mathbf{f}, \boldsymbol{\theta_j})|f] = \beta(f)$ *and* $\Pr[\psi_{S_1}(\mathbf{f}, \boldsymbol{\theta_j}) < \psi_{T_1}(\mathbf{f}, \boldsymbol{\theta_j})|f] = \gamma(f)$, *where*

$$\alpha(f) = \frac{\arccos \frac{f_2(f_2-1)+(1-f_1)^2}{\sqrt{[f_2^2+(1-f_1)^2][(1-f_2)^2+(1-f_1)^2]}}}{2\pi}, \qquad \beta(f) = \frac{\arccos \frac{f_1^2+f_2^2-2f_2}{\sqrt{[f_1^2+f_2^2][f_1^2+(2-f_2)^2]}}}{2\pi},$$

$$\gamma(f) = \frac{\arccos \frac{f_1^2+f_2^2-f_1}{\sqrt{[f_1^2+f_2^2][(1-f_1)^2+f_2^2]}} + \arccos \frac{f_1^2+f_2^2-f_1-3f_2+2}{\sqrt{[(1-f_2)^2+(1-f_1)^2][f_1^2+(2-f_2)^2]}}}{2\pi}$$

Proof. Since $\boldsymbol{\theta_j}$ $(j = 1, \cdots, k)$ are i.i.d., we only need to prove the result for some j. For simplicity, we supress the index j here and prove it for some ray $r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$.

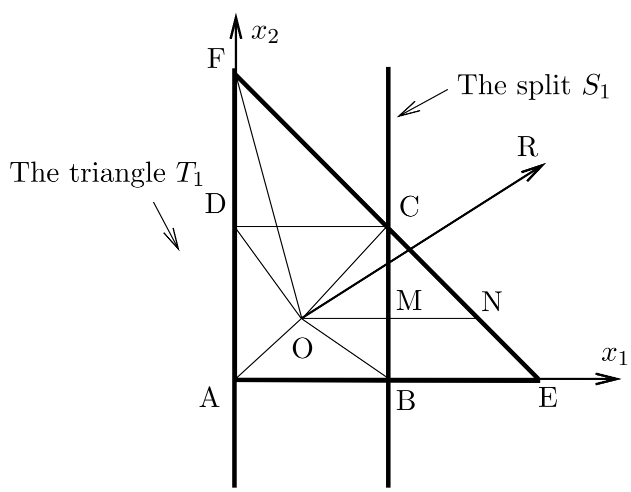


Figure 2: Computing $\Pr[\psi_S(\mathbf{f}, \boldsymbol{\theta}) < \psi_{T_1}(\mathbf{f}, \boldsymbol{\theta})]$

As shown in Figure 2, U is the unit square with vertices A, B, C and D and O is the fractional point f. Let OR be the ray defined by $f + \lambda r$. Let OM be the line parallel to the x_1-axis that intersects S and T_1 at M and N respectively. Then θ is the angle between OM and OR in the counterclockwise direction. Let the symbol $\angle$ denote an angle less than π. Since the probability density function of $\boldsymbol{\theta}$ is $\frac{1}{2\pi}I(\theta \in [0, 2\pi))$, by the law of total probability,

$$\Pr[\psi_{S_1}(\mathbf{f}, \boldsymbol{\theta}) < \psi_{T_1}(\mathbf{f}, \boldsymbol{\theta})|f] = \int_0^{2\pi} \frac{I(\psi_{S_1}(f,\theta) < \psi_{T_1}(f,\theta))}{2\pi} d\theta, \tag{5}$$

where $I(A)$ is the indicator function of event A.

By (3), $\psi_{S_1}(f,\theta) = \frac{1}{\lambda_{S_1}}$, where $f + \lambda_{S_1}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \in \text{boundary}(S)$, and $\psi_{T_1}(f,\theta) = \frac{1}{\lambda_{T_1}}$ where $f + \lambda_{T_1}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \in \text{boundary}(T_1)$. Therefore, $\psi_{S_1}(f,\theta) < \psi_{T_1}(f,\theta)$ if the ray $f + \lambda\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$ hits the boundary of T_1 first, and $\psi_{T_1}(f,\theta) < \psi_{S_1}(f,\theta)$ if the ray $f + \lambda\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$ hits the boundary of S_1 first. When $\theta \in [0, \angle MOC)$ or $\theta \in (2\pi - \angle MOB, 2\pi)$, OR is contained in the cone bounded by OB and OC, and hits the boundary of S first, so $\psi_{T_1}(f,\theta) < \psi_{S_1}(f,\theta)$. Similarly, when $\theta \in (\angle MOC, \angle MOF)$ or $\theta \in (2\pi - \angle MOA, 2\pi - \angle MOB)$, $\psi_{S_1}(f,\theta) < \psi_{T_1}(f,\theta)$; when $\theta \in [\angle MOF, 2\pi - \angle MOA]$ or θ is equal to $\angle MOC$ or $2\pi - \angle MOB$, $\psi_{S_1}(f,\theta) = \psi_{T_1}(f,\theta)$. Therefore, by (5),

$$\Pr[\psi_{S_1}(\mathbf{f}, \boldsymbol{\theta}) < \psi_{T_1}(\mathbf{f}, \boldsymbol{\theta})|f] = \frac{\angle AOB + \angle COF}{2\pi}, \quad \Pr[\psi_{S_1}(\mathbf{f}, \boldsymbol{\theta}) = \psi_{T_1}(\mathbf{f}, \boldsymbol{\theta})|f] = \frac{\angle AOF}{2\pi},$$

$$\Pr[\psi_{T_1}(\mathbf{f}, \boldsymbol{\theta}) < \psi_{S_1}(\mathbf{f}, \boldsymbol{\theta})|f] = \frac{\angle BOC}{2\pi}.$$

In $\triangle BOC$, $|\overline{OB}| = \sqrt{(1-f_1)^2 + f_2^2}$, $|\overline{OC}| = \sqrt{(1-f_1)^2 + (1-f_2)^2}$ and $|\overline{BC}| = 1$. By the law of cosines,

$$\cos\angle BOC = \frac{|\overline{OB}|^2 + |\overline{OC}|^2 - |\overline{BC}|^2}{2|\overline{OB}||\overline{OC}|} = \frac{f_2(f_2-1) + (1-f_1)^2}{\sqrt{[f_2^2 + (1-f_1)^2][(1-f_2)^2 + (1-f_1)^2]}} = 2\pi\alpha(f).$$

Therefore,

$$\Pr[\psi_{T_1}(\mathbf{f}, \boldsymbol{\theta}) < \psi_{S_1}(\mathbf{f}, \boldsymbol{\theta})|f] = \alpha(f).$$

Similarly, $\angle AOF = 2\pi\beta(f)$ and $\angle AOB + \angle COF = 2\pi\gamma(f)$. Therefore,

$$\Pr[\psi_{S_1}(\mathbf{f}, \boldsymbol{\theta}) = \psi_{T_1}(\mathbf{f}, \boldsymbol{\theta})|f] = \beta(f), \quad \Pr[\psi_{S_1}(\mathbf{f}, \boldsymbol{\theta}) < \psi_{T_1}(\mathbf{f}, \boldsymbol{\theta})|f] = \gamma(f).$$

□

Lemma 1 provides the probabilities that a single coefficient of the split cut C_{S_1} is smaller than, equal to, and larger than that of the triangle cut C_{T_1} as a function of f. To compare the other split and type 1 triangles in Figure 1, we only need to change f_1 to $1 - f_1$ or f_2 to $1 - f_2$ in $\alpha(f)$, $\beta(f)$ and $\gamma(f)$ by symmetry. The following theorem gives the conditional probability that the split cut C_{S_1} dominates the triangle cut C_{T_1} with respect to f and the number of continuous variables k.

Theorem 1.

$$\Pr[C_{S_1} \succ_D C_{T_1}|f] = [\beta(f) + \gamma(f)]^k - [\beta(f)]^k.$$

Proof.

$$\begin{aligned}&\Pr[C_{S_1} \succ_D C_{T_1}|f]\\&= \Pr[\psi_{S_1}(\mathbf{f},\boldsymbol{\theta_j}) \leq \psi_{T_1}(\mathbf{f},\boldsymbol{\theta_j}), \forall j|f] - \Pr[\psi_{S_1}(\mathbf{f},\boldsymbol{\theta_j}) = \psi_{T_1}(\mathbf{f},\boldsymbol{\theta_j}), \forall j|f]\\&= \Pr[\psi_{S_1}(\mathbf{f},\boldsymbol{\theta_j}) \leq \psi_{T_1}(\mathbf{f},\boldsymbol{\theta_j})|f]^k - \Pr[\psi_{S_1}(\mathbf{f},\boldsymbol{\theta_j}) = \psi_{T_1}(\mathbf{f},\boldsymbol{\theta_j})|f]^k\\&= [\beta(f)+\gamma(f)]^k - [\beta(f)]^k,\end{aligned}$$

where the second equality follows from the assumption that $\boldsymbol{\theta_j}$ $(j = 1, \cdots, k)$ are i.i.d.. □

Given integer free bodies L_1 and L_2, let

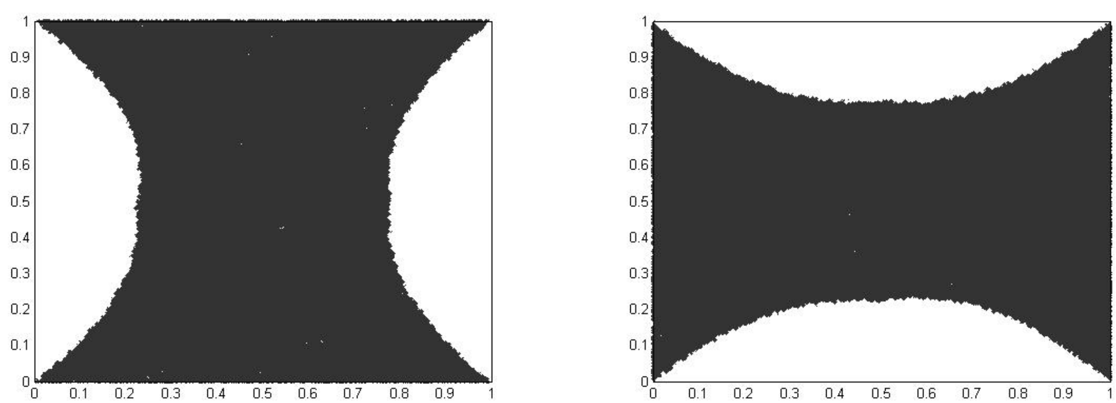
$$R_D(L_1, L_2) = \{f \in U : \Pr[C_{L_1} \succ_D C_{L_2}|f] > \Pr[C_{L_2} \succ_D C_{L_1}|f]\},$$

i.e. the region within U where the cut C_{L_1} is more likely to dominate the cut C_{L_2}. The following corollary follows from Theorem 1.

Corollary 1.

$$R_D(S_1, T_1) = \{f \in U : \gamma(f) > \alpha(f)\} \quad \textit{and} \quad R_D(T_1, S_1) = \{f \in U : \alpha(f) > \gamma(f)\}.$$

By symmetry, after appropriately translating f, we can similarly describe the regions $R_D(S_i, T_j)$ and $R_D(T_j, S_i)$ for $i = 1, 2$ and $j = 1, 2, 3, 4$ corresponding to any of the two splits and four type 1 triangles in Figure 1. Figures 3(a) and 3(b) show the regions $\cap_{j=1}^4 R_D(S_1, T_j)$ and $\cap_{j=1}^4 R_D(S_2, T_j)$, respectively shaded in black. The white regions in these figures indicate $\cup_{j=1}^4 R_D(T_j, S_1)$ and $\cup_{j=1}^4 R_D(T_j, S_2)$, respectively. Since the union of the two black regions covers the unit square, there is no f for which a type 1 triangle cut is more likely to dominate both splits S_1 and S_2. Thus if we are only allowed to add one cut, when $f \in \cap_{j=1}^4 R_D(S_1, T_j)$, we would select S_1 because the cut C_{S_1} is more likely to dominate any other type 1 triangle cut, and when $f \in \cup_{j=1}^4 R_D(T_j, S_1)$, we would select S_2 because the cut C_{S_2} is then more likely to dominate any other triangle cut.



(a) The region $\cap_{j=1}^4 R_D(S_1, T_j)$

(b) The region $\cap_{j=1}^4 R_D(S_2, T_j)$

Figure 3: The region.

3.2 Volume comparison

In this section, we compare cuts based on the volume cut off from the linear relaxation of system (1). First we describe how the volume cut off is computed.

Suppose that $C : \sum_{j=1}^{k} a_j y_j \geq 1$, with $a_j \geq 0$ for all j, is a valid inequality for system (1). Consider the linear relaxation of (1)

$$\begin{aligned} & x = f + \sum_{j=1}^{k} r^j y_j \\ & x \in \mathbb{R}^2,\ y_j \geq 0. \end{aligned} \tag{6}$$

Let X_{LP} be the set of feasible solutions of system (6) and $S_C = X_{LP} \cap \left\{(x, y) : \sum_{j=1}^{k} a_j y_j \leq 1\right\}$. Let $\text{vol}(S_C)$ denote the volume of the polyhedron S_C, which is also the volume cut off from S by the valid inequality C. The following lemma gives the volume of S_C.

Lemma 2.

$$vol(S_C) = \begin{cases} +\infty & \text{if } \exists j \text{ such that } a_j = 0 \\ \frac{\alpha}{n! \prod_{j=1}^{k} a_j} & \text{otherwise} \end{cases} \tag{7}$$

where α is a constant depending on the rays $r^1, \cdots, r^k$.

Proof. When $a_j = 0$ for some j, S_C is an unbounded polyhedron, and $\text{vol}(S_C) = +\infty$. When $a_j > 0$ for all j, S_C is a k-dimensional polytope containing $(f, 0)$. Let

$$\text{Proj}_y(S_C) = \left\{y \in \mathbb{R}^k : \exists x \in \mathbb{R}^2 \text{ such that } (x, y) \in S_C\right\}$$

be the projection of S_C onto the y space. $\text{Proj}_y(S_C)$ is a k-dimensional simplex with $0, \frac{1}{a_1}e^1, \cdots, \frac{1}{a_k}e^k$ as its $(k+1)$ vertices, where e^j is the j-th unit vector. Therefore,

$$\text{vol}(\text{Proj}_y(S_C)) = \frac{1}{n!}\frac{1}{a_1} \cdots \frac{1}{a_k} = \frac{1}{n! \prod_{j=1}^{k} a_j}.$$

Each point in S_C is just an affine transformation of a point in the simplex $\text{Proj}_y(S_C)$, so $\text{vol}(S_C)$ and $\text{vol}(\text{Proj}_y(S_C))$ only differ by a factor α depending on the rays $r^1, \cdots, r^k$. Thus $\text{vol}(S_C) = \frac{\alpha}{n! \prod_{j=1}^{k} a_j}$. □

By Lemma 2, it suffices to compute the product of cut coefficients when we compare cuts based on the volume cut off from the linear relaxation.

Now consider the split S_1 and type 1 triangle T_1 as in Section 3.1. As before, the analysis easily extends to another pair of split and type 1 triangle bodies by symmetry. Note that for fixed $f \in (0, 1)^2$, $\psi_{T_1}(f, \boldsymbol{\theta_j}) > 0$ w.p.1. Moreover, since $\boldsymbol{\theta_j}$ is continuously distributed, $\Pr[\exists j \text{ s.t. } \psi_{S_1}(f, \boldsymbol{\theta_j}) = 0] = \Pr[\exists j \text{ s.t. } \boldsymbol{\theta_j} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}] = 0$.

Theorem 2.

$$\Pr[C_{S_1} \succ_V C_{T_1} | f] = \Pr[\sum_{j=1}^{k} \ln \frac{\psi_{S_1}(f, \boldsymbol{\theta_j})}{\psi_{T_1}(f, \boldsymbol{\theta_j})} < 0].$$

Proof. From Definition 2, Lemma 2 and the fact that $\psi_{S_1}(f, \boldsymbol{\theta_j}) > 0$ and $\psi_{T_1}(f, \boldsymbol{\theta_j}) > 0$ w.p.1, we have that

$$\begin{aligned} \Pr[C_{S_1} \succ_V C_{T_1} | f] &= \Pr[\text{vol}(S_{C_{S_1}}) > \text{vol}(S_{C_{T_1}}) | f] \\ &= \Pr[\frac{\alpha}{n! \prod_{j=1}^{k} \psi_{S_1}(f, \boldsymbol{\theta_j})} > \frac{\alpha}{n! \prod_{j=1}^{k} \psi_{T_1}(f, \boldsymbol{\theta_j})}] = \Pr[\sum_{j=1}^{k} \ln \frac{\psi_{S_1}(f, \boldsymbol{\theta_j})}{\psi_{T_1}(f, \boldsymbol{\theta_j})} < 0]. \end{aligned}$$

□

Next we analyze the asymptotic behavior of the probability $\Pr[C_{S_1} \succ_V C_{T_1}|f]$ as the number of continuous variables k increases. Before presenting further results, we give two technical lemmas.

Lemma 3.

$$\int_0^{\frac{\pi}{2}} \ln\cos x dx = -\frac{\pi \ln 2}{2} \quad \textit{and} \quad \int_0^{\frac{\pi}{2}} (\ln\cos x)^2 dx < \infty.$$

Proof. See the appendix. □

Lemma 4.

$$|E[\ln \frac{\psi_{S_1}(f, \boldsymbol{\theta_j})}{\psi_{T_1}(f, \boldsymbol{\theta_j})}]| < \infty \quad \textit{and} \quad \textit{Var}[\ln \frac{\psi_{S_1}(f, \boldsymbol{\theta_j})}{\psi_{T_1}(f, \boldsymbol{\theta_j})}] < \infty.$$

Proof. See the appendix. □

Now we present the asymptotic result on the probability that a split cut cuts off more volume than a type 1 triangle cut as the number of continuous variables increases.

Theorem 3.

$$\lim_{k\to\infty} \Pr[C_{S_1} \succ_V C_{T_1}|f] = \begin{cases} 1 & \textit{if } E[\ln \frac{\psi_{S_1}(f,\boldsymbol{\theta_j})}{\psi_{T_1}(f,\boldsymbol{\theta_j})}] < 0 \\ 1/2 & \textit{if } E[\ln \frac{\psi_{S_1}(f,\boldsymbol{\theta_j})}{\psi_{T_1}(f,\boldsymbol{\theta_j})}] = 0 \\ 0 & \textit{if } E[\ln \frac{\psi_{S_1}(f,\boldsymbol{\theta_j})}{\psi_{T_1}(f,\boldsymbol{\theta_j})}] > 0. \end{cases}$$

Proof. From Theorem 2, we know $\Pr[C_{S_1} \succ_V C_{T_1}|f] = \Pr[\sum_{j=1}^k \ln \frac{\psi_{S_1}(f,\boldsymbol{\theta_j})}{\psi_{T_1}(f,\boldsymbol{\theta_j})} < 0]$. Note that for a fixed $f \in (0,1)^2$, the random variable $\ln \frac{\psi_{S_1}(f,\boldsymbol{\theta_j})}{\psi_{T_1}(f,\boldsymbol{\theta_j})}$ is uniquely determined by $\boldsymbol{\theta_j}$. The assumption that $\boldsymbol{\theta_j}$, for $j = 1, \ldots, k$, are i.i.d. implies that $\ln \frac{\psi_{S_1}(f,\boldsymbol{\theta_j})}{\psi_{T_1}(f,\boldsymbol{\theta_j})}$, for $j = 1, \ldots, k$, are also i.i.d.. Therefore, we can apply the Weak Law of Large Numbers and the Central Limit Theorem. To simplify the notation, let $X_j = \ln \frac{\psi_{S_1}(f,\boldsymbol{\theta_j})}{\psi_{T_1}(f,\boldsymbol{\theta_j})}$ and $\overline{X}_k = \frac{\sum_{j=1}^k X_j}{k}$. Since $\mathrm{E}[X_j]$ is finite (Lemma 4), by the Weak Law of Large Numbers, $\lim_{k\to\infty} \Pr[|\overline{X}_k - \mathrm{E}[X_j]| < \epsilon] = 1$ for all $\epsilon > 0$. We consider three cases: $\mathrm{E}[X_j] < 0$, $\mathrm{E}[X_j] > 0$ and $\mathrm{E}[X_j] = 0$.

(1) $\mathrm{E}[X_j] < 0$. Choose $\epsilon = -\frac{\mathrm{E}[X_j]}{2}$. Then

$$\Pr[\sum_{j=1}^k X_j < 0] = \Pr[\overline{X}_k < 0] \geq \Pr[\overline{X}_k - \mathrm{E}[X_j] < \epsilon] \geq \Pr[|\overline{X}_k - \mathrm{E}[X_j]| < \epsilon].$$

Thus, $\liminf_{k\to\infty} \Pr[\sum_{j=1}^k X_j < 0] \geq \liminf_{k\to\infty} \Pr[|\overline{X}_k - \mathrm{E}[X_j]| < \epsilon] = \lim_{k\to\infty} \Pr[|\overline{X}_k - \mathrm{E}[X_j]| < \epsilon] = 1$. Since $\limsup_{k\to\infty} \Pr[\sum_{j=1}^k X_j < 0] \leq 1$, $\lim_{k\to\infty} \Pr[\sum_{j=1}^k X_j < 0] = 1$.

(2) $\mathrm{E}[X_j] > 0$. Choose $\epsilon = \frac{\mathrm{E}[X_j]}{2}$. Then we can show $\lim_{k\to\infty} \Pr[\sum_{j=1}^k X_j < 0] = 0$ analogous to the case $\mathrm{E}[X_j] < 0$.

(3) $\mathrm{E}[X_j] = 0$. From Lemma 4, $\mathrm{Var}[X_j]$ is finite. By the Central Limit Theorem, $\frac{\overline{X}_k - \mathrm{E}[X_j]}{\sqrt{\mathrm{Var}(X_j)/k}}$ converges to the standard normal random variable $\mathcal{N}(0,1)$ in distribution. Thus

$$\lim_{k\to\infty} \Pr[\sum_{j=1}^{k} X_j < 0] = \lim_{k\to\infty} \Pr[\frac{\overline{X}_k - \mathrm{E}[X_j]}{\sqrt{\mathrm{Var}(X_j)/k}} < 0] = \frac{1}{2}.$$

□

Define

$$R_V(S_1, T_1) = \left\{ f \in U : \mathrm{E}[\ln \frac{\psi_{S_1}(f, \boldsymbol{\theta_j})}{\psi_{T_1}(f, \boldsymbol{\theta_j})}] < 0 \right\} \text{ and } R_V(T_1, S_1) = \left\{ f \in U : \mathrm{E}[\ln \frac{\psi_{S_1}(f, \boldsymbol{\theta_j})}{\psi_{T_1}(f, \boldsymbol{\theta_j})}] > 0 \right\}.$$

Then, $R_V(S_1, T_1)$ indicates the region where the split cut C_{S_1} cuts off more volume than the type 1 triangle cut C_{T_1} with probability close to 1 when k is large, and $R_V(T_1, S_1)$ indicates the region where the type 1 triangle cut C_{T_1} cuts off more volume than the split cut C_{S_1} with probability close to 1 when k is large. Even though $\boldsymbol{\theta_j}$ has a simple distribution, it is difficult to analytically compute $\mathrm{E}[\ln \frac{\psi_{S_1}(f,\boldsymbol{\theta_j})}{\psi_{T_1}(f,\boldsymbol{\theta_j})}]$. However we can estimate $\mathrm{E}[\ln \frac{\psi_{S_1}(f,\boldsymbol{\theta_j})}{\psi_{T_1}(f,\boldsymbol{\theta_j})}]$ by Monte Carlo simulation for a given value of f, and identify the regions $R_V(S_1, T_1)$ and $R_V(T_1, S_1)$. The black and white regions in Figure 4 indicate $R_V(S_1, T_1)$ and $R_V(T_1, S_1)$, respectively. These have been identified as follows. First we randomly generate 10^5 fractional points f in U; then for each f, we independently generate 1000 θ_j uniformly from $[0, 2\pi)$ and check if the sample mean of $\ln \frac{\psi_{S_1}(f,\theta_j)}{\psi_{T_1}(f,\theta_j)}$ is less or greater than zero to identify if the corresponding f is in $R_V(S_1, T_1)$ or $R_V(T_1, S_1)$. The area of the black region is approximately 0.9. Unless f_1 is close to 1, the split cut C_{S_1} cuts off more volume than the type 1 triangle cut C_{T_1} with probability close to 1 when k is large, and therefore C_{S_1} is preferred.

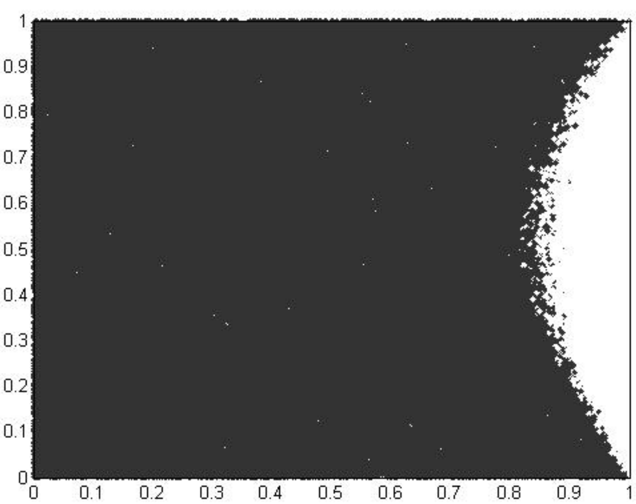


Figure 4: The shape of $R_V(S_1, T_1)$ and $R_V(T_1, S_1)$.

4 Total Probabilities

In this section, we use the conditional probabilities from the previous section to compute coefficient dominance and volume cut off probabilities for split and type 1 triangle cuts when f is random. As before, we focus on the split cut C_{S_1} and the type 1 triangle cut C_{T_1} and note that the analysis and conclusions extend to another pair of split and type 1 triangle bodies by symmetry. The total probability analysis provides some insight on how these cuts are likely to perform when no information about the instance is available.

4.1 Cut coefficient comparison

By the law of total probability

$$\Pr[C_{S_1} \succ_D C_{T_1}] = \Pr[\psi_{S_1}(\mathbf{f}, \boldsymbol{\theta_j}) < \psi_{T_1}(\mathbf{f}, \boldsymbol{\theta_j}),\ \forall j]$$
$$= \oint_U \Pr[\psi_{S_1}(\mathbf{f}, \boldsymbol{\theta_j}) < \psi_{T_1}(\mathbf{f}, \boldsymbol{\theta_j}),\ \forall j|f] d\Phi(f) = \oint_U \{\Pr[\psi_{S_1}(\mathbf{f}, \boldsymbol{\theta_j}) < \psi_{T_1}(\mathbf{f}, \boldsymbol{\theta_j})|f]\}^k d\Phi(f),$$

where $\Phi(f)$ is the cumulative distribution function of $\mathbf{f}$ and the last inequality follows from the fact that $\boldsymbol{\theta_j}$ are i.i.d. for $j = 1, \ldots, k$. Recall that the conditional probability $\Pr[\psi_{S_1}(\mathbf{f}, \boldsymbol{\theta_j}) < \psi_{T_1}(\mathbf{f}, \boldsymbol{\theta_j})|f]$ is given in Lemma 1. The following theorem describes the performance of the split cut C_{S_1} and type 1 triangle cut C_{T_1} when there is only one continuous variable.

Theorem 4. *If* $k = 1$ *then* $\Pr[C_{S_1} \succ_D C_{T_1}] \approx 0.426 > 0.25 = \Pr[C_{T_1} \succ_D C_{S_1}]$.

Proof. Note that $\angle BOC$, $\angle AOB$ and $\angle COF$ are shown in Figure 2. Then

$$\Pr[C_{T_1} \succ_D C_{S_1}] = \oint_U \Pr[\psi_{T_1}(\mathbf{f}, \boldsymbol{\theta}) < \psi_{S_1}(\mathbf{f}, \boldsymbol{\theta})] d\Phi(f) = \oint_U \frac{\angle BOC}{2\pi} d\Phi(f).$$

Similarly,

$$\Pr[C_{S_1} \succ_D C_{T_1}] = \oint_U \frac{\angle AOB + \angle COF}{2\pi} d\Phi(f).$$

The proof then follows from Lemma 5. □

Lemma 5.

$$\oint_U \frac{\angle BOC}{2\pi} d\Phi(f) = \oint_U \frac{\angle COD}{2\pi} d\Phi(f) = \oint_U \frac{\angle DOA}{2\pi} d\Phi(f) = \oint_U \frac{\angle AOB}{2\pi} d\Phi(f) = 0.25,$$

and

$$\oint_U \frac{\angle COF}{2\pi} d\Phi(f) \approx 0.176.$$

Proof. See the appendix. □

Now we consider the case $k > 1$.

Theorem 5. *For any* k*,* $\Pr[C_{S_1} \succ_D C_{T_1}] > \Pr[C_{T_1} \succ_D C_{S_1}]$.

Proof.

$$\begin{aligned}\Pr[C_{S_1} \succ_D C_{T_1}] &= \oint_U \left(\frac{\angle AOB + \angle COF}{2\pi}\right)^k d\Phi(f)\\ &> \oint_U \left(\frac{\angle AOB}{2\pi}\right)^k d\Phi(f) = \oint_U \left(\frac{\angle BOC}{2\pi}\right)^k d\Phi(f)\\ &= \Pr[C_{T_1} \succ_D C_{S_1}].\end{aligned}$$

The second equality follows from symmetry since f is uniformly distributed in $(0, 1)^2$. □

Theorem 5 states that a single split cut is more likely to dominate a single type 1 triangle cut under our probabilistic model no matter how many continuous variables there are in system (1). We also use Monte Carlo simulation to estimate the magnitude of the probabilities that one cut dominates another. The result is shown in Figure 5.

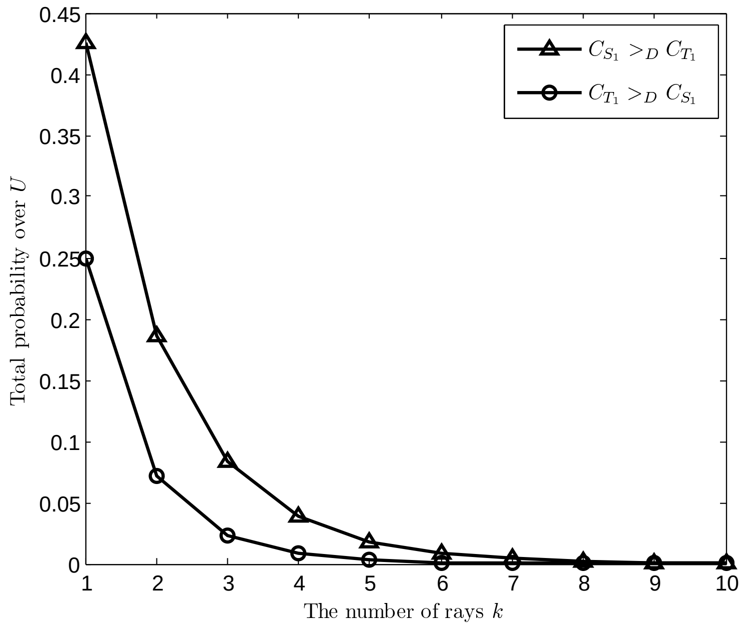


Figure 5: $\Pr[C_{S_1} \succ_D C_{T_1}]$ and $\Pr[C_{T_1} \succ_D C_{S_1}]$ wrt the number of rays k.

From Figure 5, although $\Pr[C_{S_1} \succ_D C_{T_1}] > \Pr[C_{T_1} \succ_D C_{S_1}]$ for all k, both probabilities are very small when $k \geq 5$ indicating that it is unlikely that one cut totally dominates another when there are many continuous variables.

4.2 Volume comparison

In this section we estimate $\Pr[C_{S_1} \succ_V C_{T_1}]$ with respect to the number of continuous variables k. Recall that $\Pr[C_{S_1} \succ_V C_{T_1}] = \Pr[\prod_{j=1}^{k} \frac{\psi_{S_1}(\mathbf{f}, \boldsymbol{\theta_j})}{\psi_{T_1}(\mathbf{f}, \boldsymbol{\theta_j})} < 1]$. We use Monte Carlo simulation to estimate the above probabilities as follows. For each $k \in \{1, \ldots, 1000\}$, we randomly generate $N = 10^5$ samples of $f_1, f_2, \theta_1, \cdots, \theta_k$ according to our probabilistic input model. The probability $\Pr[\prod_{j=1}^{k} \frac{\psi_S(\mathbf{f}, \boldsymbol{\theta_j})}{\psi_{T_1}(\mathbf{f}, \boldsymbol{\theta_j})} < 1]$ is then estimated by the proportion of the N samples with $\prod_{j=1}^{k} \frac{\psi_S(\mathbf{f}, \boldsymbol{\theta_j})}{\psi_{T_1}(\mathbf{f}, \boldsymbol{\theta_j})} < 1$. The estimated probabilities with respect to k are shown in Figure 6. The estimated probability that C_{S_1} cuts off more volume from the linear relaxation than C_{T_1} increases as the number of continuous variables increases, converging to approximately 0.9. To explain this, note that

$$\lim_{k\to\infty} \Pr[C_{S_1} \succ_V C_{T_1}] = \lim_{k\to\infty} \oint_U \Pr[C_{S_1} \succ_V C_{T_1} | f] d\Phi(f).$$

Since $\Pr[C_{S_1} \succ_V C_{T_1} | f]$ is bounded, by interchanging limit and integral and applying Theorem 2 we have

$$\lim_{k\to\infty} \Pr[C_{S_1} \succ_V C_{T_1}] = \oint_U \{I(\mathrm{E}[\ln \frac{\psi_{S_1}(f,\boldsymbol{\theta_j})}{\psi_{T_1}(f,\boldsymbol{\theta_j})}] < 0) + \frac{1}{2} I(\mathrm{E}[\ln \frac{\psi_{S_1}(f,\boldsymbol{\theta_j})}{\psi_{T_1}(f,\boldsymbol{\theta_j})}] = 0)\} d\Phi(f)$$
$$\geq \oint_U I(\mathrm{E}[\ln \frac{\psi_{S_1}(f,\boldsymbol{\theta_j})}{\psi_{T_1}(f,\boldsymbol{\theta_j})}] < 0) d\Phi(f) = \Pr[\mathbf{f} \in R_V(S_1, T_1)]$$

where $I(A)$ is the indicator function of event A and $R_V(S_1, T_1)$ is defined in Section 3.2. Figure 6 presents $\Pr[C_{S_1} \succ_V C_{T_1}]$ with respect to the number of continuous variables k (in two different scales). Recall that, as observed in Figure 4, the area of $R_V(S_1, T_1)$ is approximately 0.9, which coincides with the observation in Figure 6. We can conclude C_{S_1} is more likely to cut off more volume than C_{T_1} when k is not too small given any instance of (1) with parameters distributed according to our probabilistic input model.

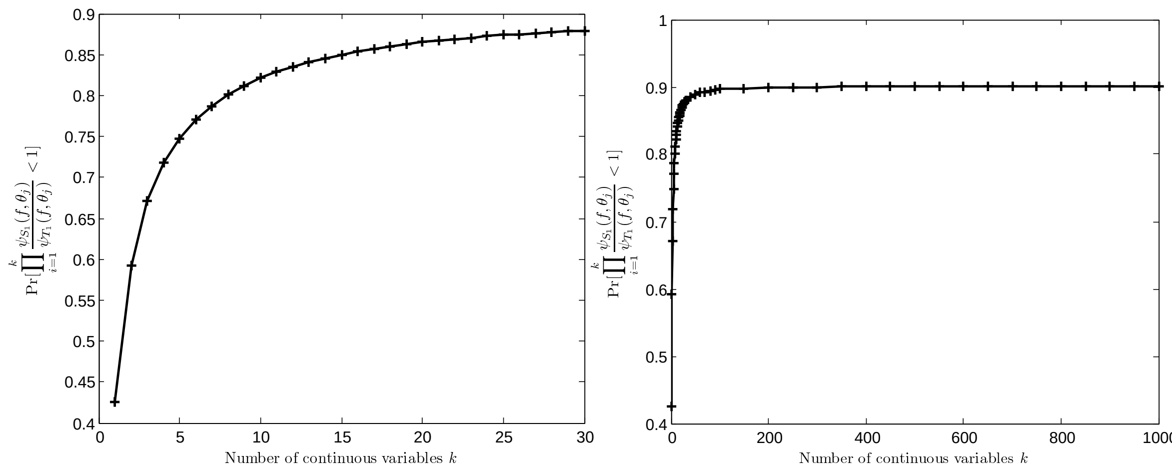


Figure 6: Estimated $\Pr[\prod_{j=1}^{k} \frac{\psi_S(\mathbf{f},\boldsymbol{\theta_j})}{\psi_{T_1}(\mathbf{f},\boldsymbol{\theta_j})} < 1]$ with respect to k.

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Appendices

A Proof of Lemma 3

Proof. By substitution of variables, $\int_0^{\frac{\pi}{2}} \ln\cos x dx = \int_0^{\frac{\pi}{2}} \ln\sin x dx$. Then,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \ln\sin x dx &= \int_0^{\frac{\pi}{2}} \ln(2\sin\frac{x}{2}\cos\frac{x}{2})dx \\ &= \int_0^{\frac{\pi}{2}} \ln 2 dx + \int_0^{\frac{\pi}{2}} \ln\sin\frac{x}{2}dx + \int_0^{\frac{\pi}{2}} \ln\cos\frac{x}{2}dx \\ &= \frac{\pi\ln 2}{2} + 2\int_0^{\frac{\pi}{4}} \ln\sin y dy + 2\int_0^{\frac{\pi}{4}} \ln\cos z dz \\ &= \frac{\pi\ln 2}{2} + 2\int_0^{\frac{\pi}{4}} \ln\sin y dy + 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\sin y dy \\ &= \frac{\pi\ln 2}{2} + 2\int_0^{\frac{\pi}{2}} \ln\sin y dy\end{aligned}$$

Therefore, $\int_0^{\frac{\pi}{2}} \ln\sin x dx = -\frac{\pi\ln 2}{2}$.
By substitution of variables, $\int_0^{\frac{\pi}{2}} (\ln\cos x)^2 dx = \int_0^{\frac{\pi}{2}} (\ln\sin x)^2 dx$. Since $0 \le \sin x \le x$ for $0 \le x \le \frac{\pi}{2}$, then $0 \le (\ln\sin x)^2 \le (\ln x)^2$. Moreover, $\int (\ln x)^2 dx = x(\ln x)^2 - 2x\ln x + 2x + d$, where d is a constant. Thus, $\int_0^{\frac{\pi}{2}} (\ln x)^2 dx = \frac{\pi}{2}(\ln\frac{\pi}{2})^2 - \pi\ln\frac{\pi}{2} + \pi < \infty$. Therefore, $\int_0^{\frac{\pi}{2}} (\ln\sin x)^2 dx$ is finite. □

B Proof of Lemma 4

Proof. To simplify the notation, let $X_j = \ln\frac{\psi_{S_1}(f,\boldsymbol{\theta_j})}{\psi_{T_1}(f,\boldsymbol{\theta_j})}$ and $\overline{X}_k = \frac{\sum_{j=1}^k X_j}{k}$. $\mathrm{E}[X_j] = \mathrm{E}[\ln\psi_{S_1}(f,\boldsymbol{\theta_j})] - \mathrm{E}[\ln\psi_{T_1}(f,\boldsymbol{\theta_j})]$. By (3), $\psi_{T_1}(f,\boldsymbol{\theta_j})$ is bounded and strictly positive for fixed $f \in (0,1)^2$. Thus

$\ln \psi_{T_1}(f, \boldsymbol{\theta_j})$ is bounded and $\mathrm{E}[\ln \psi_{T_1}(f, \boldsymbol{\theta_j})]$ is finite. By (3), $\psi_{S_1}(f, \theta_j) = \frac{1}{\lambda_{S_1}}$ where $f + \lambda_{S_1} \begin{pmatrix} \cos\theta_j \\ \sin\theta_j \end{pmatrix}$ hits the boundary of the split S_1. Thus, $f_1 + \lambda_{S_1} \cos\theta_j = 1$ when $\theta_j \in [0, \frac{\pi}{2})$ and $\theta_j \in (\frac{3\pi}{2}, 2\pi)$, and $f_1 + \lambda_{S_1} \cos\theta_j = 0$ when $\theta_j \in (\frac{\pi}{2}, \frac{3\pi}{2})$. Therefore, $\psi_{S_1}(f, \theta_j) = \frac{\cos\theta_j}{1-f_1}$ when $\theta_j \in [0, \frac{\pi}{2})$ and $\theta_j \in (\frac{3\pi}{2}, 2\pi)$, and $\psi_{S_1}(f, \theta_j) = -\frac{\cos\theta_j}{f_1}$ when $\theta_j \in (\frac{\pi}{2}, \frac{3\pi}{2})$. The probability density function of $\boldsymbol{\theta_j}$ is $\frac{1}{2\pi} I(\theta_j \in [0, 2\pi))$. Therefore,

$$
\begin{aligned}
\mathrm{E}[\ln \psi_{S_1}(f, \boldsymbol{\theta_j})] &= \int_0^{2\pi} \ln \psi_{S_1}(f, \theta_j) \frac{1}{2\pi} d\theta_j \\
&= \frac{1}{2\pi} [\int_0^{\frac{\pi}{2}} \ln \frac{\cos\theta_j}{1-f_1} d\theta_j + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \ln \frac{-\cos\theta_j}{f_1} d\theta_j + \int_{\frac{3\pi}{2}}^{2\pi} \ln \frac{\cos\theta_j}{1-f_1} d\theta_j] \\
&= \frac{1}{2\pi} [\int_0^{\frac{\pi}{2}} \ln\cos\theta_j d\theta_j - \int_0^{\frac{\pi}{2}} \ln(1-f_1) d\theta_j + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \ln(-\cos\theta_j) d\theta_j \\
&\quad - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \ln f_1 d\theta_j + \int_{\frac{3\pi}{2}}^{2\pi} \ln\cos\theta_j d\theta_j - \int_{\frac{3\pi}{2}}^{2\pi} \ln(1-f_1) d\theta_j] \\
&= \frac{1}{2\pi} [4 \int_0^{\frac{\pi}{2}} \ln\cos\theta_j d\theta_j - \pi \ln f_1(1-f_1)]
\end{aligned}
$$

By Lemma 3, $\int_0^{\frac{\pi}{2}} \ln\cos\theta_j d\theta_j = -\frac{\pi \ln 2}{2}$. Therefore, $\mathrm{E}[\ln \psi_{S_1}(f, \boldsymbol{\theta_j})]$ is finite and $\mathrm{E}[\ln \frac{\psi_{S_1}(f, \boldsymbol{\theta_j})}{\psi_{T_1}(f, \boldsymbol{\theta_j})}] < \infty$. It only remains to verify that $\mathrm{Var}(X_j)$ is finite. Since $\mathrm{Var}(X_j) = \mathrm{E}[X_j]^2 - (\mathrm{E}[X_j])^2$, we need to verify that $\mathrm{E}[X_j]^2$ is finite.

$$
\mathrm{E}[X_j]^2 = \mathrm{E}[\ln \frac{\psi_{S_1}(f, \boldsymbol{\theta_j})}{\psi_{T_1}(f, \boldsymbol{\theta_j})}] = \mathrm{E}[\ln \psi_{S_1}(f, \boldsymbol{\theta_j})]^2 - 2\mathrm{E}[\ln \psi_{S_1}(f, \boldsymbol{\theta_j}) \ln \psi_{T_1}(f, \boldsymbol{\theta_j})] + \mathrm{E}[\ln \psi_{T_1}(f, \boldsymbol{\theta_j})]^2.
$$

Since we have shown that $\ln \psi_{T_1}(f, \boldsymbol{\theta_j})$ is bounded and $\mathrm{E}[\ln \psi_{S_1}(f, \boldsymbol{\theta_j})]$ is finite for fixed f, the last two terms in the above equation are finite. For the first term $\mathrm{E}[\ln \psi_{S_1}(f, \boldsymbol{\theta_j})]^2$, substitute the formula for $\ln \psi_{S_1}(f, \theta_j)$ and expand it as an integration,

$$
\begin{aligned}
\mathrm{E}[\ln \psi_{S_1}(f, \boldsymbol{\theta_j})]^2 &= \int_0^{\frac{\pi}{2}} (\ln \frac{\cos\theta_j}{1-f_1})^2 \frac{1}{2\pi} d\theta_j + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\ln \frac{-\cos\theta_j}{f_1})^2 \frac{1}{2\pi} d\theta_j + \int_{\frac{3\pi}{2}}^{2\pi} (\ln \frac{\cos\theta_j}{1-f_1})^2 \frac{1}{2\pi} d\theta_j \\
&= \frac{1}{2\pi} [4 \int_0^{\frac{\pi}{2}} (\ln\cos\theta_j)^2 d\theta_j - 4 \ln f_1(1-f_1) \int_0^{\frac{\pi}{2}} \ln\cos\theta_j d\theta_j + \pi(\ln(1-f_1))^2 + \pi(\ln f_1)^2]
\end{aligned}
$$

By Lemma 3, $\int_0^{\frac{\pi}{2}} (\ln\cos\theta_j)^2 d\theta_j$ and $\int_0^{\frac{\pi}{2}} \ln\cos\theta_j d\theta_j$ are both finite. Thus, $\mathrm{E}[\ln \psi_{S_1}(f, \boldsymbol{\theta_j})]^2 < \infty$. Therefore, $\mathrm{Var}(X_j)$ is finite. □

C Proof of Lemma 5

Proof. Indeed, since Φ is uniformly distributed over U,

$$
\oint_U \frac{\angle COD}{2\pi} d\Phi(f) = \lim_{\epsilon \to 0} \int_\epsilon^{1-\epsilon} \int_\epsilon^{1-\epsilon} \frac{\angle COD}{2\pi} df_1 df_2
$$

In $\triangle COD$, $|\overline{OC}| = \sqrt{(1-f_1)^2 + (1-f_2)^2}$, $|\overline{OD}| = \sqrt{f_1^2 + (1-f_2)^2}$ and $|\overline{CD}| = 1$. By the law of cosines,

$$\cos \angle COD = \frac{|\overline{OD}|^2 + |\overline{OC}|^2 - |\overline{CD}|^2}{2|\overline{OD}||\overline{OC}|} = \frac{f_1(f_1-1) + (1-f_2)^2}{\sqrt{[f_1^2 + (1-f_2)^2][(1-f_1)^2 + (1-f_2)^2]}}$$

Therefore, $\angle COD = \arccos \frac{f_1(f_1-1)+(1-f_2)^2}{\sqrt{[f_1^2+(1-f_2)^2][(1-f_1)^2+(1-f_2)^2]}}$. Similarly,

$$\angle BOC = \arccos \frac{f_2(f_2-1) + (1-f_1)^2}{\sqrt{[f_2^2 + (1-f_1)^2][(1-f_2)^2 + (1-f_1)^2]}}.$$

By substitution of variables,

$$\begin{aligned}
&\int_{\epsilon}^{1-\epsilon}\int_{\epsilon}^{1-\epsilon} \frac{\angle COD}{2\pi} df_1 df_2 \\
&= \int_{\epsilon}^{1-\epsilon}\int_{\epsilon}^{1-\epsilon} \frac{\arccos \frac{f_1(f_1-1)+(1-f_2)^2}{\sqrt{[f_1^2+(1-f_2)^2][(1-f_1)^2+(1-f_2)^2]}}}{2\pi} df_1 df_2 \\
\overset{f_1=1-g_2, f_2=g_1}{\longrightarrow} &= \int_{\epsilon}^{1-\epsilon}\int_{\epsilon}^{1-\epsilon} \frac{\arccos \frac{g_2(g_2-1)+(1-g_1)^2}{\sqrt{[(1-g_2)^2+(1-g_1)^2][g_2^2+(1-g_1)^2]}}}{2\pi} dg_2 dg_1 \\
&= \int_{\epsilon}^{1-\epsilon}\int_{\epsilon}^{1-\epsilon} \frac{\arccos \frac{g_2(g_2-1)+(1-g_1)^2}{\sqrt{[g_2^2+(1-g_1)^2][(1-g_2)^2+(1-g_1)^2]}}}{2\pi} dg_1 dg_2 \\
\overset{g_1=f_1, g_2=f_2}{\longrightarrow} &= \int_{\epsilon}^{1-\epsilon}\int_{\epsilon}^{1-\epsilon} \frac{\arccos \frac{f_2(f_2-1)+(1-f_1)^2}{\sqrt{[f_2^2+(1-f_1)^2][(1-f_2)^2+(1-f_1)^2]}}}{2\pi} df_1 df_2 \\
&= \int_{\epsilon}^{1-\epsilon}\int_{\epsilon}^{1-\epsilon} \frac{\angle BOC}{2\pi} df_1 df_2
\end{aligned}$$

Similarly, we can show

$$\int_{\epsilon}^{1-\epsilon}\int_{\epsilon}^{1-\epsilon} \frac{\angle DOA}{2\pi} df_1 df_2 = \int_{\epsilon}^{1-\epsilon}\int_{\epsilon}^{1-\epsilon} \frac{\angle AOB}{2\pi} df_1 df_2 = \int_{\epsilon}^{1-\epsilon}\int_{\epsilon}^{1-\epsilon} \frac{\angle BOC}{2\pi} df_1 df_2$$

Therefore,

$$\begin{aligned}
&4\int_{\epsilon}^{1-\epsilon}\int_{\epsilon}^{1-\epsilon} \frac{\angle BOC}{2\pi} df_1 df_2 \\
&= \int_{\epsilon}^{1-\epsilon}\int_{\epsilon}^{1-\epsilon} \frac{\angle BOC + \angle COD + \angle DOA + \angle AOB}{2\pi} df_1 df_2 \\
&= \int_{\epsilon}^{1-\epsilon}\int_{\epsilon}^{1-\epsilon} \frac{2\pi}{2\pi} df_1 df_2 \\
&= (1-2\epsilon)^2
\end{aligned}$$

Thus, $\oint_U \frac{\angle BOC}{2\pi} d\Phi(f) = 0.25$.
Now we compute $\oint_U \frac{\angle COF}{2\pi} d\Phi(f)$.

$$\begin{aligned}
\oint_U \frac{\angle COF}{2\pi} d\Phi(f) &= \lim_{\epsilon\to 0}\int_{\epsilon}^{1-\epsilon}\int_{\epsilon}^{1-\epsilon} \frac{\angle COF}{2\pi} df_1 df_2 \\
&= \lim_{\epsilon\to 0}\int_{\epsilon}^{1-\epsilon}\int_{\epsilon}^{1-\epsilon} \frac{\arccos \frac{f_1^2+f_2^2-f_1-3f_2+2}{\sqrt{[(1-f_2)^2+(1-f_1)^2][f_1^2+(2-f_2)^2]}}}{2\pi} df_1 df_2 \approx 0.176.
\end{aligned}$$

In the final step, we used the Matlab function ‘dblquad’ with $\epsilon = 10^{-8}$ for the numerical calculation.

$\square$