# Non-linear approximations for solving 3D-packing MIP models: a heuristic approach

**Giorgio Fasano**

CMath FIMA CSci,

Thales Alenia Space Italia S.p.A., Str. Antica di Collegno 253, 10146 Turin, Italy

e-mail: giorgio.fasano@thalesaleniaspace.com

**Abstract**

This article extends a previous work focused on a mixed integer programming (MIP) based heuristic approach, aimed at solving non-standard three-dimensional problems with additional conditions. The paper that follows considers a mixed integer non-linear (MINLP) reformulation of the previous model, to improve the former heuristic, based on linear relaxation. The approach described herewith is addressed, in particular, to standard MINLP solvers up to exploiting linear substructures of the mathematical model.

***Key words:***

*three-dimensional packing, MIP/MINLP models, linear/non-linear approximation, heuristics*

**MSC Classification (2000):** 05B40, 90C11, 90C30, 90C59, 90C90

## 1. Introduction

A number of specialist works in the Operations Research framework deal with mixed integer programming (MIP) formulations of packing problems (see e.g. Chen, Lee, Shen 1995; Fasano 2010; Padberg 1999; Pisinger, Sigurd 2006; Onodera, Taniguchi, Tmaru 1991). The subject discussed herewith extends the approach proposed by Fasano (2008) to solve non-standard 3D-packing problems, involving *tetris*-like items, convex domains (with possible *holes* and separation planes), in the presence of additional conditions, such as balancing. An MIP-based heuristic was described there to efficiently tackle real-world instances. Its first step consists of an LP-relaxation of the non-intersection constraints between items and the adoption of an ad hoc objective function aimed at providing the further steps of the process with an approximated solution to the original problem, expressed in term of a feasibility one.

Vast literature is available nowadays on mixed integer non-linear programming and global optimization (MINLP, GO, see e.g. Borchers, Mitchell 1997; Floudas 1995; Floudas et al. 1999; Floudas**,** Pardalos 2001; Floudas et al. 2005; Grossmann, Kravanja 1997; Horst, Pardalos 1995; Horst, Tuy 1996; Horst, Pardalos 1997; Hussain et al. 2004; Kallrath, Schreieck 1995; Kallrath 1999; Kallrath, Maindl 2006; Kallrath 2008; Liberti L, Maculan N 2005; Pardalos, Resende 2002; Pardalos, Romeijn 2002; Pintér 1996; Pintér 2006; Pintér 2009; Rebennack, Kallrath, Pardalos 2009) encouraging their application to several class of challenging optimization problems, including hard packing issues (see e.g. Addis, Locatelli, Schoen 2008a; Addis, Locatelli, Schoen 2008b; Caprara, Locatelli, Monaci 2005; Castillo, Kampas, Pintér 2008; Chernov, Stoyan, Romanova 2010; Kallrath 2009; Kampas, Pintér 2005; Locatelli, Raber 2002; Stoyan, Yaskov, Scheithauer 2001; Stoyan, Zlotnick 2007).

The $\Phi$ function concept (Stoyan et al. 2004; Chernov, Stoyan, Romanova 2010) has been introduced to solve arduous irregular nesting problems, involving complex 2D-objects and, following a different point of view, Birgin et al. (2006), Cassioli and Locatelli (2010) investigated a non-linear-based approach for the packing of rectangles inside a convex region.

A novel non-linear approach is outlined herewith. It is aimed at improving the approximate initial LP-relaxed solution of the MIP-based heuristic previously introduced (Fasano 2008). The approach that is going to be proposed in the following stresses the linear structure of the packing MIP model. This approach is addressed in particular to standard MINLP solvers (see e.g. Bussieck, Vigerske 2010) with special algorithmic features aimed at exploiting the presence of linear constraint sets.

Section 2 briefly reviews the previous MIP-based approach (Fasano 2008, 2010). Section 3.1 discusses the non-linear re-formulation of its model, focusing on the non-intersection constraints. This reformulation is described in Section 3.2 to improve the previous MIP-based heuristic.

## 2. MIP formulation and LP-relaxed approximation

In the following, we refer to the (three-dimensional) loading problem taken into account in the author's previous work (Fasano 2008). It was focused on the orthogonal packing of *tetris*-like items within a convex domain (with possible separation planes and forbidden zones), subject to further additional conditions, such as balancing. Nevertheless, for the sake of simplicity, the discussion herewith will consider only items consisting of a single parallelepiped to load inside a convex domain, neglecting any possible further conditions (e.g. separation planes, forbidden zones, balancing). In the following, the only packing rules to consider are then:

- *each item (parallelepiped) side has to be parallel to an axis of a prefixed orthonormal reference frame* (*orthogonality conditions*);
- *each item has to be contained within the given domain D (domain conditions*);
- *items cannot overlap* (*non-intersection conditions*).

In the previous work, the objective function consisted of maximizing either the loaded volume or the mass. As pointed out there, the MIP formulation of the packing issue stated above gives rise, when dealing with real-world instances, to optimization problems extremely hard to solve. This is caused in particular by the *non-intersection*s conditions, consisting of *big-M* constraints (given a set of n items, *O(6n) big-M* constraints with their relative binary variables have to be generated).

In the previous work, to efficiently solve the above packing problem (or even more complex versions of it), a heuristic approach was proposed. The first step of the process considers the problem in terms of feasibility (i.e. all the given items have to be picked) and performs a linear relaxation of the *non-intersection* constraints, in order to obtain an approximate solution (with possible intersections between items). An ad hoc (linear) target function, aimed at *minimizing* the intersection between items has been introduced. The approximated solution so obtained is utilized to generate an *abstract configuration*, that is, a set of relative positions (each one for each couple of items) that would be feasible in any unbounded domain (see Fasano 2008). The *abstract configuration* is then imposed to the original problem, eliminating a number of items, if necessary. The loaded volume or mass is maximized, combining, by a recursive process, exchange and hole-filling techniques.

The feasibility model on which the first step of the heuristic process is based, is briefly outlined herewith, referring the reader to the author's previous works (Fasano 2008, 2010) for more details. The constraints relative to the basic packing rules (*orthogonality, domain* and *non-intersection* conditions) are reported here below.

$$\forall\alpha,\forall i \qquad \sum_{\beta=1}^{3}\delta_{\alpha\beta i}=1, \tag{1}$$

$$\forall\beta,\forall i \qquad \sum_{\alpha=1}^{3}\delta_{\alpha\beta i}=1, \tag{2}$$

$$\forall\beta,\forall i,\forall\eta\in E_i \qquad w_{\beta i}\pm\frac{1}{2}\sum_{\alpha=1}^{3}L_{\alpha i}\delta_{\alpha\beta i}=\sum_{\gamma}V_{\beta\gamma}\psi_{\gamma\eta i}, \tag{3}$$

$$\forall i,\forall\eta\in E_i \qquad \sum_{\gamma}\psi_{\gamma\eta i}=1, \tag{4}$$

$$\forall\beta,\forall i,\forall j,i<j \qquad w_{\beta i}-w_{\beta j}\geq\frac{1}{2}\sum_{\alpha=1}^{3}\left(L_{\alpha i}\delta_{\alpha\beta i}+L_{\alpha j}\delta_{\alpha\beta j}\right)+d^{+}_{\beta ij}-D_{\beta}, \tag{5-1}$$

$$\forall\beta,\forall i,\forall j,i<j \qquad w_{\beta j}-w_{\beta i}\geq\frac{1}{2}\sum_{\alpha=1}^{3}\left(L_{\alpha i}\delta_{\alpha\beta i}+L_{\alpha j}\delta_{\alpha\beta j}\right)+d^{-}_{\beta ij}-D_{\beta}, \tag{5-2}$$

$$\forall\beta,\forall i,\forall j,i<j \qquad d^{+}_{\beta ij}\geq\sigma^{+}_{\beta ij}D_{\beta}, \tag{6-1}$$

$$\forall\beta,\forall i,\forall j,i<j \qquad d^{-}_{\beta ij}\geq\sigma^{-}_{\beta ij}D_{\beta}, \tag{6-2}$$

$$\forall i,\forall j,i<j \qquad \sum_{\beta=1}^{3}\left(\sigma^{+}_{\beta ij}+\sigma^{-}_{\beta ij}\right)=1, \tag{7}$$

where:
given the reference frame $O$, $w_\beta$, $\beta\in\{1,2,3\}$, for each item $i\in\{1,\dots,n\}$ of sides $L_{\alpha i}$, $\alpha\in\{1,2,3\}$, the variables $\delta_{\alpha\beta i}$ $\alpha\in\{0,1\}$ have been introduced with the condition $\delta_{\alpha\beta 1}=1$ if $L_{\alpha i}$ is parallel to the $w_\beta$ axis and $\delta_{\alpha\beta i}=0$ otherwise;
$E_i$ is the set of vertexes associated to item $i$, $V_{\beta\gamma}$ are the vertexes of the convex domain $D$, $\psi_{\gamma\eta i}$ are non negative variables;
$D_\beta$ are the sides of the parallelepiped of minimum volume enveloping $D$;
$d^{+}_{\beta ij}, d^{-}_{\beta ij}\in[0,D_\beta]$, $\sigma^{+}_{\beta hij}$, $\sigma^{-}_{\beta hij}\in\{0,1\}$.
Constraints (1) and (2) represent the *orthogonality* conditions; (3) and (4) the *domain* ones; (5), (6), (7) the *non-intersection* ones.

The goal of the first step of the heuristic process is to obtain a good approximate initial solution. The MIP model described above is adopted and the integrality conditions on the $\sigma$ variables are dropped. In this case constraints (6) and (7) can be replaced by: $\sum_{\beta=1}^{3}\left(\frac{d^{+}_{\beta ij}}{D_\beta}+\frac{d^{-}_{\beta ij}}{D_\beta}\right)=1$.

The ad hoc objective function (aimed at *minimizing* the intersection between items) is the following:

$$\max\sum_{\beta,i<j}(d^{+}_{\beta ij}+d^{-}_{\beta ij}). \tag{8}$$

Figure 1 depicts an approximate solution (with possible intersections between items) obtained by the linear relaxation described above.

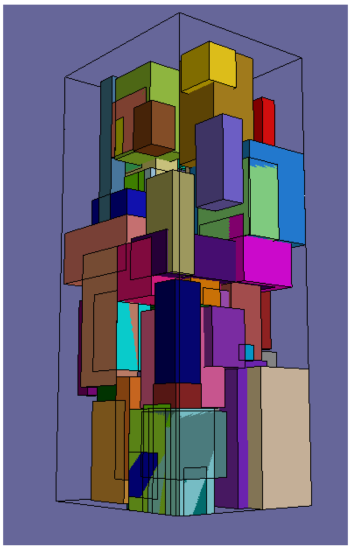

Fig. 1 Linear-relaxed approximate solution

## 3. MINLP-based approach

### 3.1 *Non-intersection* constraint non-linear formulation

We introduce here a possible non-linear reformulation of the *non intersection* constraints reported in Section 2. Keeping the same meaning of the symbols previously adopted, it is straightforward to prove that the following non-linear constraints are equivalent to (5), (6), (7):

$$\forall \beta, \forall i, \forall j, i < j \quad (w_{\beta i} - w_{\beta j})^2 - [\frac{1}{2}\sum_{\alpha=1}^{3}(L_{\alpha i}\delta_{\alpha\beta i} + L_{\alpha j}\delta_{\alpha\beta j})]^2 = s_{\beta ij} - r_{\beta ij}, \tag{9}$$

$$\forall i, \forall j, i < j \qquad \prod_{\beta=1}^{3} r_{\beta ij} = 0, \tag{10}$$

where $s_{\beta ij} \in [0, D^2{}_\beta]$ and $r_{\beta ij} \in [0, D^2{}_\beta]$.

Indeed for each couple $i,j$ ($i<j$) equations (10) guarantee that the terms $r_{\beta ij}$ are nil for at least one $\beta$ and equation (9), corresponding to such $\beta$, is equivalent to the *non-intersection* condition:

$$| w_{\beta i} - w_{\beta j} | \geq \frac{1}{2}\sum_{\alpha=1}^{3}(L_{\alpha i}\delta_{\alpha\beta i} + L_{\alpha j}\delta_{\alpha\beta j}).$$

Constraints (5) and (6) correspond then to equations (9) and vice versa, while equations (7) correspond to (10) and vice versa.

### 3.2 *Non*-linear approximation

In the following, we reformulate the feasibility problem reported in Section 2, on the basis of what was discussed in Section 3.1, by introducing a new ad hoc non-linear target function aimed at minimizing the intersection between items. As the non-intersection constraints (9) and (10) are most likely quite hard to tackle, they are considered in terms of penalty terms in the new objective function. All remaining constraints, on the contrary, as they are linear (MIP), are maintained as such. The MINLP model so obtained is then addressed, in particular, to standard (MINLP) solvers with special features to exploit the presence of linear constraints.

To reformulate the original feasibility (MIP) problem, we shall consider the following MINLP one:

$$\min\{\sum_{\beta,i<j}\{(w_{\beta i}-w_{\beta j})^2-[\frac{1}{2}\sum_{\alpha=1}^{3}(L_{\alpha i}\delta_{\alpha\beta i}+L_{\alpha j}\delta_{\alpha\beta j})]^2-s_{\beta ij}+r_{\beta ij}\}^2+\sum_{i<j}\prod_{\beta}r_{\beta ij}\}, \tag{11}$$

subject to constraints (1), (2), (3), (4) reported in Section 2.
It is immediately seen that the objective function is non-negative and that any zero global optimal solution exists if and only if the constraints (1), (2), (3), (4), (5), (6), (7) of the MIP problem reported in Section 2 delimit a feasible region.

The above target function thus minimizes the non-intersection between items, but differently from target function (8), its global optima guarantee, at least theoretically, an ultimate (non approximate) solution to the feasibility problem of Section 3.1 and its adoption is therefore a priori preferable.

Since obtaining a global optimal solution to the above MINLP problem is, in general, an extremely hard task, we do not however expect to solve it tout court. A joint use of the LP-relaxation of the MIP feasibility model reported in Section 2 and the above MINLP approach, within the heuristic process outlined in Section 2, could nevertheless be of advantage. The LP-relaxed solution can be improved by the MINLP process to further reduce the overall intersection between items (generally, in the LP-relaxed solution intersections between items are admissible). The values of variables $w$, $\delta$ in the LP-relaxed solution are then taken as initial guesses for the MINLP instance, while the initial guesses for variable $s$ and $r$ can be derived directly from them on the basis of equations 9 (with no loss of generality, we assume that, for each $\beta,i,j$, at least one of the variable $s_{\beta ij}$ , $r_{\beta ijr}$ is set to zero, as initial guess).

The MINLP model described above maintains all the MIP constraints present in the original model, so that, even its suboptimal solutions guarantee to satisfy all *orhogonality* and *domain* conditions. Further MIP constraints could be directly added to the MINLP model in order to contemplate additional conditions, such as balancing. The extension to include *tetris*-like items would also be straightforward (see Fasano 2008, 2010).

## 4. Conclusive remarks

This article refers to an author's previous work aimed at solving non-standard three-dimensional orthogonal packing problems with additional conditions. It focused on an MIP formulation and an MIP-based heuristic, introduced to efficiently solve hard real-world instances in practice.

The initial phase of this heuristic process addresses an LP-relaxation of the non-intersection constraints of the MIP model. The relaxed MIP model considers the original problem in terms of feasibility and an ad hoc (linear) objective function, aimed at minimizing the intersection between items, is adopted. The approximate solutions to this reformulated problem (with possible intersections) are taken as an initial step of the whole heuristic process.

In the article presented here, we have proposed an MINLP reformulation of the original MIP model expressed in terms of feasibility. This reformulation is aimed at improving the approximate LP-relaxed solution of the MIP-based heuristic process, by looking into sub-optima of the MINLP problem. The former MIP model has been reviewed in brief in Section 2, to make the introduction of the proposed non-linear approach comprehensive.

A non-linear reformulation of the non-intersection constraints has been described. The MINLP model does not consider any longer the non-intersection conditions in terms of constraints, but through a non-negative penalty function that is nil if and only if no overlapping between items occurs. The *orthogonality* and *domain* conditions are, on the contrary, actually still treated as model constraints.

A joint use of the MINLP formulation with the former LP-relaxed one has been proposed to improve the initial phase of the heuristic process. The MINLP formulation discussed here is addressed in particular to standard MINLP solvers up to exploiting linear sub-structures. In the author's opinion the approach proposed deserves an in-depth experimental investigation that could represent the goal of a dedicated future activity.

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