ON THE SAFETY FIRST PORTFOLIO SELECTION[1]

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Abstract. A.D.Roy's (1952) safety first (SF) approach to a financial portfolio selection is improved. Safety first means minimization of probability of poor returns. Improvement concerns a better estimation of the poor return probabilities by means of shortfall risk functions. Optimal SF-portfolio is sought similar to Roy's geometric method but with a different efficient frontier. In case of a finite number of yield scenarios the SF-portfolio selection problem is reduced to a linear mixed-Boolean programming one.

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Introduction

The paper improves Roy's (1952) [1] *safety first* (SF) approach to a financial portfolio optimization. The approach consists in minimization of a probability of a shortfall in portfolio returns subject to a constraint from below on its mean return. The improvement consists in a more accurate estimation of the probability of non-desired returns by means of threshold downside risk functions. The search of the safe optimal portfolio is reduced to construction and analysis of a new effective frontier of non-dominated with respect to the risk functions portfolios. For the case of a finite scenario set the problem of the safety first portfolio selection is reduced to a mixed-Boolean linear programming.

It is well known Markowitz' (1952) [2] model of a financial portfolio optimization through a "risk-return" criteria, where as a risk measure a variance (or a standard deviation) of returns is used. The essence of Markowitz' model consists in the construction of a frontier of effective portfolios, which have minimal return variance subject to given mean return, and in the selection of an effective portfolio, that maximizes some utility function [3]. In the same 1952 year independently a very close paper by an English economist A.D.Roy [1] was published, where he suggested to select a portfolio that minimizes probability of returns shortfall with respect to a give threshold (Safety First criterion). Since the probability optimization problem was a difficult task in 50-s of the past century, A.D.Roy suggested optimizing its upper estimate, obtained from Chebyshev inequality. For the approximate problem A.D.Roy proposed a simple and elegant geometric solution lying on the effective frontier of non-dominated portfolios in the plane "risk (return variance) – mean return". Although this work had not got much popularity, nevertheless Roy's ideas continued to develop, see, e.g., [4] (maximization of a mean return subject to a bound on the probability of returns below a given threshold), [5] (some quantile of returns is optimized), [6] (a portfolio mean return is maximized and the probability of a shortfall of returns is minimized), [7] (lexicographic SF-optimization is developed), [8] (multi-period SF-optimization is considered), [9] (extreme distributions for extreme losses are used to assess probability of the shortfall of returns), [10] (a probability of excess returns above a given threshold is maximized and a probability of returns below a given level is minimized), [11, 12] (applications of SF-approach to portfolios containing very risky assets are considered). A.I.Kibzun and Yu.S.Kan [13 - 19] study portfolio optimal control problems with quantile criterion, which is closely connected to the probability of a shortfall of returns.

For a long time Roy's work was in the shadow of the Markowitz' theory, but, possibly, the situation may change. Here we would like to quote a paragraph from paper [20].

«In autumn 2001 a famous French sociologist Bruno Latour had written a preface for a French translation of the book «Risikogesellschaft – Auf dem Weg in eine andere Moderne» («Risk society. Towards a New Modernity» [21]) by not less famous German sociologist Ulrich Beck, and pointed out on a strange coincidence. Beck's book was published just after explosion at Chernobyl power atomic station. Its French translation appeared just after tragedy 9/11. The society starts to realize its exposure to dangers, including those that appear due to its own development. The time of a good hope has passed, the history of science has no a lucky end. In

the post modern epoch, which Latour following Beck calls the epoch of risks, the science is necessary for the other, for the minimization of unavoidable losses. »

In view of recent global financial crises this notices seem even more prophetic. In any case, it seems that portfolios of pension and other social security funds have to be optimized according to SF-criteria.

In the present paper Roy's safety first approach to a financial portfolio optimization is improved in the following directions.

The original Roy's setting is supplemented with a constraint on the minimal mean portfolio return as in Markowitz' approach and by that full analogy between both approaches is established.

One of shortcomings of the Roy's model (and of Markowitz' one) is removed, namely symmetry of the variance as a risk measure with respect to a mean, by replacing the variance with a mean shortfall of returns.

In the Chebyshev inequality not a variance as in the original Roy's approach is used but a mean one-sided deviation of the return below a given threshold for the estimation of the probability of the critical downside fall of returns. The threshold then is selected in an optimal way. As a result more accurate probability estimates are obtained.

The effective frontier of portfolios is constructed not in the "variance – mean return" plane, but in another "one-sided risk – mean return" one.

Thus in the present paper SF portfolio optimization is interpreted as an approximate optimization of the probabilistic risk measure (probability of the event that returns fall below a given critical threshold) subject to a constraint from below on the mean return. Solution of the problem consists in construction and analysis of the frontier of effective portfolios.

Finally, for the case of a finite return scenario set the problem of minimization of probability of default is equivalently reduced to a linear partially Boolean programming, which in turn is solved by a branch and bound method. The obtained exact solution allows estimating the error of Roy's approximation approach and the accuracy of proposed its modifications.

1. Modeling and optimization of a financial portfolio

Financial portfolio is described by the vector $x=\left(x_1,...,x_n\right)$ of values x_i and by the vector of random returns $\omega=\left(\omega_1,...,\omega_n\right)\in\Omega$ of assets $i=1,...,n$ in some fixed time interval. Denote $X=\left\{x\in R^n:\sum_{i=1}^n x_i\le 1,\ x_i\ge c_i\ge -\infty\right\}$ a set of admissible portfolios with a unit total cost of the whole portfolio, c_i is a lower bound on the value of component i of the portfolio (e.g., limitation on borrowing of assets). In the definition of the set X the inequality $\sum_{i=1}^n x_i\le 1$ is used, which means that non-used funds $x_0=1-\sum_{i=1}^n x_i$ have zero yield. The portfolio is characterized by a random return $f(x,\omega)$, mean return $\mu(x)=E_\omega f(x,\omega)=\sum_{i=1}^n x_i E_\omega \omega_i$ for the considered period of time and by variance of return $\sigma^2(x) = E_\omega\left(f(x,\omega) - E_\omega f(x,\omega)\right)^2$, where E_ω denotes a mathematical expectation over distribution of random variable ω, ω' denotes transposition of a column vector ω into a row one.

According to Markowitz [2, 3] the portfolio is optimized by two criteria, mean return $\mu(x)$ and standard deviation $\sigma(x)$, $\left(\mu(x), -\sigma(x)\right) \to \max_{x\in X}$,. A set of non-dominated portfolios $\Gamma = \left(y \in R^1, \sigma^*(y)\right)$ such that $\left[\sigma^*(y)\right]^2 = \min_{x\in X,\mu(x)\ge y} \sigma^2(x)$, is called an effective frontier (boundary). An optimal portfolio is selected from the effective frontier by optimization of some utility function $\Phi(\mu,\sigma)$, defined in the plane "risk – return" (σ,μ).

1.1. Safety First approach to a financial portfolio selection

According to A.D.Roy [1] a portfolio optimization should be made on the basis of a safety principle (Safety-First) and be consisted in minimization of the probability

$$P_u(x) = \Pr_\omega\left\{ f(x,\omega) \le u\right\} \to \min_{x\in X} \qquad (1)$$

that portfolio return $f(x,\omega)$ is less than some given critical threshold u, for example, $u = 1$. Since the probability optimization problem is a rather difficult task, A.D.Roy [1] proposed to minimize its upper bound, obtained from the Chebyshev inequality,

$$P_u(x) = \Pr_\omega\left\{ f(x,\omega) \le u\right\} = P_\omega\left\{-f(x,\omega) \ge -u\right\} = \Pr_\omega\left\{\mu(x) - f(x,\omega) \ge \mu(x) - u\right\} \le$$

$$\le \frac{\sigma^2(x)}{\left(\mu(x)-u\right)^2} = \overline{P}_u(x) \to \min_{x\in X}. \qquad (2)$$

Roy noted that in the plane (σ,μ) the quantity $\sigma(x)/\left(\mu(x)-u\right)$ is a co-tangent of the angle between the line coming through points $(0,u)$ and $\left(\sigma(x),\mu(x)\right)$ and the horizontal axis σ. The maximal such angle corresponds to the minimal value of $\overline{P}_u(x)$. Thus solution of the approximate problem (2) is decomposed into subproblems:

1) Construction of the effective boundary $\Gamma_u = \left(y > u, \sigma^*(y)\right)$ such that
$$\left[\sigma^*(y)\right]^2 = \min_{x\in X,\mu(x)\ge y} \sigma^2(x), \qquad \forall y > u; \qquad (3)$$
2) Finding a tangent to the effective frontier Γ_u, going through the point $(0,u)$. The optimal portfolio $x^*(u)$ corresponds to the tangent point.

Remark that Roy's framework allows to take into account also a constraint from below on the mean return, $\mu(x) \ge z$, where z is a given constant, $z > u$. For this, one has to consider points on the effective frontier lying above level z, i.e. points on $\Gamma_z = \left(y \ge z, \sigma^*(y)\right)$.

Thus, in essence, the difference between Roy's approximation approach (2) and the Markowitz' one consists only in the way of selection of an optimal portfolio from the set Γ of efficient portfolios.

1.2. Exact solution of the probability optimization problem for a finite number of scenarios

Let random vector ω in (1) takes on values from a finite set Ω with probabilities $p_\omega,\ \omega\in\Omega$. Consider a problem

$$\Pi_u(x)=\sum\nolimits_{\omega\in\Omega} p_\omega I_{\{f(x,\omega)<u\}} \to \min\nolimits_{x\in X,\mu(x)\ge y}, \qquad (4)$$

where $I_{\{f(x,\omega)<u\}}$ is the indicator function of the event $\{f(x,\omega)<u\}$, i.e. $I_{\{f(x,\omega)<u\}}=1$ if $f(x,\omega)<u$, and $I_{\{f(x,\omega)<u\}}=0$ otherwise. It is easy to see that for continuous in x functions $f(x,\omega)$ discontinuous functions $I_{\{f(x,\omega)<u\}}$ and hence $\Pi_u(x)$ are lower semicontinuous in x, so for a compact set X of admissible portfolios problem (4) has a solution. Remark that functions $P_u(x)$ from (1) and $\Pi_u(x)$ from (4) are bound by the relation $\Pi_u(x)\le P_u(x)$, so any upper bound for $P_u(x)$ is valid also for $\Pi_u(x)$. Conditions of continuity of functions $P_u(x)$ and $\Pi_u(x)$ are available in [13, sec. 2.2.2], [22], [23, sec. 7.2.4].

Problem (4) in principal can be solved by a stochastic branch and bound method [24]. Further we reduce problem (4) to a mixed integer (in fact to mixed Boolean) linear programming one. Assume $\inf_{x\in X,\omega\in\Omega} f(x,\omega)\ge -M>-\infty$ and $M+u>0$. For each $\omega\in\Omega$ introduce a binary variable $z_\omega\in\{0,1\}$. Consider a minimization problem

$$\sum\nolimits_{\omega\in\Omega} p_\omega z_\omega \to \min\nolimits_{x\in X,z_\omega\in\{0,1\}} \qquad (5)$$

subject to additional constraints

$$\mu(x)\ge y, \qquad (6)$$

$$-f(x,\omega)+u\le (M+u)z_\omega, \quad \omega\in\Omega. \qquad (7)$$

Theorem 1. Let the set X is compact, and Ω is finite; functions $f(\cdot,\omega)$ are continuous for any $\omega\in\Omega$; $\inf_{x\in X,\omega\in\Omega} f(x,\omega)\ge -M>-\infty$, $M+u>0$. Then problems (4) and (5)-(7) are equivalent in the sense that their solutions exist and optimal values of the objective functions coincide. Besides, if x^* is an optimal solution for problem (4), then $\left(x^*, z^*_\omega = I_{\{f(x^*,\omega)<u\}}\right)$ is an optimal solution for (5)-(7), and conversely, if $\left(x^*, z^*_\omega\right)$ is an optimal solution of problem (5)-(7), then x^* is an optimal solution of (4).

The statement follows from a more general Theorem 2 given in the Appendix.

Problem (5)-(7) can be solved by a standard branch and bound method. The branching is performed over Boolean variables. Solution process consists in enumerative search over a tree of problems like (5)-(7), where $z_\omega\in Z_\omega\subseteq\{0,1\}$, with the use of lower and upper bounds for optimal values of the objective function (5). The lower bounds are obtained through continuous relaxation $0\le z_\omega\le 1$ of free Boolean variables $z_\omega\in Z_\omega=\{0,1\}$, and the upper bounds are obtained by fixing free Boolean variables at their upper values $z_\omega=1$.

1.3. Modifications of Roy's SF-approach by means of one-sided risk measures

As indicated by G.Markowitz himself [3], a shortcoming of the standard deviation $\sigma(x)$ as a risk measure is that $\sigma(x)$ accounts for excesses of return $f(x,\omega)$ over the mean value $\mu(x) = Ef(x,\omega)$, that have no relation to non-desirable risk. Obviously, the same shortcoming takes place in Roy's approximation approach. So our first improvement of the Roy's approach consists in the replacement of the standard deviation $\sigma(x)$ by others more exact and adequate one-sided (down side) risk measures. Following Markowitz [3] let us define standard down-side semi-deviation of the random return $f(x,\omega)$ from its mean value $\mu(x)$:

$$R_{(\mu)}(x) = E_\omega \max\{0, \mu(x) - f(x,\omega)\}. \tag{8}$$

By Chebyshev inequality, the following estimate holds true for $u > \mu(x)$,

$$\begin{aligned} P_u(x) &= \Pr_\omega\left\{ f(x,\omega) \le u\right\} = \Pr_\omega\left\{-f(x,\omega) \ge -u\right\} = \\ &\le \Pr_\omega\left\{\mu(x) - f(x,\omega) \ge \mu(x) - u\right\} \\ &\le \Pr_\omega\left\{\max\{0, \mu(x) - f(x,\omega)\} \ge \mu(x) - u\right\} \le \\ &\le \frac{E_\omega \max\{0, \mu(x) - f(x,\omega)\}^r}{\left(\mu(x) - u\right)^r} \le \frac{E_\omega \left|\mu(x) - f(x,\omega)\right|^r}{\left(\mu(x) - u\right)^r}, \qquad r > 0. \end{aligned} \tag{9}$$

Thus,

$$P_u(x) \le \frac{R_{(\mu)}(x)}{\mu(x) - u} = P_u^-(x), \tag{10}$$

and by (9) holds $P_u^-(x) \le \overline{P}_u(x)$, i.e. estimate (10) is more accurate than (2).

Following Roy's idea [1], we can minimize instead of probability $P_u(x)$ its upper estimate (10). The latter problem is reduced to the following steps:

1. Construction of the effective frontier $\Gamma_- = \left(y > u, R^*_{(\mu)}(y)\right)$ such that
$$R^*_{(\mu)}(y) = \min\nolimits_{x \in X,\, \mu(x) \ge y} R(x), \qquad \forall y > u; \tag{11}$$
2. Finding a tangent to the effective boundary Γ_-, that goes through point $\left(0, u\right)$.

Computationally problem (11) differs from (3) by

(i) that functional $R_{(\mu)}(x)$ in (11) is not expressed in a closed form through covariance matrix of assets returns, and

(ii) that $R_{(\mu)}(x)$ is a nonsmooth function.

Problem (11) belongs to a class of convex programming problems [25].

Consider one more modification of Roy's approach. Let us introduce α-quantile $q_\alpha(x) = \max\left\{q : \Pr\left\{ f(x,\omega) \le q\right\} \le \alpha\right\}$ and a related risk measure

$$R_{q_\alpha}(x) = \frac{E_\omega \max\{0, q_\alpha(x) - f(x,\omega)\}}{\Pr\left\{ f(x,\omega) \le q_\alpha(x)\right\}},$$

which is called a mean one-sided deviation (toward less values) from the α -quantile $q_\alpha(x)$. For $q_\alpha(x) > u$ by the Chebyshev inequality analogously to (9) holds

$$P_u(x) \le \frac{E_\omega \max\{0, q_\alpha(x) - f(x,\omega)\}}{q_\alpha(x) - u}. \qquad (12)$$

This estimate can be rewritten in the form

$$P_u(x) \le \frac{\Pr\{f(x,\omega) \le q_\alpha(x)\}}{q_\alpha(x) - u} \frac{E_\omega \max\{0, q_\alpha(x) - f(x,\omega)\}}{\Pr\{f(x,\omega) \le q_\alpha(x)\}} \le \frac{\alpha R_{q_\alpha}(x)}{q_\alpha(x) - u}. \qquad (13)$$

As a particular case of this estimate is the median one with $\alpha = 1/2$. Now the problem is reduced to minimization of the right hand side of relation (13), that in turn, is reduced to the construction of the effective frontier $\left\{ r(y) = \min_{x \in X, q_\alpha(x) \ge y} R_{q_\alpha}(x), y > u \right\}$, and finding a tangent to it, going through point $(0, u)$.

The outlined Roy's approximation approach and its modifications indicate the connection of SF portfolio optimization with traditional and some new one-sided risk measures. In the next section we generalize the described modifications of the Roy's approximation approach.

3. **Generalization of Roy's Safety First approach to optimization of a financial portfolio**

Let us define a quantity $R_{(y)}(x) = E_\omega \max\{0, y - f(x,\omega)\}$. Analogously to (9) for any $y > u$ holds

$$P_u(x) \le \frac{E_\omega \max\{0, y - f(x,\omega)\}}{y - u} = \frac{R_{(y)}(x)}{y - u}, \qquad (14)$$

so

$$P_u(x) \le \min_{y > u} \frac{R_{(y)}(x)}{y - u}. \qquad (15)$$

Obviously, estimate (15) is more accurate than (10), (12), which have been obtained from a weaker estimate (14) by substitution of $y = \mu(x)$ and $y = q_\alpha(x)$ respectively. A lower bound for $P_u(x)$ has the form

$$P_u(x) = E_\omega I_{f(x,\omega) \le u} = E_\omega I_{f(x,\omega) \le u} \frac{u - f(x,\omega)}{u - f(x,\omega)} \ge$$

$$\ge \frac{E_\omega \left(u - f(x,\omega)\right) I_{f(x,\omega) \le u}}{u - \min_{\{\omega \in \Omega: f(x,\omega) \le u\}} f(x,\omega)} = \frac{E_\omega \max\left\{0, u - f(x,\omega)\right\}}{u - \min_{\{\omega \in \Omega: f(x,\omega) \le u\}} f(x,\omega)}.$$

Following Roy [1] instead of $P_u(x)$ we optimize upper bound (15) over $x \in X$, and beside put a restriction from below $z \le \mu(x)$ on a mean return $\mu(x)$ of the portfolio x. Thus we obtain the following nonconvex minimization problem with parameters (u, z):

$$\frac{R_{(y)}(x)}{y - u} \to \min_{\{(x,y): x \in X, \mu(x) \ge z, y > u\}}. \qquad (16)$$

Practically, in this setting the probability of critical downfall of the portfolio return is approximately minimized subject to a bound from below on its mean return. Obviously,

$$\min_{\{(x,y):x\in X,\mu(x)\ge z,y>u\}} \frac{R_{(y)}(x)}{y-u} = \min_{y>u} \frac{\min_{x\in X,\mu(x)\ge z} R_{(y)}(x)}{y-u} = \min_{y>u} \frac{r(y,z)}{y-u},$$

where a risk function $r(y,z)$ has the form

$$r(y,z) = \min_{x\in X,\mu(x)\ge z} R_{(y)}(x) = \min_{x\in X,\mu(x)\ge z} \left[E_\omega \max\{0, y - f(x,\omega)\}\right]$$

For a fixed z let us construct an effective frontier

$$\Gamma_z = \left\{(r(y,z), y) \in R^2,\ y \ge \min_{x\in X,\ \omega\in\Omega} f(x,\omega)\right\}.$$

From geometrical considerations it is seen that minimum in problem (16) is achieved at point $(r^* = r(y^*,z), y^*)$ of tangency of the effective frontier Γ_z with a straight line going through the point $(0,u)$ and having minimal slope with respect to the vertical axis y (see Fig. 1).

Remark that if the function of random returns $f(x,\omega)$ is concave in x over a convex set X, then $\max\{0, y - f(x,\omega)\}$ is convex jointly in a pare variable (x,y), and a minimum function $r_z(y) = \min_{x\in X,\mu(x)\ge z} R_{(y)}(x)$ is convex and monotonically increasing in y. From here it follows that in the plane (r,y) the effective boundary $r_z(y)$ is concave upward with respect to a horizontal axis r and is monotonic in r. Thus if there exists a portfolio x such that with a positive probability its return $f(x,\omega)$ is greater than u, then problem (16) has a solution.

Finding the efficient front $r_z(y) = \min_{x\in X,\mu(x)\ge z} R_{(y)}(x)$, $y > u$, is a convex programming problem that can be solved, for instance, by the stochastic subgradient method [25]. A stochastic subgradient of function $R_{(y)}(\cdot)$ has the form

$$g(x,\omega) = \begin{cases} -\omega, & y - \omega^T x > 0, \\ 0, & y - \omega^T x \le 0, \end{cases}$$

and $E_\omega g(x,\omega) \in \partial R_{(y)}(x)$, where $\partial R_{(y)}(x)$ is a subdifferential of function $R_{(\mu)}(\cdot)$ at point x.

Another approach to solution of stochastic programming problems consists in approximation of the distribution of the random vector variable ω by a discrete distribution with values ω^s and probabilities p_s, $s = 1,...,S$, [26]. Then the mathematical expectation in $R_{(y)}(x) = E_\omega \max\{0, y - f(x,\omega)\}$ can be approximated by averages

$$R^S_{(y)}(x) = \sum_{s=1}^S p_s \max\left\{0, y - f(x,\omega^s)\right\},$$

And instead of $r(y) = \min_{x\in X,\mu(x)\ge z} R_{(y)}(x)$ one can solve problems

$$R^S_{(y)}(x) \to \min_{x\in X,\mu(x)\ge z}, \qquad \forall y > u.$$

In case of empirical approximations random points $\left\{\omega^s\right\}$ are independent identically distributed vectors (with the same distribution as the one of vector ω), and $p_s = 1/S$.

Remark that for linear functions $f(\cdot,\omega)$ the problem of minimization of a piece-wise linear function $R^S_{(y)}(x)$ is reduced to the linear programming.

4. Some other settings of the safety first portfolio selection

Maximization of the mean portfolio return subject to a bound on the probability of a downfall of returns below a given threshold. Telser [4] considered the following problem

$$\mu(x) = Ef(x,\omega) \to \max_{x\in X} , \quad (17)$$

$$P_{(u)}(x) = P_{\omega}\left\{ f(x,\omega) \le u \right\} \le \alpha , \quad (18)$$

where $f(x,\omega)$ denoted a return of the portfolio x with a random vector ω of component returns. Remark that a complicated probabilistic constraint (18) can be replaced by an equivalent and also a rather intricate quantile constraint $q_{\alpha}(x) \ge u$. For an approximate solution of problem (17), (18) we can exploit an upper approximation of the probability $P_{(u)}(x)$, replacing probabilistic constraint (18) by one of more strong restrictions:

$$\text{а) } \frac{R_{(\mu)}(x)}{\mu(x)-u} \le \alpha , \qquad \text{б) } \frac{\alpha R_{q_{\alpha}}(x)}{q_{\alpha}(x)-u} \le \alpha , \qquad \text{or} \qquad \text{в) } \min_{y>u} \frac{R_{(y)}(x)}{y-u} \le \alpha . \quad (19)$$

In case a) the solution of the problem is reduced to finding a maximally distant from the origin an intersection point of the effective frontier $\Gamma_{-} = \left(R^{*}_{(\mu)}(y), y > u \right)$ with a straight line going through the point $(0,u)$ and having tangent $1/\alpha$ of the slope angle to the horizontal axis.

For a finite scenario set Ω problem (17), (18) can be reduced to a mixed-integer (mixed-Boolean) programming problem. For the equiprobable scenarios constraint (18) means that at a feasible solution x no more than $\alpha|\Omega|$ of constraints $f(x,\omega) \le u$ are fulfilled, where $|\Omega|$ is the total number of scenarios. In discrete programming it is well known a method of reducing of such problems to mixed-integer ones [27, Ch. 2, § 4]. Assume $\inf_{x\in X,\omega\in\Omega} f(x,\omega) \ge -M > -\infty$. For each $\omega \in \Omega$ introduce a binary variable $z_{\omega} \in \{0,1\}$. Consider a maximization problem

$$E_{\omega} f(x,\omega) \to \max_{x\in X, \{z_{\omega}\in\{0,1\}\}} \quad (20)$$

subject to additional constraints

$$\sum\nolimits_{\omega\in\Omega} p_{\omega} z_{\omega} \le \alpha , \quad (21)$$

$$-f(x,\omega) + u \le \left(M + u \right) z_{\omega}, \qquad \omega \in \Omega . \quad (22)$$

The equivalence of problems (17), (18) and (20)-(22) follows from Theorem 2 from Appendix.

Minimization of a monetary reserve (VaR), which guarantees the fulfillment of probabilistic constraint on the total income. Kataoka [5] considered the following problem

$$v \to \min_{x\in X, v\in R^{1}} ,$$

$$P_{u,v}(x) = P_{\omega}\left\{ v + f(x,\omega) \le u \right\} \le \alpha . \quad (23)$$

Obviously, this problem is equivalent to a quantile minimization:

$$q_{\alpha}(x) = \max\left\{ q : \Pr\left\{ f(x,\omega) \le q \right\} \le \alpha \right\} \to \max_{x\in X} .$$

Analogously to (19) probabilistic constraint (23) can be replaced by a stronger one using upper bounds on the probabilities $P_{u,v}(x)$. For example, consider the problem

$$v \to \min_{x\in X, v},$$

$$\min_{y>u-v} \frac{E_\omega \max\{0, y - f(x,\omega)\}}{y-(u-v)} \le \alpha .$$

From geometric considerations it follows that the optimum in the latter problem is achieved at a portfolio x^*, corresponding to such a point (r^*, y^*) of the effective frontier $\Gamma^0 = \{(r,y)\in R^2 :\ r(y) = \min_{x\in X} R_{(y)}(x),\ y>u\}$ that a tangent to the boundary Γ^0 at this point has tangent $1/\alpha$ of the slope angle to the horizontal axis, and the optimal value of the objective function equals to $v^* = u + r^*/\alpha - y^*$.

Portfolio selection as a lexicographic optimization problem. In [7] the following lexicographic optimization problem is considered:

$$\left(\pi_{u,\alpha}(x), \mu(x)\right) \to \operatorname{Lex}\max_{x\in X},$$

where

$$\pi_{u,\alpha}(x) = \begin{cases} 1-\Pr\{ f(x,\omega)\le u\}, & \Pr\{ f(x,\omega)\le u\} > \alpha, \\ 1, & \Pr\{ f(x,\omega)\le u\} \le \alpha. \end{cases}$$

In this approach firstly the following problem is solved:

$$\Pr\{ f(x,\omega)\le u\} \to \min_{x\in X},$$

And if it has a feasible solution x^* such that $\Pr\{ f(x^*,\omega)\le u\} \le \alpha$, then the next problem is solved:

$$\mu(x) \to \min_{x\in X},$$

$$\Pr\{ f(x,\omega)\le u\} \le \alpha .$$

Here probabilities $\Pr\{ f(x,\omega)\le u\}$ also can be replaced by their upper bounds (10), (12), (15).

5. Numerical experiments

Let us compare on a numerical example, taken from [3], solutions of the safety first portfolio selection problem obtained by the original (subsection 1.1) and by a modified (Section 2) Roy's approaches with the exact solution, obtained by the branch and bound method (subsection 1.2).

In the considered example the portfolio consists of 9 securities, stocks of 9 large American companies, with annual returns for 18 years since 1937 up to 1954 given in [3, Table 1, page 13]. Thus vectors x and ω have nine components, and the set of (equiprobable) scenarios Ω contains 18 elements (row vectors from Table 1 [3]).

Figure 1 for the average return $z = 0.15$ presents an effective boundary $\Gamma_{0.15} = \{(r(y,0.15), y)\in R^2,\ y \ge -0.477\}$ described in Section 2, and a tangent to it, going through the point $(0, u = -0.07)$, and corresponding to the optimal portfolio with $u = -0.07$. The tangency happens at point $(0.0252, 0.0352)$, which corresponds to the optimal portfolio $x^* = (0,\ 0,\ 0.3951,\ 0,\ 0.2151,\ 0,\ 0.3846,\ 0,\ 0.0052)$.

Figure 2 for the average return $z = 0.1$ presents graphs of exact probability function (solid line)

$$\Pi(u) = \min_{x \in X, \mu(x) \geq z} \Pr\{ f(x,\omega) < u \},$$

its Roy's approximation (dotted line)

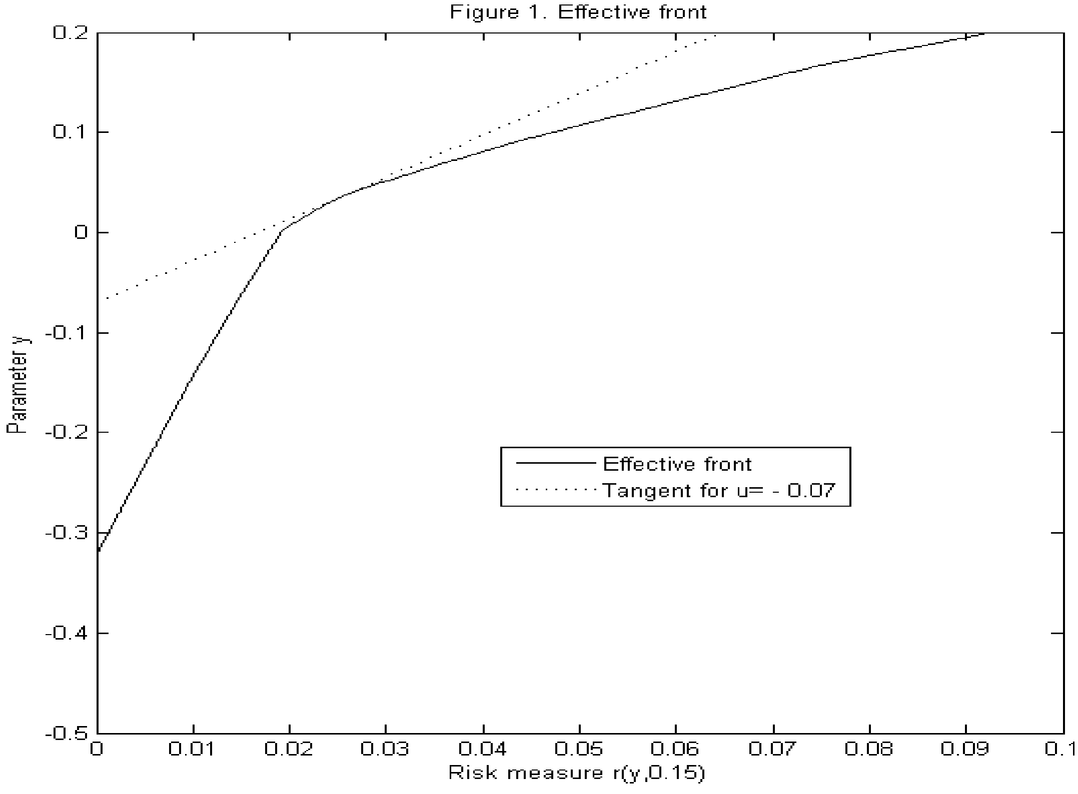
$$\bar{\Pi}(u) = \min_{x \in X, \mu(x) \geq z} \left[\sigma^2(x) \big/ (\mu(x) - u)^2 \right]$$

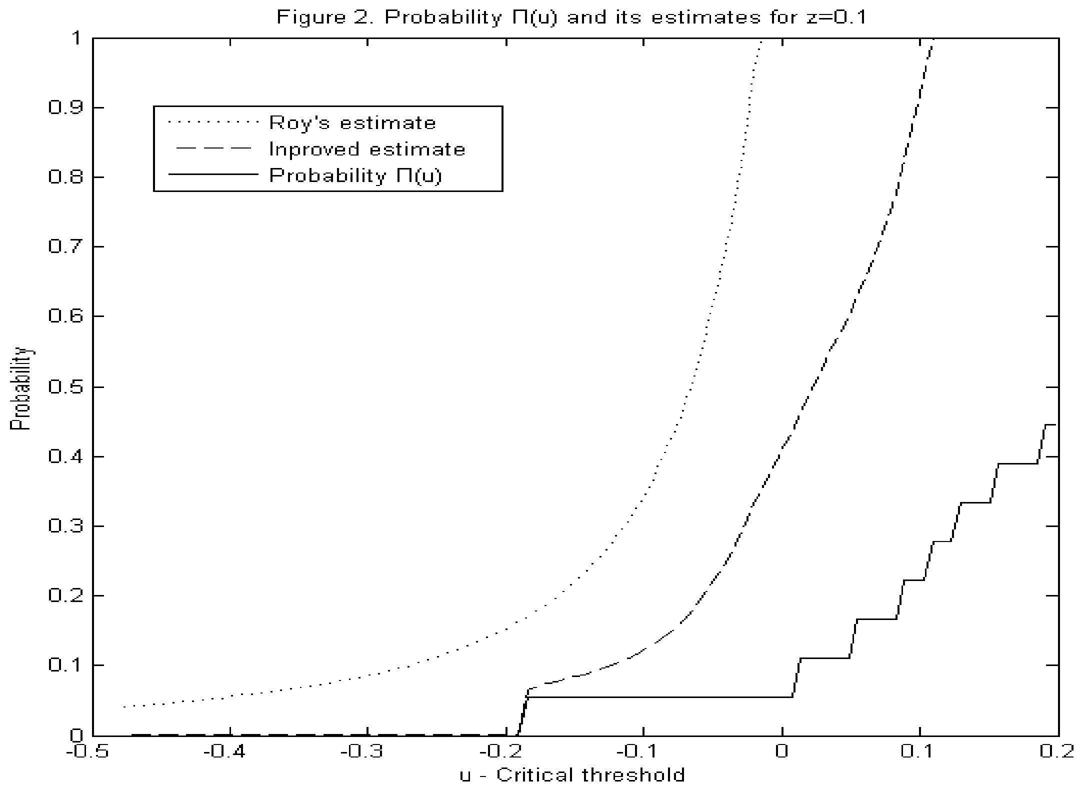
and its improved estimate from Section 2 (dashed line)

$$\tilde{\Pi}(u) = \min_{y > u} \left[r(y,z) \big/ (y - u) \right],$$

where $r(y,z) = \min_{x \in X, \mu(x) \geq z} E_\omega \max\{0, y - f(x,\omega)\}$.

Figure 1. Effective front





For other values of the average return z the picture is similar; decreasing the parameter z shifts the curves to the right, increasing moves them to the left. This example shows that Roy's approximation $\overline{\Pi}(u)$ (dotted line) gives a bad estimate of function $\Pi(u)$ in the whole range of thresholds u. The improved estimate $\tilde{\Pi}(u)$ (dashed line) is much better, but worth to apply only for negative values of u. For the critical level $u=-0.1$ and for the mean return $z=0.1$ Table 1 presents the structure of optimal portfolios, obtained by Markowitz' method, by Roy's approach (subsection 1.1), by approximation method of Section 2 and by the exact safety first method (subsection 1.2).

Table 1

	Am.T.	A.T. &T.	U.S.S.	G.M.	A.T. & Sfe	C.C.	Bdn.	Frstn.	S.S.
Markowitz	0	0	0.1152	0.0226	0.084	0	0.4907	0	0
Roy	0	0	0.1154	0.0226	0.0841	0	0.4913	0	0
Approx.SF	0	0	0.3534	0	0.0769	0.0015	0.2590	0	0
Exact SF	0.0582	0.0708	0.0417	0.0869	0.1354	0.0577	0.1843	0.2154	0.0943

Characteristics of optimal portfolios from Table 1 are given in Table 2, where $x_0 = 1 - \sum_{i=1}^{n} x_i$, $P_{-0.1}(\cdot)$ and $P_0(\cdot)$ are probabilities of portfolio returns less than -0.1 and 0 respectively, $\mu(\cdot)$ is a mean return, $\sigma(\cdot)$ is a standard deviation of returns.

Table 2

	x_0	$\bar{\Pi}$	$\tilde{\Pi}$	Π	$P_{-0.1}(\cdot)$	$P_0(\cdot)$	$\mu(\cdot)$	$\sigma(\cdot)$
Markowitz	0.2875				0.0556	0.1667	0.1	0.1174
Roy	0.2867	0.3448			0.0556	0.1667	0.1001	0.1175
Approx. SF	0		0.122		0.0556	0.0556	0.1	0.1362
Exact SF	0.0552			0.0556	0.0556	0.0556	0.1323	0.2044

Tables 1, 2 lead to the following conclusions. First two Markowitz' and Roy's portfolios differ a little. The third approximately SF-optimal portfolio is alike in its structure to Markowitz' and Roy's portfolios, but it is more safe, since the probability $P_0 = 0.0556$ of its negative returns is less. The fourth SF-optimal portfolio differs from the others considerably: it is more diversified, more safe, has greater average return, but it is less focused around the mean return.

Conclusion

In the present paper Roy's [1] safety first approach to a financial portfolio selection is improved. The approach consists in minimization of an upper bound on the probability of undesirable (e.g. negative) returns instead of exact probability, subject to a constraint from below on the portfolio mean return. Roy's and Markowitz' approaches to portfolio selection are very close and differ only in the choice of a point on the same effective frontier in the plane "risk (i.e. variance of returns) – mean return". In the present paper, first, an estimate of undesirable portfolio returns is improved by means of one-sided threshold risk measures. The search of the optimal portfolio is reduced to construction and analysis of a new effective front in the plane "threshold – threshold risk measure". The construction of the effective front consists in solution of a series of stochastic programming problems, namely, minimization of threshold risk measures. Safety first optimal portfolio is found by a simple and elegant Roy's method. Second, for the case of finite return scenario set the safety first selection problem is reduced to a mixed Boolean linear programming problem, where continuous variables correspond to portfolio components and the number of Boolean variables coincides with the number of scenarios. The latter problem is solved by a standard branch and bound method. Theoretical considerations are illustrated by a numerical example of a portfolio with nine components and eighteen return scenarios.

Appendix 1. Reduction of a probability optimization problem with discrete distribution to a mixed-integer programming problem.

Let random vector ω in (1) takes values from a finite set Ω with probabilities $p_\omega, \omega \in \Omega$. Denote $P_k(x) = \sum_{\omega\in\Omega} p_\omega I_{\{f_k(x,\omega)<0\}}$, $k = 1,...,K$, where $I_{\{f_k(x,\omega)<0\}}$ is the indicator function of the event $\{ f_k(x,\omega) < 0\}$, i.e. $I_{\{ f_k(x,\omega)<0\}} = 1$ if $f_k(x,\omega) < 0$, and $I_{\{ f_k(x,\omega)<0\}} = 0$ otherwise. Function $P_k(x)$ has a sense of probability of the event $\{ f_k(x,\omega) < 0\}$. Consider a problem:

$$G_0\left(x, P_1(x),..., P_K(x)\right) \to \min_{x\in X}, \tag{24}$$

$$G_j\left(x, P_1(x),..., P_K(x)\right) \le 0, \quad j = 1,...,m, \tag{25}$$

where functions $G_j(x,\pi_1,...,\pi_K)$, $j = 0,1,...,m$, are monotonic (non-decreasing) in their arguments $\pi_1 \ge 0,...,\pi_K \ge 0$, X is a compact set. Particular cases of problem (24), (25) are a probability optimization problem and a chance constrained one. Next we reduce problem (24), (25) to a mixed integer programming one. Assume that

$$\inf_{x\in X} f_k(x,\omega) \ge -M_{k\omega} > -\infty .$$

Let us introduce variables $z_{k\omega} \in \{0,1\}$, $z_k = \{z_{k\omega}, \omega\in\Omega\}$ and linear Boolean functions $p_k(x,z_k) = \sum_{\omega\in\Omega} p_\omega z_{k\omega}$. Consider a problem:

$$G_0\left(x, p_1(x,z_1),..., p_K(x,z_K)\right) \to \min_{x\in X, \{z_{k\omega}\}}, \tag{26}$$

$$G_j\left(x, p_1(x,z_1),..., p_K(x,z_K)\right) \le 0, \quad j = 1,...,m, \tag{27}$$

$$-f_k(x,\omega) \le M_{k\omega} z_{k\omega}, \quad z_{k\omega} \in \{0,1\}, \quad \omega\in\Omega, \qquad k = 1,...,K. \tag{28}$$

Theorem 2. Let X be a compact set, and Ω be a finite set; functions $f_k(\cdot,\omega)$ are continuous for any $\omega\in\Omega$; functions $G_j\left(x,\pi_1,...,\pi_K\right)$ are continuous jointly in its arguments and non-decreasing in $\pi_1 \ge 0,...,\pi_K \ge 0$; $\inf_{x\in X} f_k(x,\omega) \ge -M_{k\omega} > -\infty$, $M_{k\omega} > 0$. Then problems (4) and (5), (7) are equivalent in the sense that their solutions exist and their optimal values coincide; besides, if x^* is a solution of problem (24), (25), then $\left(x^*, \left\{z^*_{k\omega} = I_{\{ f_k(x^*,\omega)<0\}}\right\}\right)$ is a solution of (26)-(28), and conversely, if $\left(x^*, \{z^*_{k\omega}\}\right)$ is a solution of problem (26)-(28), then x^* is a solution of (24), (25).

Proof. Let us prove the equivalence of optimization problems (24), (25) and (26)-(28) in the following sense [28, page 131]: first we show that optimal values of the problems jointly exist or do not exist, then set up a correspondence between optimal solutions and finally prove that in case of solutions existence optimal values of the problems coincide. For this it is sufficient for each feasible point of one problem to point out a feasible point of the other with not worse objective function value.

Let $\left(x, \{z_{k\omega}\}\right)$ be a feasible solution of problem (26)-(28). If $f_k(x,\omega) < 0$, then from (28) it follows that with necessity $z_{k\omega} = 1$. Otherwise (28) does not put any restrictions on $z_{k\omega} \in \{0,1\}$. So,

$$p_k(x,z_k) = \sum_{\omega\in\Omega} p_\omega z_{k\omega} \ge \sum_{\omega\in\Omega} p_\omega I_{\{ f_k(x,\omega)<0\}} = P_k(x),$$

and by monotonicity of functions $G_j(x,\pi_1,...,\pi_K)$ in their arguments $\pi_1 \ge 0,...,\pi_K \ge 0$, holds

$$G_j\left(x, p_1(x,z_1),..., p_K(x,z_K)\right) \geq G_j\left(x, P_1(x),..., P_K(x)\right).$$

Thus, for each feasible solution $\left(x,\{z_{k\omega}\}\right)$ of problem (26)-(28) it corresponds a feasible solution x of problem (24), (25) with not greater objective function value $G_0\left(x, P_1(x),..., P_K(x)\right) \leq G_0\left(x, p_1(x,z_1),..., p_K(x,z_K)\right)$.

Conversely, let x be an arbitrary feasible solution of problem (24), (25). Calculate $z_{k\omega} = I_{\{f_k(x,\omega)<0\}}$ for all k and ω. Obviously,

$$G_j\left(x, P_1(x),..., P_K(x)\right) = G_j\left(x, p_1(x,z_1),..., p_K(x,z_K)\right).$$

Thus, for each feasible solution x of problem (24), (25) it corresponds a feasible solution $\left(x,\left\{z_{k\omega} = I_{\{f_k(x,\omega)<0\}}\right\}\right)$ of problem (26)-(28) with not greater value of the objective function.

From here it follows that problems (24), (25) and (26)-(28) simultaneously both either have or do not have feasible solutions.

In conditions of the theorem functions $I_{\{f_k(x,\omega)<0\}}$, $G_j\left(x, P_1(x),..., P_K(x)\right)$, $j = 0,1,...,m$, in (24), (25) are lower semicontinuous, and the set X is compact. So, if a feasible solution of problem (24), (25) exists, then there exits its optimal solution x^*. In problem (26)-(28) functions $\left(f_k(x,\omega) + M_{k\omega}z_{k\omega}\right)$ and $G_j\left(x, p_1(x,z_1),..., p_K(x,z_K)\right)$ are continuous, set X and the set of admissible solutions of $\{z_{k\omega}\}$ are compact, so if a feasible solution of problem (26)-(28) exists, then there exists its optimal solution $\left(x^*,\{z^*_{k\omega}\}\right)$. From here by established correspondence of feasible solutions $x \leftrightarrow \left(x,\left\{z_{k\omega} = I_{\{f_k(x,\omega)<0\}}\right\}\right)$ it follows that in case of feasibility of problems (24), (25) and (26)-(28) their optimal values coincide. Besides, if x^* is an optimal solution of problem (24), (25), then $\left(x^*,\left\{z^*_{k\omega} = I_{\{f_k(x^*,\omega)<0\}}\right\}\right)$ is an optimal solution of (26)-(28). Conversely, if $\left(x^*,\{z^*_{k\omega}\}\right)$ is an optimal solution of (26)-(28), then x^* is an optimal solution of (24), (25). The proof is complete.

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