On Doubly Positive Semidefinite Programming Relaxations

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Abstract

Recently, researchers have been interested in studying the semidefinite programming (SDP) relaxation model, where the matrix is both positive semidefinite and entry-wise nonnegative, for quadratically constrained quadratic programming (QCQP). Comparing to the basic SDP relaxation, this doubly-positive SDP model possesses additional $O(n^2)$ constraints, which makes the SDP solution complexity substantially higher than that for the basic model with $O(n)$ constraints. In this paper, we prove that the doubly-positive SDP model is equivalent to the basic one with a set of valid quadratic cuts. When QCQP is symmetric and homogeneous (which represents many classical combinatorial and nonconvex optimization problems), the doubly-positive SDP model is equivalent to the basic SDP even without any valid cut. On the other hand, the doubly-positive SDP model could help to tighten the bound up to 36%, but no more.

1 Introduction

Consider the quadratically constrained quadratic programming problem

$$\begin{array}{rl} \text{Maximize} & x^T Q_0 x + c_0^T x \\ \text{Subject to} & x^T Q_i x + c_i^T x = b_i, \ i = 1, \ldots, m, \\ & -e \le x \le e, \end{array} \tag{1}$$

where symmetric matrix $Q_i \in \Re^{n\times n}$ and vector $c_i \in \Re^n$, $i = 0, 1, \cdots, m$, and $e \in \Re^n$ is the vector of all ones. Note that any other lower and upper bounds on decision variables, $l \le x \le u$, can be transformed to $-e \le x \le e$ through scaling and linear translation. Also, the results developed in this paper are easily extendable to quadratic inequality constraints. We assume that the QP problem is known to be feasible, so that any of its relaxation models would be also feasible.

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The classical and basic semidefinite programming relaxation for problem (1) is

$$\begin{array}{ll} \text{Maximize} & Q_0 \cdot X + c_0^T x \\ \text{Subject to} & Q_i \cdot X + c_i^T x = b_i,\ i = 1, 2, \cdots, m, \\ & X_{jj} \leq 1,\ j = 1, 2, \cdots, n, \\ & \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0. \end{array} \tag{2}$$

If the SDP solution has a rank one property, that is, $X^* = x^*(x^*)^T$, then x^* solves problem (1).

Recently, there are research efforts to construct stronger or tighter SDP relaxations for QCQP [11, 15]. One particular effort is to let $y = (x + e)/2$ so that problem (1) has an equivalent form within the nonnegative domain:

$$\begin{array}{ll} \text{Maximize} & 4y^T Q_0 y + (2c_0^T - 4e^T Q_0)y + e^T Q_0 e - c_0^T e \\ \text{Subject to} & 4y^T Q_i y + (2c_i^T - 4e^T Q_i)y + e^T Q_i e - c_i^T e = b_i,\ i = 1, \ldots, m, \\ & 0 \leq y \leq e. \end{array} \tag{3}$$

Using the knowledge that all decision variables need to be nonnegative, the following SDP relaxation can be constructed:

$$\begin{array}{lll} v_p^* := & \text{Maximize} & 4Q_0 \cdot Y + (2c_0^T - 4e^T Q_0)y + e^T Q_0 e - c_0^T e \\ & \text{Subject to} & 4Q_i \cdot Y + (2c_i^T - 4e^T Q_i)y + e^T Q_i e - c_i^T e = b_i,\ i = 1, \ldots, m, \\ & & Y_{jj} \leq y_j \leq 1,\ j = 1, 2, \cdots, n, \\ & & Z := \begin{bmatrix} Y & y \\ y^T & 1 \end{bmatrix} \succeq 0, \\ & & Y_{ij} \geq 0,\ \forall 1 \leq i < j \leq n. \end{array} \tag{4}$$

Since $Z \succeq 0$ as well as $Z \geq 0$, it is called a *doubly-positive semidefinite program*; e.g., see Dong et al. [5], Burer [3] and Burer et al. [4]. It is well known that there exists a hierarchy of linear and semidefinite representable cones that approximate the co-positive and completely positive cone (see Bomze et al. [2] and Parrilo [10]), where the doubly-positive SDP is a mostly used relaxation technique due to its computability. Very Recently researchers have compared its theoretical performance with the D.C. relaxations on lower bounds for quadratic programs(Anstreicher et al. [1] and Zheng et al. [16]), and extended its applications in different areas, e.g., appointment scheduling by Kong et al. [7], order statistics by Natarajan et al. [8].

The doubly-positive SDP increases the number of constraints from $m+n$ in basic SDP model (2) to $m+2n+n(n-1)/2$ in (4). With such a sacrifice in computational complexity, (4) must be stronger or tighter than (2). In this paper, we are trying to answer this very question: when and how much is the doubly-positive SDP relaxation tighter than the basic SDP one?

2 The basic SDP relaxation with valid cuts

In fact, one can add valid cuts

$$1 + X_{ij} + x_i + x_j \geq 0,,\ \forall 1 \leq i < j \leq n;$$

into basic model (2) from the fact that binary product $(1 + x_i)(1 + x_j) \geq 0$ always in original problem (1). Then, SDP relaxation model (2) becomes

$$
\begin{array}{rll}
v_s^* := & \text{Maximize} & Q_0 \cdot X + c_0^T x \\
& \text{Subject to} & Q_i \cdot X + c_i^T x = b_i, \ i = 1, 2, \cdots, m, \\
& & X_{jj} \leq 1, \ j = 1, 2, \cdots, n, \\
& & \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0, \\
& & 1 + X_{ij} + x_i + x_j \geq 0, \ \forall 1 \leq i < j \leq n.
\end{array} \tag{5}
$$

This strengthened SDP relaxation has $m + n + n(n-1)/2$ constraints. We now show the following equivalence theorem:

Theorem 1. *SDP relaxations (5) and (4) produce exactly the same optimal objective value, that is,* $v_s^* = v_p^*$.

Proof. First, we prove $v_p^* \geq v_s^*$.

Given an optimal solution (X^*, x^*) to (5), let $Y_{ij} = \frac{1}{4}(X_{ij}^* + x_i^* + x_j^* + 1)$ and $y_j = (x_j^* + 1)/2$, $1 \leq i, j \leq n$, that is, $Y = \frac{1}{4}(X^* + e(x^*)^T + x^* e^T + ee^T)$ and $y = (x^* + e)/2$.

First, one can see, for all $i \leq j \leq n$, $Y_{ij} \geq 0$, since $X_{ij}^* + x_i^* + x_j^* + 1 \geq 0$ in (5).

Secondly,

$$
Y - yy^T = \frac{1}{4}(X^* + e(x^*)^T + x^* e^T + ee^T) - \frac{1}{4}(x^* + e)(x^* + e)^T = \frac{1}{4}(X^* - x^*(x^*)^T) \succeq 0,
$$

so that $Z \succeq 0$.

Thirdly, $Y_{jj} = \frac{1}{4}(X_{jj}^* + 2x_j^* + 1) \leq (2 + 2x_j^*)/4 = (1 + x_j^*)/2 = y_j$ for all $j = 1, 2, \cdots, n$.

Finally, for $i = 0, 1, \cdots, n$,

$$
\begin{array}{rl}
& 4Q_i \cdot Y + (2c_i^T - 4e^T Q_i)y + e^T Q_i e - c_i^T e \\
= & Q_i \cdot (X^* + e(x^*)^T + x^* e^T + ee^T) + (c_i^T - 2e^T Q_i)(x^* + e) + e^T Q_i e - c_i^T e \\
= & Q_i \cdot X^* + c_i^T x^*.
\end{array}
$$

Thus (Y, y) is a feasible solution to (4) and its objective value equals v_s^*, which implies $v_p^* \geq v_s^*$.

Next we prove $v_p^* \leq v_s^*$. Given an optimal solution (Y^*, y^*) to (4), we let $X_{ij} = 4Y_{ij}^* - 2y_i^* - 2y_j^* + 1$ and $x_j = 2y_j^* - 1$, $1 \leq i, j \leq n$, that is, $X = 4Y^* - 2e(y^*)^T - 2y^* e^T + ee^T$ and $x = 2y^* - e$.

First, one can see, for all $i \leq j \leq n$,

$$
X_{ij} + x_i + x_j + 1 = 4Y_{ij}^* \geq 0;
$$

and for all $j = 1, \cdots, n$, $X_{jj} = 4(Y_{jj}^* - y_j^*) + 1 \leq 1$.

Secondly,

$$
X - xx^T = 4Y^* - 2e(y^*)^T - 2y^* e^T + ee^T - (2y^* - e)(2y^* - e)^T = 4(Y^* - y^*(y^*)^T) \succeq 0.
$$

Finally, for $i = 0, 1, \cdots, n$,

$$
\begin{array}{rl}
& Q_i \cdot X + c_i^T x \\
= & Q_i \cdot \left(4Y^* - 2e(y^*)^T - 2y^* e^T + ee^T\right) + c_i^T (2y^* - e) \\
= & 4Q_i \cdot Y^* + (2c_i^T - 4e^T Q_i)y^* + e^T Q_i e - c_i^T e.
\end{array}
$$

Thus X is a feasible solution to (5) whose objective value equals v_p^*, which implies $v_p^* \leq v_s^*$. □

Theorem 1 implies that the doubly-positive SDP relaxation is precisely the basic SDP relaxation with additional $n(n-1)/2$ valid binary product cuts. (In fact, the latter has n constraints fewer than the former does.) It also has practical implication, since an effective rounding procedure have been developed and analyzed for basic model (2), which can be extended to strengthened relaxation model (5). To the best our knowledge, there is no effective rounding procedure currently available for SDP relaxation model (4).

3 Homogeneous and symmetric QCQP

In this section, we show that the doubly-positive SDP relaxation is precisely the basic SDP relaxation even without the valid binary quadratic cuts, when the quadratic functions are homogeneous. Here we consider the homogeneous QCQP problem:

$$\begin{array}{rl} \text{Maximize} & x^T Q_0 x \\ \text{Subject to} & x^T Q_i x = b_i,\ i = 1, \ldots, m, \\ & -e \leq x \leq e. \end{array} \tag{6}$$

The solution set of (6) is symmetric, meaning that when x is a solution, so is $-x$.

In general, problem (6) remains NP-hard, and it includes many classical combinatorial and non-convex optimization problems. Since there is no linear term in each of the quadratic function, the basic SDP relaxation can be simplified to:

$$\begin{array}{rrl} v_b^* := & \text{Maximize} & Q_0 \cdot X \\ & \text{Subject to} & Q_i \cdot X = b_i,\ i = 1, 2, \cdots, m, \\ & & X_{jj} \leq 1,\ j = 1, 2, \cdots, n, \\ & & X \succeq 0. \end{array} \tag{7}$$

Note that basic relaxation (7) has only $m + n$ constraints.

If (6) represents the max-cut problem, Goemans and Williamson [6] developed a rounding procedure to produce a feasible solution to (6) from the maximal solution matrix of (7) such that the feasible solution is 0.878-optimal. If (6) represents the max-bisection problem, Ye [14] developed a rounding procedure to produce a feasible solution that is 0.699-optimal. More generally, when coefficient matrix Q_i is diagonal for all i, Nesterov [9] and Ye [13] developed a rounding procedure to produce a feasible solution that is $\frac{2}{\pi}$-optimal.

Note now that the doubly-positive SDP relaxation of (6) becomes

$$\begin{array}{rrl} v_p^* := & \text{Maximize} & 4Q_0 \cdot Y - 4e^T Q_0 y + e^T Q_0 e \\ & \text{Subject to} & 4Q_i \cdot Y - 4e^T Q_i y + e^T Q_i e = b_i,\ i = 1, 2, \cdots, m, \\ & & Y_{jj} \leq y_j \leq 1,\ j = 1, 2, \cdots, n, \\ & & Z = \begin{bmatrix} Y & y \\ y^T & 1 \end{bmatrix} \succeq 0, \\ & & Z \geq 0. \end{array} \tag{8}$$

We prove that, in the homogeneous and symmetric case, the doubly-positive SDP relaxation adds no value to the basic relaxation:

Theorem 2. *SDP relaxations (7) and (8) produce exactly the same optimal objective value, that is, $v_b^* = v_p^*$.*

Proof. The proof is similar to the proof of Theorem 1 by treading $x^* = 0$.

Given an optimal solution X^* to (7), let $Y_{ij} = \frac{1}{4}(X_{ij}^* + 1)$ and $y_j = \frac{1}{2}$, $1 \le i, j \le n$, that is, $Y = \frac{1}{4}(X^* + ee^T)$ and $y = e/2$.

First, one can see $Y_{ij} \ge 0$ since $-1 \le X_{ij}^* \le 1$ for all i, j. Secondly, $Y - yy^T = \frac{1}{4}(X^* + ee^T) - \frac{1}{4}ee^T = \frac{1}{4}X^* \succeq 0$. Thirdly, $Y_{jj} = \frac{1}{4}(X_{jj}^* + 1) = 1/2 = y_j$. Finally, for $i = 0, 1, \cdots, n$,

$$\begin{aligned} & 4Q_i \cdot Y - 4e^T Q_i y + e^T Q_i e \\ = \; & Q_i \cdot (X^* + ee^T) - 2Q_i \cdot ee^T + Q_i \cdot ee^T \\ = \; & Q_i \cdot X^*. \end{aligned}$$

Thus (Y, y) is a feasible solution to (8) and its objective value equals v_b^*. This implies $v_p^* \ge v_b^*$.

On the other hand, given an optimal solution (Y^*, y^*) to (8), we let $X_{ij} = 4Y_{ij}^* - 2y_i^* - 2y_j^* + 1$.

First, from $X_{ij} = 4(Y_{ij}^* - y_i^* y_j^*) + 4y_i^* y_j^* - 2y_i^* - 2y_j^* + 1$, we see

$$X = 4(Y^* - y^*(y^*)^T) + (2y^* - e)(2y^* - e)^T \succeq 0.$$

Secondly, we have $X_{jj} = 4Y_{jj}^* - 4y_j^* + 1 \le 1$.

Finally, for $i = 0, 1, \cdots, n$,

$$\begin{aligned} & Q_i \cdot X \\ = \; & Q_i \cdot \left(4(Y^* - y^*(y^*)^T) + (2y^* - e)(2y^* - e)^T\right) \\ = \; & 4Q_i \cdot Y^* - 4Q_i \cdot yy^T + 4Q_i \cdot y^*(y^*)^T - 4e^T Q_i y^* + Q_i \cdot ee^T \\ = \; & 4Q_i \cdot Y^* - 4e^T Q_i y^* + e^T Q_i e. \end{aligned}$$

Thus X is a feasible solution to (7) whose objective value equals v_p^*. This implies $v_p^* \le v_b^*$. □

4 How tighter the doubly-positive SDP model is

Now we ask the question: what difference the doubly-positive SDP relaxation could make comparing to the basic one without any valid cut. Due to Theorem 1, the question is as the same as: what difference strengthened SDP relaxation (5) could make comparing to basic model (2). For simplicity, we consider SDP relaxation for non-homogeneous binary QP:

$$\begin{aligned} v_b^* := \quad & \text{Maximize} \quad Q \cdot \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \\ & \text{Subject to} \quad X_{jj} = 1, \; j = 1, 2, \cdots, n, \\ & \qquad\qquad\quad \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0, \end{aligned} \tag{9}$$

where the coefficient matrix $Q \in \Re^{(n+1)\times(n+1)}$.

The strengthened SDP relaxation model becomes

$$
\begin{array}{rll}
v_p^* := & \text{Maximize} & Q \cdot \begin{bmatrix} Y & y \\ y^T & 1 \end{bmatrix} \\
& \text{Subject to} & Y_{jj} = 1,\ j = 1, 2, \cdots, n, \\
& & \begin{bmatrix} Y & y \\ y^T & 1 \end{bmatrix} \succeq 0, \\
& & 1 + Y_{ij} + y_i + y_j \geq 0,\ \forall 1 \leq i < j \leq n.
\end{array} \tag{10}
$$

Our next theorem is

Theorem 3. *For SDP problems (9) and (10), if Q is positive semidefinite then*

$$v_b^* \geq v_p^* \geq \frac{2}{\pi} v_b^*;$$

if Q is a graph Laplacian matrix with nonnegative edge weights then

$$v_b^* \geq v_p^* \geq 0.878 v_b^*.$$

Proof. It is clear that $v_b^* \geq v_p^*$.

Let (X^*, x^*) be a maximal solution of SDP (9), and let

$$Y_{ij} = \frac{2}{\pi} \arcsin(X_{ij}^*) \quad \text{and} \quad y_j = \frac{2}{\pi} \arcsin(x_j^*)$$

for all $1 \leq i \leq j \leq n$. That is,

$$\begin{bmatrix} Y & y \\ y^T & 1 \end{bmatrix} = \frac{2}{\pi} \arcsin \left[\begin{bmatrix} X^* & x^* \\ (x^*)^T & 1 \end{bmatrix} \right].$$

Since for all $1 \leq i \leq j \leq n$,

$$\begin{bmatrix} 1 & X_{ij}^* & x_i^* \\ X_{ij}^* & 1 & x_j^* \\ x_i^* & x_j^* & 1 \end{bmatrix} \succeq 0,$$

we claim

$$1 + \frac{2}{\pi}(\arcsin(X_{ij}^*) + \arcsin(x_i^*) + \arcsin(x_j^*)) \geq 0.$$

This is because that, when we let

$$\hat{x}_j = \begin{cases} 1 & \text{if } u_j \geq 0, \\ -1 & \text{if } u_j < 0; \end{cases}$$

where u is a multivariate normal random vector with 0 mean and the covariance matrix $\begin{bmatrix} X^* & x^* \\ (x^*)^T & 1 \end{bmatrix}$, then let $\hat{x}_j = \hat{x}_{n+1} \hat{x}_j$ for $1 \leq j \leq n$; we have always

$$1 + \hat{x}_i + \hat{x}_j + \hat{x}_i \hat{x}_j \geq 0,$$

so that

$$\mathrm{E}[1+\hat{x}_i\hat{x}_j+\hat{x}_i+\hat{x}_j]\geq 0.$$

Furthermore,

$$\mathrm{E}[1+\hat{x}_i\hat{x}_j+\hat{x}_i+\hat{x}_j]=1+\frac{2}{\pi}(\arcsin(X^*_{ij})+\arcsin(x^*_i)+\arcsin(x^*_j))$$

from Sheppard [12], which proves the claim.

Thus, (Y,y) is a feasible solution to SDP (10), so that

$$v^*_p\geq Q\cdot\begin{bmatrix}Y & y\\ y^T & 1\end{bmatrix}=Q\cdot\frac{2}{\pi}\arcsin\left[\begin{bmatrix}X^* & x^*\\ (x^*)^T & 1\end{bmatrix}\right]\geq\frac{2}{\pi}Q\cdot\begin{bmatrix}X^* & x^*\\ (x^*)^T & 1\end{bmatrix}=\frac{2}{\pi}v^*_b.$$

Here, we have used a fact that $\arcsin[X]\succeq X$ when X is a feasible solution to (9); see Nesterov [9].

If Q is a Laplacian matrix with nonnegative weights, Goemans and Williamson [6] have showed that

$$Q\cdot\frac{2}{\pi}\arcsin\left[\begin{bmatrix}X^* & x^*\\ (x^*)^T & 1\end{bmatrix}\right]\geq 0.878Q\cdot\begin{bmatrix}X^* & x^*\\ (x^*)^T & 1\end{bmatrix}=0.878v^*_b.$$

□

Below is a small example to show the tightness of the bound. Consider an example where the Laplacian matrix

$$Q=\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}$$

in SDPs (9) and (10). One can see that $v^*_b=9$ and $v^*_p=8$. In fact, the optimal solution matrix for the latter has rank one.

We remark that Theorem 3 applies to a more strengthened SDP model where more valid (triangle) quadratic constraints

$$1+X_{ij}+X_{jk}+X_{ik}\geq 0$$

into (2) when original decision variables are either 1 or -1; or

$$2+Y_{ii}+Y_{jj}+Y_{kk}+2(Y_{ij}+Y_{jk}+Y_{ik})-3(y_i+y_j+y_k)\geq 0$$

are added into (4) when original decision variables are either 1 or 0, for all $1\leq i<j<k\leq n$. These cuts actually have $O(n^3)$ many.

We also remark that Theorem 3 applies to general SDP (2) where Q_i is diagonal and $c_i=0$ for $i=1,\cdots,m$ described in [13], that is, the doubly-positive SDP model, or the strengthened SDP model, could help to tighten the SDP upper bound up to $1-\frac{2}{\pi}$, or about 36%, but no more.

References

[1] K. Anstreicher and S. Burer, D.C. versus copositive boundeds for standard QP, *J. Global Optim.*, 43:471-484, 2009.

[2] I. M. Bomze and E. de Klerk, Solving standard quadratic optimization problems via linear, semidefinite and copositive programming. *J. Global Optim.*, 24(2):163-185, 2002. Dedicated to Professor Naum Z. Shor on his 65th birthday.

[3] S. Burer, On the copositive representation of binary and continuous nonconvex quadratic programs, *Math. Program.*, 479-495, 2009.

[4] S. Burer, K. M. Anstreicher and M. Dür, The difference between 5×5 doubly nonnegative and completely positive matrices, *Linear Algebra Appl*, 431: 1539-1552, 2009.

[5] H. Dong and K. Anstreicher, Separating Doubly Nonnegative and Completely Positive Matrices, *Working Paper*, 2010.

[6] M. X. Goemans and D. P. Williamson, Improved approximation algorithms for Maximum Cut and Satisfiability problems using semidefinite programming, *J. ACM* 42:1115–1145, 1995.

[7] Q. Kong, C. Lee, C. P. Teo and Z. Zheng, Scheduling Arrivals to a Stochastic Service Delivery System using Copositive Cones, *Manuscript*, 2010.

[8] K. Natarajan, C. P. Teo and Z. Zheng, Mixed zero-one linear programs under objective uncertainty: a completely positive representation, *Manuscript*, 2010.

[9] Yu.E. Nesterov, Semidefinite relaxation and nonconvex quadratic optimization, *Optimization Methods and Software* 9:141–160, 1998. Special Issue Celebrating the 60th Birthday of Professor Naum Shor.

[10] P. Parrilo, Structured Semidefinite Programs and Semi-algebraic Geometry Methods in Robustness and Optimization, *PhD thesis*, California Institute of Technology, 2000.

[11] S. Poljak, F. Rendl and H. Wolkowicz. A recipe for semidefinite relaxation for $(0, 1)$-quadratic programming, *J. Global Optim.*, 7(1):51-73, 1995.

[12] W. F. Sheppard, On the calculation of the double integral expressing normal correlation, *Transactions of the Cambridge Philosophical Society* 19: 23-66, 1900.

[13] Y. Ye, Approximating quadratic programming with bound and quadratic constraints, *Math. Program.*,84:219-226, 1999.

[14] Y. Ye, A .699-approximation algorithm for Max-Bisection, *Math. program.*, 90(1):101-111, 2001.

[15] X. J. Zheng, X. L. Sun and D. Li, Nonconvex Quadratically Constrained Quadratic Programming: Best D.C. Decompositions and Their SDP Representations, *J. Global Optim.*, In press, 2010.

[16] X. J. Zheng, X. L. Sun and D. Li, Convex Relaxations for Nonconvex Quadratically Constrained Quadratic Programming: Cone-Specific Best Decomposition and Polyhedral Convexification, *Manuscript*, 2010.