New formulas for the Fenchel subdifferential of the conjugate function*

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Abstract

Following [13] we provide new formulas for the Fenchel subdifferential of the conjugate of functions defined on locally convex spaces. In particular, this allows deriving expressions for the minimizers set of the lower semicontinuous convex hull of such functions. These formulas are written by means of primal objects related to the subdifferential of the initial function, namely a new enlargement of the Fenchel subdifferential operator.

Key words. Fenchel conjugate and biconjugate, Fenchel subdifferential, Lower semicontinuous convex hull, Argmin sets.

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1 Introduction

Duality is a basic tool in nonlinear (convex or not) analysis and optimization theory in which the concept of Fenchel transformation plays a central role. The main feature of this transformation is that the resulting function which is called the Fenchel conjugate is always convex. Therefore, by the abundance of results and tools from convex analysis, one disposes of much information on the behavior of this new function, namely in view of the nice properties of the Fenchel subdifferential in the convex setting ([16, 19]). For instance, differentiability or Gâteaux differentiability of the conjugate function, which is in fact related to the geometry of the subdifferential set, allows information on the convexity/strict convexity of the function itself; e.g., [2, 4, 12, 16, 17]. To benefit from this transformation in an efficient way one needs an inversion process to go back to the original function and to be able, for example, to know about its minimizers over a set, the integration of its subdifferential, its convex and lower semicontinuous hulls, etc. Roughly speaking, in order to go back and forth between the (possibly nonconvex) primal and the convex dual settings, it is necessary to have in hand formulas relating the Fenchel subdifferential of a function and its conjugate. At this point, the purpose of this paper is to provide such formulas which are valid in more general settings. Let us mention that this study enters in the intensive research on subdifferential calculus rules for pointwise supremum functions, see [3, 5, 6, 7, 10, 11, 13, 14, 18] among others.

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Explicit formulas for the Fenchel subdifferential of conjugate functions have been recently established by M.A. López and V. Volle [13]. The most general one, given in Theorem 2 of [13] in terms of the approximate subdifferentials of the initial function, applies for any extended real-valued function defined on a locally convex space X and gives us

$$\partial f^*(x^*) = \bigcap_{\substack{\varepsilon>0 \\ y^*\in \operatorname{dom} f^*}} \overline{\operatorname{co}}\left[(\partial_\varepsilon f)^{-1}(x^*) + \{y^* - x^*\}^-\right], \tag{1}$$

provided that the domain $\operatorname{dom} f^*$ of f^* is not empty. As a consequence, it is proven in Theorem 6 and Corollary 2 of [13] that

$$\partial f^*(x^*) = \bigcap_{\substack{\varepsilon>0 \\ L\in\mathcal{F}(x^*)}} \overline{\operatorname{co}}\left[(\partial_\varepsilon f)^{-1}(x^*) + \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)\right], \tag{2}$$

where $\mathcal{F}(x^*)$ either stands for $\widetilde{\mathcal{F}}_{x^*} := \{L \subset X^* \mid L \text{ convex},\ x^* \in L,\ \operatorname{rint}(\operatorname{cone}(L\cap \operatorname{dom} f^*)) \neq \emptyset\}$ or

$$\mathcal{F}_{x^*} := \{L \subset X^* \mid L \text{ is a finite-dimensional linear subspace containing } x^*\}. \tag{3}$$

It is the purpose of this paper to introduce another enlargement ∂_L^* of the Fenchel subdifferential of f (instead of $\partial_\varepsilon f$) in order to obtain new formulas modelled on (1) and (2). We shall prove (see Theorem 4) that

$$\partial f^*(x^*) = \bigcap_{L\in\mathcal{F}(x^*)} \overline{\operatorname{co}}\left[(\partial_L^* f)^{-1}(x^*) + \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)\right], \tag{4}$$

where $\mathcal{F}(x^*)$ either stands for $\mathcal{F}_{x^*}$ defined in (3) or

$$\widehat{\mathcal{F}}_{x^*} := \{L \subset X^* \text{ convex} \mid x^* \in L,\ \operatorname{rint}(L\cap \operatorname{dom} f^*) \neq \emptyset,\ f^*_{|\operatorname{rint}(L\cap \operatorname{dom} f^*)} \text{ is continuous}\}. \tag{5}$$

To fix the ideas, if the conjugate f^* is finite and continuous at some point in the interior of its domain and f is weakly lsc, then the formula in (4) simplifies to

$$\partial f^*(x^*) = \mathrm{N}_{\operatorname{dom} f^*}(x^*) + \overline{\operatorname{co}}\left[(\partial f)^{-1}(x^*)\right]. \tag{6}$$

As an immediate consequence, if we take $x^* = \theta$ in (6) then we derive a formula for the argmin set of the lower semicontinuous convex hull of f

$$\operatorname{Argmin}(\overline{\operatorname{co}} f) = \mathrm{N}_{\operatorname{dom} f^*}(\theta) + \overline{\operatorname{co}}\,[\operatorname{Argmin} f].$$

It is the place to mention that many formulas of other types for $\operatorname{Argmin}(\overline{\operatorname{co}} f)$ can be found in [1, 9, 14].

The remainder of the paper is organized as follows: in Section 2, we fix the notation which will be used throughout the paper. In Section 3, we investigate an enlargement of the Fenchel subdifferential. In Section 4, we establish in Theorem 4 the desired formulas for the subdifferential of the conjugate function. After that, some important special cases are treated in a series of corollaries, namely the formula of the Argmin set of the lower semicontinuous convex hull is provided in Corollary 8. The paper is ended up by a couple of examples.

2 Notation

In this paper, X and X^* are real locally convex (lc, for short) spaces paired in duality by a bilinear form $(x^*, x) \in X^* \times X \mapsto \langle x^*, x\rangle = \langle x, x^*\rangle = x^*(x)$. For instance, this setting includes the following situations: X is a Hilbert space, X^* is its topological dual, and $\langle\cdot,\cdot\rangle$ is the inner product; X is a reflexive Banach space, X^* is the dual normed space, and $\langle\cdot,\cdot\rangle$ is the dual product; X is endowed with the weak topology $\sigma(X, X^*)$ (Mackey topology $\tau(X, X^*)$, respectively) and X^* is endowed with the weak topology $\sigma(X^*, X)$ (Mackey topology $\tau(X^*, X)$, respectively), etc. The null vector in the involved spaces are all denoted by θ, and the convex symmetric neighborhoods of θ are called θ–neighborhoods. We use the notation $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.

The following notation and preliminary results are standard in convex analysis [8, 16, 19]. Given two nonempty sets A and B in X (or in X^*) and $\Lambda \subset \mathbb{R}$, we define

$$A + B := \{a + b \mid a \in A,\ b \in B\},\ \Lambda A := \{\lambda a \mid \lambda \in \Lambda,\ a \in A\},\ A + \emptyset := \emptyset + A := \emptyset,\ \Lambda\emptyset := \emptyset.$$

By $\operatorname{co} A$, $\operatorname{cone} A$, and $\operatorname{aff} A$, we denote the *convex hull,* the *conic hull,* and the *affine hull* of the set A, respectively. We use $\operatorname{int} A$ and $\operatorname{cl} A$ (or, indistinctly, $\overline{A}$) to respectively denote the *interior* of A and the *closure* of A. Hence, $\overline{\operatorname{co}}A := \operatorname{cl}(\operatorname{co} A)$, $\overline{\operatorname{aff}}A := \operatorname{cl}(\operatorname{aff} A)$, and $\overline{\operatorname{cone}}A := \operatorname{cl}(\operatorname{cone} A)$. We use $\operatorname{rint} A$ to denote the (topological) *relative interior* of A (i.e. the interior of A relative to $\operatorname{aff} A$ if $\operatorname{aff} A$ is closed, and the empty set otherwise (see [19])).

By A° and A^-, we respectively denote the (one-sided) *polar* and the *negative dual cone* of A given by $A^\circ := \{x^* \in X^* \mid \langle x^*, x\rangle \le 1\ \forall\ x \in A\}$ and $A^- := (\operatorname{cone} A)^\circ$. In particular, by the *bipolar theorem* we have the equalities

$$A^{\circ\circ} := (A^\circ)^\circ = \overline{\operatorname{co}}(A \cup \{\theta\}),\ A^{--} := (A^-)^- = \overline{\operatorname{cone}}(\operatorname{co} A).$$

The *normal cone* to A at x is defined as

$$\mathrm{N}_A(x) := \begin{cases} (A - x)^- & \text{if } x \in A, \\ \emptyset & \text{if } x \in X \setminus A. \end{cases}$$

The *support* and the *indicator* functions of A are respectively $\sigma_A : X^* \to \overline{\mathbb{R}}$ and $\mathrm{I}_A : X \to \overline{\mathbb{R}}_+$ defined by

$$\sigma_A(x^*) := \sup\{\langle x^*, a\rangle \mid a \in A\},\ \mathrm{I}_A(x) := 0 \text{ if } x \in A;\ \ +\infty \text{ if } x \in X \setminus A,$$

with the convention that $\sigma_\emptyset = -\infty$.

If $f : X \to \overline{\mathbb{R}}$ (or $f : X^* \to \overline{\mathbb{R}}$) is a given function, we use $\operatorname{dom} f$ and $\operatorname{epi} f$ to denote respectively the (effective) *domain* and the *epigraph* of f

$$\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\},\ \ \operatorname{epi} f := \{(x, \lambda) \in X \times \mathbb{R} \mid f(x) \le \lambda\}.$$

We say that f is *proper* if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. If $A \subset X$ (or X^*), we denote $f_{|A}$ the restriction of the function f to the subset A.

The *lower semicontinuous (lsc) hull* and the *lsc convex hull* of f, respectively written $\operatorname{cl} f$, $\overline{\operatorname{co}}f : X \longrightarrow \overline{\mathbb{R}}$, are defined so that

$$\operatorname{epi}(\operatorname{cl} f) := \operatorname{cl}(\operatorname{epi} f),\ \ \operatorname{epi}(\overline{\operatorname{co}}f) := \overline{\operatorname{co}}(\operatorname{epi} f).$$

If $\overline{\operatorname{co}}f = f$ and f is proper then we write $f \in \Gamma_0(X)$. If f equals its *weak lsc hull* $\operatorname{cl}^w f$, then we say that f is weakly lsc (or $\sigma(X, X^*)$-lsc).

The *(Fenchel) conjugate* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ given by

$$f^*(x^*) := \sup\{\langle x, x^*\rangle - f(x), x \in X\},$$

while $f^{**} : X \to \overline{\mathbb{R}}$ given by

$$f^{**}(x) := \sup\{\langle x, x^*\rangle - f^*(x^*), x^* \in X\}$$

is the *(Fenchel) biconjugate* of f. It is known that $f^{**} = \overline{\text{co}} f$ if and only if f is minorized by a continuous affine function.

If $M : X \rightrightarrows X^*$ (or $X^* \rightrightarrows X$) is a set-valued operator, $M^{-1} : X^* \rightrightarrows X$ denotes its inverse set-valued operator defined as

$$M^{-1}(x^*) := \{x \in X \mid x^* \in Mx\};$$

in the sequel, we identify M to its graph $\{(x, x^*) \in X \times X^* \mid x^* \in Mx\}$. Finally, when $\inf_X f \in \mathbb{R}$ and $\varepsilon \geq 0$, a vector x is said to be a *global ε-minimum* of f, and we write $x \in \varepsilon\text{-Argmin} f$ (or, simply, $\text{Argmin} f$ if $\varepsilon = 0$) if $f(x) \leq \inf_X f + \varepsilon$.

3 Enlargement of the Fenchel subdifferential operator

In this section, we investigate a new enlargement of the Fenchel subdifferential operator. We recall that if $f : X \to \overline{\mathbb{R}}$ is an extended real-valued function, for $\varepsilon \geq 0$ the $\varepsilon-$subdifferential of f is the set-valued map $\partial_\varepsilon f : X \rightrightarrows X^*$ which assigns to $x \in X$ the (possibly empty) w^*–closed convex set

$$\partial_\varepsilon f(x) := \{x^* \in X^* \mid f^*(x^*) + f(x) \leq \langle x^*, x\rangle + \varepsilon\}$$

(with the convention that $\partial_\varepsilon f(x) = \emptyset$ if $f(x) \notin \mathbb{R}$). In particular, $\partial f(x) := \partial_0 f(x)$ is the usual Fenchel subdifferential of f at x. We shall say that f is subdifferentiable at x if $\partial f(x) \neq \emptyset$.

It is well-known that for functions $f \in \Gamma_0(X)$, the subdifferential operator of f^* is completely characterized by the subdifferential of f in view of the straightforward relationship

$$\partial f^* = (\partial f)^{-1}.$$

But, for functions not necessarily in $\Gamma_0(X)$ the operator ∂f is small enough to build up the whole ∂f^* as the following simple example shows.

Example 1 Let $f(x) := e^{-|x|}$. Then, direct computations yield $f^* = \mathrm{I}_{\{0\}}$,

$$\partial f^*(0) = \mathrm{N}_{\{0\}}(0) = \mathbb{R}, \text{ and } (\partial f)^{-1}(0) = (\partial f)^{-1}(0) + \mathrm{N}_{\text{dom} f^*}(0) = \emptyset.$$

This example suggests to enlarge the concept of the Fenchel subdifferential by taking into account the geometry of $\text{dom} f^*$.

Definition 1 *Given a function $f : X \to \overline{\mathbb{R}}$, $L \subset X^*$ and $(x, x^*) \in X \times X^*$, we say that $(x, x^*) \in \partial_L^* f$ if and only if $x^* \in L$, $f^*(x^*) \in \mathbb{R}$, and there exists a net $(x_\gamma)_{\gamma \in D} \subset X$ (where D is a directed set) such that*

$$\lim_{\gamma \in D} \langle x_\gamma - x, y^*\rangle = 0 \quad \forall y^* \in \overline{\text{aff}}\,(L \cap \text{dom} f^* - x^*)\,, \text{ and}$$

$$\lim_{\gamma \in D} (f(x_\gamma) - \langle x_\gamma, x^*\rangle) = -f^*(x^*).$$

If $\operatorname{dom} f^* \subset L$, *for simplicity we denote* $\partial^* f := \partial_L^* f$.

In particular, if $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$ it follows from the definition above that

$$\partial^* f = \partial(\operatorname{cl}^w f), \tag{7}$$

and so both ∂^* and ∂ coincide if the function f is in addition weakly lsc. Furthermore, by Corollary 2.3 and Remark 2.4 in [18] (giving formulas for the subdifferential of the lower semi-continuous hull), (7) allows the following representation of the operator ∂^* by means of the ε-subdifferentials of the function f

$$\partial^* f(x) = \bigcap_{\substack{\varepsilon>0 \\ U\in\mathcal{N}}} \bigcup_{y\in x+U} \partial_\varepsilon f(y),$$

where $\mathcal{N}$ denotes the collection of the θ-neighborhoods in $(X, \sigma(X, X^*))$.

The following proposition gives first comparisons between ∂_L^*, ∂f, and ∂f^*.

Proposition 1 *Let* $f : X \to \overline{\mathbb{R}}$ *be a function. Then, the following statements hold*:

(i) *for every* $x \in X$ *and* $L \subset X^*$,

$$L \cap \partial f(x) \subset \partial_L^* f(x);$$

(ii) *for every* $\varepsilon > 0$, $L \subset X^*$, *and* $x^* \in L \cap \operatorname{dom} f^*$,

$$\overline{\operatorname{co}} \left\{ (\partial_L^* f)^{-1} (x^*) \right\} \subset \overline{\operatorname{co}} \left\{ (\partial_\varepsilon f)^{-1}(x^*) + \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*) \right\};$$

(iii) *for every closed subspace* $L \subset X^*$ *and* $x^* \in L \cap \operatorname{dom} f^*$,

$$\overline{\operatorname{co}} \left\{ (\partial_L^* f)^{-1} (x^*) + \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*) \right\} \subset \partial(f^* + \mathrm{I}_L)(x^*);$$

(iv) *for every* $x^* \in X^*$,

$$\bigcap_{F\in\mathcal{F}_{x^*}} \overline{\operatorname{co}} \left\{ (\partial_F^* f)^{-1} (x^*) + \mathrm{N}_{F\cap \operatorname{dom} f^*}(x^*) \right\} \subset \partial f^*(x^*),$$

where $\mathcal{F}_{x^*}$ *is defined in* (3).

Proof. (i) The proof is immediate if $L \cap \partial f(x) = \emptyset$. Otherwise, if $z^* \in L \cap \partial f(x)$ $(\subset L \cap \operatorname{dom} f^*)$, we get that $f^*(z^*) \in \mathbb{R}$ and

$$f(x) - \langle x, z^* \rangle = -f^*(z^*).$$

Hence, by taking $x_\gamma = x$ in Definition 1 it follows that $z^* \in \partial_L^* f(x)$.

(ii) The proof is obvious if $(\partial_L^* f)^{-1} (x^*) = \emptyset$ and, so, we suppose that $(\partial_L^* f)^{-1} (x^*) \neq \emptyset$ which in particular yields $(\partial_\varepsilon f)^{-1}(x^*) \neq \emptyset$. Consequently, it suffices to show that

$$\sigma_{(\partial_L^* f)^{-1}(x^*)}(v^*) \leq \sigma_{(\partial_\varepsilon f)^{-1}(x^*)+\mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)}(v^*) \quad \text{for all } v^* \in X^*. \tag{8}$$

Indeed, if $v^* \in X^* \backslash \overline{\operatorname{cone}}(L \cap \operatorname{dom} f^* - x^*)$ $(= X^* \backslash [\mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)]^-)$, we choose $v \in \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)$

such that $\langle v^*, v\rangle > 0$. Therefore, recalling that $\sigma_{(\partial_\varepsilon f)^{-1}(x^*)}(v^*) > -\infty$,

$$\sigma_{(\partial_\varepsilon f)^{-1}(x^*)+\mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*)}(v^*) \geq \sup_{\lambda>0}\langle \lambda v, v^*\rangle + \sigma_{(\partial_\varepsilon f)^{-1}(x^*)}(v^*) = +\infty;$$

i.e. (8) holds. If $v^* \in \overline{\mathrm{cone}}(L \cap \mathrm{dom}\, f^* - x^*)$, we fix $x \in (\partial_L^* f)^{-1}(x^*)$ and $\alpha > 0$. By invoking Definition 1, $f^*(x^*) \in \mathbb{R}$ and we find a net $(x_\gamma)_{\gamma\in D} \subset X$ such that, for each $\gamma \in D$,

$$\langle x_\gamma - x, v^*\rangle \leq \alpha,\ f(x_\gamma) + f^*(x^*) \leq \langle x_\gamma, x^*\rangle + \varepsilon.$$

Hence, $x_\gamma \in (\partial_\varepsilon f)^{-1}(x^*)$ and so we write

$$\langle x, v^*\rangle \leq \sigma_{(\partial_\varepsilon f)^{-1}(x^*)}(v^*) + \langle x - x_\gamma, v^*\rangle \leq \sigma_{(\partial_\varepsilon f)^{-1}(x^*)+\mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*)}(v^*) + \alpha.$$

Consequently, (8) follows by taking the supremum over $x \in (\partial_L^* f)^{-1}(x^*)$, and next letting $\alpha \to 0$.

(iii) Let us first show that

$$\partial(f^* + \mathrm{I}_L)(x^*) = \bigcap_{\substack{\varepsilon>0\\ F\in\mathcal{F}_{x^*}}} \overline{\mathrm{co}}\left\{(\partial_\varepsilon f)^{-1}(x^*) + \mathrm{N}_{F\cap L\cap \mathrm{dom}\, f^*}(x^*)\right\}. \tag{9}$$

Indeed, writing the (lsc convex) function $f^* + \mathrm{I}_L : X^* \to \overline{\mathbb{R}}$ as

$$(f^* + \mathrm{I}_L)(z^*) = \sup\{\langle z^*, z\rangle + \mathrm{I}_L(z^*) - f(z),\ z \in X\},$$

by applying Theorem 4 in [7] to the family of lsc convex functions $\{f_z(\cdot) := \langle \cdot, z\rangle + \mathrm{I}_L(\cdot) - f(z),\ z \in X\}$ we obtain that

$$\partial(f^* + \mathrm{I}_L)(x^*) = \bigcap_{\substack{\varepsilon>0\\ F\in\mathcal{F}_{x^*}}} \overline{\mathrm{co}}\{\{z + \partial_\varepsilon \mathrm{I}_L(x^*) : \langle x^*, z\rangle - f(z) \geq f^*(x^*) - \varepsilon\} + \mathrm{N}_{F\cap L\cap \mathrm{dom}\, f^*}(x^*)\}.$$

Therefore, (9) follows since that $\partial_\varepsilon \mathrm{I}_L(x^*) = L^\perp$, $L^\perp + \mathrm{N}_{F\cap L\cap \mathrm{dom}\, f^*}(x^*) = \mathrm{N}_{F\cap L\cap \mathrm{dom}\, f^*}(x^*)$, and

$$(\partial_\varepsilon f)^{-1}(x^*) = \{z \in X \mid \langle x^*, z\rangle - f(z) \geq f^*(x^*) - \varepsilon\}.$$

Now, in view of (9), the statement of (ii) leads us to

$$\overline{\mathrm{co}}\left\{(\partial_L^* f)^{-1}(x^*)\right\} \subset \partial(f^* + \mathrm{I}_L)(x^*). \tag{10}$$

Consequently, if $\partial(f^* + \mathrm{I}_L)(x^*) = \emptyset$, then $(\partial_L^* f)^{-1}(x^*) = \emptyset$ and, so,

$$\overline{\mathrm{co}}\left\{(\partial_L^* f)^{-1}(x^*)\right\} = \overline{\mathrm{co}}\left\{(\partial_L^* f)^{-1}(x^*) + \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*)\right\} = \partial(f^* + \mathrm{I}_L)(x^*) = \emptyset;$$

that is (iii) holds. If $\partial(f^* + \mathrm{I}_L)(x^*) \neq \emptyset$, by [19, Exer. 2.23] we have that $\mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*) = (\partial(f^* + \mathrm{I}_L)(x^*))_\infty$ and so, again by (10),

$$\overline{\mathrm{co}}\left\{(\partial_L^* f)^{-1}(x^*)\right\} + \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*) \subset \partial(f^* + \mathrm{I}_L)(x^*) + \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*) = \partial(f^* + \mathrm{I}_L)(x^*),$$

which leads us, as $\partial(f^* + \mathrm{I}_L)(x^*)$ is closed and convex, to

$$\overline{\mathrm{co}}\left\{(\partial_L^* f)^{-1}(x^*) + \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(x^*)\right\} \subset \partial(f^* + \mathrm{I}_L)(x^*),$$

showing that (iii) also holds in this case.

(iv) This statement follows from (iii) in view of the relationship $\bigcap_{F\in\mathcal{F}_{x^*}} \partial(f^* + \mathrm{I}_F)(x^*) = \partial f^*(x^*)$. ■

In the following Proposition we show that in the convex case ∂_L^* almost agree with the usual Fenchel subdifferential.

Proposition 2 *Let $f : X \to \overline{\mathbb{R}}$ be a function and fix $x \in X$.*

(i) *If f is convex, then for every closed subspace $L \subset X^*$ satisfying $L \cap \partial f(x) \neq \emptyset$, we have that*

$$\partial_L^* f(x) = L \cap \partial f(x).$$

(ii) *if $f \in \Gamma_0(X)$, then $\partial f(x)$ and $\partial^* f(x)$ coincide, i.e. $\partial f(x)$ admits the following equivalent representation*

$$\begin{aligned}\partial f(x) = \{&x^* \in X^* \mid f^*(x^*) \in \mathbb{R} \text{ and } \exists \text{ a net } (x_\gamma)_{\gamma\in D} \subset X, \text{ s.t.}\\ &\lim_{\gamma\in D}\langle x_\gamma - x, y^*\rangle \to 0 \;\; \forall y^* \in \overline{\mathrm{aff}}(\mathrm{dom}\, f^* - x^*), \;\; \lim_{\gamma\in D}(f(x_\gamma) - \langle x_\gamma, x^*\rangle) = -f^*(x^*)\}.\end{aligned}$$

Proof. (i) We fix a closed subspace $L \subset X^*$. By Proposition 1(i) it is sufficient to show that

$$\partial_L^* f(x) \subset L \cap \partial f(x).$$

This inclusion is obvious if $\partial_L^* f(x) = \emptyset$. Otherwise, if $z^* \in \partial_L^* f(x)$, Proposition 1(iii) gives us $x \in \partial(f^* + \mathrm{I}_L)(z^*) = \partial(f\Box\sigma_L)^*(z^*)$, where $f\Box\sigma_L(\cdot) := \inf_{y'\in X}\{f(y') + \sigma_L(\cdot - y')\}$ is the inf-convolution of f and σ_L. So, $z^* \in \partial(f\Box\sigma_L)^{**}(x)$. Since that $L \cap \partial f(x) \neq \emptyset$, by assumption, we obtain that $\partial(f\Box\sigma_L)(x) = L \cap \partial f(x) \neq \emptyset$. Therefore, $z^* \in \partial(f\Box\sigma_L)^{**}(x) = \partial(f\Box\sigma_L)(x) = L \cap \partial f(x)$, as we wanted to prove.

(ii) As in (i) it suffices to show that $\partial^* f(x) \subset \partial f(x)$. We fix $z^* \in \partial^* f(x)$ $(= \partial_{X^*}^* f(x))$. By Proposition 1(iii) we get that $x \in \partial(f^* + \mathrm{I}_{X^*})(z^*) = \partial f^*(z^*)$. Hence, as $f \in \Gamma_0(X)$ we deduce that $z^* \in \partial f^{**}(x) = \partial f(x)$. ■

Let us once again verify Example 1 by using now the new operator ∂_L^*.

Example 2 (Continuation of Example 1) As $\mathrm{aff}(\mathrm{dom}\, f^*) = \{0\}$ and $f^*(0) = 0$, it follows that $x \in (\partial^* f)^{-1}(0)$ if and only if there exists a sequence $(x_k)_k \subset \mathbb{R}$ such that

$$f(x_k) - 0x_k = f(x_k) \to -f^*(0) = 0.$$

Therefore, observing that $\lim_{k\to+\infty} e^{-k} = 0$, $(\partial^* f)^{-1}(0) = \mathbb{R}$ and so we write

$$\partial f^*(0) = (\partial^* f)^{-1}(0) = \mathbb{R};$$

in other words, in view of Proposition 2 we have the formula

$$\partial^* f^*(0) = (\partial^* f)^{-1}(0).$$

Next, we give a couple of simple examples illustrating Definition 1; the one in (a) is Example 4.1 in [1].

Example 3 (a) Let $f(a,b) := \sqrt{a^2 + e^{-|b|}}$, for $a, b \in \mathbb{R}$. Then, because $\overline{\text{co}} f(a,b) = |a|$ it follows that

$$f^*(a,b) = (\overline{\text{co}} f)^*(a,b) = \mathrm{I}_{[-1,1]\times\{0\}}(a,b);$$

hence, $\operatorname{dom} f^* = [-1,1] \times \{0\}$ and we easily verify that $\partial f = \emptyset$. Let us calculate $\partial^* f(a,b)$ for fixed $a, b \in \mathbb{R}$. Following Definition 1 we write

$$\partial^* f(a,b) = \left\{ (\alpha, 0) \in [-1,1] \times \{0\} \mid \exists (b_k)_k \text{ s.t. } \sqrt{a^2 + e^{-|b_k|}} \to a\alpha \right\}.$$

Thus, $(\alpha, \beta) \in \partial^* f(a,b)$ if and only if $\beta = 0$ and $\alpha \in \partial |\cdot| (a)$; that is,

$$\partial^* f(a,b) = \begin{cases} \{(-1,0)\} & \text{if } a < 0 \\ [-1,1] \times \{0\} & \text{if } a = 0 \\ \{(1,0)\} & \text{if } a > 0. \end{cases}$$

(b) Let $f(a) := \sqrt{a}$ if $a \geq 0$ and $f(a) := +\infty$ if $a < 0$. Then, we have that

$$f^*(a) = 0 \text{ if } a \leq 0 \text{ and } f^*(a) = +\infty \text{ if } a > 0,$$

so that $\operatorname{dom} f^* = (-\infty, 0]$ and, for every $a \leq 0$,

$$\partial^* f(a) = \mathbb{R}_+ \text{ if } a = 0; \ \{0\} \text{ if } a < 0.$$

We end up this section by the following proposition which will be used in the proof of Theorem 4. It may be of independent interest since it sheds more light on the operators ∂_L^*.

Proposition 3 *Let be given a function $f : X \to \overline{\mathbb{R}}$, an $x^* \in X^*$, and a subset $L \in \widehat{\mathcal{F}}_{x^*}$ (see (5)). If f^* is subdifferentiable at x^*, then $\left(\partial_L^* f\right)^{-1}(x^*) \neq \emptyset$ and for every $z^* \in \operatorname{rint}(L \cap \operatorname{dom} f^* - x^*)$ we have that*

$$\limsup_{\varepsilon \to 0^+} \sigma_{(\partial_\varepsilon f)^{-1}(x^*) + \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)}(z^*) \leq \sigma_{\left(\partial_L^* f\right)^{-1}(x^*)}(z^*).$$

Proof. Let us first suppose that $x^* = \theta$ and $f^*(x^*) = f^*(\theta) = 0$ (which is possible since that f^* is subdifferentiable at x^* and $f^*(x^*) \in \mathbb{R}$). We let $L \subset X^*$ be as in the proposition (in particular, $\theta \in L$ and $\operatorname{aff}(L \cap \operatorname{dom} f^*)$ is closed), and fix $z^* \in \operatorname{rint}(L \cap \operatorname{dom} f^*)$. Thus, by the current assumption on f^*, there exists a θ-neighborhood $U \subset X^*$ such that $(z^* + U) \cap \operatorname{aff}(L \cap \operatorname{dom} f^*) \subset L \cap \operatorname{dom} f^*$ and, for all $u \in U \cap \operatorname{aff}(L \cap \operatorname{dom} f^* - z^*)$ $(= U \cap \operatorname{aff}(L \cap \operatorname{dom} f^*)$, as $\theta \in L \cap \operatorname{dom} f^*$ by assumption),

$$f^*(z^* + u) \leq f^*(z^*) + 1. \tag{11}$$

We denote

$$\beta_\varepsilon := \inf_{y \in (\partial_\varepsilon f)^{-1}(\theta) + \mathrm{N}_{L\cap \operatorname{dom} f^*}(\theta)} \langle -z^*, y \rangle \quad \text{for } \varepsilon > 0.$$

According to (2), we have that $\emptyset \neq \partial f^*(\theta) \subset (\partial_\varepsilon f)^{-1}(\theta) + \mathrm{N}_{L\cap \operatorname{dom} f^*}(\theta)$ and, so,

$$\beta_\varepsilon \leq \max\left\{1, \inf_{y \in \partial f^*(\theta)} \langle -z^*, y \rangle \right\}.$$

Moreover, due to the fact that $z^* \in L \cap \operatorname{dom} f^*$ it holds that $-\langle z^*, y \rangle \geq 0$ for all $y \in \mathrm{N}_{L\cap \operatorname{dom} f^*}(\theta)$ and, so,

$$\beta_\varepsilon \geq \inf_{y \in (\partial_\varepsilon f)^{-1}(\theta)} \langle -z^*, y \rangle \geq - \sup_{y \in (\partial_\varepsilon f)^{-1}(\theta)} \langle z^*, y \rangle - f(y) + \varepsilon \geq -f^*(z^*) - \varepsilon,$$

where in the second inequality we used the definition of $\partial_\varepsilon f$ (i.e. $f(y) - \varepsilon \le -f^*(\theta) = 0$). Consequently, we obtain that

$$-\infty < -f^*(z^*) - \varepsilon \le \beta_\varepsilon \le \sup_{\varepsilon'>0} \beta_{\varepsilon'} \le \max\left\{1, \inf_{y\in\partial f^*(\theta)} \langle -z^*, y\rangle\right\} < +\infty \quad \text{for all } \varepsilon > 0.$$

We pick $\varepsilon_k \to 0^+$ and denote $\beta_k := \beta_{\varepsilon_k}$. So, by the definition of the β_k's, and taking into account the above inequality, there are sequences $(y_k)_{k\ge1}$, $(z_k)_{k\ge1} \subset X$ such that, for each $k \ge 1$,

$$y_k \in (\partial_{\varepsilon_k} f)^{-1}(\theta),\ z_k \in \mathrm{N}_{L\cap \operatorname{dom} f^*}(\theta),\ \langle -z^*, y_k + z_k\rangle \le \beta_k + \frac{1}{k} \le \sup_{\varepsilon>0} \beta_\varepsilon + 1 < +\infty. \tag{12}$$

Hence, for every $u \in U \cap \operatorname{aff}(L \cap \operatorname{dom} f^* - z^*)$,

$$\begin{aligned}
\langle u, y_k\rangle &= \langle u + z^*, y_k\rangle + \langle y_k, z^*\rangle - 2\langle z^*, z_k + y_k\rangle + 2\langle z^*, z_k\rangle \\
&\le \langle u + z^*, y_k\rangle + \langle y_k, z^*\rangle - 2\langle z^*, z_k + y_k\rangle && (\text{as } z_k \in \mathrm{N}_{L\cap\operatorname{dom} f^*}(\theta)) \\
&\le \langle u + z^*, y_k\rangle + \langle y_k, z^*\rangle + 2\sup\nolimits_{\varepsilon>0} \beta_\varepsilon + 2 && (\text{by (12)}) \\
&\le f^*(z^* + u) + f^*(z^*) + 2f(y_k) + 2\sup\nolimits_{\varepsilon>0} \beta_\varepsilon + 2 && (\text{by Fenchel inequality}) \\
&\le f^*(z^* + u) + f^*(z^*) + 2\varepsilon_k + 2\sup\nolimits_{\varepsilon>0} \beta_\varepsilon + 2 && (\text{as } y_k \in (\partial_{\varepsilon_k} f)^{-1}(\theta)) \\
&\le \max\{1, 2f^*(z^*) + 2\varepsilon_k + 2\sup\nolimits_{\varepsilon>0} \beta_\varepsilon + 3\} =: r && (\text{by (11)}).
\end{aligned}$$

Consequently, $(y_k)_k \subset [r^{-1}(U \cap \operatorname{aff}(L \cap \operatorname{dom} f^*))]^\circ$ and, so, by Alaoglu-Bourbaki Theorem applied in the lc space $\operatorname{aff}(L \cap \operatorname{dom} f^*)$, whose topology is induced by the one of X^*, there are a subnet of $(y_k)_k$, denoted in the same way, and $y \in X$ such that

$$\langle y_k, w\rangle \to \langle y, w\rangle \quad \text{for all } w \in \operatorname{aff}(L \cap \operatorname{dom} f^*).$$

On another hand, using again the fact that $y_k \in (\partial_{\varepsilon_k} f)^{-1}(\theta)$,

$$0 = -f^*(\theta) = \inf_X f \le f(y_k) = f(y_k) + f^*(\theta) \le \varepsilon_k \quad \forall k \ge 1;$$

that is, $\lim_k f(y_k) = 0 = -f^*(\theta)$ and so $y \in (\partial_L^* f)^{-1}(\theta)$. Hence, $(\partial_L^* f)^{-1}(\theta) \ne \emptyset$ and the first conclusion follows.

Now, as $z^* \in L \cap \operatorname{dom} f^*$, and $z_k \in \mathrm{N}_{L\cap\operatorname{dom} f^*}(\theta)$ for all $k \ge 1$, from (12) we get that

$$-\sigma_{(\partial_L^* f)^{-1}(\theta)}(z^*) \le \langle -z^*, y\rangle = \lim_k \langle -z^*, y_k\rangle \le \lim_k \langle -z^*, y_k + z_k\rangle \le \liminf_{k\to\infty} \beta_k,$$

establishing the desired inequality.

To study the general case, when x^* is not necessarily null, we define the function $\tilde f : X \to \overline{\mathbb{R}}$ as

$$\tilde f(x) := f(x) - \langle x^*, x\rangle + f^*(x^*)$$

(we remember that $f^*(x^*) \in \mathbb{R}$). By direct computation it follows that $\tilde f^* := f^*(\cdot + x^*) - f^*(x^*)$ and, so, $\tilde f^*(\theta) = 0$, $(\partial_\varepsilon \tilde f)^{-1}(\theta) = (\partial_\varepsilon f)^{-1}(x^*)$ for all $\varepsilon \ge 0$, $\partial \tilde f^*(\theta) = \partial f^*(x^*)$, $\operatorname{dom} \tilde f^* =$

$\operatorname{dom} f^* - \{x^*\}$, $\operatorname{rint}((L - x^*) \cap \operatorname{dom} \tilde{f}^*) = \operatorname{rint}(L \cap \operatorname{dom} f^* - x^*)$, and

$$\mathrm{N}_{(L-x^*)\cap \operatorname{dom} \tilde{f}^*}(\theta) = \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*),\ (\partial^*_{(L-x^*)}\tilde{f})^{-1}(\theta) = (\partial^*_L f)^{-1}(x^*). \tag{13}$$

It is also clear that $\operatorname{rint}((L - x^*) \cap \operatorname{dom} \tilde{f}^*) \neq \emptyset$ and $\tilde{f}^*_{|\operatorname{rint}((L-x^*)\cap \operatorname{dom} \tilde{f}^*)}$ is continuous so that, by the first part of the proof, $(\partial^*_L f)^{-1}(x^*) = (\partial^*_{(L-x^*)}\tilde{f})^{-1}(\theta) \neq \emptyset$ and, for every $z^* \in \operatorname{rint}(L \cap \operatorname{dom} f^* - x^*) = \operatorname{rint}((L - x^*) \cap \operatorname{dom} \tilde{f}^*)$,

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} \sup_{y\in(\partial_\varepsilon f)^{-1}(x^*)+\mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)} \langle z^*, y\rangle &= \limsup_{\varepsilon\to 0^+} \sup_{y\in(\partial_\varepsilon \tilde{f})^{-1}(\theta)+\mathrm{N}_{(L-x^*)\cap \operatorname{dom} \tilde{f}^*}(\theta)} \langle z^*, y\rangle \\ &\leq \sup_{y\in\left(\partial^*_{(L-x^*)}\tilde{f}\right)^{-1}(\theta)} \langle z^*, y\rangle = \sup_{y\in\left(\partial^*_L f\right)^{-1}(x^*)} \langle z^*, y\rangle,\end{aligned}$$

completing the proof of the proposition. ■

4 Explicit formulas for the subdifferential of the conjugate function

We provide in this section the desired formulas for the subdifferential operator of the conjugate function, by means of primal objects built upon the subdifferential of the initial function, namely the operators ∂^*_L introduced in Section 3.

In the following theorem, we use the notation $\mathcal{F}_{x^*}$ and $\widehat{\mathcal{F}}_{x^*}$ defined in (3) and (5), respectively; that is,

$$\mathcal{F}_{x^*} = \{L \subset X^* \mid L \text{ is a finite-dimensional linear subspace containing } x^*\},$$

$$\widehat{\mathcal{F}}_{x^*} = \{L \subset X^* \text{ convex} \mid x^* \in L,\ \operatorname{rint}(L \cap \operatorname{dom} f^*) \neq \emptyset,\ f^*_{|\operatorname{rint}(L\cap \operatorname{dom} f^*)} \text{ is continuous}\},$$

where $f^*_{|\operatorname{rint}(L\cap \operatorname{dom} f^*)}$ is the restriction of f^* to $\operatorname{rint}(L \cap \operatorname{dom} f^*)$.

Theorem 4 *Given a function $f : X \to \overline{\mathbb{R}}$, for every $x^* \in X^*$ we have the formula*

$$\partial f^*(x^*) = \bigcap_{L\in\mathcal{F}(x^*)} \overline{\operatorname{co}} \left\{(\partial^*_L f)^{-1}(x^*) + \mathrm{N}_{L\cap \operatorname{dom} f^*}(x^*)\right\},$$

where $\mathcal{F}(x^*)$ either stands for $\mathcal{F}_{x^*}$ or $\widehat{\mathcal{F}}_{x^*}$. In particular, provided that $\operatorname{rint}(\operatorname{dom} f^*) \neq \emptyset$ and $f^*_{|\operatorname{rint}(\operatorname{dom} f^*)}$ is continuous, we have that

$$\partial f^*(x^*) = \overline{\operatorname{co}} \left\{(\partial^* f)^{-1}(x^*) + \mathrm{N}_{\operatorname{dom} f^*}(x^*)\right\}.$$

Proof. We begin by proving the first statement. The inclusion " $\supset$ " with $\mathcal{F}(x^*) = \mathcal{F}_{x^*}$ is established in Proposition 1(iv). Moreover, in view of the fact that $\mathcal{F}_{x^*} \subset \widehat{\mathcal{F}}_{x^*}$, this inclusion also holds when $\mathcal{F}(x^*) = \widehat{\mathcal{F}}_{x^*}$. So, we only need to establish the direct inclusion " $\subset$ " with $\mathcal{F}(x^*) = \widehat{\mathcal{F}}_{x^*}$ in the non trivial case $\partial f^*(x^*) \neq \emptyset$; hence, $f^*(x^*) \in \mathbb{R}$ and f^* is proper. Thus, as shown at the end of the proof of Proposition 3 (see (13)), we may assume that $x^* = \theta$ and $f^*(x^*) = f^*(\theta) = 0$; thus, in particular, $\inf_X f = 0$.

Proceeding by contradiction, if $x \notin \bigcap_{L\in\widehat{\mathcal{F}}_\theta} \overline{\operatorname{co}} \left\{(\partial^*_L f)^{-1}(\theta) + \mathrm{N}_{L\cap \operatorname{dom} f^*}(\theta)\right\}$, we find some

$L \in \widehat{\mathcal{F}}_\theta$ such that

$$x \notin \overline{\mathrm{co}}\left\{(\partial_L^* f)^{-1}(\theta) + \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(\theta)\right\}.$$

Since this last set is obviously closed, and nonempty according to Proposition 3 (as we assumed $\partial f^*(\theta) \neq \emptyset$), by the separation theorem (e.g., [19]) there exist $\tilde{x}^* \in X^* \setminus \{\theta\}$ and $\tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$ such that

$$\langle x, \tilde{x}^*\rangle > \tilde{c}_1 > \tilde{c}_2 > \langle y, \tilde{x}^*\rangle \text{ for all } y \in (\partial_L^* f)^{-1}(\theta) + \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(\theta). \tag{14}$$

In particular, we have that $\langle y, \tilde{x}^*\rangle \leq 0$, for all $y \in \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(\theta)$, which in turn yields $\tilde{x}^* \in [\mathrm{N}_{L\cap \mathrm{dom}\, f^*}(\theta)]^- = \overline{\mathrm{cone}}(L \cap \mathrm{dom}\, f^*)$. We denote $\varphi \in \Gamma_0(X^*)$ the positively homogeneous function given by

$$\varphi(z^*) := \sigma_{(\partial_L^* f)^{-1}(\theta)}(z^*) - \langle x, z^*\rangle,$$

so that (14) reads $\varphi(\tilde{x}^*) < 0$. Let us show that $\mathrm{cone}(L \cap \mathrm{dom}\, f^*) \subset \mathrm{dom}\, \varphi$. For we let $z^* \in L \cap \mathrm{dom}\, f^*$ be given, and pick $z \in (\partial_L^* f)^{-1}(\theta)$ (this set is shown above to be non-empty). Then, in view of the definition of $\partial_L^* f$ we find a net $(z_\gamma)_{\gamma\in D}$ such that $\lim_{\gamma\in D}\langle z_\gamma, z^*\rangle = \langle z, z^*\rangle$ and $\lim_{\gamma\in D} f(z_\gamma) = -f^*(\theta) = 0$. Hence, some $\gamma_0 \in D$ exists so that, invoking Fenchel inequality,

$$\langle z, z^*\rangle \leq \langle z_\gamma, z^*\rangle + 1 \leq f(z_\gamma) + f(z^*) + 1 \leq f(z^*) + 2 \text{ for all } \gamma \geq \gamma_0.$$

So, by taking the supremum over $z \in (\partial_L^* f)^{-1}(\theta)$,

$$\varphi(z^*) = \sigma_{(\partial_L^* f)^{-1}(\theta)}(z^*) - \langle x, z^*\rangle \leq f(z^*) - \langle x, z^*\rangle + 2 < +\infty,$$

showing that $z^* \in \mathrm{dom}\, \varphi$. Then, the desired inclusion $\mathrm{cone}(L \cap \mathrm{dom}\, f^*) \subset \mathrm{dom}\, \varphi$ follows from the positive homogeneity of φ.

Now, knowing that $\emptyset \neq \mathrm{rint}(L \cap \mathrm{dom}\, f^*)$ $(\subset \mathrm{rint}(\mathrm{cone}(L \cap \mathrm{dom}\, f^*)))$ and $\tilde{x}^* \in \overline{\mathrm{cone}}(L \cap \mathrm{dom}\, f^*)$, we pick $z^* \in \mathrm{rint}(L \cap \mathrm{dom}\, f^*)$ and apply the accessibility Lemma (e.g., [16]) to get that $\alpha z^* + (1-\alpha)\tilde{x}^* \in \mathrm{rint}(\mathrm{cone}(L \cap \mathrm{dom}\, f^*))$, for every $\alpha \in (0, 1]$. Consequently, invoking the convexity of φ and the facts that $\varphi(z^*) \in \mathbb{R}$ and $\varphi(\tilde{x}^*) < 0$ (from (14)), we can choose $\alpha \in (0, 1]$ small enough so that the vector $\bar{x}^* := \alpha z^* + (1-\alpha)\tilde{x}^*$ satisfies

$$\varphi(\bar{x}^*) \leq \alpha\varphi(z^*) + (1-\alpha)\varphi(\tilde{x}^*) < 0;$$

hence, $\bar{x}^* \neq \theta$ and, by the positive homogeneity of φ, we may assume that $\bar{x}^* \in L \cap \mathrm{dom}\, f^*$. In particular, the last inequalities yield the existence of $c_1, c_2 \in \mathbb{R}$ such that

$$\langle x, \bar{x}^*\rangle > c_1 > c_2 > \sigma_{(\partial_L^* f)^{-1}(\theta)}(\bar{x}^*) \geq \langle y, \bar{x}^*\rangle \quad \text{for all } y \in (\partial_L^* f)^{-1}(\theta) + \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(\theta).$$

Now, by applying Proposition 3, the last inequalities above lead us, for some $\varepsilon_0 > 0$, to

$$\sup_{y\in(\partial_\varepsilon f)^{-1}(\theta)+\mathrm{N}_{L\cap \mathrm{dom}\, f^*}(\theta)} \langle y, \bar{x}^*\rangle \leq c_1 < \langle x, \bar{x}^*\rangle \quad \text{for all } \varepsilon \in (0, \varepsilon_0],$$

which in turn implies that $x \notin \overline{\mathrm{co}}\left\{(\partial_{\varepsilon_0} f)^{-1}(\theta) + \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(\theta)\right\}$. In other words, we have proved that

$$\bigcap_{\varepsilon>0} \overline{\mathrm{co}}\left\{(\partial_\varepsilon f)^{-1}(\theta) + \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(\theta)\right\} \subset \overline{\mathrm{co}}\left\{\partial_L^* f(\theta) + \mathrm{N}_{L\cap \mathrm{dom}\, f^*}(\theta)\right\},$$

which in view of (2) yields the desired inclusion " $\subset$ " and, so, finishes the proof of the main conclusion.

To prove the last conclusion, we observe that the supplementary conditions on $\mathrm{rint}(\mathrm{dom}\, f^*)$ and f^* imply that $L := \mathrm{dom}\, f^*$ is in $\widehat{\mathcal{F}}_\theta$. Thus, by the first part of the theorem, and the fact that $\partial^* f = \partial^*_{\operatorname{dom} f^*} f$, it follows that

$$\partial f^*(\theta) \subset \overline{\text{co}} \left\{ (\partial^* f)^{-1}(\theta) + \mathrm{N}_{\operatorname{dom} f^*}(\theta) \right\}.$$

The converse inclusion is also immediate as we obviously have that $(\partial^* f)^{-1}(\theta) \subset \partial f^*(\theta)$ and $\mathrm{N}_{\operatorname{dom} f^*}(\theta)$ is the recession cone of $\partial f^*(\theta)$ (this last fact holds if $\partial f^*(\theta) \neq \emptyset$). This completes the proof of the theorem. ■

The following result gives an alternative representation of the main conclusion of Theorem 4. In it, for $y^* \in \operatorname{dom} f^*$ we denote $\partial_{y^*} f : X \rightrightarrows X^*$ the operator such that $(x, x^*) \in \partial_{y^*} f$ if and only if $f^*(x^*) \in \mathbb{R}$ and there exists a net $(x_\gamma)_{\gamma \in D} \subset X$ satisfying

$$\lim_{\gamma \in D} \langle x_\gamma - x, y^* - x^* \rangle = 0 \text{ and}$$

$$\lim_{\gamma \in D} f(x_\gamma) - \langle x_\gamma, x^* \rangle = -f^*(x^*).$$

Corollary 5 *Given a function $f : X \to \overline{\mathbb{R}}$, for every $x^* \in X^*$ we have the formula*

$$\partial f^*(x^*) = \bigcap_{y^* \in \operatorname{dom} f^*} \overline{\text{co}} \left\{ (\partial_{y^*} f)^{-1}(x^*) + [y^* - x^*]^- \right\}.$$

Proof. Given a vector $y^* \in \operatorname{dom} f^*$, we denote $L_{y^*} := x^* + \mathbb{R}\{y^* - x^*\}$ so that $L_{y^*} \in \mathcal{F}_{x^*}$ (see (3)), $\mathrm{N}_{L_{y^*} \cap \operatorname{dom} f^*}(x^*) = [y^* - x^*]^-$, and $\operatorname{aff}(L_{y^*} \cap \operatorname{dom} f^* - x^*) = \mathbb{R}\{y^* - x^*\}$. We also easily check that $(\partial^*_{L_{y^*}} f)^{-1}(x^*) = (\partial_{y^*} f)^{-1}(x^*)$ so that, by appealing to Theorem 4,

$$\partial f^*(x^*) \subset \overline{\text{co}} \left\{ (\partial_{y^*} f)^{-1}(x^*) + [y^* - x^*]^- \right\},$$

yielding the inclusion " $\subset$ " after intersecting over $y^* \in \operatorname{dom} f^*$. On the other hand, by Proposition 1(iii) we get

$$\overline{\text{co}} \left\{ (\partial_{y^*} f)^{-1}(x^*) + \mathrm{N}_{L_{y^*} \cap \operatorname{dom} f^*}(x^*) \right\} \subset \partial (f^* + \mathrm{I}_{L_{y^*}})(x^*)$$

so that, by intersecting over $y^* \in \operatorname{dom} f^*$,

$$\bigcap_{y^* \in \operatorname{dom} f^*} \overline{\text{co}} \left\{ (\partial_{y^*} f)^{-1}(x^*) + [y^* - x^*]^- \right\} \subset \bigcap_{y^* \in \operatorname{dom} f^*} \partial (f^* + \mathrm{I}_{L_{y^*}})(x^*).$$

Consequently, the inclusion " $\subset$ " holds in view of the relationship $\bigcap_{y^* \in \operatorname{dom} f^*} \partial (f^* + \mathrm{I}_{L_{y^*}})(x^*) = \partial f^*(x^*)$. ■

Next, we give a series of corollaries devoted to some important special situations relying on the topology of $\operatorname{dom} f^*$ and/or the dimension of the underlying space X.

Corollary 6 *Let $f : X \to \overline{\mathbb{R}}$ be such that* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$ *and f^* is continuous in* $\operatorname{int}(\operatorname{dom} f^*)$. *Then, for every $x^* \in X^*$ we have the formula*

$$\partial f^*(x^*) = \mathrm{N}_{\operatorname{dom} f^*}(x^*) + \overline{\text{co}} \left\{ (\partial^* f)^{-1}(x^*) \right\}.$$

In particular, provided that f is weakly lsc we have that

$$\partial f^*(x^*) = \mathrm{N}_{\operatorname{dom} f^*}(x^*) + \overline{\text{co}} \left\{ (\partial f)^{-1}(x^*) \right\}.$$

Proof. We suppose without loss of generality that $\partial f^*(x^*) \neq \emptyset$, $x^* = \theta$, and $f^*(\theta) = \inf_X f = 0$;

hence, $f^{**} = \overline{\text{co}} f$ and both functions f and f^{**} are proper. Also, taking into account the current assumption, we let $\bar{z}^* \in \text{int}(\text{dom}\, f^*)$ and θ-neighborhoods $U \subset (X, \sigma(X, X^*))$, $V \subset X^*$ be such that

$$\langle z, \bar{z}^* \rangle \leq 1, \;\; f^*(\bar{z}^* + z^*) \leq f^*(\bar{z}^*) + 1 \quad \text{for all } z \in U, \;\; z^* \in V. \tag{15}$$

To establish the main conclusion, according to Theorem 4 it is sufficient to show that the convex subset of X denoted by $A := \text{N}_{\text{dom}\, f^*}(\theta) + \overline{\text{co}} \left\{ (\partial^* f)^{-1}(\theta) \right\}$ is closed. We pick $x \in \overline{A}$. Then, for each $m \in \mathbb{N}^*$ there exist $k_m \in \mathbb{N}^*$, $v_m \in \text{N}_{\text{dom}\, f^*}(\theta)$, $\lambda_{i,m} \in [0, 1]$, and $x_{i,m} \in (\partial^* f)^{-1}(\theta)$, for $i = 1, \cdots, k_m$, such that $(\lambda_{1,m}, \cdots, \lambda_{k_m,m}) \in \Delta_{k_m}$ and

$$\lambda_{1,m} x_{1,m} + \cdots + \lambda_{k_m,m} x_{k_m,m} + v_m - x \in m^{-1} U,$$

where

$$\Delta_{k_m} := \{ (\alpha_1, \cdots, \alpha_{km}) \in \mathbb{R}^{k_m} \mid \alpha_1, \cdots, \alpha_{k_m} \geq 0, \alpha_1 + \cdots + \alpha_{k_m} = 1 \}. \tag{16}$$

Also, in view of the definition of $(\partial^* f)^{-1}(\theta)$ we may suppose that

$$f(x_{i,m}) \leq 1 \quad \text{for all } i = 1, \cdots, k_{m_m}.$$

Then, by taking into account (15) (that $\bar{z}^* + V \subset \text{dom}\, f^*$) the last relationship above gives us, for every $z^* \in V$,

$$\begin{aligned}
\langle v_m, z^* \rangle &\leq -\langle v_m, \bar{z}^* \rangle && (\text{as } v_m \in \text{N}_{\text{dom}\, f^*}(\theta)) \\
&\leq -\langle x, \bar{z}^* \rangle + \langle \Sigma_{1 \leq i \leq k_m} \lambda_{i,m} x_{i,m}, \bar{z}^* \rangle + m^{-1} \\
&\leq -\langle x, \bar{z}^* \rangle + \sum_{1 \leq i \leq k_m} \lambda_{i,m} (f(x_{i,m}) + f^*(\bar{z}^*)) + 1 && (\text{by Fenchel inequality}) \\
&\leq \max\{1, -\langle x, \bar{z}^* \rangle + f^*(\bar{z}^*) + 2\} =: r && (\text{as } \sup_{1 \leq i \leq k_m} f(x_{i,m}) \leq 1);
\end{aligned}$$

that is, $(v_m)_m \subset (r^{-1} V)^\circ$. Then, we may suppose that the nets $(v_m)_m$, $(\lambda_{1,m} x_{1,m} + \cdots + \lambda_{k_m,m} x_{k_m,m})_m$ converge to some $v \in \text{N}_{\text{dom}\, f^*}(\theta)$ and $x - v \in \overline{\text{co}} \left\{ (\partial^* f)^{-1}(\theta) \right\}$, respectively, showing that $x = v + (x - v) \in A$, as we wanted to prove.

The last conclusion is immediate in view of the relationship $\partial^* f = \partial f$ (see (7)). ■

In the following corollary we address the finite-dimensional counterpart of Corollary 6.

Corollary 7 *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be such that* $\text{int}(\text{dom}\, f^*) \neq \emptyset$. *Then, for every $x^* \in X^*$ we have the formula*

$$\partial f^*(x^*) = \text{N}_{\text{dom}\, f^*}(x^*) + \text{co} \left\{ (\partial^* f)^{-1}(x^*) \right\}.$$

In addition, if f is lsc then

$$\partial f^*(x^*) = \text{N}_{\text{dom}\, f^*}(x^*) + \text{co} \left\{ (\partial f)^{-1}(x^*) \right\}.$$

Proof. We shall denote $B_\gamma(z)$ (B_γ if $z = 0$) the ball of radius $\gamma > 0$ centered at z. As before, we suppose without loss of generality that $\partial f^*(x^*) \neq \emptyset$, $x^* = \theta$, and $f^*(x^*) = 0$; thus, f^* is proper and $\inf_X f = 0$. By assumption, we fix $x_0 \in \text{int}(\text{dom}\, f^*)$ and $\rho > 0$ such that

$$f^*(x_0 + v) \leq f^*(x_0) + 1 \quad \text{for all } v \in B_\rho. \tag{17}$$

Then, to prove the first conclusion it suffices, according to Corollary 6, to show that

$$\overline{\text{co}}\left\{(\partial^* f)^{-1}(\theta)\right\} \subset \text{co}\left\{(\partial^* f)^{-1}(\theta)\right\}. \tag{18}$$

For we pick a sequence $(x_k)_k$ in $\text{co}\left\{(\partial^* f)^{-1}(\theta)\right\}$ which converges to a given x. By taking into account Carathéodory's Theorem, for each $k \geq 1$ there are $(\lambda_{k,1}, \cdots, \lambda_{k,n+1}) \in \Delta_{n+1}$ (see (16)) and $x_{k,1}, \cdots, x_{k,n+1} \in (\partial^* f)^{-1}(\theta)$ such that

$$x_k = \lambda_{k,1} x_{k,1} + \cdots + \lambda_{k,n+1} x_{k,n+1}; \tag{19}$$

without loss of generality we suppose that $\langle x_0, x_k \rangle \geq \langle x_0, x \rangle - 1$ for all k, $\lambda_{k,i} > 0$, for all i, k, and the sequence $(\lambda_{k,1}, \cdots, \lambda_{k,n+1})_k$ converges to some $(\lambda_1, \cdots, \lambda_{n+1}) \in \Delta_{n+1}$ (recall (16)). By the definition of $(\partial^* f)^{-1}(\theta)$, for each $i \in \{1, \cdots, n+1\}$ there exists $y_{k,i} \in B_{\frac{1}{k}}(x_{k,i})$ such that $f(y_{k,i}) \leq \frac{1}{k}$ and, so, by Fenchel inequality,

$$\langle x_0, x_{k,i} \rangle \leq \langle x_0, y_{k,i} \rangle + k^{-1}\sqrt{\langle x_0, x_0 \rangle} \leq f^*(x_0) + \sqrt{\langle x_0, x_0 \rangle} + 1 \quad \text{for all } k. \tag{20}$$

So, by multiplying (19) by x_0 it follows that, for each k,

$$\begin{aligned}
\langle x_0, x_{k,i} \rangle &= \langle x_0, x_k \rangle - \sum_{\substack{j \neq i \\ 1 \leq j \leq n+1}} \lambda_{k,1} \langle x_0, x_{k,j} \rangle \\
&\geq \langle x_0, x \rangle - 1 - \max\{f^*(x_0) + \sqrt{\langle x_0, x_0 \rangle} + 1, 1\} =: \alpha.
\end{aligned}$$

Hence, by invoking Fenchel inequality together with (17) and (20), for every $v \in B_\rho$ we get that

$$\begin{aligned}
\langle v, x_{k,i} \rangle &= \langle x_0 + v, y_{k,i} \rangle - \langle x_0, x_{k,i} \rangle + \langle x_0 + v, x_{k,i} - y_{k,i} \rangle \\
&\leq f^*(x_0 + v) + f(y_{k,i}) - \alpha + k^{-1}\sqrt{\langle x_0, x_0 \rangle} + \rho \\
&\leq f^*(x_0) - \alpha + \sqrt{\langle x_0, x_0 \rangle} + \rho + 2 < +\infty;
\end{aligned}$$

that is, by subsequencing if necessary, the sequence $(x_{k,i})_k$ converges to some x_i. Thus, the corresponding sequence $(y_{k,i})_k$ also converges to x_i. But, $\lim_{k \to +\infty} f(y_{k,i}) = 0$ and so $x_i \in (\partial^* f)^{-1}(\theta)$. Consequently, (18) follows and establishes the first conclusion. The last conclusion immediately fellows from the first one in view of (7). ■

Now, in view of the relationship $\text{Argmin}\, f^{**} = \partial f^*(\theta)$, we derive from Theorem 4 and Corollaries 6 and 7 new formulas for the set $\text{Argmin}(\overline{\text{co}} f)$ by means of $\text{Argmin}\, f$. We recall that (see (3) and (5), respectively)

$$\mathcal{F}_\theta = \{L \subset X^* \mid L \text{ is a finite-dimensional linear subspace}\},$$

$$\widehat{\mathcal{F}}_\theta = \{L \subset X^* \text{ convex} \mid \theta \in L,\ \text{rint}(L \cap \text{dom}\, f^*) \neq \emptyset,\ f^*_{|\text{rint}(L \cap \text{dom}\, f^*)} \text{ is continuous}\}.$$

Corollary 8 *Given a function $f : X \to \overline{\mathbb{R}}$ having a proper conjugate, we have the formula*

$$\text{Argmin}\, f^{**} = \bigcap_{L \in \mathcal{F}(\theta)} \overline{\text{co}}\left\{(\partial_L^* f)^{-1}(\theta) + \text{N}_{L \cap \text{dom}\, f^*}(\theta)\right\},$$

where $\mathcal{F}(\theta)$ either stands for $\mathcal{F}_\theta$ or $\widehat{\mathcal{F}}_\theta$. In particular, the following statements hold:

(i) *if* $\operatorname{rint}(\operatorname{dom} f^*) \neq \emptyset$ *and* $f^*_{|\operatorname{rint}(\operatorname{dom} f^*)}$ *is continuous,*

$$\operatorname{Argmin} f^{**} = \overline{\operatorname{co}} \left\{ (\partial^* f)^{-1}(\theta) + \mathrm{N}_{\operatorname{dom} f^*}(\theta) \right\};$$

(ii) *if* $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$ *and* $f^*_{|\operatorname{int}(\operatorname{dom} f^*)}$ *is continuous,*

$$\operatorname{Argmin} f^{**} = \mathrm{N}_{\operatorname{dom} f^*}(\theta) + \overline{\operatorname{co}} \left\{ \operatorname{Argmin} (\operatorname{cl}^w f) \right\};$$

(iii) *if, additionally to* (ii), f *is weakly lsc,*

$$\operatorname{Argmin} f^{**} = \mathrm{N}_{\operatorname{dom} f^*}(\theta) + \overline{\operatorname{co}} \left\{ \operatorname{Argmin} f \right\};$$

(iv) *if* $X = \mathbb{R}^n$, $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, *and* f *is lsc,*

$$\operatorname{Argmin} f^{**} = \mathrm{N}_{\operatorname{dom} f^*}(\theta) + \operatorname{co} \left\{ \operatorname{Argmin} f \right\}.$$

We end up this article by a couple of examples given in a Hilbert space $(H, \|\cdot\|, \langle \cdot, \cdot \rangle)$.

Example 4 Given a subset $C \subset H$, we consider the function $f : H \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x) := \begin{cases} \frac{1}{2} \|x\|^2 & \text{if } x \in C \\ +\infty & \text{otherwise.} \end{cases}$$

Then, for every $x^* \in H$ we have the formula

$$\partial f^*(x^*) = \overline{\operatorname{co}}\{x \in \overline{C}^w \mid \exists\, x_k \rightharpoonup x,\ x_k \in C,\ \lim_k \|x_k - x^*\| = d_C(x^*)\},$$

where $\overline{C}^w$ denotes the weak closure of C, and d_C denotes the usual distance to C. Consequently, if C is weakly closed, then

$$\partial f^*(x^*) = \overline{\operatorname{co}}\{x \in C \mid \|x - x^*\| = d_C(x^*)\}.$$

Proof. To begin with we observe that the weak lsc hull $\operatorname{cl}^w f$ of f is given by

$$\operatorname{cl}^w f(x) := \begin{cases} \frac{1}{2} \liminf_{y \to x} \|y\|^2 & \text{if } x \in \overline{C}^w \\ +\infty & \text{otherwise.} \end{cases}$$

By Asplund's formula one has

$$(\operatorname{cl}^w f)^*(x^*) = f^*(x^*) = \frac{1}{2}(\|x^*\|^2 - d_C^2(x^*)). \tag{21}$$

We notice that $\operatorname{dom} f^* = H$ and, so, (the proper lsc convex function) f^* is continuous on H. Therefore, Corollary 6 applies and yields to

$$\partial f^*(x^*) = \overline{\operatorname{co}} \left\{ (\partial (\operatorname{cl}^w f))^{-1}(x^*) \right\} \quad \text{for all } x^* \in H.$$

To conclude, it suffices to observe that $x \in (\partial(\mathrm{cl}^w f))^{-1}(x^*)$ if and only if

$$(\mathrm{cl}^w f)(x) + \frac{1}{2}(\|x^*\|^2 - d_C^2(x^*)) = \langle x, x^* \rangle,$$

if and only if $x \in \overline{C}^w$ and there exists a sequence $C \ni x_k \rightharpoonup x$ such that

$$\frac{1}{2}\lim_k \left(\|x_k\|^2 - 2\langle x_k, x^*\rangle\right) + \frac{1}{2}(\|x^*\|^2 - d_C^2(x^*)) = \frac{1}{2}\lim_k \left(\|x_k - x^*\|^2\right) - \frac{1}{2}d_C^2(x^*) = 0,$$

thus, if and only if $x \in \overline{C}^w$ and the sequence $(x_k)_k \subset C$ weakly converges to x and satisfies $\|x_k - x^*\| \to d_C(x^*)$.

■

Example 5 Given a bounded set $C \subset H$, we consider the function $f : H \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x) := \begin{cases} -\frac{1}{2}\|x\|^2 & \text{if } x \in -C \\ +\infty & \text{otherwise.} \end{cases}$$

Then, for every $x^* \in H$ we have the formula

$$\partial f^*(x^*) = -\overline{\mathrm{co}}\{x \in C \mid \exists x_k \rightharpoonup x \ s.t. \ x_k \in C, \ \lim_k \|x_k - x^*\| = \sup_{y \in C}\|y - x^*\|\}.$$

Consequently, if H is finite-dimensional,

$$\partial f^*(x^*) = -\,\mathrm{co}\{F_{\overline{C}}(x^*)\},$$

where $F_{\overline{C}}(x^*)$ is the set of furthest points in $\overline{C}$ from x^*.

Proof. In this case the conjugate of f is

$$f^*(x^*) = \frac{1}{2}(\sup_{y \in C}\|y - x^*\|^2 - \|x^*\|^2),$$

which is then continuous on H. We also can check that

$$(\partial^* f)^{-1}(x^*) = -\{x \in H \mid \exists x_k \rightharpoonup x \text{ s.t. } x_k \in C, \ \lim_k \|x_k - x^*\| = \sup_{y \in C}\|y - x^*\|\}.$$

Therefore, the conclusion follows by Corollary 6, while the last consequence uses Corollary 7. ■

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