Penalty Decomposition Methods for Rank Minimization *

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Abstract

In this paper we consider general rank minimization problems with rank appearing in either objective function or constraint. We first show that a class of matrix optimization problems can be solved as lower dimensional vector optimization problems. As a consequence, we establish that a class of rank minimization problems have closed form solutions. Using this result, we then propose penalty decomposition methods for general rank minimization problems in which each subproblem is solved by a block coordinate descend method. Under some suitable assumptions, we show that any accumulation point of the sequence generated by our method when applied to the rank constrained minimization problem is a stationary point of a nonlinear reformulation of the problem. Finally, we test the performance of our methods by applying them to matrix completion and nearest low-rank correlation matrix problems. The computational results demonstrate that our methods generally outperform the existing methods in terms of solution quality and/or speed.

Key words: rank minimization, penalty decomposition methods, matrix completion, nearest low-rank correlation matrix

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1 Introduction

In this paper we consider the following rank minimization problems:

$$\min_X \{f(X) : \ \mathrm{rank}(X) \le r, \ X \in \mathcal{X} \cap \Omega\}, \tag{1}$$

$$\min_X \{f(X) + \nu \ \mathrm{rank}(X) : X \in \mathcal{X} \cap \Omega\} \tag{2}$$

for some r, $\nu \ge 0$, where $\mathcal{X}$ is a closed convex set, Ω is a closed unitarily invariant set in $\Re^{m\times n}$, and $f : \Re^{m\times n} \to \Re$ is a continuously differentiable function (for the definition of unitarily invariant set, see Section 2). In literature, there are numerous application problems in the form of (1) or (2). For example, several well-known combinatorial optimization problems such as maximal cut (MAXCUT) and maximal stable set can be formulated as problem (1) (see, for example, [11, 1, 5]). More generally, nonconvex quadratic programming problems can also be cast into (2) (see, for example, [1]). Recently, some image recovery problems are formulated as (1) or (2) (see, for example, [28]). In addition, the problem of

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finding nearest low-rank correlation matrix is in the form of (1), which has important application in finance (see, for example, [4, 30, 37, 39, 26, 31, 12]).

Several approaches have recently been developed for solving problems (1) and (2) or their special cases. In particular, for those arising in combinatorial optimization (e.g., MAXCUT), one novel method is to first solve the semidefinite programming (SDP) relaxation of (1) and then obtain an approximate solution of (1) by applying some heuristics to the solution of the SDP (see, for example, [11]). Despite the remarkable success on those problems, it is not clear about the performance of this method when extended to solve more general problem (1). In addition, the nuclear norm relaxation approach has been proposed for problems (1) or (2). For example, Fazel et al. [10] considered a special case of problem (2) with $f \equiv 0$ and $\Omega = \Re^{m\times n}$. In their approach, a convex relaxation is applied to (1) or (2) by replacing the rank of X by the nuclear norm of X and numerous efficient methods can then be applied to solve the resulting convex problems. Recently, Recht et al. [28] showed that under some suitable conditions, such a convex relaxation is tight when $\mathcal{X}$ is an affine manifold. The quality of such a relaxation, however, remains unknown when applied to general problems (1) and (2). Additionally, for some application problems, the nuclear norm stays constant in feasible region. For example, as for nearest low-rank correlation matrix problem (see Subsection 5.2), any feasible point is a symmetric positive semidefinite matrix with all diagonal entries equal to one. For those problems, nuclear norm relaxation approach is obviously inappropriate. Finally, nonlinear programming (NLP) reformulation approach has been applied for problem (1) (see, for example, [5]). In this approach, problem (1) is cast into an NLP problem by replacing the constraint $\mathrm{rank}(X) \le r$ by $X = UV$ where $U \in \Re^{m\times r}$ and $V \in \Re^{r\times n}$, and then numerous optimization methods can be applied to solve the resulting NLP. It is not hard to observe that such an NLP has infinitely many local minima, and moreover it can be highly nonlinear, which might be challenging for all existing numerical optimization methods for NLP. Also, it is not clear whether this approach can be applied to problem (2).

In this paper we consider general rank minimization problems (1) and (2). We first show that a class of matrix optimization problems can be solved as lower dimensional vector optimization problems. As a consequence, we establish that a class of rank minimization problems have closed form solutions. Using this result, we then propose penalty decomposition methods for general rank minimization problems in which each subproblem is solved by a block coordinate descend method. Under some suitable assumptions, we show that any accumulation point of the sequence generated by our method when applied to the rank constrained minimization problem (1) is a stationary point of a nonlinear reformulation of the problem. Finally, we test the performance of our methods by applying them to matrix completion and nearest low-rank correlation matrix problems. The computational results demonstrate that our methods generally outperform the existing methods in terms of solution quality and/or speed.

The rest of this paper is organized as follows. In Subsection 1.1, we introduce the notation that is used throughout the paper. In Section 2, we establish some technical results on a class of rank minimization problems that are used to develop the penalty decomposition methods for problems (1) and (2) in Sections 3 and 4. The convergence of our penalty decomposition methods for problem (1) is also established in those sections. In Section 5, we conduct numerical experiments to test the performance of our penalty decomposition methods for solving matrix completion and nearest low-rank correlation matrix problems. Finally, we present some concluding remarks in section 6.

1.1 Notation

In this paper, the symbol $\Re^n$ denotes the n-dimensional Euclidean space, and the set of all $m \times n$ matrices with real entries is denoted by $\Re^{m\times n}$. The spaces of $n \times n$ diagonal and symmetric matrices will be denoted by $\mathcal{D}^n$ and $\mathcal{S}^n$, respectively. If $X \in \mathcal{S}^n$ is positive semidefinite, we write $X \succeq 0$. The cone of positive semidefinite matrices is denoted by $\mathcal{S}^n_+$. Given matrices X and Y in $\Re^{m\times n}$, the standard inner product is defined by $\langle X, Y\rangle := \mathrm{Tr}(XY^T)$, where $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix. The Frobenius norm of a real matrix X is defined as $\|X\|_F := \sqrt{\mathrm{Tr}(XX^T)}$, and the nuclear norm of X, denoted by $\|X\|_*$, is defined as the sum of all singular values of X. The rank of a matrix X is denoted by $\mathrm{rank}(X)$. We denote by I the identity matrix, whose dimension should be clear from the context. For a real symmetric matrix X, $\lambda(X)$ denotes the vector of all eigenvalues of X arranged in nondecreasing order and $\Lambda(X)$ is the diagonal matrix whose ith diagonal entry is $\lambda_i(X)$ for all i. Similarly, for any $X \in \Re^{m\times n}$, $\sigma(X)$ denotes the q-dimensional vector consisting of all singular values of X arranged in nondecreasing order, where $q = \min(m, n)$, and $\Sigma(X)$ is the $m \times n$ matrix whose ith diagonal entry is $\sigma_i(X)$ for all i and all off-diagonal entries are 0, that is, $\Sigma_{ii}(X) = \sigma_i(X)$ for $1 \le i \le q$ and $\Sigma_{ij}(X) = 0$ for all $i \ne j$. We define the operator $\mathscr{D} : \Re^q \to \Re^{m\times n}$ as follows:

$$\mathscr{D}_{ij}(x) = \begin{cases} x_i & \text{if } i = j; \\ 0 & \text{otherwise} \end{cases} \qquad \forall x \in \Re^q, \tag{3}$$

where $q = \min(m, n)$. Given an $n \times n$ matrix X, $\widetilde{\mathscr{D}}(X)$ denotes a diagonal matrix whose ith diagonal element is X_{ii} for $i = 1, \ldots, n$. For any real vector, $\|\cdot\|_0$, $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the cardinality (i.e., the number of nonzero entries), the standard 1-norm and the Euclidean norm of the vector, respectively. Given a real vector space $\mathcal{U}$ and a closed set $C \subseteq \mathcal{U}$, let $\mathrm{dist}(\cdot, C) : \mathcal{U} \to \Re_+$ denote the distance function to C measured in terms of $\|\cdot\|$, that is,

$$\mathrm{dist}(u, C) := \inf_{\tilde{u}\in C} \|u - \tilde{u}\| \qquad \forall u \in \mathcal{U}.$$

Finally, $\mathcal{N}_C(x)$ and $\mathcal{T}_C(x)$ denote the normal and tangent cones of C at any $x \in C$, respectively.

2 Technical results on special rank minimization

In this section we first show that a class of matrix optimization problems can be solved as lower dimensional vector optimization problems. As a consequence, we establish a result that a class of rank minimization problems have closed form solutions, which will be used to develop penalty decomposition methods in Sections 3 and 4. Before proceeding, we introduce some definitions that will be used subsequently.

Let $\mathcal{U}^n$ denote the set of all unitary matrices in $\Re^{n\times n}$. A norm $\|\cdot\|$ is a *unitarily invariant norm* on $\Re^{m\times n}$ if $\|UXV\| = \|X\|$ for all $U \in \mathcal{U}^m$, $V \in \mathcal{U}^n$, $X \in \Re^{n\times n}$. More generally, a function $F : \Re^{m\times n} \to \Re$ is a *unitarily invariant function* if $F(UXV) = F(X)$ for all $U \in \mathcal{U}^m$, $V \in \mathcal{U}^n$, $X \in \Re^{m\times n}$. A set $\mathcal{X} \subseteq \Re^{m\times n}$ is a *unitarily invariant set* if

$$\{UXV : U \in \mathcal{U}^m, V \in \mathcal{U}^n, X \in \mathcal{X}\} = \mathcal{X}.$$

Similarly, a function $F : \mathcal{S}^n \to \Re$ is a *unitary similarity invariant function* if $F(UXU^T) = F(X)$ for all $U \in \mathcal{U}^n$, $X \in \mathcal{S}^n$. A set $\mathcal{X} \subseteq \mathcal{S}^n$ is a *unitary similarity invariant set* if

$$\{UXU^T : U \in \mathcal{U}^n, X \in \mathcal{X}\} = \mathcal{X}.$$

The following result establishes that a class of matrix optimization problems over a subset of $\Re^{m\times n}$ can be solved as lower dimensional vector optimization problems.

Proposition 2.1 *Let $\|\cdot\|$ be a unitarily invariant norm on $\Re^{m\times n}$, and let $F : \Re^{m\times n} \to \Re$ be a unitarily invariant function. Suppose that $\mathcal{X} \subseteq \Re^{m\times n}$ is a unitarily invariant set. Let $A \in \Re^{m\times n}$ be given, $q = \min(m, n)$, and let ϕ be a non-decreasing function on $[0, \infty)$. Suppose that $U\Sigma(A)V^T$ is the singular value decomposition of A. Then, $X^* = U\mathscr{D}(x^*)V^T$ is an optimal solution of the problem*

$$\begin{array}{ll} \min & F(X) + \phi(\|X - A\|) \\ s.t. & X \in \mathcal{X}, \end{array} \tag{4}$$

where $x^* \in \Re^q$ is an optimal solution of the problem

$$\begin{array}{ll} \min & F(\mathscr{D}(x)) + \phi(\|\mathscr{D}(x) - \Sigma(A)\|) \\ s.t. & \mathscr{D}(x) \in \mathcal{X}. \end{array} \tag{5}$$

Proof. Since $\|\cdot\|$ is a unitarily invariant norm, we know from exercise 18 on page 215 of [13] that

$$\|X - A\| \geq \|\Sigma(X) - \Sigma(A)\| \quad \forall X \in \Re^{m\times n}. \tag{6}$$

It then follows from (6), the monotonicity of ϕ and the relation $\Sigma(X) = \mathscr{D}(\sigma(X))$ that

$$\phi(\|X - A\|) \geq \phi(\|\mathscr{D}(\sigma(X)) - \Sigma(A)\|) \quad \forall X \in \Re^{m\times n}.$$

Since $\mathcal{X}$ is a unitarily invariant set and F is a unitarily invariant function, we have

$$\mathscr{D}(\sigma(X)) \in \mathcal{X}, \quad F(\mathscr{D}(\sigma(X))) = F(X) \quad \forall X \in \mathcal{X}.$$

Using the above relations, we immediately obtain that

$$F(X) + \phi(\|X - A\|) \geq F(\mathscr{D}(\sigma(X))) + \phi(\|\mathscr{D}(\sigma(X)) - \Sigma(A)\|) \quad \forall X \in \mathcal{X},$$

which together with $\mathscr{D}(\sigma(X)) \in \mathcal{X}$ implies that the optimal value of problem (4) is minorized by that of problem (5). Further, by the definition of x^*, we know that $\mathscr{D}(x^*) \in \mathcal{X}$, which along with the assumption that $\mathcal{X}$ is a unitarily invariant set, implies that $X^* \in \mathcal{X}$, that is, X^* is a feasible solution of problem (4). Moreover, we observe that

$$F(X^*) = F(\mathscr{D}(x^*)), \quad \phi(\|X^* - A\|) = \phi(\|U\mathscr{D}(x^*)V^T - A\|) = \phi(\|\mathscr{D}(x^*) - \Sigma(A)\|).$$

Thus, the objective function of (4) reaches the optimal value of problem (5) at X^*. It then immediately follows that problems (5) and (4) share the same optimal value, and hence X^* is an optimal solution of (4). ■

As some consequences of Proposition 2.1, we next state that a class of rank minimization problems on a subset of $\Re^{m\times n}$ can be solved as lower dimensional vector minimization problems.

Corollary 2.2 *Let $\nu \geq 0$ and $A \in \Re^{m\times n}$ be given, and let $q = \min(m, n)$. Suppose that $\mathcal{X} \subseteq \Re^{m\times n}$ is a unitarily invariant set, and $U\Sigma(A)V^T$ is the singular value decomposition of A. Then, $X^* = U\mathscr{D}(x^*)V^T$ is an optimal solution of the problem*

$$\min\{\nu \ \mathrm{rank}(X) + \frac{1}{2}\|X - A\|_F^2 : \ X \in \mathcal{X}\}, \tag{7}$$

where $x^* \in \Re^q$ is an optimal solution of the problem

$$\min\{\nu\|x\|_0 + \frac{1}{2}\|x - \sigma(A)\|_2^2 : \ \mathscr{D}(x) \in \mathcal{X}\}. \tag{8}$$

Proof. Let $\|\cdot\| := \|\cdot\|_F$, $F(X) := \nu \operatorname{rank}(X)$ for all X, and $\phi(t) := t^2/2$ for all t. Clearly, the assumptions of Proposition 2.1 are satisfied for such $\|\cdot\|$, F and ϕ. Further, notice that $\operatorname{rank}(\mathscr{D}(x)) = \|x\|_0$ and $\|\mathscr{D}(x) - \Sigma(A)\|_F = \|x - \sigma(A)\|_2$ for all $x \in \Re^q$. It immediately follows from Proposition 2.1 that the conclusion holds. ∎

Corollary 2.3 *Let $r \geq 0$ and $A \in \Re^{m\times n}$ be given, and let $q = \min(m,n)$. Suppose that $\mathcal{X} \subseteq \Re^{m\times n}$ is a unitarily invariant set, and $U\Sigma(A)V^T$ is the singular value decomposition of A. Then, $X^* = U\mathscr{D}(x^*)V^T$ is an optimal solution of the problem*

$$\min\{\|X - A\|_F : \ \operatorname{rank}(X) \leq r, \ X \in \mathcal{X}\}, \tag{9}$$

where $x^* \in \Re^q$ is an optimal solution of the problem

$$\min\{\|x - \sigma(A)\|_2 : \ \|x\|_0 \leq r, \ \mathscr{D}(x) \in \mathcal{X}\}. \tag{10}$$

Proof. Its proof is similar to that of Corollary 2.2. ∎

Remark. When $\mathcal{X}$ is simple enough, problems (7) and (9) have closed form solutions. In many applications, $\mathcal{X} = \{X \in \Re^{m\times n} : a \leq \sigma_i(X) \leq b \ \forall i\}$ for some $0 \leq a < b \leq \infty$. For such $\mathcal{X}$, one can see that $\mathscr{D}(x) \in \mathcal{X}$ if and only if $a \leq |x_i| \leq b$ for all i. In this case, it is not hard to observe that problems (8) and (10) have closed form solutions (see Lu and Zhang [21]). It thus follows from Corollaries 2.2 and 2.3 that problems (7) and (9) also have closed form solutions.

The following results are heavily used in [6, 23, 35] for developing algorithms for solving the nuclear norm relaxation of matrix completion problems. They can be immediately obtained from Proposition 2.1.

Corollary 2.4 *Let $\nu \geq 0$ and $A \in \Re^{m\times n}$ be given, and let $q = \min(m,n)$. Suppose that $U\Sigma(A)V^T$ is the singular value decomposition of A. Then, $X^* = U\mathscr{D}(x^*)V^T$ is an optimal solution of the problem*

$$\min \nu\|X\|_* + \frac{1}{2}\|X - A\|_F^2,$$

where $x^* \in \Re^q$ is an optimal solution of the problem

$$\min \nu\|x\|_1 + \frac{1}{2}\|x - \sigma(A)\|_2^2.$$

Corollary 2.5 *Let $r \geq 0$ and $A \in \Re^{m\times n}$ be given, and let $q = \min(m,n)$. Suppose that $U\Sigma(A)V^T$ is the singular value decomposition of A. Then, $X^* = U\mathscr{D}(x^*)V^T$ is an optimal solution of the problem*

$$\min\{\|X - A\|_F : \ \|X\|_* \leq r\},$$

where $x^* \in \Re^q$ is an optimal solution of the problem

$$\min\{\|x - \sigma(A)\|_2 : \ \|x\|_1 \leq r\}.$$

We next show that a class of matrix optimization problems over a subset of $\mathcal{S}^n$ can be solved as lower dimensional vector optimization problems. Before proceeding, we establish a lemma that will be used subsequently.

Lemma 2.6 *Let $\|\cdot\|$ be a unitarily invariant norm on $\Re^{n\times n}$. The following holds:*

$$\|X-Y\| \geq \|\Lambda(X)-\Lambda(Y)\| \quad \forall X, Y \in \mathcal{S}^n.$$

Proof. Let X, $Y \in \mathcal{S}^n$ be arbitrarily given. Let t be sufficiently large such that $X+tI$ and $Y+tI$ are both positive semidefinite. Then, we have

$$\|\Sigma(X+tI)-\Sigma(Y+tI)\| = \|\Lambda(X+tI)-\Lambda(Y+tI)\| = \|\Lambda(X)-\Lambda(Y)\|.$$

It follows from (6) and the above relation that

$$\|X-Y\| = \|(X+tI)-(Y+tI)\| \geq \|\Sigma(X+tI)-\Sigma(Y+tI)\| = \|\Lambda(X)-\Lambda(Y)\|,$$

and thus the conclusion holds. ∎

The following result establishes that a class of matrix optimization problems over a subset of $\mathcal{S}^n$ can be solved as lower dimensional vector optimization problems.

Proposition 2.7 *Let $\|\cdot\|$ be a unitarily invariant norm on $\Re^{n\times n}$, and let $F:\mathcal{S}^n \to \Re$ be a unitary similarity invariant function. Suppose that $\mathcal{X} \subseteq \mathcal{S}^n$ is a unitary similarity invariant set. Let $A \in \mathcal{S}^n$ be given, and let ϕ be a non-decreasing function on $[0,\infty)$. Suppose that $U\Lambda(A)U^T$ is the eigenvalue value decomposition of A. Then, $X^* = U\mathscr{D}(x^*)U^T$ is an optimal solution of the problem*

$$\begin{array}{ll} \min & F(X)+\phi(\|X-A\|) \\ s.t. & X \in \mathcal{X}, \end{array} \tag{11}$$

where $x^* \in \Re^n$ is an optimal solution of the problem

$$\begin{array}{ll} \min & F(\mathscr{D}(x))+\phi(\|\mathscr{D}(x)-\Lambda(A)\|) \\ s.t. & \mathscr{D}(x) \in \mathcal{X}. \end{array} \tag{12}$$

Proof. The conclusion of this proposition follows from Lemma 2.6 and a similar argument as for Proposition 2.1. ∎

As some consequences of Proposition 2.7, we next show that a class of rank minimization problems on a subset of $\mathcal{S}^n$ can be solved as lower dimensional vector minimization problems.

Corollary 2.8 *Let $\nu \geq 0$ and $A \in \mathcal{S}^n$ be given. Suppose that $\mathcal{X} \subseteq \mathcal{S}^n$ is a unitary similarity invariant set, and $U\Lambda(A)U^T$ is the eigenvalue decomposition of A. Then, $X^* = U\mathscr{D}(x^*)U^T$ is an optimal solution of the problem*

$$\min\{\nu \ \mathrm{rank}(X)+\frac{1}{2}\|X-A\|_F^2: \ X \in \mathcal{X}\}, \tag{13}$$

where $x^* \in \Re^q$ is an optimal solution of the problem

$$\min\{\nu\|x\|_0+\frac{1}{2}\|x-\lambda(A)\|_2^2: \ \mathscr{D}(x) \in \mathcal{X}\}. \tag{14}$$

Proof. The conclusion of this corollary immediately follows from Proposition 2.7, and the relations $\mathrm{rank}(\mathscr{D}(x)) = \|x\|_0$ and $\|\mathscr{D}(x)-\Lambda(A)\|_F = \|x-\lambda(A)\|$ for all $x \in \Re^n$. ∎

Corollary 2.9 *Let $r \geq 0$ and $A \in \mathcal{S}^n$ be given. Suppose that $\mathcal{X} \subseteq \mathcal{S}^n$ is a unitary similarity invariant set, and $U\Lambda(A)U^T$ is the eigenvalue decomposition of A. Then, $X^* = U\mathscr{D}(x^*)U^T$ is an optimal solution of the problem*

$$\min\{\|X - A\|_F : \ \mathrm{rank}(X) \leq r, \ X \in \mathcal{X}\}, \tag{15}$$

where $x^* \in \Re^q$ is an optimal solution of the problem

$$\min\{\|x - \lambda(A)\|_2 : \ \|x\|_0 \leq r, \ \mathscr{D}(x) \in \mathcal{X}\}. \tag{16}$$

Proof. Its proof is similar to that of Corollary 2.8. ∎

Remark. When $\mathcal{X}$ is simple enough, problems (13) and (15) have closed form solutions. In many applications, $\mathcal{X} = \{X \in \mathcal{S}^n : a \leq \lambda_i(X) \leq b \ \forall i\}$ for some $a < b \leq \infty$. For such $\mathcal{X}$, one can see that $\mathscr{D}(x) \in \mathcal{X}$ if and only if $a \leq x_i \leq b$ for all i (e.g., $\mathcal{X} = \mathcal{S}^n_+$). In this case, it is not hard to observe that problems (14) and (10) have closed form solutions (see Lu and Zhang [21]). It thus follows from Corollaries 2.8 and 2.9 that problems (13) and (15) also have closed form solutions.

3 Penalty decomposition method for rank minimization of asymmetric matrices

In this section, we consider the rank minimization problems (1) and (2). In particular, we first propose a penalty decomposition (PD) method for solving problem (1), and then extend it to solve problem (2) at end of this section. Throughout this section, we make the following assumption for problems (1) and (2).

Assumption 1 *Problems (1) and (2) are feasible, and moreover, at least a feasible solution, denoted by X^{feas}, is known.*

Clearly, problem (1) can be equivalently reformulated as

$$\min_{X,Y}\{f(X) : \ X - Y = 0, \ X \in \mathcal{X}, \ Y \in \mathcal{Y}\}, \tag{17}$$

where

$$\mathcal{Y} := \{Y \in \Omega | \ \mathrm{rank}(Y) \leq r\}. \tag{18}$$

Given a penalty parameter $\varrho > 0$, the associated quadratic penalty function for (17) is defined as

$$Q_\varrho(X, Y) := f(X) + \frac{\varrho}{2}\|X - Y\|_F^2. \tag{19}$$

In addition, we define

$$\widetilde{Q}_\varrho(X, U, V) := Q_\varrho(X, UV) \quad \forall X \in \Re^{m\times n}, U \in \Re^{m\times r}, V \in \Re^{r\times n}. \tag{20}$$

We now propose a PD method for solving problem (17) (or, equivalently, (1)) in which each penalty subproblem is approximately solved by a block coordinate descent (BCD) method.

Penalty decomposition method for (17) (asymmetric matrices):

Let $\{\epsilon_k\}$ be a positive decreasing sequence. Let $\varrho_0 > 0$, $\sigma > 1$ be given. Choose an arbitrary $Y_0^0 \in \mathcal{Y}$ and a constant $\Upsilon \geq \max\{f(X^{\mathrm{feas}}), \min_{X\in\mathcal{X}} Q_{\varrho_0}(X, Y_0^0)\}$. Set $k = 0$.

1) Set $l = 0$ and apply the BCD method to find an approximate solution $(X^k, Y^k) \in \mathcal{X} \times \mathcal{Y}$ for the penalty subproblem

$$\min\{Q_{\varrho_k}(X, Y) : \; X \in \mathcal{X}, \; Y \in \mathcal{Y}\} \tag{21}$$

by performing steps 1a)-1d):

1a) Solve $X^k_{l+1} \in \operatorname{Arg}\min\limits_{X\in\mathcal{X}} Q_{\varrho_k}(X, Y^k_l)$.

1b) Solve $Y^k_{l+1} \in \operatorname{Arg}\min\limits_{Y\in\mathcal{Y}} Q_{\varrho_k}(X^k_{l+1}, Y)$.

1c) Set $(X^k, Y^k) := (X^k_{l+1}, Y^k_{l+1})$. If (X^k, Y^k) satisfies

$$\operatorname{dist}\left(-\nabla_X Q_{\varrho_k}(X^k, Y^k), \mathcal{N}_{\mathcal{X}}(X^k)\right) \;\leq\; \epsilon_k, \tag{22}$$

$$\|\nabla_U \widetilde{Q}_{\varrho_k}(X^k, U^k, V^k) + Z^k_Y (V^k)^T\|_F \;\leq\; \epsilon_k, \tag{23}$$

$$\|\nabla_V \widetilde{Q}_{\varrho_k}(X^k, U^k, V^k) + (U^k)^T Z^k_Y\|_F \;\leq\; \epsilon_k \tag{24}$$

for some $Z^k_Y \in \mathcal{N}_\Omega(Y^k)$, $U^k \in \Re^{m\times r}$, $V^k \in \Re^{r\times n}$ such that

$$(U^k)^T U^k = I, \quad Y^k = U^k V^k, \tag{25}$$

then go to step 2).

1d) Set $l \leftarrow l + 1$ and go to step 1a).

2) Set $\varrho_{k+1} := \sigma \varrho_k$.

3) If $\min\limits_{X\in\mathcal{X}} Q_{\varrho_{k+1}}(X, Y^k) > \Upsilon$, set $Y^{k+1}_0 := X^{\text{feas}}$. Otherwise, set $Y^{k+1}_0 := Y^k$.

4) Set $k \leftarrow k + 1$ and go to step 1).

end

Remark. The conditions (22)-(24) are mainly used to establish global convergence for the above method. Nevertheless, it may be hard to verify them practically unless $\mathcal{X}$ and Ω are simple. On the other hand, we observe that the sequence $\{Q_{\varrho_k}(X^k_l, Y^k_l)\}$ is non-increasing for any fixed k. In practical implementation, it is thus reasonable to terminate the BCD method based on the progress of $\{Q_{\varrho_k}(X^k_l, Y^k_l)\}$. In particular, given accuracy parameter $\epsilon_I > 0$, one can terminate the BCD method if

$$\frac{|Q_{\varrho_k}(X^k_l, Y^k_l) - Q_{\varrho_k}(X^k_{l-1}, Y^k_{l-1})|}{\max(|Q_{\varrho_k}(X^k_l, Y^k_l)|, 1)} \;\leq\; \epsilon_I. \tag{26}$$

It is also reasonable to terminate the BCD method based on the progress of the sequence $\{(X^k_l, Y^k_l)\}$. Similarly, we can terminate the outer iterations of the above method once

$$\max_{ij} |X^k_{ij} - Y^k_{ij}| \;\leq\; \epsilon_O \tag{27}$$

for some $\epsilon_O > 0$. In addition, given that problem (21) is nonconvex, the BCD method may converge to a stationary point. To enhance the quality of approximate solutions, one may execute the BCD method multiple times starting from a suitable perturbation of the current approximate solution. In detail, at the kth outer iteration, let (X^k, Y^k) be a current approximate solution of (21) obtained by the BCD method, and let $r_k = \text{rank}(Y^k)$. Assume that $r_k > 1$. Before starting the $(k+1)$th outer iteration, one can apply the BCD method again starting from $Y_0^k \in \text{Arg}\min\{\|Y - Y^k\|_F : \text{rank}(Y) \le r_k - 1\}$ (namely, a rank-one perturbation of Y^k) and obtain a new approximate solution $(\tilde{X}^k, \tilde{Y}^k)$ of (21). If $Q_{\varrho_k}(\tilde{X}^k, \tilde{Y}^k)$ is "sufficiently" smaller than $Q_{\varrho_k}(X^k, Y^k)$, one can set $(X^k, Y^k) := (\tilde{X}^k, \tilde{Y}^k)$ and repeat the above process. Otherwise, one can terminate the kth outer iteration and start the next outer iteration. Finally, in view of Corollary 2.3, the subproblem in step 1a) can be reduced to the problem in form of (10), which has closed form solution when Ω is simple enough.

We next establish a global convergence result regarding the outer iterations of the above method for solving problem (17). After that, we will study the convergence of its inner iterations.

Theorem 3.1 *Assume that $\epsilon_k \to 0$. Let $\{(X^k, Y^k)\}$ be the sequence generated by the above PD method, and $\{(U^k, V^k, Z_Y^k)\}$ be the associated sequence satisfying (22)-(25). Suppose that the level set $\mathcal{X}_\Upsilon := \{X \in \mathcal{X} | f(X) \le \Upsilon\}$ is compact. Then, the following statements hold:*

- *(a) The sequence $\{(X^k, Y^k, U^k, V^k)\}$ is bounded;*
- *(b) Suppose that a subsequence $\{(X^k, Y^k, U^k, V^k)\}_{k\in K}$ converges to (X^*, Y^*, U^*, V^*). Then, (X^*, Y^*) is a feasible point of problem (17). Moreover, if the following condition*

$$\left\{ \begin{pmatrix} d_X - d_U V^* - U^* d_V \\ d_U V^* + U^* d_V - d_Y \end{pmatrix} : \begin{array}{l} d_X \in \mathcal{T}_{\mathcal{X}}(X^*), d_U \in \Re^{m\times r}, \\ d_V \in \Re^{r\times n}, d_Y \in \mathcal{T}_\Omega(X^*) \end{array} \right\} = \Re^{m\times n} \times \Re^{m\times n} \tag{28}$$

holds, then the subsequence $\{(Z_X^k, Z_Y^k)\}_{k\in K}$ is bounded, where $Z_X^k := \varrho_k(X^k - Y^k)$, and each accumulation point (Z_X^*, Z_Y^*) of $\{(Z_X^k, Z_Y^k)\}_{k\in K}$ together with (X^*, U^*, V^*) satisfies

$$\begin{array}{rcl} -\nabla f(X^*) - Z_X^* & \in & \mathcal{N}_{\mathcal{X}}(X^*), \\ (Z_X^* - Z_Y^*)(V^*)^T & = & 0, \\ (U^*)^T (Z_X^* - Z_Y^*) & = & 0, \\ X^* - U^* V^* = 0, \quad Z_Y^* & \in & \mathcal{N}_\Omega(X^*). \end{array} \tag{29}$$

Proof. In view of (19) and our choice of Y_0^k that is specified in step 3), one can observe that

$$f(X^k) + \frac{\varrho_k}{2}\|X^k - Y^k\|_F^2 = Q_{\varrho_k}(X^k, Y^k) \le \min_{X\in\mathcal{X}} Q_{\varrho_k}(X, Y_0^k) \le \Upsilon \quad \forall k. \tag{30}$$

It immediately implies that $\{X^k\} \subseteq \mathcal{X}_\Upsilon$, and hence, $\{X^k\}$ is bounded. Moreover, we can obtain from (30) that

$$\|X^k - Y^k\|_F^2 \le 2[\Upsilon - f(X^k)]/\varrho_k \le 2[\Upsilon - \min_{X\in\mathcal{X}_\Upsilon} f(X)]/\varrho_0, \tag{31}$$

which together with the boundedness of $\{X^k\}$ yields that $\{Y^k\}$ is bounded. In addition, it follows from (25) that $\{U^k\}$ is bounded and $V^k = (U^k)^T Y^k$, which implies that $\{V^k\}$ is also bounded. Thus, statement (a) holds.

We next show that statement (b) also holds. Suppose that $\{(X^k, Y^k, U^k, V^k)\}_{k\in K}$ converges to (X^*, Y^*, U^*, V^*). Notice that $\varrho_k \to \infty$ as $k \to \infty$. Upon taking limits on both sides of (31) as $k \in K \to \infty$, we have $X^* - Y^* = 0$. In addition, it is not hard to show that $\mathcal{Y}$ is closed. We then see that $X^* \in \mathcal{X}$ and $Y^* \in \mathcal{Y}$ due to the closedness of $\mathcal{X}$ and $\mathcal{Y}$. It thus follows that (X^*, Y^*) is a feasible point of problem (17). Now, let us prove the second part of statement (b). In view of (19), (20), (22)-(25) and the definition of Z_X^k, we have

$$\begin{array}{rcl} \mathrm{dist}(-\nabla f(X^k) - Z_X^k, \mathcal{N}_{\mathcal{X}}(X^k)) & \leq & \epsilon_k, \\ \|(Z_Y^k - Z_X^k)(V^k)^T\|_F & \leq & \epsilon_k, \\ \|(U^k)^T(Z_Y^k - Z_X^k)\|_F & \leq & \epsilon_k. \end{array} \tag{32}$$

We now claim that the subsequence $\{(Z_X^k, Z_Y^k)\}_{k\in K}$ is bounded. Suppose not, by passing to a subsequence if necessary, we can assume that $\{(Z_X^k, Z_Y^k)\}_{k\in K} \to \infty$. Let

$$(\bar{Z}_X^k, \bar{Z}_Y^k) = (Z_X^k, Z_Y^k)/\|(Z_X^k, Z_Y^k)\|_F \quad \forall k.$$

Without loss of generality, assume that $\{(\bar{Z}_X^k, \bar{Z}_Y^k)\}_{k\in K} \to (\bar{Z}_X, \bar{Z}_Y)$ (otherwise, one can consider its convergent subsequence). Clearly, $\|(\bar{Z}_X, \bar{Z}_Y)\|_F = 1$. Recall that $\{(X^k, Y^k, U^k, V^k)\}_{k\in K} \to (X^*, Y^*, U^*, V^*)$ and $Y^* = X^*$. Dividing both sides of the inequalities in (32) by $\|(Z_X^k, Z_Y^k)\|_F$, taking limits as $k \in K \to \infty$, and using the relation $Z_Y^k \in \mathcal{N}_\Omega(Y^k)$ and the semicontinuity of $\mathcal{N}_{\mathcal{X}}(\cdot)$ and $\mathcal{N}_\Omega(\cdot)$ (see Lemma 2.42 of [32]) we obtain that

$$\begin{array}{c} -\bar{Z}_X \in \mathcal{N}_{\mathcal{X}}(X^*), \quad \bar{Z}_Y \in \mathcal{N}_\Omega(X^*) \\ (\bar{Z}_Y - \bar{Z}_X)(V^*)^T = 0, \quad (U^*)^T(\bar{Z}_Y - \bar{Z}_X) = 0. \end{array} \tag{33}$$

By (28), there exist $d_X \in \mathcal{T}_{\mathcal{X}}(X^*)$, $d_Y \in \mathcal{T}_\Omega(X^*)$, $d_U \in \Re^{m\times r}$, $d_V \in \Re^{r\times n}$ such that

$$\begin{array}{rcl} -\bar{Z}_X & = & d_X - d_U V^* - U^* d_V, \\ -\bar{Z}_Y & = & d_U V^* + U^* d_V - d_Y. \end{array}$$

It then follows from these equalities that

$$\begin{array}{rcl} \|(\bar{Z}_X, \bar{Z}_Y)\|_F^2 & = & -\langle \bar{Z}_X, -\bar{Z}_X\rangle - \langle \bar{Z}_Y, -\bar{Z}_Y\rangle, \\ & = & -\langle \bar{Z}_X, d_X - d_U V^* - U^* d_V\rangle - \langle \bar{Z}_Y, d_U V^* + U^* d_V - d_Y\rangle, \\ & = & \langle \bar{Z}_X - \bar{Z}_Y, d_U V^* + U^* d_V\rangle - \langle \bar{Z}_X, d_X\rangle + \langle \bar{Z}_Y, d_Y\rangle, \end{array}$$

which together with (33) and the fact that $d_X \in \mathcal{T}_{\mathcal{X}}(X^*)$ and $d_Y \in \mathcal{T}_\Omega(X^*)$ implies that $\|(\bar{Z}_X, \bar{Z}_Y)\|_F^2 \leq 0$, which contradicts the identity $\|(\bar{Z}_X, \bar{Z}_Y)\|_F = 1$. Thus, $\{(Z_X^k, Z_Y^k)\}_{k\in K}$ is bounded. Now let (Z_X^*, Z_Y^*) be an accumulation point of $\{(Z_X^k, Z_Y^k)\}_{k\in K}$. By passing to a subsequence if necessary, we can assume that $\{(Z_X^k, Z_Y^k)\}_{k\in K} \to (Z_X^*, Z_Y^*)$. Recall that $\{(X^k, U^k, V^k)\}_{k\in K} \to (X^*, U^*, V^*)$. Taking limits on both sides of the inequalities in (32) as $k \in K \to \infty$, and using the semicontinuity of $\mathcal{N}_{\mathcal{X}}(\cdot)$, we immediately see that the first three relations of (29) hold. In addition, the last two relations of (29) hold due to the identities $Y^k = U^k V^k$, $Y^* = X^*$ and the semicontinuity of $\mathcal{N}_\Omega(\cdot)$. ∎

Remark. From Theorem 3.1 (b), we see that under condition (28), any accumulation point $(X^*, U^*, V^*, Z_X^*, Z_Y^*)$ of $\{(X^k, U^k, V^k, Z_X^k, Z_Y^k)\}_{k\in K}$ satisfies (29). Thus, (X^*, U^*, V^*) together with (Z_X^*, Z_Y^*) satisfies the first-order optimality (i.e., KKT) conditions of the following reformulation of (17) (or, equivalently, (1)):

$$\min_{X,U,V} \{f(X): \; X - UV = 0, \; UV \in \Omega, \; X \in \mathcal{X}, \; U \in \Re^{m\times r}, \; V \in \Re^{r\times n}\}.$$

We next establish a convergence result regarding the inner iterations of the above PD method. In particular, we will show that an approximate solution $(X^k, Y^k) \in \mathcal{X} \times \mathcal{Y}$ for problem (21) satisfying

(22)-(24) can be found by the BCD method described in steps 1a)-1d). For convenience of presentation, we omit the index k from (21) and consider the BCD method for solving the following problem:

$$\min_{X,Y}\{Q_\varrho(X,Y):\ X\in\mathcal{X},\ Y\in\mathcal{Y}\}. \tag{34}$$

Accordingly, we relabel and simplify the BCD method described in step 1a)-1d) above and present it as follows.

Block coordinate descent method for (34):

Choose an arbitrary initial point $Y^0\in\mathcal{Y}$. Set $k=0$.

1) Solve $X^{k+1}\in \operatorname{Arg}\min\limits_{X\in\mathcal{X}} Q_\varrho(X,Y^k)$.

2) Solve $Y^{k+1}\in \operatorname{Arg}\min\limits_{Y\in\mathcal{Y}} Q_\varrho(X^{k+1},Y)$.

3) Set $k\leftarrow k+1$ and go to step 1).

end

We now establish a convergence result regarding the above BCD method.

Theorem 3.2 *Let $\{(X^k,Y^k)\}\subseteq\mathcal{X}\times\mathcal{Y}$ be generated by the above BCD method, and let $U^k\in\Re^{m\times r}$, $V^k\in\Re^{r\times n}$ be such that $(U^k)^TU^k=I$ and $Y^k=U^kV^k$. Suppose that a subsequence $\{(X^k,Y^k,U^k,V^k)\}_{k\in K}$ converges to $(\bar X,\bar Y,\bar U,\bar V)$. If the following condition*

$$\left\{d_U\bar V+\bar U d_V-d_Y: d_U\in\Re^{m\times r}, d_V\in\Re^{r\times n}, d_Y\in\mathcal{T}_\Omega(\bar Y)\right\}=\Re^{m\times n} \tag{35}$$

holds, then $(\bar X,\bar Y,\bar U,\bar V)$ satisfies

$$-\nabla_X Q_\varrho(\bar X,\bar Y)\in\mathcal{N}_{\mathcal{X}}(\bar X),\quad \nabla_U\widetilde{Q}_\varrho(\bar X,\bar U,\bar V)+\bar Z_Y\bar V^T=0,\quad \nabla_V\widetilde{Q}_\varrho(\bar X,\bar U,\bar V)+\bar U^T\bar Z_Y=0 \tag{36}$$

for some $\bar Z_Y\in\mathcal{N}_\Omega(\bar Y)$.

Proof. We observe from the first two steps of the BCD method that

$$\begin{array}{r}Q_\varrho(X^{k+1},Y^k)\le Q_\varrho(X,Y^k)\quad \forall X\in\mathcal{X},\\ Q_\varrho(X^k,Y^k)\le Q_\varrho(X^k,Y)\quad \forall Y\in\mathcal{Y}.\end{array} \tag{37}$$

It follows that

$$Q_\varrho(X^{k+1},Y^{k+1})\le Q_\varrho(X^{k+1},Y^k)\le Q_\varrho(X^k,Y^k)\quad \forall k\ge 1. \tag{38}$$

Hence, the sequence $\{Q_\varrho(X^k,Y^k)\}$ is non-increasing. Further, by the continuity of Q_ϱ and $\{(X^k,Y^k)\}_{k\in K}$ $\to(\bar X,\bar Y)$, we know that $\{Q_\varrho(X^k,Y^k)\}_{k\in K}\to Q_\varrho(\bar X,\bar Y)$. We thus have $\{Q_\varrho(X^k,Y^k)\}\to Q_\varrho(\bar X,\bar Y)$, which together with (38) implies that $\{Q_\varrho(X^{k+1},Y^k)\}\to Q_\varrho(\bar X,\bar Y)$. Upon taking limits on both sides of (37) as $k\in K\to\infty$, one can obtain

$$\begin{array}{rcl}Q_\varrho(\bar X,\bar Y)&\le& Q_\varrho(X,\bar Y)\quad \forall X\in\mathcal{X},\\ Q_\varrho(\bar X,\bar Y)&\le& Q_\varrho(\bar X,Y)\quad \forall Y\in\mathcal{Y},\end{array}$$

which, together with (18), (20) and the relation $\bar{Y} = \bar{U}\bar{V}$ due to $Y^k = U^k V^k$, imply that

$$\begin{aligned} \bar{X} &\in \operatorname{Arg}\min_{X \in \mathcal{X}} Q_\varrho(X, \bar{Y}), && (39) \\ (\bar{U}, \bar{V}) &\in \operatorname{Arg}\min_{U,V} \{\widetilde{Q}_\varrho(\bar{X}, U, V) : \ UV \in \Omega, \ U \in \Re^{m\times r}, \ V \in \Re^{r \times n}\}. && (40) \end{aligned}$$

The first relation of (36) immediately follows from the first-order optimality condition of (39). In view of (40), (35) and Theorem 3.38 on page 134 of [32], we also see that the last two relations of (36) hold. ■

Using the semicontinuity of $\mathcal{N}_\mathcal{X}(\cdot)$ and $\mathcal{N}_\Omega(\cdot)$, and the continuous differentiability of Q_ϱ and $\widetilde{Q}_\varrho$, we immediately conclude from Theorem 3.2 that the following corollary holds.

Corollary 3.3 *Suppose that condition (35) holds for any $\bar{U} \in \Re^{m\times r}$, $\bar{V} \in \Re^{r\times n}$ such that $\bar{U}^T\bar{U} = I$ and $\bar{Y} = \bar{U}\bar{V} \in \Omega$. The approximate solution $(X^k, Y^k) \in \mathcal{X} \times \mathcal{Y}$ for problem (21) satisfying (22)-(25) can be found by the BCD method described in steps 1a)-1d) within a finite number of iterations.*

Remark. When $\Omega = \Re^{m\times n}$, condition (35) clearly holds at any $(\bar{U}, \bar{V}, \bar{Y}) \in \Re^{m\times r} \times \Re^{r\times n} \times \Re^{m\times n}$.

Before ending this section, we next extend the PD method proposed above to solve problem (2). Clearly, (2) can be equivalently reformulated as

$$\min_{X,Y} \{f(X) + \nu \ \mathrm{rank}(Y) : \ X - Y = 0, \ X \in \mathcal{X}, \ Y \in \Omega\}. \tag{41}$$

Given a penalty parameter $\varrho > 0$, the associated quadratic penalty function for (41) is defined as

$$P_\varrho(X, Y) := f(X) + \nu \ \mathrm{rank}(Y) + \frac{\varrho}{2}\|X - Y\|_F^2. \tag{42}$$

We are now ready to present the PD method for solving (41) (or, equivalently, (2)) in which each penalty subproblem is approximately solved by a BCD method.

Penalty decomposition method for (41):

Let $\varrho_0 > 0$, $\sigma > 1$ be given. Choose an arbitrary $Y_0^0 \in \Omega$ and a constant Υ such that $\Upsilon \geq \max\{f(X^{\mathrm{feas}}) + \nu \ \mathrm{rank}(X^{\mathrm{feas}}), \min_{X\in\mathcal{X}} P_{\varrho_0}(X, Y_0^0)\}$. Set $k = 0$.

1) Set $l = 0$ and apply the BCD method to find an approximate solution $(X^k, Y^k) \in \mathcal{X} \times \Omega$ for the penalty subproblem
$$\min\{P_{\varrho_k}(X, Y) : \ X \in \mathcal{X}, \ Y \in \Omega\} \tag{43}$$
by performing steps 1a)-1c):
   1a) Solve $X_{l+1}^k \in \operatorname{Arg}\min_{X\in\mathcal{X}} P_{\varrho_k}(X, Y_l^k)$.
   1b) Solve $Y_{l+1}^k \in \operatorname{Arg}\min_{Y\in\Omega} P_{\varrho_k}(X_{l+1}^k, Y)$.
   1c) Set $l \leftarrow l + 1$ and go to step 1a).
2) Set $\varrho_{k+1} := \sigma\varrho_k$.
3) If $\min_{X\in\mathcal{X}} P_{\varrho_{k+1}}(X, Y^k) > \Upsilon$, set $Y_0^{k+1} := X^{\mathrm{feas}}$. Otherwise, set $Y_0^{k+1} := Y^k$.

4) Set $k \leftarrow k+1$ and go to step 1).

end

Remark. In view of Corollary 2.2, the BCD subproblem in step 1a) can be reduced to the problem in form of (8), which has closed form solution when Ω is simple enough. In addition, the practical termination criteria proposed for the previous PD method can be suitably applied to this method. Moreover, given that problem (43) is nonconvex, the BCD method may converge to a stationary point. To enhance the quality of approximate solutions, one may apply a similar strategy as described for the previous PD method by executing the BCD method multiple times starting from a suitable perturbation of the current approximate solution. Finally, by a similar argument as in the proof of Theorem 3.1, we can show that every accumulation point of the sequence $\{(X^k, Y^k)\}$ is a feasible point of (41). Nevertheless, it is not clear whether a similar convergence result as in Theorem 3.1 (b) can be established due to the discontinuity and non-convexity of the objective function of (2).

4 Penalty decomposition method for rank minimization of symmetric matrices

In this section we modify the PD methods proposed in Section 3 to solve rank minimization of general symmetric matrices and positive symmetric matrices, respectively.

4.1 Penalty decomposition method for rank minimization of general symmetric matrices

Throughout this subsection, assume that $\mathcal{X}$ is a closed convex set in $\mathcal{S}^n$, Ω is a closed unitary similarity invariant set in $\mathcal{S}^n$, and $f : \mathcal{S}^n \to \Re$ is a continuously differentiable function. Our aim is to modify the PD methods proposed in Section 3 to solve problem (1) and (2) with $\mathcal{X}$, Ω and f specified above. As in Section 3, we also assume that problems (1) and (2) are feasible, and moreover, at least a feasible solution, denoted by $X^{\rm feas}$, is known.

Let $\mathcal{Y}$ and $Q_\varrho(\cdot,\cdot)$ be similarly defined as in (18) and (19), respectively. In addition, we define

$$\widetilde{Q}^s_\varrho(X,U,D) := Q_\varrho(X, UDU^T) \quad \forall X \in \mathcal{S}^n, U \in \Re^{n\times r}, D \in \mathcal{D}^r. \tag{44}$$

Similarly as for asymmetric matrices, problem (1) can also be equivalently reformulated as (17) in the context of symmetric matrices.

We are now ready to modify the PD method proposed in Section 3 to solve problem (17) (or, equivalently, (1)) in the context of symmetric matrices.

Penalty decomposition method for (17) (symmetric matrices):

All steps of this method are the same as those of the PD method proposed in Section 3 except that step 1c) is replaced by the following one:

1c) Set $(X^k, Y^k) := (X^k_{l+1}, Y^k_{l+1})$. If (X^k, Y^k) satisfies

$$\begin{array}{rcl} \mathrm{dist}\left(-\nabla_X Q_{\varrho_k}(X^k,Y^k), \mathcal{N}_{\mathcal{X}}(X^k)\right) & \leq & \epsilon_k, \\ \|\nabla_U \widetilde{Q}^s_{\varrho_k}(X^k,U^k,D^k) + 2Z^k_Y U^k D^k\|_F & \leq & \epsilon_k, \\ \|\nabla_D \widetilde{Q}^s_{\varrho_k}(X^k,U^k,D^k) + \widetilde{\mathscr{D}}\left((U^k)^T Z^k_Y U^k\right)\|_F & \leq & \epsilon_k \end{array} \tag{45}$$

for some $Z_Y^k \in \mathcal{N}_\Omega(Y^k)$, $U^k \in \Re^{n\times r}$, $D^k \in \mathcal{D}^r$ such that

$$(U^k)^T U^k = I, \quad Y^k = U^k D^k (U^k)^T, \tag{46}$$

then go to step 2).

Remark. The conditions (45)-(46) are mainly used to establish global convergence for the above method and they may be hard to verify practically unless $\mathcal{X}$ and Ω are simple. Clearly, the practical termination criteria proposed for the previous PD methods can be suitably applied to this method. Moreover, given that problem (21) is nonconvex, the BCD method may converge to a stationary point. To enhance the quality of approximate solutions, one may apply a similar strategy as described for the previous PD methods by executing the BCD method multiple times starting from a suitable perturbation of the current approximate solution. Finally, in view of Corollary 2.9, the BCD subproblem in step 1a) of the above method can be reduced to the problem in form of (16), which has closed form solution when Ω is simple enough.

We now state a convergence result regarding the outer iterations of the above method for solving problem (17) in the context of symmetric matrices. Its proof is similar to the one of Theorem 3.1.

Theorem 4.1 *Assume that $\epsilon_k \to 0$. Let $\{(X^k, Y^k)\}$ be the sequence generated by the above PD method, and $\{(U^k, D^k, Z_Y^k)\}$ be the associated sequence satisfying (45) and (25). Suppose that the level set $\mathcal{X}_\Upsilon := \{X \in \mathcal{X} | f(X) \le \Upsilon\}$ is compact. Then, the following statements hold:*

(a) The sequence $\{(X^k, Y^k, U^k, D^k)\}$ is bounded;

(b) Suppose that a subsequence $\{(X^k, Y^k, U^k, D^k)\}_{k\in K}$ converges to (X^*, Y^*, U^*, D^*). Then, (X^*, Y^*) is a feasible point of problem (17). Moreover, if the following condition

$$\left\{ \begin{pmatrix} d_X - d_U D^*(U^*)^T - U^* d_D (U^*)^T - U^* D^* d_U^T \\ d_U D^*(U^*)^T + U^* d_D (U^*)^T + U^* D^* d_U^T - d_Y \end{pmatrix} : \begin{array}{l} d_X \in \mathcal{T}_{\mathcal{X}}(X^*), d_U \in \Re^{n\times r}, \\ d_D \in \mathcal{D}^r, d_Y \in \mathcal{T}_\Omega(X^*) \end{array} \right\} = \mathcal{S}^n \times \mathcal{S}^n \tag{47}$$

holds, then the subsequence $\{(Z_X^k, Z_Y^k)\}_{k\in K}$ is bounded, where $Z_X^k := \varrho_k(X^k - Y^k)$, and each accumulation point (Z_X^*, Z_Y^*) of $\{(Z_X^k, Z_Y^k)\}_{k\in K}$ together with (X^*, U^*, D^*) satisfies

$$\begin{array}{rcl} -\nabla f(X^*) - Z_X^* & \in & \mathcal{N}_{\mathcal{X}}(X^*), \\ (Z_X^* - Z_Y^*)U^* D^* & = & 0, \\ \widetilde{\mathscr{D}}\left((U^*)^T (Z_X^* - Z_Y^*) U^*\right) & = & 0, \\ X^* - U^* D^* (U^*)^T = 0, \quad Z_Y^* & \in & \mathcal{N}_\Omega(X^*). \end{array} \tag{48}$$

Remark. From Theorem 4.1 (b), we see that under condition (47), any accumulation point $(X^*, U^*, D^*, Z_X^*, Z_Y^*)$ of $\{(X^k, U^k, D^k, Z_X^k, Z_Y^k)\}_{k\in K}$ satisfies (48). Thus, (X^*, U^*, V^*) together with (Z_X^*, Z_Y^*) satisfies the first-order optimality (i.e., KKT) conditions of the following reformulation of (17) (or, equivalently, (1)) in the context of symmetric matrices:

$$\min_{X,U,D} \{f(X) : \; X - UDU^T = 0, \; UDU^T \in \Omega, \; X \in \mathcal{X}, \; U \in \Re^{n\times r}, \; D \in \mathcal{D}^r\}.$$

We next state a convergence result regarding the inner iterations of the above PD method, whose proof is similar to the one of Theorem 3.2. For ease of of presentation, we simplify the BCD method by omitting the index k from (21) and the resulting BCD method is the same as the one presented in Section 3 except in the context of symmetric matrices.

Theorem 4.2 *Let* $\{(X^k, Y^k)\} \subseteq \mathcal{X} \times \mathcal{Y}$ *be generated by the above BCD method, and let* $U^k \in \Re^{n\times r}$, $D^k \in \mathcal{D}^r$ *be such that* $(U^k)^T U^k = I$ *and* $Y^k = U^k D^k (U^k)^T$. *Suppose that a subsequence* $\{(X^k, Y^k, U^k, D^k)\}_{k\in K}$ *converges to* $(\bar{X}, \bar{Y}, \bar{U}, \bar{D})$. *If the following condition*

$$\left\{d_U \bar{D}\bar{U}^T + \bar{U} d_D \bar{U}^T + \bar{U}\bar{D}d_U^T - d_Y : \ d_U \in \Re^{n\times r}, d_D \in \mathcal{D}^r, d_Y \in \mathcal{T}_\Omega(\bar{Y})\right\} = \mathcal{S}^n \tag{49}$$

holds, then $(\bar{X}, \bar{Y}, \bar{U}, \bar{D})$ *satisfies*

$$-\nabla_X Q_\varrho(\bar{X}, \bar{Y}) \in \mathcal{N}_\mathcal{X}(\bar{X}), \quad \nabla_U \widetilde{Q}^s_\varrho(\bar{X}, \bar{U}, \bar{D}) + 2\bar{Z}_Y \bar{U}\bar{D} = 0, \quad \nabla_D \widetilde{Q}^s_\varrho(\bar{X}, \bar{U}, \bar{D}) + \widetilde{\mathscr{D}}\left(\bar{U}^T \bar{Z}_Y \bar{U}\right) = 0$$

for some $\bar{Z}_Y \in \mathcal{N}_\Omega(\bar{Y})$.

Using the semicontinuity of $\mathcal{N}_\mathcal{X}(\cdot)$ and $\mathcal{N}_\Omega(\cdot)$, and the continuous differentiability of Q_ϱ and $\widetilde{Q}^s_\varrho$, we immediately conclude from Theorem 4.2 that the following corollary holds.

Corollary 4.3 *Suppose that condition (49) holds for any* $\bar{U} \in \Re^{n\times r}$, $\bar{D} \in \mathcal{D}^r$ *such that* $\bar{U}^T\bar{U} = I$ *and* $\bar{Y} = \bar{U}\bar{D}\bar{U}^T \in \Omega$. *The approximate solution* $(X^k, Y^k) \in \mathcal{X} \times \mathcal{Y}$ *for problem (21) satisfying (45) and (46) can be found by the BCD method described in steps 1a)-1d) within a finite number of iterations.*

Remark. When $\Omega = \mathcal{S}^n$, condition (49) clearly holds at any $(\bar{U}, \bar{D}, \bar{Y}) \in \Re^{m\times r} \times \mathcal{D}^r \times \mathcal{S}^n$. In addition, it is obvious that the above PD method can be extended to solve problem (41) in the context of symmetric matrices by the same manner as described in Section 3. Moreover, it follows from Corollary 2.8 that the corresponding BCD subproblems can be reduced to the problems in form of (14), which have closed form solutions when Ω is simple enough.

4.2 Penalty decomposition method for rank minimization of positive semidefinite matrices

Throughout this subsection, assume that $\mathcal{X}$ is a closed convex set in $\mathcal{S}^n$, Ω is a closed unitary similarity invariant set in $\mathcal{S}^n_+$, and $f : \mathcal{S}^n \to \Re$ is a continuously differentiable function. We aim to modify the PD methods proposed in Section 3 to solve problems (1) and (2) with $\mathcal{X}$, Ω and f specified above. As in Section 3, we also assume that problems (1) and (2) are feasible, and moreover, at least a feasible solution, denoted by X^{feas}, is known.

Let $\mathcal{Y}$ and $Q_\varrho(\cdot,\cdot)$ be defined as in (18) and (19), respectively. In addition, we define

$$\widetilde{Q}^{s+}_\varrho(X, U) := Q_\varrho(X, UU^T) \quad \forall X \in \mathcal{S}^n, U \in \Re^{n\times r}.$$

Similarly as for asymmetric matrices, problem (1) can also be equivalently reformulated as (17) in the context of positive semidefinite matrices. Further, since $\Omega \subseteq \mathcal{S}^n_+$, it can be represented as

$$\Omega = \mathcal{S}^n_+ \cap \widetilde{\Omega}$$

for some $\widetilde{\Omega} \subseteq \mathcal{S}^n$. For example, when $\Omega = \{X \in \mathcal{S}^n_+ : \text{Tr}(X) = 1\}$, we can choose $\widetilde{\Omega} = \{X \in \mathcal{S}^n : \text{Tr}(X) = 1\}$ accordingly.

We are now ready to modify the PD method proposed in Section 3 to solve problem (17) (or, equivalently, (1)) in the context of positive semidefinite matrices.

Penalty decomposition method for (17) (positive semidefinite matrices):

All steps of this method are the same as those of the PD method proposed in Section 3 except that step 1c) is replaced by the following one:

1c) Set $(X^k, Y^k) := (X^k_{l+1}, Y^k_{l+1})$. If (X^k, Y^k) satisfies

$$\begin{array}{rcl} \mathrm{dist}\left(-\nabla_X Q_{\varrho_k}(X^k, Y^k), \mathcal{N}_{\mathcal{X}}(X^k)\right) & \leq & \epsilon_k, \\ \|\nabla_U \widetilde{Q}^{s+}_{\varrho_k}(X^k, U^k) + 2Z^k_U U^k\|_F & \leq & \epsilon_k \end{array} \tag{50}$$

for some $Z^k_U \in \mathcal{N}_{\widetilde{\Omega}}(Y^k)$, $U^k \in \Re^{n\times r}$ such that

$$Y^k = U^k(U^k)^T, \tag{51}$$

then go to step 2).

Remark. The conditions (50)-(51) are mainly used to establish a global convergence result for the above method and they may be hard to verify practically unless $\mathcal{X}$ and $\widetilde{\Omega}$ are simple. Clearly, the practical termination criteria proposed for the previous PD methods can be suitably applied to this method. Moreover, given that problem (21) is nonconvex, the BCD method may converge to a stationary point. To enhance the quality of approximate solutions, one may apply a similar strategy as described for the previous PD methods by executing the BCD method multiple times starting from a suitable perturbation of the current approximate solution.

We now state a convergence result regarding the outer iterations of the above method for solving problem (17) in the context of positive semidefinite matrices. Its proof is similar to the one of Theorem 3.1.

Theorem 4.4 *Assume that $\epsilon_k \to 0$. Let $\{(X^k, Y^k)\}$ be the sequence generated by the above PD method, and $\{(U^k, Z^k_U)\}$ be the associated sequence satisfying (50) and (51). Suppose that the level set $\mathcal{X}_\Upsilon := \{X \in \mathcal{X} | f(X) \leq \Upsilon\}$ is compact. Then, the following statements hold:*

(a) The sequence $\{(X^k, Y^k, U^k)\}$ is bounded;

(b) Suppose that a subsequence $\{(X^k, Y^k, U^k)\}_{k\in K}$ converges to (X^*, Y^*, U^*). Then, (X^*, Y^*) is a feasible point of problem (17). Moreover, if the following condition

$$\left\{\begin{pmatrix} d_X - d_U(U^*)^T - U^* d_U^T \\ d_U(U^*)^T + U^* d_U^T - d_Y \end{pmatrix} : \begin{array}{l} d_X \in \mathcal{T}_{\mathcal{X}}(X^*), d_U \in \Re^{n\times r}, \\ d_Y \in \mathcal{T}_{\widetilde{\Omega}}(X^*) \end{array}\right\} = \mathcal{S}^n \times \mathcal{S}^n \tag{52}$$

holds, then the subsequence $\{(Z^k_X, Z^k_Y)\}_{k\in K}$ is bounded, where $Z^k_X := \varrho_k(X^k - Y^k)$, and each accumulation point (Z^*_X, Z^*_Y) of $\{(Z^k_X, Z^k_Y)\}_{k\in K}$ together with (X^*, U^*) satisfies

$$\begin{array}{rcl} -\nabla f(X^*) - Z^*_X & \in & \mathcal{N}_{\mathcal{X}}(X^*), \\ (Z^*_X - Z^*_Y)U^* & = & 0, \\ X^* - U^*(U^*)^T = 0, \quad Z^*_Y & \in & \mathcal{N}_{\widetilde{\Omega}}(X^*). \end{array} \tag{53}$$

Remark. From Theorem 4.1 (b), we see that under condition (52), any accumulation point (X^*, U^*, Z^*_X, Z^*_Y) of $\{(X^k, U^k, Z^k_X, Z^k_Y)\}_{k\in K}$ satisfies (53). Thus, (X^*, U^*) together with (Z^*_X, Z^*_Y) satisfies the first-order optimality (i.e., KKT) conditions of the following reformulation of (17) (or, equivalently, (1)) in the context of positive semidefinite matrices:

$$\min_{X,U}\{f(X) : \; X - UU^T = 0, \; UU^T \in \widetilde{\Omega}, \; X \in \mathcal{X}, \; U \in \Re^{n\times r}\}.$$

We next state a convergence result regarding the inner iterations of the above PD method, whose proof is similar to the one of Theorem 3.2. For ease of of presentation, we simplify the BCD method by omitting the index k from (21) and the resulting BCD method is the same as the one presented in Section 3 except in the context of positive semidefinite matrices.

Theorem 4.5 *Let $\{(X^k, Y^k)\} \subseteq \mathcal{X} \times \mathcal{Y}$ be generated by the above BCD method, and let $U^k \in \Re^{n\times r}$ be such that $Y^k = U^k(U^k)^T$. Suppose that a subsequence $\{(X^k, Y^k, U^k)\}_{k\in K}$ converges to $(\bar{X}, \bar{Y}, \bar{U})$. If the following condition*

$$\left\{d_U \bar{U}^T + \bar{U} d_U^T - d_Y : \ d_U \in \Re^{n\times r}, d_Y \in \mathcal{T}_{\widetilde{\Omega}}(\bar{Y})\right\} = \mathcal{S}^n \tag{54}$$

holds, then $(\bar{X}, \bar{Y}, \bar{U})$ satisfies

$$-\nabla_X Q_\varrho(\bar{X}, \bar{Y}) \in \mathcal{N}_{\mathcal{X}}(\bar{X}), \quad \nabla_U \widetilde{Q}_\varrho^{s+}(\bar{X}, \bar{U}) + 2\bar{Z}_Y \bar{U} = 0$$

for some $\bar{Z}_Y \in \mathcal{N}_{\widetilde{\Omega}}(\bar{Y})$.

Using the semicontinuity of $\mathcal{N}_{\mathcal{X}}(\cdot)$ and $\mathcal{N}_{\Omega}(\cdot)$, and the continuous differentiability of Q_ϱ and $\widetilde{Q}_\varrho^{s+}$, we immediately conclude from Theorem 4.5 that the following corollary holds.

Corollary 4.6 *Suppose that condition (54) holds for any $\bar{U} \in \Re^{n\times r}$ such that $\bar{Y} = \bar{U}\bar{U}^T \in \widetilde{\Omega}$. The approximate solution $(X^k, Y^k) \in \mathcal{X} \times \mathcal{Y}$ for problem (21) satisfying (50) and (51) can be found by the BCD method described in steps 1a)-1d) within a finite number of iterations.*

Remark. When $\widetilde{\Omega} = \mathcal{S}^n$, condition (54) clearly holds at any $(\bar{U}, \bar{Y}) \in \Re^{m\times r} \times \mathcal{S}^n$. In addition, it is obvious that the above PD method can be extended to solve problem (41) in the context of positive semidefinite matrices by the same manner as described in Section 3. Moreover, it follows from Corollary 2.8 that the corresponding BCD subproblems can be reduced to the problems in form of (14), which have closed form solutions when $\widetilde{\Omega}$ is simple enough.

5 Numerical results

In this section, we conduct numerical experiments to test the performance of our penalty decomposition (PD) methods proposed in Sections 3 and 4 by applying them to solve matrix completion and nearest low-rank correlation matrix problems. All computations below are performed on an Intel Xeon E5410 CPU (2.33GHz) and 8GB RAM running Red Hat Enterprise Linux (kernel 2.6.18).

5.1 Matrix completion problem

In this subsection, we apply our PD method proposed in Section 3 to the matrix completion problem, which has numerous applications in control and systems theory, image recovery and data mining (see, for example, [34, 25, 9, 18]). It can be formulated as

$$\begin{array}{rl} \min\limits_{X\in\Re^{m\times n}} & \mathrm{rank}(X) \\ \mathrm{s.t.} & X_{ij} = M_{ij}, \ (i,j) \in \Theta, \end{array} \tag{55}$$

where $M \in \Re^{m\times n}$ and Θ is a subset of index pairs (i,j). Recently, numerous methods were proposed to solve the nuclear norm relaxation or the variant of (55) (see, for example, [20, 6, 23, 8, 14, 16, 22, 24, 33, 19, 38, 36]).

It is not hard to see that problem (55) is a special case of the general rank minimization problem (2) with $f(X) \equiv 0$, $\nu = 1$, $\Omega = \Re^{m\times n}$, and

$$\mathcal{X} = \{X \in \Re^{m\times n} : X_{ij} = M_{ij},\ (i,j) \in \Theta\}.$$

Thus, the PD method proposed in Section 3 for problem (2) can be suitably applied to (55). Recall that the main computational parts of this method lie in solving the subproblems in steps 1a) and 1b). In the context of (55), they are in the form of

$$\min_X \{\|X - A\|_F^2 : X \in \mathcal{X}\}, \tag{56}$$

$$\min_Y \{\operatorname{rank}(Y) + \varrho\|Y - B\|_F^2 : Y \in \Re^{m\times n}\} \tag{57}$$

for some $\varrho > 0$, $A, B \in \Re^{m\times n}$, respectively. In view of the above definition of $\mathcal{X}$, we easily see that the solution of (56) is given by

$$X = A + \mathcal{P}_\Theta(M - A),$$

where $\mathcal{P}_\Theta$ is the projection onto the subspace of matrices with nonzeros restricted to the index subset Θ. In addition, it follows from Corollary 2.2 that problem (57) also has a closed form solution.

We now address the initialization and termination criteria for our PD method when applied to (55). In particular, we choose X^{feas} to be the $m \times n$ matrix satisfying $X^{\text{feas}}_{ij} = M_{ij}$ for all $(i,j) \in \Theta$ and $X^{\text{feas}}_{ij} = 0$ for all $(i,j) \notin \Theta$, and set $Y^0_0 = X^{\text{feas}}$. In addition, we choose the initial penalty parameter ϱ_0 to be 0.1, and set the parameter $\sigma = \sqrt{10}$. We use (26) and (27) as the inner and outer termination criteria for the PD method and set their associated accuracy parameters $\epsilon_I = 10^{-7}$ and $\epsilon_O = 10^{-5}$.

Next we conduct numerical experiments to test the performance of our PD method for solving matrix completion problem (55) on random and real data. We also compare the results of our method with the other two methods, that is, the FPCA method [23] and the LMaFit method [36]. The codes of all the methods are written in Matlab.

In the first experiment, we aim to recover a random matrix $M \in \Re^{m\times n}$ with rank r based on a subset of entries $\{M_{ij}\}_{(i,j)\in\Theta}$. For this purpose, we randomly generate M and Θ by a similar procedure as described in [23]. In detail, we first generate random matrices $M_L \in \Re^{m\times r}$ and $M_R \in \Re^{r\times n}$ with i.i.d. standard Gaussian entries and let $M = M_L M_R^T$. We then sample a subset Θ of p entires uniformly at random. As observed in [23], the recoverability for M based on its partial entries is closely related to two ratios, that is, the sampling ratio $SR = p/(mn)$ (i.e., the number of measurements divided by the number of entries of the matrix) and another ratio $FR = r(m+n-r)/p$ (i.e., the dimension of the set of rank r matrices divided by the number of measurements). Evidently, when SR gets larger, it is easier to recover M. Additionally, as mentioned in [23], if $FR > 1$, there is always an infinite number of matrices of rank r with the given entries $\{M_{ij}\}_{(i,j)\in\Theta}$. Thus in this situation it is impossible to recover M based on its partial entries. Moreover, even if $FR \le 1$, it still becomes harder to recover M as FR gets closer to 1. In our experiment, we set $m = n = 40$, $p = 800$ (or, equivalently, $SR = 0.5$) and randomly generate 50 copies of M for each $r = 1, \ldots, 10$. Clearly, their FR varies in $(0, 1)$.

Given an approximate recovery X^* for M, we define the relative error as

$$\text{rel\_err} := \frac{\|X^* - M\|_F}{\|M\|_F}.$$

Table 1: Computational results for $m = 40$, $n = 40$ and $p = 800$

Problems		FPCA			LMaFit			PD		
Rank	FR	NS	rel_err	Time	NS	rel_err	Time	NS	rel_err	Time
1	0.0988	50	7.20e-6	2.4	50	1.38e-4	0.004	50	1.17e-5	0.1
2	0.1950	50	1.34e-5	2.4	50	1.76e-4	0.005	50	1.57e-5	1.2
3	0.2888	50	2.20e-5	2.6	50	2.15e-4	0.008	50	1.90e-5	2.4
4	0.3800	50	2.87e-5	2.6	50	2.94e-4	0.014	50	2.34e-5	2.8
5	0.4688	50	3.52e-5	2.7	50	3.54e-4	0.024	50	3.27e-5	3.5
6	0.5550	50	3.98e-5	2.9	48	4.41e-4	0.034	50	3.77e-5	5.4
7	0.6388	50	6.13e-5	3.0	46	6.23e-4	0.045	48	5.64e-5	6.1
8	0.7200	48	1.22e-4	3.2	26	7.57e-4	0.066	47	9.21e-5	11.5
9	0.7987	47	1.50e-4	3.5	6	9.24e-4	0.107	40	1.31e-4	19.7
10	0.8750	33	6.24e-4	4.6	0	–	–	25	2.07e-4	48.5

We adopt the same criterion as used in [28, 7], and say a matrix M is *successfully recovered* by X^* if the corresponding relative error is less than 10^{-3}. For each rank r, we apply our PD method, the FPCA method [23] and the LMaFit method [36] to recover M on 50 instances randomly generated above. As in [23, 36], the parameter setting for LMaFit is tol $= 10^{-4}$, $K = \lfloor 1.25r \rfloor$, est_rank $= 1$, rank_max $= 40$ while the one for FPCA is tol $= 10^{-4}$ and $\mu = 10^{-4}$. All other parameters are set to the default values for these two methods. For convenience, we use NS to denote the number of matrices that are successfully recovered. The computational results are presented in Table 1. In detail, the rank r and FR of the problems are given in the first two columns. The results of all three methods in terms of NS, the average rel_err and CPU time on the successfully recovered instances are reported in columns three to eleven, respectively. Table 1 shows that the recoverability of three methods are similar for the instances with $r \leq 7$, but FPCA and PD are significantly better than LMaFit for the instances with $r \geq 8$. In addition, though FPCA successfully recovers slightly more matrices than PD when FR gets closer to one, most of recovery matrices by PD are closer to the original matrices M on the average since their average relative errors are smaller. Overall, PD is comparable to FPCA, which has better recoverability than the other existing methods as demonstrated in [23, 36]. Finally, we observe that our method is slower compared to the other two methods because it requires a full singular value decomposition at each inner iteration while the other two methods require partial or none singular value decompositions. Despite this drawback, our method sometimes outperforms the other two methods in terms of solution quality as demonstrated in the next test.

In the second experiment, our goal is to recover a high-rank matrix $M \in \Re^{n\times n}$, whose most of singular values are nearly zero, by a low-rank matrix based on a subset of entries $\{M_{ij}\}_{(i,j)\in\Theta}$. To this aim, we randomly generate M and Θ by a similar procedure as described in [36]. In particular, we first generate random matrices $M_L \in \Re^{n\times n}$ and $M_R \in \Re^{n\times n}$ with i.i.d. standard Gaussian entries. Then we obtain matrices U and V by orthonormalizing the columns of M_L and M_R, respectively. Let Σ be a diagonal matrix whose diagonal elements are $\sigma_i = i^{-4}$ for all i or $\sigma_i = 9.9^{-(i-1)}$ for all i. Finally, we set $M = U\Sigma V^T$, and sample a subset Θ of p entires uniformly at random. We generate 50 instances for each one of the sample ratios SR that vary from 0.5 to 0.9.

For each sample ratio SR, we apply our PD method, the FPCA method [23] and the LMaFit method [36] to recover M on 50 instances randomly generated above. As in [36], the parameter setting for LMaFit is tol $= 10^{-4}$, $K = 1$, est_rank $= 2$, rk_inc $= 1$, rank_max $= 40$ while the one for FPCA is tol $= 10^{-4}$ and $\mu = 10^{-4}$. All other parameters are set to the default values for these two methods. The computational results are presented in Tables 2 and 3. In particular, Table 2 reports the results for the instances with $n = 40$ and $\sigma_i = i^{-4}$ for all i while Table 3 presents the results for the instances

Table 2: Computational results for $n = 40$ and $\sigma_i = i^{-4}$

	FPCA		LMaFit		PD	
SR	Rank	rel_err	Rank	rel_err	Rank	rel_err
0.5	2.0	1.45e-2	9.2	4.73e-4	5.0	9.91e-4
0.6	2.0	1.42e-2	9.9	2.73e-4	5.0	9.56e-4
0.7	2.0	1.34e-2	10.0	1.89e-4	5.0	9.37e-4
0.8	2.0	1.31e-2	10.8	1.42e-4	5.0	9.25e-4
0.9	2.1	1.24e-2	10.5	1.16e-4	5.0	9.01e-4

Table 3: Computational results for $n = 40$ and $\sigma_i = 9.9^{-(i-1)}$

	FPCA		LMaFit		PD	
SR	Rank	rel_err	Rank	rel_err	Rank	rel_err
0.5	2.0	1.10e-2	4.2	2.66e-4	4.0	1.27e-4
0.6	2.0	1.07e-2	4.0	1.86e-4	4.0	1.15e-4
0.7	2.1	1.00e-2	4.5	1.46e-4	4.0	1.08e-4
0.8	2.1	1.02e-2	4.6	1.36e-4	4.0	1.04e-4
0.9	2.1	1.01e-2	4.4	1.08e-4	4.0	1.02e-4

with $n = 40$ and $\sigma_i = 9.9^{-(i-1)}$ for all i. In each table, the sample ratio SR of the test problems is given in the first column. The average rank and rel_err of the solutions given by these three methods over every 50 randomly generated instances are reported in columns two to seven, respectively. From Table 2, we see that the average rank of FPCA is much smaller than that of LMaFit and PD, but FPCA has much larger average rel_err than LMaFit and PD. Given that the average rel_err for FPCA is far above 10^{-3}, it follows from the above commonly adopted criterion that the original matrix M is not successfully recovered by FPCA. On the other hand, M is successfully recovered by LMaFit and PD as their average rel_err is far below 10^{-3}. In addition, we observe that LMaFit has much smaller average rel_err than PD, but PD has much smaller average rank. Now, one natural question is whether there exists a matrix X^* with a smaller rank than the one given by PD for successfully recovering such a M. The answer is actually not. Indeed, let X^* be a matrix of rank at most four with smallest rel_err, that is,

$$X^* \in \operatorname{Arg\,min}\{\|X - M\|_F : \ \operatorname{rank}(X) \leq 4\}.$$

Using Corollary 2.3 and the fact that $\sigma_i(M) = i^{-4}$ for all i, we have

$$\frac{\|X^* - M\|_F}{\|M\|_F} = \frac{\sqrt{\sum_{i=5}^{40} \sigma_i^2(M)}}{\sqrt{\sum_{i=1}^{40} \sigma_i^2(M)}} = \frac{\sqrt{\sum_{i=5}^{40} i^{-8}}}{\sqrt{\sum_{i=1}^{40} i^{-8}}} \approx 1.8e-3 > 10^{-3}.$$

Thus, according to the above criterion, any matrix of rank at most four cannot successfully recover such a M. The similar phenomenon as above can be observed in Table 3 for FPCA. We also see from Table 3 that PD has equal or smaller average rank than LMaFit, and moreover, PD has smaller average rel_err. One can also observe that the original matrix M is successfully recovered by PD and LMaFit , but not by FPCA. In addition, by a similar argument as above, we can show that any matrix of rank less than the one given by PD has rel_err above 10^{-3} and thus it cannot successfully recover such a M. From these two examples, we see that our method PD can find the recovery matrix of smallest rank, but the other two methods generally cannot.

In the last experiment, we aim to test the performance of our PD method for solving a real data problem, that is, grayscale image inpainting problem [2]. This problem has been used in [23, 36] to test FPCA and LMaFit, respectively. For an image inpainting problem, our goal is to fill the missing pixel values of the image at given pixel locations. The missing pixel positions can be either randomly distributed or not. As shown in [34, 25], this problem can be solved as a matrix completion problem if the image is of low-rank. In our test, the original 512×512 grayscale image is shown in Figure 1(a). To obtain the data for problem (55), we first apply the singular value decomposition to the original image and truncate the resulting decomposition to get an image of rank 40 shown in Figure 1(b). Figures 1(c) and 1(e) are then constructed from Figures 1(a) and 1(b) by sampling half of their pixels uniformly at random, respectively. Figure 1(g) is generated by masking 6% of the pixels of Figure 1(b) in a non-random fashion. We now apply our PD method to solve problem (55) with the data given in Figures 1(c), 1(e) and 1(g), and the resulting recovered images are presented in Figures 1(d), 1(f) and 1(h), respectively. In addition, we shall mention that the relative errors of three recovered images to the original images by our method are 6.72e-2, 6.43e-2 and 6.77e-2, respectively, which are all smaller than those reported in [23, 36].

5.2 Nearest low-rank correlation matrix problem

In this subsection, we apply our PD method proposed in Subsection 4.2 to find the nearest low-rank correlation matrix, which has important applications in finance (see, for example, [4, 30, 37, 39, 31]). It can be formulated as

$$\begin{array}{cl} \min\limits_{X \in S^n} & \frac{1}{2}\|X - C\|_F^2 \\ \text{s.t.} & \text{diag}(X) = e, \\ & \text{rank}(X) \le r, \ X \succeq 0 \end{array} \tag{58}$$

for some correlation matrix $C \in S_+^n$ and some integer $r \in [1, n]$, where diag(X) denotes the vector consisting of the diagonal entries of X and e is the all-ones vector. Recently, a few methods have been proposed for solving problem (58) (see, for example, [29, 27, 3, 26, 12, 17]).

It is not hard to see that problem (58) is a special case of the general rank constraint problem (2) with $f(X) = \frac{1}{2}\|X - C\|_F^2$, $\Omega = \mathcal{S}_+^n$, and

$$\mathcal{X} = \{X \in \mathcal{S}^n : \text{diag}(X) = e\}.$$

Thus, the PD method proposed in Section 4 for problem (2) can be suitably applied to (58). Recall that the main computational parts of this method lie in solving the subproblems in steps 1a) and 1b). In the context of (58), they are in the form of

$$\begin{array}{l} \min\limits_X \{\|X - A\|_F^2 : X \in \mathcal{X}\}, \\ \min\limits_Y \{\|Y - B\|_F^2 : \text{rank}(Y) \le r, \ Y \succeq 0\} \end{array} \tag{59}$$

for some $A, B \in \mathcal{S}^n$, respectively. In view of the above definition of $\mathcal{X}$ and Corollary 2.9, we easily see that the above two problems have closed form solutions.

We now address the initialization and termination criteria for our PD method when applied to (58). In particular, we choose $X^{\text{feas}} = ee^T$, and Y_0^0 to be the solution of problem (59) by replacing B by C. In addition, we choose the initial penalty parameter ϱ_0 to be 1, and set the parameter $\sigma = \sqrt{10}$. We use (26) and (27) as the inner and outer termination criteria for the PD method and set their associated accuracy parameters $\epsilon_I = 10^{-7}$ and $\epsilon_O = 10^{-5}$.

Next we conduct numerical experiments to test the performance of our method for solving (58) on four classes of benchmark testing problems. These problems are widely used in literature (see, for example, [3, 30, 26, 17]) and their corresponding data matrices C are defined as follows:

Table 4: Statistics of γ_i, $i = 1, \ldots, 4$

	γ_1	γ_2	γ_3	γ_4
Estimate	0.000	0.480	1.511	0.186
Standard error	-	0.099	0.289	0.127

(P1) $C_{ij} = 0.5 + 0.5 \exp(-0.05|i-j|)$ for all i, j (see [3]).

(P2) $C_{ij} = \exp(-|i-j|)$ for all i, j (see [3]).

(P3) $C_{ij} = \text{LongCorr} + (1 - \text{LongCorr}) \exp(\kappa|i-j|)$ for all i, j, where LongCorr $= 0.6$ and $\kappa = -0.1$ (see [30]).

(P4)

$$C_{ij} = \exp\left(-\gamma_1|i-j| - \frac{\gamma_2|i-j|}{\max(i,j)^{\gamma_3}} - \gamma_4|\sqrt{i} - \sqrt{j}|\right)$$

for all i, j, where $\gamma_i > 0$ for $i = 1, 2, 4$, and the estimate and standard error of each γ_i is given in Table 4 (see [26]).

We generate two instances for each (P1)-(P3) by letting $n = 100$, 500, respectively. In addition, for each $n = 100$, 500, we generate 10 instances for problem (P4) by sampling γ_i's according to their mean and standard deviation detailed in Table 4.

We now apply our PD method and the method named as Major developed in [26] to solve problem (58) on the instances generated above. The codes for both methods are written in Matlab. To fairly compare their performance, we choose the termination criterion for Major to be the one based on the relative error rather than the (default) absolute error. More specifically, it terminates once the relative error is less than 10^{-5}. In addition, our method needs to solve problem (59) at each iteration, which requires computing r leading eigenvalues and the associated eigenvectors. We use the package PROPACK [15] to compute them, which is much more efficient than the standard eigenvalue decomposition when r is relatively small. The computational results of both methods on the instances generated above with $r = 5, 10, \ldots, 25$ are presented in Table 5. The names of all problems are given in column one and they are labeled in the same manner as described in [17]. For example, P1n100r5 means that it corresponds to problem (P1) with $n = 100$ and $r = 5$. The results of both methods in terms of number of iterations, objective function value and CPU time are reported in columns two to seven of Table 5, respectively. For those problems prefixed by "P4", the results are obtained by taking the average over 10 random instances. We observe that the objective function values for both methods are comparable though the ones for Major are slightly better on some instances. In addition, for small r (say, $r = 5$), Major generally outperforms PD in terms of speed, but PD substantially outperforms Major as r gets larger (say, $r = 15$).

6 Concluding remarks

In this paper we proposed penalty decomposition (PD) methods for general rank minimization problems in which each subproblem is solved by a block coordinate descend method. Under some suitable assumptions, we showed that any accumulation point of the sequence generated by our method when applied to the rank constrained minimization problem is a stationary point of a nonlinear reformulation of the problem. The computational results on matrix completion and nearest low-rank correlation

Table 5: Comparison of Major and PD

Problem	Major			PD		
	Iter	Obj	Time	Iter	Obj	Time
P1n100r5	243	15.0	1.4	976	15.0	8.2
P1n100r10	1698	1.9	12.6	596	1.9	7.8
P1n100r15	5581	0.5	52.5	403	0.5	7.6
P1n100r20	10667	0.2	128.4	208	0.2	5.2
P1n100r25	17430	0.1	259.9	116	0.1	2.4
P1n500r5	488	3107.0	22.9	2514	3107.2	80.7
P1n500r10	836	748.2	51.5	1220	748.2	48.4
P1n500r15	1690	270.2	137.0	804	270.2	37.3
P1n500r20	3106	123.4	329.1	581	123.4	31.5
P1n500r25	5444	65.5	722.0	480	65.5	29.4
P2n100r5	528	852.3	3.1	963	852.3	7.7
P2n100r10	180	356.6	1.4	573	356.6	7.1
P2n100r15	122	196.4	1.3	421	196.4	7.0
P2n100r20	105	121.0	1.2	318	121.0	6.5
P2n100r25	102	79.4	1.6	231	79.4	5.9
P2n500r5	2126	24248.5	97.8	3465	24248.5	112.3
P2n500r10	3264	11749.5	199.6	1965	11749.5	76.6
P2n500r15	5061	7584.4	409.9	1492	7584.4	70.4
P2n500r20	4990	5503.2	532.0	1216	5503.2	67.2
P2n500r25	2995	4256.0	404.1	1022	4256.0	69.2
P3n100r5	279	29.4	1.6	937	29.4	7.0
P3n100r10	1232	4.4	9.2	601	4.4	7.1
P3n100r15	3685	1.3	34.6	505	1.3	8.6
P3n100r20	7903	0.6	95.2	317	0.6	7.3
P3n100r25	13472	0.3	201.5	176	0.3	5.2
P3n500r5	2541	2869.3	116.4	2739	2869.4	90.4
P3n500r10	2357	981.8	144.2	1410	981.8	55.4
P3n500r15	2989	446.9	241.9	923	446.9	41.6
P3n500r20	4086	234.7	438.4	662	234.7	33.0
P3n500r25	5923	135.9	788.3	504	135.9	29.5
P4n100r5	387	9.5	2.2	1573	9.5	12.9
P4n100r10	2062	1.2	15.1	1977	1.2	25.7
P4n100r15	6545	0.4	61.5	1480	0.4	27.8
P4n100r20	13439	0.2	157.5	1089	0.2	27.6
P4n100r25	23470	0.1	350.9	875	0.1	28.8
P4n500r5	160	845.7	8.1	2389	846.2	79.1
P4n500r10	971	109.6	60.8	2044	109.7	80.6
P4n500r15	2674	32.8	215.4	1608	32.9	72.6
P4n500r20	5249	13.9	572.0	1322	14.0	69.0
P4n500r25	8248	7.2	1107.8	1203	7.2	74.7

matrix problems demonstrate that our methods generally outperform the existing methods in terms of solution quality and/or speed.

We remark that the methods proposed in this paper can be straightforwardly extended to solve more general rank minimization problems:

$$\begin{array}{ll} \min\limits_{X} & f(X) \\ \text{s.t.} & g_i(X) \le 0,\ i = 1,\ldots,p, \\ & h_i(X) = 0,\ i = 1,\ldots,q, \\ & \text{rank}(X) \le r,\ X \in \mathcal{X} \cap \Omega, \end{array} \qquad \begin{array}{ll} \min\limits_{X} & f(X) + \nu\ \text{rank}(X) \\ \text{s.t.} & g_i(X) \le 0,\ i = 1,\ldots,p, \\ & h_i(X) = 0,\ i = 1,\ldots,q, \\ & X \in \mathcal{X} \cap \Omega \end{array}$$

for some $r,\ \nu \geq 0$, where $\mathcal{X}$ is a closed convex set, Ω is a closed unitarily invariant set in $\Re^{m\times n}$, $f : \Re^{m\times n} \to \Re$, $g_i : \Re^{m\times n} \to \Re$, $i = 1, \ldots, p$, and $h_i : \Re^{m\times n} \to \Re$, $i = 1, \ldots, q$, are continuously differentiable functions. In addition, we can also develop augmented Lagrangian decomposition methods for solving these problems simply by replacing the quadratic penalty functions of the PD methods by augmented Lagrangian functions.

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(a) original image (b) rank 40 image

(c) 50% masked original image (d) recovered image by PD

(e) 50% masked rank 40 image (f) recovered image by PD

(g) 6.34% masked rank 40 image (h) recovered image by PD

Figure 1: Image inpainting