Implicit Multifunction Theorems in complete metric spaces

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Abstract

In this paper, we establish some new characterizations of the metric regularity of implicit multifunctions in complete metric spaces by using the lower semicontinuous envelopes of the distance functions for set-valued mappings. Through these new characterizations it is possible to investigate implicit multifunction theorems based on coderivatives and on contingent derivatives as well as the perturbation stability of implicit multifunctions.

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1 Introduction

Let X and Y be metric spaces endowed with metrics both denoted by $d(\cdot,\cdot)$. The open ball with center x and radius $r>0$ is denoted by $B(x,r)$. We recall that a set-valued (multivalued) mappping $F : X \rightrightarrows Y$ is a mapping which assigns to every $x \in X$ a subset (possibly empty) $F(x)$ of Y. As usual, we use the notation $\mathrm{gph}\ F := \{(x,y) \in X \times Y \ : \ y \in F(x)\}$ for the graph of F, $\mathrm{Dom}\ F := \{x \in X \ : \ F(x) \neq \emptyset\}$ for the domain of F and $F^{-1} : Y \rightrightarrows X$ for the inverse of F. This inverse (which always exists) is defined by $F^{-1}y := \{x \in X \ : \ y \in F(x)\},\ y \in Y$ and satisfies

$$(x,y) \in \mathrm{gph}\ F \iff (y,x) \in \mathrm{gph}\ F^{-1}.$$

It is well known that a large amount of problems, such that for instance, inequalities and equalities systems, variational inequalities or systems of optimality conditions are covered by the solvability of a generalized equation (in the terminology of Robinson):

For a given $y \in Y$, determine $x \in X$ such that $y \in F(x)$.

In general F is is given in the form $f + T$, where $f : X \to Y$ is a mapping and $T : X \rightrightarrows Y$ is a set-valued mapping. An important subcase is furnished by variational inequalities, that is the problem of finding a solution to the equation $y \in f(x) + N_C(x)$ where $T = N_C$ is the normal-cone operator. For each $x \in \mathbb{R}^n$, the set $N_C(x)$ is the normal cone (in the sense of convex analysis) to a closed convex set C of $\mathbb{R}^n$ at x.

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A central issue in variational analysis is to investigate the behavior of the solution set of a generalized equation associated to F, that is the behavior of the set $F^{-1}(y)$ when y and/or F itself are perturbed. A key to this is the concept of metric regularity. Using the standard notation $d(x, C) = \inf_{z \in C} d(x, z)$, with the convention that $d(x, S) = +\infty$ whenever S is empty, recall that a mapping F is said to be *metrically regular* at some $\bar{x} \in X$ with respect to $\bar{y} \in F(\bar{x})$ with modulus $\tau > 0$ if there exists a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \quad \text{for all} \quad (x, y) \in U \times V. \tag{1}$$

The infimum of all modulus τ is denoted by reg $F(\bar{x}, \bar{y})$ ([22]). In the case for example of a set-valued mapping F with closed and convex graph, the Robinson-Ursescu Theorem ([53] and [55]), says that F is metrically regular at (x_0, y_0), if and only if y_0 is an interior point to the range of F, i.e., to Dom F^{-1}.

According to the long history of metric regularity there is an abundant literature on conditions ensuring this property. This concept goes back to the surjectivity of a linear continuous mapping in the Banach Open Mapping Theorem and to its extension to nonlinear operators known as the Lyusternik & Graves Theorem ([40], [27], see also [15]) and [21]). For a detailed account the reader is referred to the books or works of many researchers, [3], [5], [9], [10], [11], [12], [13], [17], [18], [20], [22], [29], [30], [32], [34], [36], [37], [40], [42], [43], [41], [44], [45], [46], [50], [51], [52], [57] and the references given therein for many theoretical results on the metric regularity as well as its various applications. Metric regularity or its equivalent notions (covering at a linear rate) [38] or Aubin property of the inverse [1] is now considered as a central concept in modern variational analysis (see the survey paper by Ioffe [34]).

Along with the metric regularity of a multifunction, the study of conditions ensuring the metric regularity of parametric multifunctions, that is, *implicit multifunction theorems,* plays a crucial rule in investigating problems of sensitivity analysis with respect to parameters. The metric regularity of implicit multifunctions has been extensively studied by several authors (see, for example, [2], [5], [8], [25], [24], [39], [46], [56] and the references given therein). Almost all implicit multifunction results in the literature are based on conditions on some tangent cones or on some normal cones to the graphs of the multifunctions involved. Another approach used recently in Azé & Benhamed [8], consists in perturbing a metrically regular mapping by a pseudo-Lipschitzian multifunction. Our main objective in this paper is to use the theory of error bounds to study the metric regularity of implicit multifunctions. The approach based on error bounds to investigate the metric regularity has been recently used in [6] and in [46] to study implicit multifunctions in smooth spaces. Especially in the survey paper [5], it was shown that this approach is powerful to provide a unified theory of the metric regularity. In this work, we use a different way for deriving implicit multifunction results. Our approach is based on the error bound property of the lower semicontinuous envelope of distance functions to the images of set-valued mappings. This approach which is used in Ngai & Théra [48] allows to avoid the completeness of the image space.

The organization of this paper is as follows. In Section 2, we prove new characterizations of the error bound for parametric inequality systems in complete metric spaces. Using this result, we derive in Section 3 a new criterion assuring the metric regularity of implicit multifunctions by using the lower semicontinuous envelope of distance functions to the images of set-valued mappings. In connection with this criterion, we establish new implicit multifunction theorems based on conditions on strong slope, contingent cones as well as coderatives. In the final section, a result on the perturbation stability of implicit multifunctions is reported.

2 Characterizations of error bounds for parametric inequality systems

Let X be a metric space. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function. As usual, $\mathrm{dom} f := \{x \in X : f(x) < +\infty\}$ denotes the domain of f. We set

$$S := \{x \in X : \quad f(x) \leq 0\}. \tag{2}$$

We use the symbol $[f(x)]_+$ to denote $\max(f(x), 0)$. We shall say that the system (2) admits *an error bound* if there exists a real $c > 0$ such that

$$d(x, S) \leq c\big[f(x)\big]_+ \quad \text{for all} \quad x \in X. \tag{3}$$

For $x_0 \in S$, we shall say that the system (2) has an error bound at x_0, when there exist reals $c > 0$ and $\varepsilon > 0$ such that relation (3) is satisfied for all x around x_0, i.e., in an open ball $B(x_0, \varepsilon)$ with center x_0 and radius ε.

Several conditions using subdifferential operators or directional derivatives and ensuring the error bound in Banach spaces have been established, for example, in [16], [35], [49], [46], [58]. Recently, Azé [4], Azé & Corvellec [7] have used the so-called strong slope introduced by De Giorgi, Marino & Tosques in [19] to prove criteria for error bounds in complete metric spaces.

In the sequel, we will need the following result established in [48], which gives an estimation for the distance $d(\bar{x}, S)$ from a given point $\bar{x}$ outside of S to the set S in complete metric spaces. Such an estimation using the Fréchet subdifferential in Asplund spaces has been established by Ngai & Théra [47].

Theorem 1 *Let X be a complete metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $\bar{x} \notin S$. Then, setting*

$$m(\bar{x}) := \inf \left\{ \sup_{y \in X, y \neq x} \frac{f(x) - [f(y)]_+}{d(x, y)} : \qquad \begin{array}{l} d(x, \bar{x}) < d(\bar{x}, S) \\ f(x) \leq f(\bar{x}) \end{array} \right\}, \tag{4}$$

one has

$$m(\bar{x}) d(\bar{x}, S) \leq f(\bar{x}). \tag{5}$$

Here and in what follows the convention $0 \cdot (+\infty) = 0$ is used.

We now consider the parametric inequality system, that is, the problem of finding $x \in X$ such that

$$f(x, p) \leq 0, \tag{6}$$

where $f : X \times P \to \mathbb{R} \cup \{+\infty\}$ is an extended-real-valued function, X is a complete metric space and P is a topological space. Denote by $S(p)$ the set of solutions of system (6):

$$S(p) := \{x \in X : \quad f(x, p) \leq 0\}.$$

The following theorem gives characterizations of the error bound for the parametric system (6).

Theorem 2 *Let X be a complete metric space and P be a topological space. Suppose that the mapping $f : X \times P \to \mathbb{R} \cup \{+\infty\}$ satisfies the following conditions for some $(\bar{x}, \bar{p}) \in X \times P$:*

(a) $\bar{x} \in S(\bar{p})$;

(b) the mapping $p \mapsto f(\bar{x}, p)$ *is upper semicontinuous at* $\bar{p}$;

(c) for any p *near* $\bar{p}$, *the mapping* $x \mapsto f(x, p)$ *is lower semicontinuous near* $\bar{x}$.

Let $\tau > 0$ *be given and consider the following statements:*

(i) There exists a neighborhood $V \times W \subseteq X \times P$ *of* $(\bar{x}, \bar{p})$ *such that for any* $p \in W$, *we have* $V \cap S(p) \neq \emptyset$ *and*

$$d(x, S(p)) \leq \tau[f(x,p)]_+ \quad \textit{for all} \quad (x,p) \in V \times W. \tag{7}$$

(ii) There exists a neighborhood $V \times W \subseteq X \times P$ *of* $(\bar{x}, \bar{p})$ *such that for each* $(x,p) \in V \times W$ *with* $f(x,p) \in (0, \gamma)$ *and for any* $\varepsilon > 0$, *we can find* $z \in X$ *such that*

$$0 < d(x,z) < (\tau + \varepsilon)(f(x,p) - [f(z,p)]_+). \tag{8}$$

(iii) There exist a neighborhood $V \times W \subseteq X \times P$ *of* $(\bar{x}, \bar{p})$ *such that for each* $(x,p) \in V \times W$ *with* $f(x,p) \in (0, \gamma)$ *and for any* $\varepsilon > 0$, *we can find* $z \in X$ *with* $f(z,p) \geq 0$ *such that (8) holds.*

(iv) There exists a neighborhood $V \times W \subseteq X \times P$ *of* $(\bar{x}, \bar{p})$ *such that for each* $(x,p) \in V \times W$ *with* $f(x,p) \in (0, \gamma)$ *and for any* $\varepsilon > 0$, *we can find* $z \in X$ *with* $f(z,p) > 0$ *such that (8) holds.*

Then, one has ((iv) $\Rightarrow$ *(iii)* $\Rightarrow$ *(ii)* $\Leftrightarrow$ *(i). In addition, if* X *is a Banach space and the mapping* $x \mapsto f(x,p)$ *is continuous near* $\bar{x}$ *for all* p *near* $\bar{p}$ *then the four statements above are equivalent.*

Proof. The implications $(iv) \Rightarrow (iii) \Rightarrow (ii)$ are obvious. For $(i) \Rightarrow (ii)$, suppose that (i) holds for some neighborhood $V \times W$ of $(\bar{x}, \bar{p})$. Let $(x,p) \in V \times W$ with $x \notin S(p)$ and $\varepsilon > 0$ be given. Obviously, (8) holds trivially if $f(x,p) = +\infty$. Assume now $f(x,p) < +\infty$. Then $S(p) \neq \emptyset$, and by taking $z \in S(p)$ such that $d(x,z) < (1 + \varepsilon/\tau) d(x, S(p))$, one has

$$d(x,z) \leq (\tau + \varepsilon) f(x,p) = (\tau + \varepsilon)(f(x,p) - [f(z,p)]_+).$$

Let us now prove $(ii) \Rightarrow (i)$. Suppose that (8) is satisfied for $V \times W := B(\bar{x}, \alpha) \times W$ for some $\alpha > 0$. Let $\varepsilon \in (0, \tau/2)$ be given. Since the assumption (a) and (b), by setting $\delta := \min\{\alpha, \frac{\alpha}{6(\tau+\varepsilon)}, \gamma/4\}$, there exists an open set $W_1 \subseteq W$ containing $\bar{p}$ such that

$$f(\bar{x}, p) < f(\bar{x}, \bar{p}) + \delta \leq \delta \quad \text{for all} \quad p \in W_1.$$

It follows that for $p \in W_1$ arbitrary fixed, one has

$$[f(\bar{x}, p)]_+ \leq \inf_{x \in X} [f(x,p)]_+ + \delta.$$

By virtue of the Ekeland variational principle [26] applied to the function $x \mapsto [f(x,p)]_+$ on X, we can select $z \in X$ satisfying $d(\bar{x}, z) \leq \delta(\tau + 2\varepsilon)$ and $[f(z,p)]_+ \leq [f(\bar{x}, p)]_+ (< \delta)$ such that

$$[f(z,p)]_+ \leq [f(x,p)]_+ + \frac{1}{\tau + 2\varepsilon} d(x,z) \quad \text{for all } x \in X.$$

Consequently, $z \in V$ and

$$[f(z,p)]_+ - [f(x,p)]_+ \leq \frac{d(z,x)}{\tau + \varepsilon} \quad \text{for all } x \in X.$$

Therefore, by assumption we must have $z \in S(p)$. Consequently, $B(\bar{x}, 2\delta\tau) \cap S(p) \neq \emptyset$ for all $p \in W_1$.

Let now $(x, p) \in B(\bar{x}, 2\delta\tau) \times W_1$ be given. If $f(x, p) \geq \gamma$, then by $B(\bar{x}, 2\delta\tau) \cap S(p) \neq \emptyset$, one has

$$d(x, S(p)) \leq d(x, \bar{x}) + d(\bar{x}, S(p)) < 2\delta\tau + 2\delta\tau = 4\delta\tau \leq \tau f(x, p). \tag{9}$$

Let us now consider the case of $0 < f(x, p) < \gamma$. Then for any $z \in X$ with $d(x, z) < d(x, S(p))$; $f(z, p) \leq f(x, p)$, one has

$$d(z, \bar{x}) \leq d(z, x) + d(x, \bar{x}) \leq d(\bar{x}, S(p)) + 2d(x, \bar{x}) < 6\delta\tau.$$

Thus, $z \in V$. Therefore, according to (8), one has

$$m(x) := \inf \left\{ \sup_{u \in X, u \neq z} \frac{f(z, p) - [f(u, p)]_+}{d(z, u)} : \quad \begin{array}{l} d(z, x) < d(x, S(p)) \\ f(z, p) \leq f(x, p) \end{array} \right\} > \frac{1}{\tau + \varepsilon}.$$

By virtue of Theorem 1 and as ε is arbitrarily small, we obtain

$$d(x, S(p)) \leq \tau [f(x, p)]_+ \quad \text{for all } (x, p) \in B(\bar{x}, 2\delta\tau) \times W_1,$$

where $\delta := \min\{\alpha, \alpha\tau^{-1}/6, \gamma/4\}$.

Let now X be a Banach space and for each p near $\bar{p}$, the function $x \mapsto f(x, p)$ be continuous. For *(ii)* $\Rightarrow$ *(iii)*, let (x, p) be sufficiently close to $(\bar{x}, \bar{p})$ with $f(x, p) \in (0, \gamma)$ such that there exists $z \in X$ verifying (8). When $f(z, p) < 0$, since $f(x, p) > 0$ and the function $f(\cdot, p)$ is continuous, we can find $y \in [x, z] := \{tx + (1 - t)z : \ t \in (0, 1]\}$ such that $f(y, p) = 0$. Hence,

$$0 < d(x, y) \leq d(x, z) < (\tau + \varepsilon) f(x, p) = (\tau + \varepsilon)(f(x, p) - f(y, p)).$$

Finally, for *(iii)* $\Rightarrow$ *(iv)*, let $z \in X$ with $f(z, p) \geq 0$ satisfying (8). If $f(z, p) > 0$ then the conclusion holds obviously. Suppose that $f(z, p) = 0$. Let $A \subseteq \mathbb{R}$ be defined by $A = \{t \in [0, 1] : f(tx + (1 - t)z, p) \leq 0\}$. Since A is nonempty, closed and bounded in $\mathbb{R}$, we may define $\max A := t_0$ with $t_0 \in [0, 1)$. For each $t \in (t_0, 1)$ if $y_t := tx + (1 - t)z$, then $f(y_t, p) > 0$. Pick a real $\delta > 0$ such that $\frac{1 - t_0}{1 - \delta(\tau + \varepsilon)} < 1$. Noticing that $f(y_{t_0}, p) = 0$ and using the continuity of $f(\cdot, p)$, we can find $t_1 \in (t_0, 1)$ such that $f(y_t, p) < \delta d(x, z)$ for all $t \in (t_0, t_1)$. Then for all $t \in (t_0, t_1)$, one has

$$d(x, y_t) = (1 - t) d(x, z) < (1 - t)(\tau + \varepsilon) f(x, p) < (1 - t)(\tau + \varepsilon)(f(x, p) - f(y_t, p)) + (1 - t)(\tau + \varepsilon)\delta d(x, z).$$

Thus,

$$d(x, y_t) < \frac{(1 - t)(\tau + \varepsilon)}{1 - \delta(\tau + \varepsilon)} (f(x, p) - f(y_t, p)) < (\tau + \varepsilon)(f(x, p) - f(y_t, p)),$$

which completes the proof. □

Recall from [19], [7] that the strong slope $|\nabla f|(x)$ of a lower semicontinuous function f at $x \in \mathrm{dom} f$ is the quantity defined by $|\nabla f|(x) = 0$ if x is a local minimum of f, otherwise

$$|\nabla f|(x) = \limsup_{y \to x} \frac{f(x) - f(y)}{d(x, y)}.$$

For $x \notin \mathrm{dom} f$, we set $|\nabla f|(x) = +\infty$. Theorem 1 implies directly the following result, which is a slight improvement of Azé ([5], Theorem 2.13).

Corollary 3 *Let X be a complete metric space, let P be a topological space. Suppose that the mapping $f : X \times P \to \mathbb{R} \cup \{+\infty\}$ satisfies conditions (a), (b), (c) of Theorem 2. If there exist a neighborhood $V \times W$ of $(\bar{x}, \bar{p})$ and a real $m > 0$ such that $|\nabla f(\cdot, p)|(x) \geq m$ for all $(x, p) \in V \times W$ with $f(x, p) \in (0, \gamma)$ then*

$$md(x, S(p)) \leq [f(x, p)]_+ \quad \textit{for all} \quad (x, p) \in V \times W.$$

Let a real $\gamma > 0$ be given. By noting that for all $x \in X$ with $f(x) > 0$, $|\nabla f^\alpha|(x) = \alpha f^{\alpha-1}(x)|\nabla f|(x)$, one obtains the following more general error bound with an exponent.

Corollary 4 *Let X be a complete metric space, let P be a topological space and suppose that the mapping $f : X \times P \to \mathbb{R} \cup \{+\infty\}$ satisfies the conditions (a), (b), (c) of Theorem 2. For a given $\alpha > 0$, if there exist a neighborhood $V \times W$ of $(\bar{x}, \bar{p})$ and a real $m > 0$ such that*

$$\alpha f^{\alpha-1}(x, p)|\nabla f(\cdot, p)|(x) \geq m \quad \textit{for all } (x, p) \in V \times W \textit{ with } f(x, p) \in (0, \gamma)$$

then

$$md(x, S(p)) \leq [f(x, p)]_+^\alpha \quad \textit{for all} \quad (x, p) \in V \times W.$$

3 Implicit multifunction theorems

Let X, Y be metric spaces and let P be a topological space. Let $F : X \times P \rightrightarrows Y$ be a multifunction. For a given $\bar{y} \in Y$, we next consider the implicit multifunction (set-valued) problem

$$S(\bar{y}, p) := \{x \in X : \ \bar{y} \in F(x, p)\}. \tag{10}$$

In what follows, we make use of the lower semicontinuous envelope $(x, y) \mapsto \varphi_p(x, y)$ of the function $(x, y) \mapsto d(y, F(x, p))$ for each $p \in P$, i.e., for $(x, y) \in X \times Y$,

$$\varphi_p(x, y) := \liminf_{(u,v)\to(x,y)} d(v, F(u, p)) = \liminf_{u\to x} d(y, F(u, p)).$$

We establish in the following theorem the characterizations of the implicit multifunction (10).

Theorem 5 *Let X be a complete metric space and Y be a metric space. Let P be a topological space and suppose that the set-valued mapping $F : X \times P \rightrightarrows Y$ satisfies the following conditions for some $(\bar{x}, \bar{y}, \bar{p}) \in X \times Y \times P$:*

(a) $\bar{x} \in S(\bar{y}, \bar{p})$;

(b) the multifunction $p \rightrightarrows F(\bar{x}, p)$ is lower semicontinuous at $\bar{p}$;

(c) for any p near $\bar{p}$, the set-valued mapping $x \rightrightarrows F(x, p)$ is a closed multifunction (i.e., its graph is closed).

Let $\tau \in (0, +\infty)$, be fixed. Then, the following statements are equivalent:

(i) There exists a neighborhhood $V \times W \subseteq X \times P$ of $(\bar{x}, \bar{p})$ such that $V \cap S(\bar{y}, p) \neq \emptyset$ for any $p \in W$ and

$$d(x, S(\bar{y}, p)) \leq \tau d(\bar{y}, F(x, p)) \quad \textit{for all } (x, p) \in V \times W;$$

(ii) There exists a neighborhhood $V \times W \subseteq X \times P$ of $(\bar{x}, \bar{p})$ such that $V \cap S(\bar{y}, p) \neq \emptyset$ for any $p \in W$ and

$$d(x, S(\bar{y}, p)) \leq \tau \varphi_p(x, \bar{y}) \quad \textit{for all } (x, p) \in V \times W;$$

(iii) There exists a neighborhood $V \times W \subseteq X \times P$ of $(\bar{x}, \bar{p})$ such that for any $(x,p) \in V \times W$ with $\bar{y} \notin F(x,p)$, $\varepsilon > 0$ any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converging to x with

$$\limsup_{n\to\infty} d(\bar{y}, F(x_n, p)) \leq d(\bar{y}, F(x,p)),$$

there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ with $\lim_{n\to\infty} d(u_n, x) > 0$ such that

$$\limsup_{n\to\infty} \frac{d(\bar{y}, F(x_n,p)) - d(\bar{y}, F(u_n,p))}{d(x_n,u_n)} > \frac{1}{\tau+\varepsilon}; \tag{11}$$

(iv)There exist a neighborhood $V \times W \subseteq X \times P$ of $(\bar{x}, \bar{p})$ and a real $\gamma \in (0, +\infty)$ such that for any $(x,p) \in V \times W$ with $\bar{y} \notin F(x,p)$ and $\varphi_p(x, \bar{y}) < \gamma$ and any $\varepsilon > 0$, then for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converging to x with

$$\lim_{n\to\infty} d(\bar{y}, F(x_n,p)) = \liminf_{u\to x} d(\bar{y}, F(u,p)),$$

we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ with $\lim_{n\to\infty} d(u_n, x) > 0$ such that (11) holds.

The next lemma is useful.

Lemma 6 *For each $y \in Y$, and each p near $\bar{p}$,*

$$S(y,p) = \{x \in X : \quad \varphi_p(x,y) = 0\}.$$

Proof.

Indeed, let $(x,y) \in X \times Y$ and let p near $\bar{p}$ be such that the mapping $x \rightrightarrows F(x,p)$ is a closed multifunction. Obviously, if $x \in S(y,p)$ then $\varphi_p(x,y) = 0$. Conversely, suppose $\varphi_p(x,y) = 0$. There exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ with limit x such that $d(y, F(x_n,p))$ converges to 0. Then, we can find a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq Y$ such that $z_n \in F(x_n,p)$ and $d(y, z_n) \to 0$. Since the graph of $F(\cdot,p)$ is closed, then $(x,y) \in \text{gph } F(\cdot,p)$, i.e., $x \in S(y,p)$. □

Proof of Theorem 5. For $(i) \Rightarrow (iii)$, let $V \times W$ be a open neighborhood of $(\bar{x}, \bar{y})$ such that $\text{gph}F(\cdot,p)$ is closed for $p \in W$ and that

$$d(x, S(\bar{y},p)) \leq \tau d(\bar{y}, F(x,p)) \ \ \forall (x,p) \in V \times W.$$

Let $(x,p) \in V \times W$, $\bar{x} \notin F(x,p)$ and $\varepsilon > 0$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence converging to x. When n is sufficiently large, say $n \geq n_0$, then $x_n \in V$ as well as $\bar{y} \notin F(x_n,p)$. Hence $d(x_n, S(\bar{y},p)) \leq \tau d(\bar{y}, F(x_n,p))$. For each $n \geq n_0$, pick $u_n \in S(\bar{y},p)$ such that $d(x_n,u_n) < (1+\varepsilon/2\tau) d(x_n, S(\bar{y},p))$. On relabeling if necessary, we can assume that $\lim_{n\to\infty} d(x_n,u_n)$ exists. Then by the closedness of $\text{gph}F(\cdot,p)$, one has $\lim_{n\to\infty} d(x_n,u_n) > 0$ and moreover for all $n \geq n_0$,

$$d(x_n,u_n) < (1+\varepsilon/2\tau) d(x_n, S(\bar{y},p)) \leq (\tau+\varepsilon/2)[d(\bar{y}, F(x_n,p)) - d(\bar{y}, F(u_n,p))].$$

This shows that (11) holds.

$(iii) \Rightarrow (iv)$ is obvious. To complete the proof, let us prove $(iv) \Rightarrow (ii)$. Since the multifunction $p \rightrightarrows F(\bar{x},p)$ is assumed to be lower semicontinuous at $\bar{p}$, then the function $p \mapsto d(\bar{y}, F(\bar{x},p))$ is upper semicontinuous at $\bar{p}$ (see, e.g., Cor. 20 in [1]). Therefore,

$$\limsup_{p\to\bar{p}} \varphi_p(\bar{x}, \bar{y}) \leq \limsup_{p\to\bar{p}} d(\bar{y}, F(\bar{x},p)) \leq d(\bar{y}, F(\bar{x},\bar{p})) + \varepsilon = \varepsilon = \varphi_{\bar{p}}(\bar{x}, \bar{y}) + \varepsilon.$$

That is, the function $p \mapsto \varphi_p(\bar{x}, \bar{y})$ is upper semicontinuous at $\bar{p}$. Therefore, by virtue of Theorem 2, it suffices to show that statement (ii) of Theorem 2 is verified. Let $(x,p) \in V \times W$ with $\bar{y} \notin F(x,p)$ and $\varphi_p(x, \bar{y}) < \gamma$ and let $\varepsilon > 0$ be given. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence converging to x with

$$\lim_{n\to\infty} d(\bar{y}, F(x_n, p)) = \varphi_p(x, \bar{y}).$$

By (iv), we can find a sequence $\{u_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} d(u_n, x) > 0$ such that

$$\limsup_{n\to\infty} \frac{d(\bar{y}, F(x_n, p)) - d(\bar{y}, F(u_n, p))}{d(x_n, u_n)} > \frac{1}{\tau + \varepsilon}.$$

Consequently,

$$\limsup_{n\to\infty} \frac{\varphi_p(x, \bar{y}) - \varphi_p(u_n, \bar{y})}{d(x, u_n)} > \frac{1}{\tau + \varepsilon};$$

and statement (ii) of Theorem 2 follows directly. The proof is complete. □

By combining this theorem and Corollary 3, we obtain the following implicit multifunction theorem.

Theorem 7 *Let X be a complete metric space; let Y be a metric space and let P be a topological space. Suppose that the multifunction $F : X \times P \rightrightarrows Y$ satisfies the conditions (a), (b), (c) around $(\bar{x}, \bar{p}, \bar{y})$ of Theorem 5. If there exist a neighborhood $U \subseteq X \times P$ of $(\bar{x}, \bar{p})$ and reals $m, \gamma > 0$ such that $|\nabla\varphi_p(\cdot, \bar{y})|(x) \geq m$ for all $(x,p) \in U$ with $\varphi_p(x, \bar{y}) \in (0, \gamma)$ then*

$$md(x, S(\bar{y}, p)) \leq d(\bar{y}, F(x,p)) \quad \textit{for all} \quad (x,p) \in V \times W.$$

The next theorem gives the metric regularity of implicit multifunctions by using the strong slope of the lower semicontinuous envelope function $x \mapsto \varphi_p(x, y)$.

Theorem 8 *Let X be a complete metric space; let Y be a metric space and let P be a topological space. Suppose that the multifunction $F : X \times P \rightrightarrows Y$ verifies conditions (a), (b), (c) in Theorem 5 around $(\bar{x}, \bar{p}, \bar{y}) \in gphF$. Let $m > 0$ be given. If there exist a neighborhood $U \times W \times V$ of $(\bar{x}, \bar{p}, \bar{y})$ and a real $\gamma > 0$ such that*

$$|\nabla\varphi_p(\cdot, y)|(x) \geq m \quad \textit{for all } (x,p,y) \in U \times W \times V \quad \textit{with } \varphi_p(x,y) \in (0,\gamma), \tag{12}$$

then there exists a neighborhood $\tilde{U} \times \tilde{W} \times \tilde{V}$ of $(\bar{x}, \bar{p}, \bar{y})$ such that

$$d(x, F_p^{-1}(y)) \leq d(y, F(x,p))/m \quad \forall (x,p,y) \in \tilde{U} \times \tilde{W} \times \tilde{V}. \tag{13}$$

Moreover, the converse holds if Y is assumed to be a normed linear space.

Proof. The first part follows directly from Theorem 7. Conversely, assume that Y is a normed linear space. Let $r > 0$ and an open neighborhood W of $\bar{p}$ be such that

$$d(x, F_p^{-1}(y)) \leq d(y, F(x,p))/m \quad \forall (x,p,y) \in B(\bar{x}, 2r) \times W \times B(\bar{y}, 2r).$$

Let $(x,p,y) \in B(\bar{x}, r) \times W \times B(\bar{y}, r)$ be given with $y \notin F(x,p)$; $\varphi_p(x,y) < r$. Let (u_n) be a sequence of X such that

$$d(u_n, x) < n^{-1}\varphi_p(x,y); \; d(y, F(u_n, p)) \leq (1 + 1/n)\varphi_p(x,y) \text{ (therefore, } \lim_{n\to\infty} d(y, F(u_n,p)) = \varphi_p(x,y)).$$

For each $n \in \mathbb{N}^*$, there exists $y_n \in F(u_n, p)$ such that

$$d(y, F(u_n, p)) \le \|y - y_n\| < (1 + n^{-1})d(y, F(u_n, p)).$$

Setting $z_n := \dfrac{1 + n^{1/2}}{n + 1} y + \dfrac{n(1 - n^{-1/2})}{n + 1} y_n$, one has

$$\|y - z_n\| = \frac{n(1 - n^{-1/2})}{n + 1}\|y - y_n\| < \frac{n(1 - n^{-1/2})}{n + 1}(1 + n^{-1})d(y, F(u_n, p)) < (1 + 1/n)\varphi_p(x, y).$$

Therefore, $z_n \notin F(u_n, p)$ and $\|z_n - \bar{y}\| \le \|y - \bar{y}\| + \|y - z_n\| < 2r$ when n is sufficiently large. Hence, we can select $x_n \in F_p^{-1}(z_n)$ such that

$$\begin{aligned} d(u_n, x_n) < (1 + n^{-1/2})d(u_n, F_p^{-1}(z_n)) &\le (1 + n^{-1/2})d(z_n, F(u_n, p))/m \\ &\le (1 + n^{-1/2})(1 + n^{1/2})(n + 1)^{-1}\|y - y_n\|/m. \end{aligned} \tag{14}$$

Consequently, $\lim_{n\to\infty} d(x, x_n) = 0$. Next, one has the following estimation for n sufficiently large,

$$\begin{aligned} \varphi_p(x, y) - \varphi_p(x_n, y) &\ge \frac{n}{n + 1}d(y, F(u_n, p)) - d(y, F(x_n, p)) \\ &\ge \frac{n}{(n + 1)}\big((1 + n^{-1})^{-1} - (1 - n^{-1/2})\big)\|y - y_n\| = \frac{n^{1/2}(1 - n^{-1/2} + n^{-1})}{(n + 1)(1 + n^{-1})}\|y - y_n\| \end{aligned} \tag{15}$$

From this relation and (14),

$$\frac{\varphi_p(x, y) - \varphi_p(x_n, y)}{d(x, x_n)} \ge \frac{\varphi_p((x, y) - \varphi_p((x_n, y)}{d(x, u_n) + d(u_n, x_n)} \ge \frac{mn^{1/2}(1 - n^{-1/2} + n^{-1})}{n^{-1}(n + 1)r(1 + n^{-1}) + (1 + n^{-1})(1 + n^{-1/2})(1 + n^{1/2})}.$$

Thus,

$$|\nabla\varphi_p(\cdot, y)|(x) \ge \liminf_{n\to\infty} \frac{\varphi(x, y) - \varphi(x_n, y)}{d(x, x_n)} \ge m,$$

which completes the proof. □

From Theorem 8 we can derive as a corollary an exact formula for the metric regularity modulus on metric spaces.

Corollary 9 *Let X be a complete metric space and let Y be a normed linear space. Suppose that the multifunction $F : X \rightrightarrows Y$ is closed and $(\bar{x}, \bar{y}) \in gphF$. Denote by $\varphi(x, y)$ the lower semicontinuous envelope of the function $d(y, F(x))$. Then, one has*

$$1/regF(\bar{x}, \bar{y}) = \liminf_{(x,y)\to_\varphi(\bar{x},\bar{y}), y\notin F(x)} |\nabla\varphi(\cdot, y)|(x).$$

Let us remind that the notation $(x, y) \underset{\varphi}{\to} (\bar{x}, \bar{y})$ means that $(x, y) \to (\bar{x}, \bar{y})$ with $\varphi(x, y) \to 0$.

Proof. For any $0 < m < \liminf_{(x,y)\underset{\varphi}{\to}(\bar{x},\bar{y}), y\notin F(x)} |\nabla\varphi(\cdot, y)|(x)$, by the first part of Theorem 8, one has $1/m \ge \mathrm{reg}F(\bar{x}, \bar{y})$. Thus

$$1/\mathrm{reg}F(\bar{x}, \bar{y}) \ge \liminf_{(x,y)\underset{\varphi}{\to}(\bar{x},\bar{y}), y\notin F(x)} |\nabla\varphi(\cdot, y)|(x).$$

For the opposite inequality, if $\mathrm{reg}F(\bar{x},\bar{y}) = +\infty$, we are done. Let $\mathrm{reg}F(\bar{x},\bar{y}) < \tau < +\infty$. By the converse part of Theorem 8,

$$\liminf_{(x,y)\underset{\varphi}{\to}(\bar{x},\bar{y}),y\notin F(x)} |\nabla\varphi(\cdot,y)|(x) \leq 1/\tau.$$

Since τ can be arbitrary close to $\mathrm{reg}F(\bar{x},\bar{y})$, one obtains

$$\liminf_{(x,y)\underset{\varphi}{\to}(\bar{x},\bar{y}),y\notin F(x)} |\nabla\varphi(\cdot,y)|(x) \leq 1/\mathrm{reg}F(\bar{x},\bar{y}),$$

which completes the proof. □

Remark 10 *When X, Y are both complete metric spaces and Y is a locally coherent space (see [33]). An estimation for* reg $F(\bar{x},\bar{y})$ *has been established by Ioffe in [33].*

3.1 Coderivative conditions for implicit multifunctions

Let X be a Banach space. We use the symbol ∂ to denote any *abstract subdifferentials, that is any set-valued mapping which associates to every function defined on X and every $x \in X$ the set $\partial f(x) \subset X^\star$ (possibly empty), in such a way that*

(C1) If $f : X \to \mathbb{R} \cup \{+\infty\}$ is a l.s.c convex function, then ∂f coincides with the Fenchel-Moreau-Rockafellar subdifferential:

$$\partial f(x) := \{x^* \in X^* : \langle x^*, y - x\rangle \leq f(y) - f(x) \quad \forall y \in X\};$$

(C2) $\partial f(x) = \partial g(x)$ if $f(y) = g(y)$ for all y in a neighborhood of x.

(C3) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c function and $g : X \to \mathbb{R}$ be convex and Lipschitz. If $f + g$ attains a local minimum at x_0, then for any $\varepsilon > 0$, there exist $x_1, x_2 \in x_0 + \varepsilon B_X$, $x_1^* \in \partial f(x_1)$, $x_2^* \in \partial g(x_2)$, such that $|f(x_1) - f(x_0)| < \varepsilon$ and $\|x_1^* + x_2^*\| < \varepsilon$.

It is well known that the class of abstract subdifferentials includes Fréchet subdifferentials in Asplund spaces, viscosity subdifferentials in smooth Banach spaces as well as the Ioffe and the Clarke-Rockafellar subdifferentials in Banach spaces. For a closed subset C of X, the normal cone to C with respect to a subdifferential operator ∂ at $x \in C$ is defined by $N_\partial(C,x) = \partial\delta_C(x)$, where δ_C is the indicator function of C given by $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ otherwise and we assume here that $\partial\delta_C(x)$ is *a cone* for any closed subset C of X.

Let X, Y be Banach spaces, and let ∂ be a subdifferential on $X \times Y$. Let $F : X \rightrightarrows Y$ be a closed multifunction (graph-closed) and let $(\bar{x},\bar{y}) \in \mathrm{gph}F$. The multifunction $D^*F(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$ defined by

$$D^*F(\bar{x},\bar{y})(y^*) = \{x^* \in X^* : (x^*, -y^*) \in N_\partial(\mathrm{gph}F, (\bar{x},\bar{y}))\}$$

is called *the $\partial-$coderivative* of F at $(\bar{x},\bar{y})$.

Let $F : X \rightrightarrows Y$ be a closed multifunction and denote by $\varphi(x,y)$, $(x,y) \in X \times Y$ the lower semicontinuous envelope of the function $(x,y) \mapsto d(y, F(x))$. The following lemma gives an estimation for the strong slopes of the function $\varphi(\cdot,y)$ by using abstract subdifferential operators on $X \times Y$, which is of independent interests, see also a recent work by Chuong, Kruger and Yao [14].

Lemma 11 *Let ∂ be a subdifferential on $X \times Y$. Then for each $(x, y) \in X \times Y$ with $y \notin F(x)$, one has*

$$|\nabla\varphi(\cdot, y)|(x) \geq \liminf_{\eta \downarrow 0} \left\{ \|x^*\| : \quad \begin{array}{l} (u, v) \in gphF,\ x^* \in D^*F(u, v)(y^*),\ \|y^*\| = 1,\ u \in B(x, \eta), \\ d(y, F(u)) \leq \varphi(x, y) + \eta,\ \|v - y\| \leq d(y, F(u)) + \eta \\ |\langle y^*, y - v\rangle - d(y, F(u))| < \eta \end{array} \right\}. \tag{16}$$

Proof. Let $(x, y) \in X \times Y$ be such that $y \notin F(x)$ and set $|\nabla\varphi(\cdot, y)|(x) := m$. By the lower semicontinuity of φ as well as the definition of the strong slope, for each $\varepsilon \in (0, \varphi(x, y))$, there is $\eta \in (0, \varepsilon)$ with $2\eta + \varepsilon < \varphi(x, y)$ and $1 - (m + \varepsilon + 2)\eta > 0$ such that $d(y, F(u)) \geq \varphi(x, y) - \varepsilon$, $\forall u \in B(x, 4\eta)$ and

$$m + \varepsilon \geq \frac{\varphi(x, y) - \varphi(z, y)}{\|x - z\|} \quad \text{for all } z \in \bar{B}(x, \eta).$$

Equivalently,

$$\varphi(x, y) \leq \varphi(z, y) + (m + \varepsilon)\|z - x\| \quad \text{for all } z \in \bar{B}(x, \eta).$$

Take $u \in B(x, \eta^2/4)$, $v \in F(u)$ such that $\|y - v\| \leq \varphi(x, y) + \eta^2/4$. Then,

$$\|y - v\| \leq \varphi(z, y) + (m + \varepsilon)\|z - x\| + \eta^2/4 \ \forall z \in \bar{B}(x, \eta).$$

Therefore,

$$\|y - v\| \leq \|y - w\| + \delta_{\mathrm{gph}F}(z, w) + (m + \varepsilon)\|z - u\| + (m + \varepsilon + 1)\eta^2/4 \quad \forall (z, w) \in \bar{B}(x, \eta) \times Y.$$

By applying the Ekeland variational principle to the function

$$(z, w) \mapsto \|y - w\| + \delta_{\mathrm{gph}F}(z, w) + (m + \varepsilon)\|z - u\|$$

on $\bar{B}(x, \eta) \times Y$, we can select $(u_1, v_1) \in (u, v) + \frac{\eta}{4}B_{X \times Y}$ with $(u_1, v_1) \in \mathrm{gph}F$ such that

$$\|y - v_1\| \leq \|y - v\|(\leq \varphi(x, y) + \eta^2/4); \tag{17}$$

and that the function

$$(z, w) \mapsto \|y - w\| + \delta_{\mathrm{gph}F}(z, w) + (m + \varepsilon)\|z - u\| + (m + \varepsilon + 1)\eta\|(z, w) - (u_1, v_1)\|$$

attains a minimum on $\bar{B}(x, \eta) \times Y$ at (u_1, v_1). Hence, by (C3), we can find

$$v_2 \in B_Y(v_1, \eta);\ (u_3, v_3) \in B_{X \times Y}((u_1, v_1), \eta) \cap \mathrm{gph}F;\ v_2^* \in \partial\|y - \cdot\|(v_2);\ (u_3^*, -v_3^*) \in N(\mathrm{gph}F, (u_3, v_3))$$

satisfying

$$\|v_2^* - v_3^*\| < (m + \varepsilon + 2)\eta \quad \text{and} \quad \|u_3^*\| \leq m + \varepsilon + (m + \varepsilon + 2)\eta. \tag{18}$$

Since $v_2^* \in \partial\|y - \cdot\|(v_2)$ (note that $\|y - v_2\| \geq \|y - v\| - \|v_2 - v\| \geq \varphi(x, y) - \varepsilon - 2\eta > 0$), then $\|v_2^*\| = 1$ and $\langle v_2^*, y - v_2\rangle = \|y - v_2\|$. Thus, from the first relation in (18) it follows that

$$\|v_3^*\| \geq \|v_2^*\| - (m + \varepsilon + 2)\eta = 1 - (m + \varepsilon + 2)\eta, \text{ and } \|v_3^*\| \leq \|v_2^*\| + (m + \varepsilon + 2)\eta = 1 + (m + \varepsilon + 2)\eta.$$

By setting $y^* = v_3^*/\|v_3^*\|$; $x^* = u_3^*/\|v_3^*\|$, one derives that $x^* \in D^*F(u_3, v_3)(y^*)$ with $\|y^*\| = 1$ and (by the second relation of (18))

$$\|x^*\| = \|u_3^*\|/\|v_3^*\| \leq \frac{m + \varepsilon + (m + \varepsilon + 2)\eta}{1 - (m + \varepsilon + 2)\eta}. \tag{19}$$

On the other hand, relation (17)follows that

$$\varphi(x,y)-\varepsilon \le d(y,F(u_3)) \le \|y-v_3\| \le \|y-v_1\|+\eta \le \varphi(x,y)+\eta^2/4+\eta. \tag{20}$$

Consequently,

$$\langle y^*, y-v_3\rangle \le \|y-v_3\| \le d(y,F(u_3))+\eta^2/4+\eta+\varepsilon. \tag{21}$$

Furthermore,

$$\langle y^*, y-v_3\rangle = \frac{\langle v_2^*, y-v_2\rangle + \langle v_2^*, v_2-v_3\rangle + \langle v_3^*-v_2^*, y-v_3\rangle}{\|v_3^*\|} \ge \frac{\|y-v_2\|-2\eta-(m+\varepsilon+2)\eta\|y-v_3\|}{1+(m+\varepsilon+2)\eta};$$

it follows that

$$\langle y^*, y-v_3\rangle \ge \frac{d(y,F(u_3))(1-m+\varepsilon+2)\eta)-4\eta}{1+(m+\varepsilon+2)\eta}. \tag{22}$$

As $\varepsilon, \eta > 0$ are arbitrary small, by combining relations (19)-(22), we complete the proof. □

The following implicit multifunction theorem generalizes the ones established by Ledyaev and Zhu ([39]) in the context of Fréchet smooth spaces and by Ngai-Théra ([46]) in general smooth spaces. It is worth to note that this result is also sharper than the ones mentioned in [39], [46].

Theorem 12 *Let ∂ be a subdiffential operator on $X \times Y$ which satisfies the three conditions $(C1)-(C3)$. Suppose that the multifunction $F : X \times P \rightrightarrows Y$ verifies the conditions (a), (b), (c) in Theorem 5.*

If there exist a neighborhood U of $(\bar{x}, \bar{p})$ and reals $m, \gamma > 0$ such that for any $(x,p) \in U$ with $\bar{y} \notin F(x,p)$,

$$m \le \liminf_{\eta \downarrow 0} \left\{ \|x^*\| : \begin{array}{l} v \in F(u,p);\ x^* \in D^*F_p(u,v)(y^*),\ \|y^*\|=1,\ u \in B(x,\eta) \\ d(\bar{y}, F(u,p)) \le \gamma+\eta,\ \|v-\bar{y}\| \le d(\bar{y},F(u,p))+\eta \\ |\langle y^*, y-v\rangle - d(\bar{y},F(u,p))| < \eta \end{array} \right\}. \tag{23}$$

Then there exists a neighborhood $V \times W$ of $(\bar{x}, \bar{p})$ such that $V \cap S(\bar{y}, p) \neq \emptyset$ for any $p \in W$ and

$$d(x, S(\bar{y},p)) \le d(\bar{y}, F(x,p))/m \quad \forall (x,p) \in V \times W.$$

Proof. The proof follows immediately from Theorem 7 and Lemma 11. □

Theorem 12 yields the following corollary whose first part has been established by Azé, Corvellec and Lucchetti in [5] (see also [6], Corollary 5.7).

Corollary 13 *Let X, Y be Banach spaces and let ∂ be a subdiffential operator on $X \times Y$ which satisfies the three conditions $(C1)-(C3)$. Suppose that the multifunction $F : X \times P \rightrightarrows Y$ verifies the conditions (a), (b), (c) in Theorem 5 around $(\bar{x}, \bar{p}, \bar{y}) \in gphF$. Assume that*

$$\liminf_{(x,p,y) \to_F (\bar{x},\bar{p},\bar{y})} d_*(0, D^*F_p(x,y)(S_{Y^*})) > m > 0,$$

where S_{Y^*} denotes the unit sphere in Y^* and the notation $(x,p,y) \to_F (\bar{x},\bar{p},\bar{y})$ means that $(x,p,y) \to (\bar{x},\bar{p},\bar{y})$ with $(x,p,y) \in gphF$. Then there exist a neighborhood $V \times W \times U$ of $(\bar{x},\bar{p},\bar{y})$ such that

$$d(x, F_p^{-1}(y)) \le d(y, F(x,p))/m \quad \forall (x,p,y) \in U \times W \times V. \tag{24}$$

Conversely, assume that X, Y are Asplund spaces and ∂ is the Fréchet subdifferential. If (24) holds true for some neighborhood, then

$$\liminf_{(x,p,y)\to_F(\bar{x},\bar{p},\bar{y})} d_*(0, D^*F_p(x,y)(S_{Y^*})) \geq m.$$

Proof. The first part follows directly from Theorem 12. For the converse part, let $U \times W \times V$ be an open neighborhood of $(\bar{x}, \bar{p}, \bar{y})$ such that (24) holds. Let $(x, p, y) \in \mathrm{gph}F \cap U \times W \times V$; $y^* \in S_{Y^*}$ and $x^* \in D^*F_p(x, y)(y^*)$. For any $\varepsilon \in (0, 1)$, there exists $\delta > 0$ such that

$$\langle x^*, u - x\rangle - \langle y^*, v - y\rangle \leq \varepsilon(\|u - x\| + \|v - y\|) \; \forall (u, v) \in \mathrm{gph}F_p \cap B((x, y), \delta). \tag{25}$$

Shrinking δ if necessary, we can assume that $B((x, y), \delta) \subseteq U \times V$. Take $e \in S_Y$ such that $\langle y^*, e\rangle \geq 1-\varepsilon$ and $0 < \gamma < \min\{\delta, \delta m/(1 + \varepsilon)\}$. By (24), we can select $u \in F_p^{-1}(y - \gamma e)$ such that

$$\|x - u\| \leq (1 + \varepsilon)d(y - \gamma e, F_p(x))/m \leq (1 + \varepsilon)\gamma/m < \delta.$$

From (25), one obtains

$$(1 + \varepsilon)\gamma\|x^*\|/m \geq \langle x^*, x - u\rangle \geq (1 - \varepsilon)(\|u - x\| + \gamma) \geq \gamma((1 - \varepsilon) - \varepsilon((1 + \varepsilon)/m + 1)).$$

As $\varepsilon > 0$ is arbitrarily small, one has $\|x^*\| \geq m$. □

3.2 Tangencial conditions

Let X, Y be normed linear spaces. Recall that the graphical contingent derivative of a given set-valued mapping $F : X \rightrightarrows Y$ at $(x, y) \in \mathrm{gph}F$ is the set-valued mapping $DF(x, y) : X \rightrightarrows Y$ defined by

$$v \in DF(x, y)(u) \quad \Longleftrightarrow \quad (u, v) \in T_{\mathrm{gph}F}(x, y),$$

where $T_{\mathrm{gph}F}(x, y)$ is the tangent cone to $\mathrm{gph}F$ at (x, y), that is, $(u, v) \in T_{\mathrm{gph}F}(x, y)$ *iff* there exist sequences $t_n \downarrow 0$, $u_n \to u$ and $v_n \to v$ such that for all n, $y + t_n v_n \in F(x + t_n u_n)$.

We will make use of the quantity of the *inner* norm of a mapping $H : Y \rightrightarrows X$ (see, for example, [54], [24])

$$\|H\|^- = \sup_{y\in B_Y} \inf_{x\in H(y)} \|x\|.$$

The following lemma whose proof is very similar to the one of Lemma 11, gives an estimation for the strong slope of the lower semicontinuous envelope of $d(y, F(x))$ by using the graphical contingent derivative.

Lemma 14 *Let X, Y be Banach spaces. Let $F : X \rightrightarrows Y$ be a closed multifunction. Denote by $\varphi(x, y)$ the lower semicontinuous envelope of $d(y, F(x))$. Then for each $(x, \bar{y}) \in X \times Y$ with $\bar{y} \notin F(x)$, one has*

$$|\nabla\varphi(\cdot, \bar{y})|(x) \geq \frac{1}{\tau(x, \bar{y})}.$$

where

$$\tau(x, \bar{y}) := \limsup_{\eta\downarrow 0} \left\{ \|DF(z, y)^{-1}\|^- : \begin{array}{l} (z, y) \in gphF, \; z \in B(x, \eta), \\ d(\bar{y}, F(z)) \leq \varphi(x, \bar{y}) + \eta, \; \|\bar{y} - y\| \leq d(\bar{y}, F(z)) + \eta \end{array} \right\}. \tag{26}$$

Proof. If $\tau(x,\bar{y}) = +\infty$, we are done. Let $\tau(x,\bar{y}) < c < +\infty$. Then, by the definition of $\tau(x,\bar{y})$, we can find $\eta \in (0,1)$ such that $\|DF(z,y)^{-1}\|^{-} < c$ for all $(z,y) \in \mathrm{gph}F$ satisfying $z \in B(x,\eta)$; $d(\bar{y},F(z)) \leq \varphi(x,\bar{y}) + \eta$ and $\|\bar{y}-y\| \leq d(\bar{y},F(z)) + \eta$. Set $|\nabla\varphi(\cdot,\bar{y})|(x) := m$. Then there exists $\delta \in (0,\eta/4)$ such that

$$d(\bar{y},F(\zeta)) > \varphi(x,\bar{y}) - \eta/4 \quad \text{and} \quad m+\eta \geq \frac{\varphi(x,\bar{y})-\varphi(\zeta,\bar{y})}{\|x-\zeta\|} \quad \text{for all } \zeta \in \bar{B}(x,\delta). \tag{27}$$

Equivalently,

$$\varphi(x,\bar{y}) \leq \varphi(\zeta,\bar{y}) + (m+\eta)\|\zeta - x\| \quad \text{for all } \zeta \in \bar{B}(x,\delta).$$

Take $z \in B(x,\delta^2/4)$, $y \in F(z)$ such that $\|\bar{y}-y\| \leq \varphi(x,\bar{y}) + \delta^2/4$. Then,

$$\|\bar{y}-y\| \leq \varphi(\zeta,\bar{y}) + (m+\eta)\|\zeta - x\| + \delta^2/4 \; \forall \zeta \in \bar{B}(x,\delta).$$

Therefore,

$$\|\bar{y}-y\| \leq \|\bar{y}-\xi\| + \delta_{\mathrm{gph}F}(\zeta,\xi) + (m+\eta)\|\zeta - z\| + (m+\eta+1)\delta^2/4 \quad \forall (\zeta,\xi) \in \bar{B}(x,\delta)\times Y.$$

By applying the Ekeland variational principle to the function

$$(\zeta,\xi) \mapsto \|\bar{y}-\xi\| + \delta_{\mathrm{gph}F}(\zeta,\xi) + (m+\eta)\|\zeta - z\|$$

on $\bar{B}(x,\delta)\times Y$, we can select $(z_1,y_1) \in (z,y) + \frac{\delta}{4}B_{X\times Y}$ with $(z_1,y_1) \in \mathrm{gph}F$ such that

$$\|\bar{y}-y_1\| \leq \|\bar{y}-y\| (\leq \varphi(x,\bar{y}) + \delta^2/4); \tag{28}$$

and that

$$\|\bar{y}-y_1\| + (m+\eta)\|z-z_1\| \leq \|\bar{y}-\xi\| + (m+\eta)\|\zeta - z\| + (m+\eta+1)\delta\|(\zeta,\xi)-(z_1,y_1)\| \tag{29}$$

for all $(\zeta,\xi) \in \mathrm{gph}F \cap \bar{B}(x,\delta)\times Y$. Since $z \in B(x,\delta^2/4)$, and $z_1 \in B(z,\delta/4)$, then $z_1 \in B(x,\eta)$. Moreover, from relations (27), (28), one has $\|\bar{y}-y_1\| < d(\bar{y},F(z_1)) + \eta$ as well as $d(\bar{y},F(z_1)) < \varphi(x,\bar{y}) + \eta$. Hence $\|DF(z_1,y_1)^{-1}\|^{-} < c$. Consequently, for $v := y - y_1$, there exist sequences $t_n \downarrow 0$, $u_n \to u$, $v_n \to v$ such that

$$y_1 + t_n v_n \in F(z_1 + t_n u_n) \text{ for all } n \quad \text{and} \quad \|u\| < c\|v\|.$$

By taking $\zeta := z_1 + t_n u_n$; $\xi := y_1 + t_n v_n$ into account of (29), one obtains

$$\|v\| - \|v - t_n v_n\| \leq (m+\eta)(\|z - z_1 - t_n u_n\| - \|z - z_1\|) + (m+\eta+1)\delta t_n \|(u_n,v_n)\| \; \forall n.$$

Hence,

$$\|v\| - \|v - v_n\| \leq (m+\eta)\|u_n\| + (m+\eta+1)\delta\|(u_n,v_n)\|.$$

Passing to limit with $n \to \infty$, we derive that

$$m+\eta \geq (\|v\| - (m+\eta+1)\delta\|(u,v)\|)\|u\| \geq (\|v\| - (m+\eta+1)\delta\|(u,v)\|)(c\|v\|)^{-1}.$$

As η, δ are arbitrary small and c is arbitrary close to $\tau(x,\bar{y})$, we obtain $m \geq 1/\tau(x,\bar{y})$ and finish the proof of the lemma. □

The following theorem provides an implicit multifunction theorem by making use of the graphical contingent derivative.

Theorem 15 *Let X, Y be Banach spaces and let P be a topological space. Suppose that the multifunction $F : X \times P \rightrightarrows Y$ verifies the conditions (a), (b), (c) in Theorem 5 around $(\bar{x}, \bar{p}, \bar{y}) \in X \times P \times Y$.*

If there exist a neighborhood U of $(\bar{x}, \bar{p})$ and reals $\tau, \gamma > 0$ such that for any $(x, p) \in U$ with $\bar{y} \notin F(x, p)$,

$$\lim_{\eta \downarrow 0} \sup \left\{ \|DF_p(z, y)^{-1}\|^- : \begin{array}{l} y \in F(z, p);\ z \in B(x, \eta) \\ d(\bar{y}, F(z, p)) \leq \gamma + \eta,\ \|y - \bar{y}\| \leq d(\bar{y}, F(z, p)) + \eta \end{array} \right\} < \tau, \tag{30}$$

then there exist a neighborhood $V \times W$ of $(\bar{x}, \bar{p})$ such that $V \cap S(\bar{y}, p) \neq \emptyset$ for any $p \in W$ and

$$d(x, S(\bar{y}, p)) \leq \tau d(\bar{y}, F(x, p)) \quad \forall (x, p) \in V \times W.$$

Proof. The proof follows immediately from the preceding lemma and Theorem 7. □

The preceding theorem yields directly the following result of Dontchev, Quincampoix and Zlateva [24].

Theorem 16 *([24], Theorem 2.1) Let X, Y be Banach spaces and let P be a topological space. Suppose that the multifunction $F : X \times P \rightrightarrows Y$ verifies conditions (a), (b), (c) in Theorem 5 around $(\bar{x}, \bar{p}, \bar{y}) \in X \times P \times Y$. Assume that*

$$\liminf_{(x,p,y) \to_F (\bar{x}, \bar{p}, \bar{y})} \|DF_p(x, y)^{-1}\|^- < \tau < +\infty.$$

Then there exist a neighborhood $V \times W \times U$ of $(\bar{x}, \bar{p}, \bar{y})$ such that

$$d(x, F_p^{-1}(y)) \leq d(y, F(x, p))/m \quad \forall (x, p, y) \in U \times W \times V.$$

4 Metric regularity of implicit multifunctions under perturbation

In this section, we apply the results established in Section 3 to study the perturbation stability of metric regularity of implicit multifunctions. As in the preceding section, let $F : X \times P \rightrightarrows Y$ be a given multifunctions. Let us consider the implicit multifunction S_F associated with F defined by $S_F : Y \times P \rightrightarrows X$,

$$S_F(y, p) := \{x \in X : \ y \in F(x, p)\}, \ (y, p) \in Y \times P.$$

Let $(\bar{x}, \bar{p}, \bar{y}) \in \text{gph } F$. The implicit multifunction S_F is said to be metrically regular at $\bar{x}$ with respect to $(\bar{y}, \bar{p})$ with modulus $\tau \in (0, +\infty)$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ and W of $\bar{p}$ such that

$$d(x, S_F(y, p)) \leq \tau d(y, F(x, p)) \quad \text{for all } (x, p, y) \in U \times W \times V. \tag{31}$$

Similarly to Section 2, let $\varphi_p(\cdot, \cdot)$ denote the lower semicontionuous envelope functions of the mapping $d(\cdot, F(\cdot, p))$.

Let now X be a complete metric space and let Y be a normed linear space. Let $F, \Phi : X \times P \rightrightarrows Y$ be set-valued mappings and $(\bar{x}, \bar{p}, \bar{y}) \in \text{gph } F \cap \text{gph } \Phi$ be given. We will make use of the following quantity which is regarded as a measure of "closenes" between the two mappings $F_p := F(\cdot, p)$ and $\Phi_p := \Phi(\cdot, p)$ for some $p \in P$, (see [31], [48])

$$\Sigma_{F_p, \Phi_p}(x, r) := \sup_{\eta \in \Phi_p(x)} \inf_{v \in F_p(x)} \sup_{d(u,x) < r, w \in F_p(u)} \inf_{\xi \in \Phi_p(u)} \|\eta - v + w - \xi\|, \tag{32}$$

with $x \in X$.

Theorem 17 *Let X be a complete metric space and Y be a normed linear space. Let $F, \Phi : X \rightrightarrows Y$ be set-valued mappings which satisfy conditions (a), (b), (c) of Theorem 5 at $(\bar{x}, \bar{p}, \bar{y}) \in \operatorname{gph} F$ and $(\bar{x}, \bar{p}, \bar{z}) \in \operatorname{gph} \Phi$, respectively. Suppose that S_F is metrically regular with modulus $\tau > 0$ around $\bar{x}$ with respect to $(\bar{p}, \bar{y})$ and that the following two conditions are satisfied.*

(i) There exist positive reals s, λ, δ with $\lambda \in (0, \tau^{-1})$ and a neighborhood W of $\bar{p}$ such that

$$\Sigma_{F_p, \Phi_p}(x, r) \leq \lambda r \;\; \textit{for all}\; (x, p) \in B(\bar{x}, \delta) \times W, \;\; r \in (0, s); \tag{33}$$

(ii) $\limsup_{(x,p)\to(\bar{x},\bar{p})} e(F(x,p) - \bar{y}, \Phi(x,p) - \bar{z}) = 0$, *where,*

$$e(F(x,p) - \bar{y}, \Phi(x,p) - \bar{z}) = \sup_{u \in F(x,p) - \bar{y}} d(u, \Phi(x,p) - \bar{z}).$$

Then S_Φ is metrically regular around $\bar{x}$ with respect to $(\bar{p}, \bar{z})$ with modulus $(\tau^{-1} - \lambda)^{-1}$.

Proof. By translation, considering $\Phi + \bar{y} - \bar{z}$ instead of Φ, we can assume that $\bar{z} = \bar{y}$. Let $\alpha, \beta > 0$ and Ω such that

$$d(x, S_F(y, p)) \leq \tau d(y, F(x, p)) \quad \text{for all } (x, y, p) \in B(\bar{x}, \alpha) \times B(\bar{y}, \beta) \times \Omega. \tag{34}$$

By (ii), we can find $\delta_1 \in (0, \delta/2)$ and a neighborhood, say U of $\bar{p}$ such that

$$e(F(x,p), \Phi(x,p)) < \beta/4 \quad \text{for all } (x, p) \in B(\bar{x}, \delta_1) \times U. \tag{35}$$

Set

$$\gamma = \min\{\delta_1/2, \beta/4, s\tau^{-1}, \beta\alpha^{-1}/4\}; \;\; a = \min\{\alpha, \delta_1/2\}; \;\; b = \beta/4.$$

It suffices to show that the statement (iv) of Theorem 5 is satisfied for the mapping Φ around $(\bar{x}, \bar{p}, \bar{y})$. Indeed, Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $d(x_n, x) \to 0$ and $d(y, \Phi(x_n, p)) \to \liminf_{u\to x} d(y, \Phi(u, p))$. Without loss of generality, we can assume that $x_n \in B(\bar{x}, a)$ and $d(y, \Phi(x_n, p)) < \gamma$ for all $n \in \mathbb{N}$. Pick a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals converging to zero and satisfying $(1 + \varepsilon_n) d(y, \Phi(x_n, p)) < \gamma$ for all $n \in \mathbb{N}$. For each integer n take $\eta_n \in \Phi(x_n, p)$ such that

$$\|y - \eta_n\| < (1 + \varepsilon_n) d(y, \Phi(x_n, p)). \tag{36}$$

If $r_n := (1+\varepsilon_n)\tau d(y, \Phi(x_n, p))$, then, $r_n \in (0, s)$. Therefore by (33), for each n there exists $v_n \in F(x_n, p)$ such that

$$\sup_{d(u,x)<r_n, w\in F(u,p)} \; \inf_{\xi\in\Phi(u,p)} \|\eta_n - v_n + w - \xi\| < (1 + \varepsilon_n)\lambda r_n. \tag{37}$$

Then, by relations (33) and (35), one has

$$\|\eta_n - v_n\| < \sup_{d(u,x)<r_n, w\in F(u,p)} \; \inf_{\xi\in\Phi(u,p)} \|w - \xi\| + (1 + \varepsilon_n)\lambda r_n < \beta/4 + \beta/4 = \beta/2.$$

Consequently, $z_n := y - \eta_n + v_n \in B(\bar{y}, \beta)$. According to relation (34), we can select $u_n \in S_F(z_n, p)$ such that

$$d(x_n, u_n) \leq (1 + \varepsilon_n)\tau d(z_n, F(x_n, p)) \leq (1 + \varepsilon_n)\tau \|z_n - v_n\| < (1 + \varepsilon_n)\tau d(y, \Phi(x_n, p)) := r_n < \tau\gamma \leq s.$$

Therefore, by (37),

$$\inf_{\xi\in\Phi(u_n,p)} \|\eta_n - v_n + z_n - \xi\| < (1 + \varepsilon_n)\lambda r_n,$$

i.e., $d(y, \Phi(u_n, p)) < (1+\varepsilon_n)\lambda r_n$. It implies that

$$\limsup_{n\to\infty} d(y, \Phi(u_n, p)) \le \lambda\tau \liminf_{u\to x} d(y, \Phi(u, p)) < \liminf_{u\to x} d(y, \Phi(u, p)),$$

and consequently, $\liminf_{n\to\infty} d(u_n, x_n) > 0$. Hence, we obtain $\lim_{n\to\infty} d(x, u_n) > 0$ and that

$$\limsup_{n\to\infty} \frac{d(y, \Phi(x_n, p)) - d(y, \Phi(u_n, p))}{d(x_n, u_n)} > \tau^{-1} - \lambda.$$

By virtue of Theorem 5, we derive that there exists a neighborhood $U \times W' \times V$ such that

$$d(x, S_\Phi(y, p)) \le (\tau^{-1} - \lambda)^{-1} d(y, \Phi(x, p)) \quad \text{for all } (x, p, y) \in U \times W' \times V;$$

which completes the proof. □

Noticing that if $G : X \times P \rightrightarrows Y$ is uniformly locally Lipschitz arround $\bar{x}$ for p near $\bar{p}$ then (ii) holds trivially for $\Phi := F + G$. We obtain the following corollary.

Corollary 18 *Let X be a complete metric space and let Y be a normed linear space. Let $F, G : X \times P \rightrightarrows Y$ be set-valued mappings and let $(\bar{x}, \bar{p}, \bar{y}) \in \operatorname{gph} F$ and $(\bar{x}, \bar{p}, \bar{z}) \in \operatorname{gph} G$. Suppose that F and $\Phi := F + G$ satisfy the conditions (a), (b), (c) of Theorem 5 at $(\bar{x}, \bar{p}, \bar{y})$ and $(\bar{x}, \bar{p}, \bar{z})$, respectively. Let If S_F is metrically regular at $\bar{x}$ with respect to $(\bar{y}, \bar{p})$ with modulus $\tau > 0$ and $G(\cdot, p)$ is uniformly locally Lipschitz arround $\bar{x}$ for p near $\bar{p}$ with constant $\lambda \in (0, \tau^{-1})$, then S_Φ is metrically regular at $\bar{x}$ with respect to $(\bar{y} + \bar{z}, \bar{p})$ with modulus $(\tau^{-1} - \lambda)^{-1}$.*

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