On mixed-integer sets with two integer variables

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Abstract

We show that every facet-defining inequality of the convex hull of a mixed-integer polyhedral set with two integer variables is a crooked cross cut (which we defined recently in [3]). We then extend this observation to show that crooked cross cuts give the convex hull of mixed-integer sets with more integer variables provided that the coefficients of the integer variables form a matrix of rank 2. We also present an alternative characterization of the crooked cross cut closure of mixed-integer sets similar to the one about the equivalence of different definitions of split cuts presented in Cook, Kannan, and Schrijver [4]. This characterization implies that crooked cross cuts dominate the 2-branch split cuts defined by Li and Richard [6]. Finally, we extend our results to mixed-integer sets that are defined as the set of points (with some components being integral) inside a closed, bounded, convex set.

1 Introduction

Given a polyhedral mixed-integer set

$$P = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \ : \ Ax + Gy = b, \ y \geq 0\},$$

where A, G and b have m rows and rational components, let P^{LP} denote its continuous relaxation. For fixed $\pi_1, \pi_2 \in \mathbb{Z}^{n_1}$, and $\gamma_1, \gamma_2 \in \mathbb{Z}$, define the sets

$$D_1(\pi_1, \pi_2, \gamma_1, \gamma_2) = \{x \in \mathbb{R}^{n_1} \ : \ \pi_1 x \leq \gamma_1, \ (\pi_2 - \pi_1)x \leq \gamma_2 - \gamma_1\}, \tag{1}$$
$$D_2(\pi_1, \pi_2, \gamma_1, \gamma_2) = \{x \in \mathbb{R}^{n_1} \ : \ \pi_1 x \leq \gamma_1, \ (\pi_2 - \pi_1)x \geq \gamma_2 - \gamma_1 + 1\}, \tag{2}$$
$$D_3(\pi_1, \pi_2, \gamma_1, \gamma_2) = \{x \in \mathbb{R}^{n_1} \ : \ \pi_1 x \geq \gamma_1 + 1, \ \pi_2 x \leq \gamma_2\}, \text{ and} \tag{3}$$
$$D_4(\pi_1, \pi_2, \gamma_1, \gamma_2) = \{x \in \mathbb{R}^{n_1} \ : \ \pi_1 x \geq \gamma_1 + 1, \ \pi_2 x \geq \gamma_2 + 1\}. \tag{4}$$

Note that $\mathbb{Z}^{n_1} \subseteq \bigcup_{k \in \{1,2,3,4\}} D_k(\pi_1, \pi_2, \gamma_1, \gamma_2)$, and denote the latter set by $D(\pi_1, \pi_2, \gamma_1, \gamma_2)$. The *extension* of $D(\pi_1, \pi_2, \gamma_1, \gamma_2)$ is defined to be $\bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2) = \{(x, y) \in \mathbb{R}^{n_1+n_2} \ : x \in D(\pi_1, \pi_2, \gamma_1, \gamma_2)\}$ and it is called a *crooked cross (CC) disjunction* for $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$, see [3]. The extensions of the sets in (1)-(4), are defined similarly and each such extension is called an *atom* of the disjunction $\bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2)$. Furthermore, the set $C = \mathbb{R}^{n_1+n_2} \setminus$ *int*$(\bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2))$ is called the *CC set* associated with the disjunction, where *int*$(\cdot)$ of a given set means the interior of that set. Notice that *int*$(C) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$.

If a linear inequality is valid for $P^{LP} \cap \bar{D}_k(\pi_1, \pi_2, \gamma_1, \gamma_2)$ for $k = 1, \ldots, 4$, then it is called a *CC cut* for P obtained from the disjunction $\bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2)$, see [3]. In other words a linear inequality is called a

CC cut if it is valid for $P^{LP} \setminus int(C)$. Note that multiple cuts can be derived from the same disjunction. As $P \subseteq \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \subseteq \bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2)$, CC cuts are valid for all points in P.

Define $P_{CC}(\pi_1, \pi_2, \gamma_1, \gamma_2)$ as the convex hull of $P^{LP} \cap \bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2)$, i.e.,

$$P_{CC}(\pi_1, \pi_2, \gamma_1, \gamma_2) = \textit{conv}\left(\bigcup_{i \in \{1,2,3,4\}} P^{LP} \cap \bar{D}_i(\pi_1, \pi_2, \gamma_1, \gamma_2) \right).$$

By definition, this set equals the convex hull of points in P^{LP} not contained in the interior of the CC set associated with the disjunction $\bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2)$, and is the set of points in P^{LP} satisfying all CC cuts from this disjunction. The *CC closure* of P, denoted by P_{CC}, is the set of points in P^{LP} that satisfy all CC cuts obtained from all possible disjunctions for P. Clearly,

$$P_{CC} = \bigcap_{\pi_1, \pi_2 \in \mathbb{Z}^{n_1}} \bigcap_{\gamma_1, \gamma_2 \in \mathbb{Z}} P_{CC}(\pi_1, \pi_2, \gamma_1, \gamma_2).$$

Next we summarize the results presented in this paper. When the matrix A in the definition of P has two columns (i.e., $n_1 = 2$ or P is defined by only two integer variables), we show that every facet-defining inequality of *conv* (P) is a crooked cross cut in Section 2. This is a generalization of a similar result we proved in [3] when $n_1 = 2$ and $m = 2$, i.e., P is in addition defined only by two constraints. More generally, we show in Section 4 that crooked cross cuts give the convex hull of P when A has rank 2. Cook, Kannan, and Schrijver [4] define the split closure of P in two different ways. In a similar vein, we give an alternative characterization of the crooked cross cut closure of mixed-integer sets in Section 3. In [3], we showed that CC cuts dominate the "2-branch split cuts", defined in Li and Richard [6], when the matrix A has full row-rank. A consequence of the alternative characterization in Section 3 is that this dominance relationship holds for arbitrary A. We extend some of the above results to the case when P is a mixed-integer set with a convex relaxation $\mathcal{C}$, where $\mathcal{C}$ is a closed, bounded and convex set in Section 5. Finally, we discuss relationships with other results in the literature in Section 6.

2 Mixed-integer polyhedral sets with two integer variables

As noted in [3], CC cuts generalize *split cuts* [4], which are defined by Cook, Kannan, and Schrijver [4] as inequalities valid for $P^{LP} \cap \{(x, y) : \pi x \leq \gamma\}$ and $P^{LP} \cap \{(x, y) : \pi x \geq \gamma + 1\}$ for some $\pi \in \mathbb{Z}^{n_1}$ and $\gamma \in \mathbb{Z}$. Thus the CC closure of P is contained in its split closure. Moreover, it is known that when $n_1 = 1$, the split closure of P equals *conv* (P). We next prove a similar result for the facet-defining inequalities of P when $n_1 = 2$.

Lemma 2.1. *If $n_1 = 2$, then any valid inequality for conv (P) is a CC cut. Consequently $P_{CC} =$ conv (P).*

Proof. Let $cx + dy \geq f$ be a valid inequality for *conv* (P) and let $S \subseteq \mathbb{R}^{2+n_2}$ be the points in P^{LP} that violate this inequality. That is,

$$S = \{(x, y) \in \mathbb{R}^{2+n_2} \ : \ cx + dy < f, \ Ax + Gy = b, \ y \geq 0\}.$$

If S is empty, then the inequality $cx + dy \geq f$ is valid for P^{LP} and therefore it is a CC cut. We therefore assume that $S \neq \emptyset$. We will next show that there exists a CC set C that contains S in its interior. This would imply that the inequality $cx + dy \geq f$ is valid for all points in $P^{LP} \setminus int(C)$ and therefore it is a CC cut.

As $cx + dy \geq f$ is valid for P, S does not contain any integral points, that is, $S \cap \mathbb{Z}^2 \times \mathbb{R}^{n_2} = \emptyset$. Let $S_x = \textit{proj}_x(S)$ denote the projection of S in the space of x variables and note that $S_x \cap \mathbb{Z}^2 = \emptyset$, and therefore S_x is a convex lattice-free set in $\mathbb{R}^2$, though not necessarily closed.

We will next construct a lattice-free convex set that contains S_x in its interior. First note that $S_x = \{x \in \mathbb{R}^2 : A_1 x \leq b_1, A_2 x < b_2\}$ for some rational matrices A_1, A_2, b_1, b_2 of appropriate dimension (by Lemma A.1 in the Appendix). By scaling, we can further assume that these matrices all have integral entries. Let $px \leq q$ be one of the inequalities in $A_1 x \leq b_1$, and note that every integer point satisfying $px < q + 1$ also satisfies $px \leq q$, and vice-versa. Therefore, $T \cap \mathbb{Z}^2 = S_x \cap \mathbb{Z}^2 = \emptyset$ where

$$T = \{x \in \mathbb{R}^2 : A_1 x < b_1 + \mathbf{1}, A_2 x < b_2\},$$

and $\mathbf{1}$ is a column vector with all components 1 and the same number of rows as b_1. Note that T is full-dimensional as it is not empty (it contains $S_x \neq \emptyset$) and it is defined by strict inequalities only. Let $T^c = closure(T)$ and note that T^c is a lattice free convex set. Now, we have

$$S_x \subseteq T \subseteq T^c = \{x \in \mathbb{R}^2 : A_1 x \leq b_1 + \mathbf{1}, A_2 x \leq b_2\},$$

and note that $T = \textit{int}(T^c)$. As any convex lattice-free set in $\mathbb{R}^2$ is contained in a CC set (see [3]), we have $T^c \subseteq C$ for some CC set $C = \mathbb{R}^2 \setminus \textit{int}(D(\pi_1, \pi_2, \gamma_1, \gamma_2))$. Consequently, $T = \textit{int}(T^c) \subseteq \textit{int}(C)$ and therefore, S_x is contained in the interior of the CC set C.

Consider the CC set $\bar{C}$ obtained by extending C. As $S_x \subseteq \textit{int}(C)$, and $\textit{int}(C) \times \mathbb{R}^{n_2} = \textit{int}(\bar{C})$ we have $S \subseteq \textit{int}(\bar{C})$ and therefore $cx + dy \geq f$ is valid for $P^{LP} \setminus \textit{int}(\bar{C})$. In other words, $cx + dy \geq f$ is a CC cut for P. ∎

In [3], it is shown that the convex hull of P is given by CC cuts when $n_1 = 2$ and $m = 2$. Lemma 2.1 generalizes this result to arbitrary m. It is also known [2] that cuts derived from two row relaxations using lattice-free convex sets in $\mathbb{R}^2$ (called 2D lattice-free cuts in [3]) give the convex hull of P when $n_1 = 2$ and $m = 2$. However, it is an open question if 2D lattice-free cuts give the convex hull of P when $n_1 = 2$ and $m > 2$. By results in [3], this is equivalent to deciding if CC cuts strictly dominate 2D lattice-free cuts or not.

3 An alternative characterization of the CC closure

Cook, Kannan, and Schrijver give an alternative definition of split cuts: they observe that the class of split cuts for P is equivalent to the class of inequalities valid for $P^{LP} \cap \{(x, y) : \pi x \in \mathbb{Z}\}$ for all integral vectors π. We next present an alternative characterization of the CC closure of P similar to the result on split cuts above.

Let

$$P_{\Pi} = \bigcap_{\pi_1, \pi_2 \in \mathbb{Z}^{n_1}} \textit{conv}\left(\{(x, y) \in P^{LP} \ : \ \pi_1 x \in \mathbb{Z}, \pi_2 x \in \mathbb{Z}\}\right).$$

Theorem 3.1. *For a polyhedral mixed-integer set P, $P_{CC} = P_{\Pi}$.*

Proof. For fixed $\pi_1, \pi_2 \in \mathbb{Z}^{n_1}$, let

$$P_{\pi_1, \pi_2} = \textit{conv}\left(\{(x, y) \in P^{LP} \ : \ \pi_1 x \in \mathbb{Z}, \pi_2 x \in \mathbb{Z}\}\right) \tag{5}$$

and consider a point $p \in P_{\pi_1, \pi_2}$. Clearly, p is a convex combination of points $p^k = (x^k, y^k),\ k \in K$ (for some set K), such that $p^k \in P^{LP}$ and $\pi_1 x^k, \pi_2 x^k \in \mathbb{Z}$ for all k. Then for any choice of $\gamma_1, \gamma_2 \in \mathbb{Z}$, it is clear that p^k does not belong to the interior of the CC set associated with the CC disjunction $\bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2)$, as it belongs to one of the atoms. In other words

$$p^k \in \bigcap_{\gamma_1, \gamma_2 \in \mathbb{Z}} P_{CC}(\pi_1, \pi_2, \gamma_1, \gamma_2)$$

for all $k \in K$ and therefore, $p \in \bigcap_{\gamma_1,\gamma_2 \in \mathbb{Z}} P_{CC}(\pi_1, \pi_2, \gamma_1, \gamma_2)$. Consequently,

$$P_{\pi_1,\pi_2} \subseteq \bigcap_{\gamma_1, \gamma_2 \in \mathbb{Z}} P_{CC}(\pi_1, \pi_2, \gamma_1, \gamma_2) \tag{6}$$

for any fixed $\pi_1, \pi_2 \in \mathbb{Z}^{n_1}$. Therefore

$$P_{\Pi} \;=\; \bigcap_{\pi_1,\pi_2 \in \mathbb{Z}^{n_1}} P_{\pi_1,\pi_2} \subseteq \bigcap_{\pi_1,\pi_2 \in \mathbb{Z}^{n_1}} \bigcap_{\gamma_1,\gamma_2 \in \mathbb{Z}} P_{CC}(\pi_1, \pi_2, \gamma_1, \gamma_2) \;=\; P_{CC}.$$

We will now prove the reverse inclusion, $P_{CC} \subseteq P_{\Pi}$, by showing that $P_{CC} \subseteq P_{\pi_1,\pi_2}$, for every choice of $\pi_1, \pi_2 \in \mathbb{Z}^{n_1}$. To prove this, note that P_{π_1,π_2} is the projection of the set *conv* (S) on the x, y variables where S is defined as

$$S = \{(x, y, z) \;:\; (x, y) \in P^{LP}, z \in \mathbb{Z}^2, z_1 = \pi_1 x, z_2 = \pi_2 x\}.$$

As S is a mixed-integer polyhedral set with only two integer variables, by Lemma 2.1, *conv* (S) equals S_{CC}. We will next show that $P_{CC} \subseteq \textit{proj}_{x,y}(S_{CC}) = \textit{proj}_{x,y}(\textit{conv}\,(S))$.

Consider a CC cut for S, say

$$\alpha_1 z_1 + \alpha_2 z_2 + cx + dy \geq f, \tag{7}$$

derived from a CC disjunction $\bar{D}(\mu_1, \mu_2, \gamma_1, \gamma_2)$ on the z variables. Substituting out the z variables, we obtain the inequality

$$(\alpha_1 \pi_1 + \alpha_2 \pi_2 + c)x + dy \geq f \tag{8}$$

which is valid for $\textit{proj}_{x,y}(S_{CC})$. This inequality is a CC cut for P obtained from the CC disjunction $\bar{D}(\pi'_1, \pi'_2, \gamma_1, \gamma_2)$ in the x, y space, where

$$\pi'_1 = \mu_1 \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \pi'_2 = \mu_2 \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}.$$

To see this, consider an atom of the disjunction $\bar{D}(\mu_1, \mu_2, \gamma_1, \gamma_2)$, say

$$\bar{D}_4(\mu_1, \mu_2, \gamma_1, \gamma_2) = \{(x, y, z) : \mu_1 z \geq \gamma_1 + 1, \mu_2 z \geq \gamma_2 + 1\}.$$

By definition, inequality (7) is valid for $S^{LP} \cap \bar{D}_4(\mu_1, \mu_2, \gamma_1, \gamma_2)$. Suppose inequality (8) is not valid for $P^{LP} \cap \bar{D}_4(\pi'_1, \pi'_2, \gamma_1, \gamma_2)$. By definition, there is a point $(\hat{x}, \hat{y})$ such that

$$(\hat{x}, \hat{y}) \in P^{LP},\; \pi'_1 \hat{x} \geq \gamma_1 + 1,\; \pi'_2 \hat{x} \geq \gamma_2 + 1 \text{ and } (\alpha_1 \pi_1 + \alpha_2 \pi_2 + c)\hat{x} + d\hat{y} < f.$$

Consider the point $(\hat{x}, \hat{y}, \hat{z})$ defined by setting $\hat{z}_1 = \pi_1 \hat{x}$ and $\hat{z}_2 = \pi_2 \hat{x}$. Clearly, this point satisfies

$$(\hat{x}, \hat{y}, \hat{z}) \in S^{LP},\; \mu_1 \hat{z} \geq \gamma_1 + 1,\; \mu_2 \hat{z} \geq \gamma_2 + 1 \text{ and } \alpha_1 \hat{z}_1 + \alpha_2 \hat{z}_2 + c\hat{x} + d\hat{y} < f,$$

which is a contradiction. Thus inequality (8) is a CC cut for P.

Finally, if $P_{CC} \not\subseteq \mathit{proj}_{x,y}(\mathit{conv}\,(S))$, then there exists $(\hat{x}, \hat{y}) \in P_{CC}$ such that $(\hat{x}, \hat{y}) \notin \mathit{proj}_{x,y}(\mathit{conv}\,(S))$. Consider the point $(\hat{x}, \hat{y}, \hat{z})$, where $\hat{z}_1 = \pi_1\hat{x}$ and $\hat{z}_2 = \pi_2\hat{x}$; clearly this point does not belong to *conv* (S). Since *conv* (S) equals S_{CC}, there exists a CC cut for S with the form (7) violated by $(\hat{x}, \hat{y}, \hat{z})$. Then the inequality (8), which is a CC cut for P, is violated by $(\hat{x}, \hat{y})$, a contradiction. ∎

Let t be a fixed integer, and consider a disjunctive cut obtained by modifying the sets in (1) and (2) as follows:

$$D_1^t(\pi_1, \pi_2, \gamma_1, \gamma_2) = \{x \in \mathbb{R}^{n_1} \;:\; \pi_1 x \leq \gamma_1,\; (\pi_2 - t\pi_1)x \leq \gamma_2 - t\gamma_1\}, \tag{9}$$

$$D_2^t(\pi_1, \pi_2, \gamma_1, \gamma_2) = \{x \in \mathbb{R}^{n_1} \;:\; \pi_1 x \leq \gamma_1,\; (\pi_2 - t\pi_1)x \geq \gamma_2 - t\gamma_1 + 1\}. \tag{10}$$

Such cuts are referred to as parametric cross cuts in [3]. Note that when $t = 1$, parametric cross cuts are just CC cuts, and when $t = 0$, they reduce to the 2-branch split cuts of Li and Richard [6] (also referred to as *cross cuts* in [3]). Let $P^t(\pi_1, \pi_2, \gamma_1, \gamma_2)$ be the set of points in P^{LP} that satisfy all parametric cross cuts derived from the above disjunction. Observe that

$$P_{\pi_1,\pi_2} \subseteq \bigcap_{\gamma_1,\gamma_2 \in \mathbb{Z}} P^t(\pi_1, \pi_2, \gamma_1, \gamma_2),$$

where P_{π_1,π_2} is defined in (5). Therefore, arguing as in the proof of Theorem 3.1, one obtains that

$$P_{\Pi} \subseteq \bigcap_{\pi_1,\pi_2 \in \mathbb{Z}^{n_1}} \bigcap_{\gamma_1,\gamma_2 \in \mathbb{Z}} P^t(\pi_1, \pi_2, \gamma_1, \gamma_2).$$

Therefore, we have the following corollary of Theorem 3.1.

Corollary 3.2. *P_{CC} equals the set of points in P^{LP} that satisfy all (i.e., for any t) parametric cross cuts for P. In particular, P_{CC} is contained in the 2-branch split closure of P.*

4 Mixed-integer sets with simple structure

We next extend Lemma 2.1 to show that CC cuts are sufficient to define the convex hull of the mixed-integer set P for $n_1 > 2$ provided that the coefficients of the integer variables form a matrix of rank 2.

Theorem 4.1. *If rank$(A) = 2$, then any facet-defining inequality for conv (P) is a CC cut and consequently $P_{CC} =$ conv (P).*

Proof. We will show that *conv* $(P) = P_{\Pi}$, and by Theorem 3.1 the result will follow. As $P \subseteq P_{\Pi}$, clearly *conv* $(P) \subseteq P_{\Pi}$. We will next show the reverse inclusion.

As A is rational, we can assume, without loss of generality, that A, G are scaled such that A is an integral matrix. As *rank*$(A) = 2$, there exists a unimodular matrix $U \in \mathbb{Z}^{n_1 \times n_1}$ with the property that $AU = \begin{bmatrix} T & 0 \end{bmatrix}$ where $T \in \mathbb{Z}^{m \times 2}$ and has rank 2; see [8]. Let

$$Q = \{(z, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \;:\; AUz + Gy = b,\; y \geq 0\}.$$

As only the first two columns of AU are nonzero, it follows that the variables $z_3, \ldots, z_{n_1}$ are not restricted in any way. Rewriting $Ax + Gy = b$ as $AUU^{-1}x + Gy = b$, it follows that there is a one-to-one correspondence between the points in P^{LP} and Q^{LP} via the mapping $(x, y) \to (U^{-1}x, y)$, whose inverse mapping is

$(z, y) \to (Uz, y)$. We denote the latter mapping by h. Furthermore, as U is unimodular, the same one-to-one correspondence holds between the points in P and Q as well.

Consider a point $(\bar{x}, \bar{y}) \in P_{\Pi}$. Let π_1 and π_2 stand for the first and second rows of U^{-1}, respectively. By the definition of P_{Π}, $(\bar{x}, \bar{y})$ is in the convex hull of points in P^{LP} satisfying $\pi_1 x \in \mathbb{Z}$, and $\pi_2 x \in \mathbb{Z}$. In other words,

$$(\bar{x}, \bar{y}) = \sum_{i=1}^{t} \lambda_i (x^i, y^i) \text{ where } \sum_{i=1}^{t} \lambda_i = 1, \text{ and } \lambda_i \geq 0 \text{ for } i = 1, \ldots, t,$$
$$(x^i, y^i) \in P^{LP} \text{ and } \pi_1 x^i \in \mathbb{Z}, \pi_2 x^i \in \mathbb{Z} \text{ for } i = 1, \ldots, t.$$

Therefore,

$$(U^{-1}\bar{x}, \bar{y}) = \sum_{i=1}^{t} \lambda_i (U^{-1} x^i, y^i) \in Q^{LP}, \text{ and } (U^{-1} x^i, y^i) \in Q^{LP} \text{ for } i = 1, \ldots, t. \tag{11}$$

Now let $(z^i, y^i) = (U^{-1} x^i, y^i)$ for any $i \in \{1, \ldots, t\}$. As $\pi_1 x^i, \pi_2 x^i \in \mathbb{Z}$ for all i, the first two components of z^i are integral, but the remaining components may not be integral. But the vector consisting of all but the first two components of z^i can be expressed as a convex combination of integral vectors in $\mathbb{Z}^{n_1 - 2}$. In other words,

$$(z^i, y^i) = \sum_{j=1}^{s_i} \mu_i^j (w^{ij}, y^i) \text{ where } \sum_{j=1}^{s_i} \mu_i^j = 1 \text{ and } w^{ij} \in \mathbb{Z}^{n_1}, \ \mu_i^j \geq 0 \text{ for } j = 1 \ldots, s_i, \tag{12}$$

and the first two components of w^{ij} equal the first two components of z^i. Now each vector (w^{ij}, y_i) is a point in Q since the variables $z_3, \ldots, z_{n_1}$ are not restricted in any way. Combining equations (11) and (12), we conclude that $(U^{-1}\bar{x}, \bar{y})$ is a convex combination of some (integral) points $q_1, \ldots, q_l \in Q$. Therefore $h(U^{-1}\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$ is a convex combination of $h(q_1), \ldots, h(q_l)$; but the latter collection of points is contained in P, and thus $(\bar{x}, \bar{y}) \in$ *conv* (P). ∎

5 Extensions to mixed-integer sets with convex relaxations

Consider a set of the form

$$Q = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \ : (x, y) \in \mathcal{C}\}, \tag{13}$$

where $\mathcal{C}$ is a closed and bounded convex set. In this setting Q_{CC} and Q_{Π} can be defined as before. We next extend Lemma 2.1 to such convex sets. We are not able to simply reuse the proof of Lemma 2.1 as there we explicitly use the polyhedrality of P (e.g., in characterizing S^x as a set defined by strict and nonstrict rational linear linequalities).

Theorem 5.1. *Let* $Q = \{(x, y) \in \mathbb{Z}^2 \times \mathbb{R}^{n_2} \ : (x, y) \in \mathcal{C}\}$ *where* $\mathcal{C}$ *is a closed and bounded convex set. Then* $Q_{CC} =$ *conv* (Q).

Proof. Observe first that *conv* (Q) can be obtained by fixing integer points to the points in the set *proj*$_x(Q)$ and then taking the convex hull of the union of these closed and bounded convex sets. Since the union of a finite number of compact sets is a compact set, and the convex hull of a compact set is compact, *conv* (Q) is compact (see [7, Theorem 17.2]).

Let $(\hat{x}, \hat{y}) \notin conv\,(Q)$. Since $conv\,(Q)$ is a closed set, there exists a strict separating hyperplane, i.e., there exists $c \in \mathbb{R}^2$, $y \in \mathbb{R}^{n_2}$ and $f \in \mathbb{R}$ such that

$$c\hat{x} + d\hat{y} < f < cx + dy \;\forall\, (x, y) \in conv\,(Q)\,. \tag{14}$$

To show the result, we will show that the valid inequality $cx + dy \geq f$ for $conv\,(Q)$ is a CC cut. Let

$$S = \{(x, y) \in \mathbb{R}^{2+n_2} \;:\; cx + dy < f, (x, y) \in \mathcal{C}\}.$$

If S is empty, then the inequality $cx + dy \geq f$ is valid for $\mathcal{C}$ and therefore it is a CC cut. We therefore assume that $S \neq \emptyset$ and let $S^x = proj_x(S)$ denote the projection of S in the space of x variables.

We next show that there exists a maximal lattice-free convex set $V \subseteq \mathbb{R}^2$ such that $S^x \subseteq int(V)$. Let

$$T = \{(x, y) \in \mathbb{R}^{2+n_2} \;:\; cx + dy \leq f, (x, y) \in \mathcal{C}\},$$

and let $T^x = proj_x(T)$. If $x^0 \in T^x \cap \mathbb{Z}^2$, then there exists $y^0 \in \mathbb{R}^{n_2}$ such that $(x^0, y^0) \in conv\,(Q)$ and $cx^0 + dy^0 \leq f$ which contradicts (14). Therefore, $T^x \cap \mathbb{Z}^2 = \emptyset$.

As T^x is bounded, there exists a polytope U that contains T^x in its interior. Let $v^1, ..., v^k$ be the (finitely many) integer points contained in U. As T^x is closed and $v^i \notin T^x$, there exist vectors c^i and numbers $\epsilon_i > 0$ for each $i \in \{1, ..., k\}$ such that $cv^i - cx > \epsilon_i \;\forall\, x \in T^x$. Therefore the set $V = U \cap \left(\bigcap_i \{x \,|\, cx \leq cv^i\}\right)$ is lattice-free and contains T^x in its interior. Since $S^x \subseteq T^x$, we obtain that $S^x \subseteq int(V)$.

As all maximal convex lattice-free sets in $\mathbb{R}^2$ are contained in CC sets (see [3]), S^x is contained in the interior of some CC set $C = \mathbb{R}^2 \setminus int(D(\pi_1, \pi_2, \gamma_1, \gamma_2))$.

Consider the CC set $\bar{C}$ obtained by extending C. As $S^x \subseteq int(C)$, and $int(C) \times \mathbb{R}^n = int(\bar{C})$ we have $S \subseteq int(\bar{C})$ and therefore $cx + dy \geq f$ is valid for $\mathcal{C} \setminus int(\bar{C})$. ∎

Note that in the previous result, we only show that an inequality strictly satisfied by $conv\,(Q)$ is a CC cut for Q. Thus if an inequality is a supporting hyperplane for $conv\,(Q)$, we cannot assert that it is a CC cut for Q, just that there is an infinite sequence of CC cuts which imply the inequality. This result is in contrast to Lemma 2.1, where every facet-defining inequality for $conv\,(P)$ is a CC cut.

Theorems 3.1 and 4.1 can also be generalized to the mixed integer convex programming setting.

Proposition 5.2. *For Q given by (13), $Q_{CC} = Q_{\Pi}$.*

To see that this result is true, modify the proof of Theorem 3.1 by replacing P^{LP} by $\mathcal{C}$ in the definition of S, and S^{LP} by S^C, where S^C is the continuous relaxation of S. All other steps of the proof are still valid.

Now consider Q given by

$$Q = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \;:\, (Ax, y) \in \mathcal{C}\}, \tag{15}$$

where $A \in \mathbb{Z}^{m \times n_1}$ and $\mathcal{C}$ is a closed and bounded convex set. In this case, note that the proof of Theorem 4.1 does not use the polyhedrality of the continuous relaxation of P. Thus, we obtain the following result.

Proposition 5.3. *Let Q be given by (15). If $rank(A) = 2$, then $Q_{CC} = conv\,(Q)$.*

6 Other extensions

We next consider implications of our results in two different settings. First consider a mixed-integer set $W = \{x \in \mathbb{Z}^n : Ax \le b\}$ such that $A = [v, T]$, where T is a totally unimodular $m \times (n-1)$ matrix, and v is an arbitrary integer vector, and A and v have m rows. Further, let b be integral. Eisenbrand, Oriolo, Stauffer, and Ventura [5] observe that *conv* (W) equals the split closure of W. Now assume that $A = [A_1, A_2]$ where A_1 is a matrix with two columns and integral components and A_2 is a totally unimodular matrix. Rewriting $Ax \le b$ as $Ax + Iy = b, y \ge 0$ where I is an $m \times m$ identity matrix, we conclude from Theorem 3.1 that CC cuts give the facet-defining inequalities of the convex hull of

$$W' = \{x \in \mathbb{R}^n : Ax \le b, x_1, x_2 \in \mathbb{Z}\}.$$

On the other hand, as *conv* (W') is the convex hull of polyhedra of the form

$$\textit{conv}\,(\{x \in \mathbb{R}^n : Ax \le b, x_1 = t_1, x_2 = t_2\})$$

for arbitrary integers t_1, t_2, and each polyhedron of the latter form is integral, *conv* (W') is an integral polyhedron and therefore *conv* (W') = *conv* (W). Therefore, all facet-defining inequalities of *conv* (W) are CC cuts, or W_{CC} = *conv* (W). More generally, if the matrix A used in the description of the mixed-integer set P can be written as $A = [A_1, A_2]$ where A_1 has two columns with arbitrary integral components and the matrix $[A_2, G]$ is totally unimodular, and if b is integral, then the above observation implies that P_{CC} = *conv* (P).

Our results can also be applied to the generalization of the *two-row continuous group relaxation* studied by Andersen, Louveaux, and Weismantel [1], where some of the continous variables have upper bounds in addition to lower bounds of zero. As the number of integer variables in this set is two, all facet-defining inequalities are given by CC cuts.

A Appendix

Lemma A.1. *Let P stand for the set of solutions of the system of rational inequalities*

$$Ax + By \le e,\ Cx + Dy < f.$$

Then proj$_x(P) = \{x : \exists y$ such that $(x, y) \in P\}$ is defined by a finite system of strict and nonstrict rational, linear inequalities.

Proof. Assume that at least one strict inequality in $Cx + Dy < f$ is irredundant and $P \neq \emptyset$; if either condition is violated the result is trivial. Let C have t rows. Let

$$S = \Big\{x \ : \ \lambda Ax \le \lambda e, \text{ for all } \lambda \ge 0 \text{ that satisfies } \lambda B = 0, \tag{16}$$

$$(\lambda A + \mu C)x < \lambda e + \mu f, \tag{17}$$

$$\text{for all } \lambda, \mu \ge 0 \text{ that satisfies } \lambda B + \mu D = 0,\ \mu_i = 1 \text{ for some } i \in \{1, \ldots, t\}\Big\}.$$

Any $x \in$ *proj*$_x(P)$ clearly satisfies the above constraints, and thus *proj*$_x(P) \subseteq S$. Now assume $\bar{x} \in S$, but $\bar{x} \notin$ *proj*$_x(P)$. Then the following system of inequalities has no solution: $By \le e - A\bar{x}, Dy < f - C\bar{x}$. By Motzkin's transposition theorem (see Corollary 7.1k in [8]), there exists $\bar{\lambda}, \bar{\mu} \ge 0$ such that either

$$\bar{\lambda}B + \bar{\mu}D = 0, \quad \bar{\lambda}(e - A\bar{x}) + \bar{\mu}(f - C\bar{x}) < 0, \quad \text{or}$$
$$\bar{\lambda}B + \bar{\mu}D = 0, \quad \bar{\lambda}(e - A\bar{x}) + \bar{\mu}(f - C\bar{x}) \le 0, \quad \bar{\mu} \neq 0.$$

Notice that if $\bar{\lambda}, \bar{\mu}$, satisfies the first condition and $\bar{\mu} \neq 0$, then the second condition is satisfied as well. Consequently, the first condition can be modified to consider $\bar{\mu} = 0$ only and it can be replaced by the following: there exists a $\bar{\lambda} \geq 0$ such that

$$\bar{\lambda}B = 0,\ \bar{\lambda}(e - A\bar{x}) < 0.$$

If there indeed exists such an $\bar{\lambda}$, the point $\bar{x}$ violates the inequality (16) associated with $\bar{\lambda}$, which contradicts the assumption that $\bar{x} \in S$. Therefore the second condition must hold for some $\bar{\mu} \neq 0$. Without loss of generality, assume that $\bar{\mu}_k = 1$ for some $k \in \{1, \ldots, t\}$. The condition $\bar{\lambda}(e - A\bar{x}) + \bar{\mu}(f - C\bar{x}) \leq 0$ can be rewritten as $(\bar{\lambda}A + \bar{\mu}C)\bar{x} \geq \bar{\lambda}e + \bar{\mu}f$ which implies that $\bar{x}$ violates the constraint (17) associated with $\bar{\lambda}, \bar{\mu}$ and $i = k$, again a contradiction.

Finally, the set of constraints in (16) is implied by a finite set of rational constraints, namely the constraints defined by the generators of the cone $\{\lambda : \lambda B = 0, \lambda \geq 0\}$. Similarly, for each $i \in \{1, \ldots, t\}$, the set of constraints in (17) is implied by a finite set of rational constraints and therefore S has a finite linear description. ∎

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