On duality gap in linear conic problems

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Abstract

In their paper "Duality of linear conic problems" A. Shapiro and A. Nemirovski considered two possible properties (A) and (B) for dual linear conic problems (P) and (D). The property (A) is "If either (P) or (D) is feasible, then there is no duality gap between (P) and (D)", while property (B) is "If both (P) and (D) are feasible, then there is no duality gap between (P) and (D) and the optimal values val(P) and val(D) are finite". They showed that (A) holds if and only if the cone K is polyhedral, and gave some partial results related to (B). Later A. Shapiro conjectured that (B) holds if and only if all the nontrivial faces of the cone K are polyhedral. In this note we mainly prove that both the "if" and "only if" parts of this conjecture are not true by providing examples of closed convex cone in $\mathbb{R}^4$ for which the corresponding implications are not valid. Moreover, we give alternative proofs for the results related to (B) established by A. Shapiro and A. Nemirovski.

Key words: Linear conic problems, duality gap.

1 Introduction

In [3] one considers the linear conic problem

$\min_{x\in\mathcal{X}} \langle c, x\rangle$ subject to $Ax + b = 0$ and $x \in K$

and its dual

$\max_{y\in\mathcal{Y}} \langle b, y\rangle$ subject to $A^*y + c \in K^*$,

where $\mathcal{X}$ and $\mathcal{Y}$ are finite dimensional vector spaces equipped with scalar products denoted $\langle\cdot,\cdot\rangle$, $A : \mathcal{X} \to \mathcal{Y}$ is a linear mapping and $K \subset \mathcal{X}$ is a closed convex cone, while $K^* := \{x^* \in \mathcal{X} \mid \langle x, x^*\rangle \geq 0\ \forall x \in K\}$ and $A^* : \mathcal{Y} \to \mathcal{X}$ is the adjoint of A. In [3] it is raised the problem of characterizing those cones K for which property (B) holds, that is, there is no duality gap whenever both problems are feasible. Professor A. Shapiro conjectured later that property (B) holds if and only if all the nontrivial faces of the cone K are polyhedral. The main aim of this note is to show that this conjecture is not true. We also gave alternative proofs for some results in [3] related to property (B).

We reformulate the above dual problems without using a linear operator A as in [1] (see also [5]). More precisely, consider X a finite dimensional normed vector space whose dual is denoted by X^*, $K \subset X$ a proper closed convex cone and $M \subset X$ a linear space. We denote $x^*(x)$ by $\langle x, x^*\rangle$ for $x \in X$ and $x^* \in X^*$ and we consider the dual cone K^+ of K and the orthogonal space $M^\perp$ of M :

$$K^+ := \{x^* \in X \mid \langle x, x^*\rangle \geq 0\ \forall x \in K\}, \quad M^\perp := \{x^* \in X \mid \langle x, x^*\rangle = 0\ \forall x \in M\}.$$

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Consider the linear conic problems

(P) $\min \langle x, \overline{x}^* \rangle$ s.t. $x \in \overline{x} + M$, $x \in K$,

(D) $\max \langle \overline{x}, x^* \rangle$ s.t. $x^* \in M^\perp$, $\overline{x}^* - x^* \in K^+$.

Denoting $\overline{x}^* - x^*$ by y^*, the problem (D) becomes

$\max (\langle \overline{x}, \overline{x}^* \rangle - \langle \overline{x}, y^* \rangle)$ s.t. $y^* \in \overline{x}^* + M^\perp$, $y^* \in K^+$,

or equivalently

$\min \langle \overline{x}, y^* \rangle$ s.t. $y^* \in \overline{x}^* + M^\perp$, $y^* \in K^+$,

that is, a problem of the same type as (P).

We denote by F_P (S_P) and F_D (S_D) the feasible (solution) sets of problems (P) and (D), respectively; hence

$$F_P = (\overline{x} + M) \cap K, \quad F_D = M^\perp \cap (\overline{x}^* - K^+). \tag{1}$$

If x is a feasible solution of (P), that is, $x \in (\overline{x} + M) \cap K$, and x^* is a feasible solution of (D) then $\langle x, \overline{x}^* \rangle \geq \langle \overline{x}, x^* \rangle$, and so $v_P \geq v_D$, where v_P and v_D are the values of the problems (P) and (D), respectively. Indeed, in such a case we have that

$$\langle x, \overline{x}^* \rangle - \langle \overline{x}, x^* \rangle = \langle x, \overline{x}^* - x^* \rangle + \langle x - \overline{x}, x^* \rangle = \langle x, \overline{x}^* - x^* \rangle \geq 0.$$

Moreover, if x is a solution of (P), x^* is a solution of (D) and $v_P = v_D$ then necessarily,

$$\langle x, \overline{x}^* - x^* \rangle = 0. \tag{2}$$

Conversely, if x is a feasible solution of (P), x^* is a feasible solution of (D) and $\langle x, \overline{x}^* \rangle = \langle \overline{x}, x^* \rangle$, then x is an optimal solution of (P) and x^* is an optimal solution of (D).

If (P) has feasible solutions we may assume that $\overline{x} \in K$. Indeed, take $\widetilde{x} \in (\overline{x} + M) \cap K$. Then $\overline{x} + M = \widetilde{x} + M$, and so $(\overline{x} + M) \cap K = (\widetilde{x} + M) \cap K$. Moreover, for $x^* \in M^\perp$ we have that $\langle \overline{x}, x^* \rangle = \langle \overline{x} - \widetilde{x}, x^* \rangle + \langle \widetilde{x}, x^* \rangle = \langle \widetilde{x}, x^* \rangle$, which proves that the objective function of (D) remains unchanged on its feasible set. Similarly, we may assume that $\overline{x}^* \in K^+$ if (D) is feasible.

Because we are interested by the case in which both (P) and (D) are feasible, we assume that $\overline{x} \in K$ and $\overline{x}^* \in K^+$, and so $\overline{x}$ is feasible for (P) and 0 is feasible for (D); hence

$$\langle \overline{x}, \overline{x}^* \rangle \geq v_P \geq v_D \geq 0. \tag{3}$$

Moreover, if (P) has optimal solutions then we may assume that $\overline{x}$ is even optimal for (P). Relation (3) shows that $v_P = v_D$ if $\langle \overline{x}, \overline{x}^* \rangle = 0$; in particular, this happens if $\overline{x} = 0$ or $\overline{x}^* = 0$.

2 Sufficient conditions for no duality gap

Throughout this section we assume that X, K and M are as in the previous section and

$$\overline{x} \in K, \quad \overline{x}^* \in K^+, \quad \langle \overline{x}, \overline{x}^* \rangle > 0. \tag{4}$$

The relation $v_P \geq v_D$ in (3) can be obtained also using perturbation functions (see [2], [4], as well as for other results and notation not explicitly mentioned in the sequel). Indeed, let us consider

$$F : X \times Y \to \overline{\mathbb{R}}, \quad F(x, y) := \begin{cases} \langle x, \overline{x}^* \rangle & \text{if } x \in \overline{x} + y + M, \ x \in K, \\ \infty & \text{otherwise,} \end{cases} \tag{5}$$

where $Y := X$. Clearly, F is a lsc proper convex function. Setting $h(y) := \inf_{x\in X} F(x,y)$, we have that $h(0) = v_P \geq 0$. Moreover,

$$\begin{aligned}
F^*(x^*,y^*) &= \sup\{\langle x,x^*\rangle + \langle y,y^*\rangle - F(x,y) \mid x,y \in X\} \\
&= \sup\{\langle x,x^*\rangle + \langle y,y^*\rangle - \langle x,\overline{x}^*\rangle \mid x\in K,\ y \in x-\overline{x}+M\} \\
&= \sup\{\langle x,x^*\rangle + \langle x-\overline{x}+z,y^*\rangle - \langle x,\overline{x}^*\rangle \mid x\in K,\ z\in M\} \\
&= -\langle \overline{x},y^*\rangle + \sup\{\langle x,x^*+y^*-\overline{x}^*\rangle + \langle z,y^*\rangle \mid x\in K,\ z\in M\} \\
&= \begin{cases} -\langle \overline{x},y^*\rangle & \text{if } y^*\in M^\perp,\ x^*+y^*-\overline{x}^*\in K^-, \\ \infty & \text{otherwise,}\end{cases}
\end{aligned}$$

where $K^- := -K^+$. Hence the problem $\max(-F^*(0,y^*))$ s.t. $y^*\in Y^*$ becomes our problem (D).

The next result will be useful in the sequel; possibly it is not new but we have not a reference for it.

Proposition 1 *Consider Y a normed space and $F : X\times Y \to \overline{\mathbb{R}}$ a lsc proper convex function. Assume that $0 \in \operatorname{Pr}_Y(\operatorname{dom} F)$ and $S := \{x\in X \mid F(x,0) \leq F(x',0)\ \forall x'\in X\}$ is nonempty and bounded. Then $h : Y\to\overline{\mathbb{R}}$, $h(y) := \inf_{x\in X} F(x,y)$, is lsc at 0. Moreover, $\inf_{x\in X} F(x,0) = \sup_{y^*\in Y^*}(-F^*(0,y^*)) \in \mathbb{R}$.*

Proof. Take $y_n \to 0$. Assume that $\liminf h(y_n) < h(0)$. Then we may assume that $h(y_n) < \mu < h(0)$ for some $\mu\in\mathbb{R}$ and every n. For $n\in\mathbb{N}$ there exists $x_n \in X$ with $F(x_n,y_n) < \mu$. If (x_n) has a bounded subsequence, because $\dim X < \infty$, we have that $x_{n_k} \to x \in X$ for some subsequence (x_{n_k}). Since F is lsc and $y_{n_k}\to 0$, we obtain that $F(x,0) \leq \liminf F(x_{n_k},y_{n_k}) \leq \mu < h(0)$, a contradiction. Hence $\|x_n\|\to\infty$. We may assume that $\|x_n\|^{-1} x_n \to u \in X\setminus\{0\}$. Then $(x_n,y_n,\mu)\in \operatorname{epi} F$ and $\|x_n\|^{-1}(x_n,y_n,\mu) \to (u,0,0)$. Therefore, $(u,0,0)\in(\operatorname{epi} F)_\infty = \operatorname{epi} F_\infty$, whence $F_\infty(u,0)\leq 0$. Taking $x\in S$, it follows that $F(x+tu,0) \leq F(x,0) + tF_\infty(u,0) \leq F(x,0)$ for every $t\geq 0$, which proves that $x+tu\in S$ for every $t\geq 0$, and so S is unbounded. This contradiction proves that h is lsc at 0.

Since $h(0)\in\mathbb{R}$ (S being nonempty) and h is convex and lower semicontinuous at 0, we obtain (by [4, Th. 2.3.4]) that $h(0) = h^{**}(0)$, and so $\inf_{x\in X} F(x,0) = \sup_{y^*\in Y^*}(-F^*(0,y^*)) \in \mathbb{R}$. □

If $[(F_P)_\infty =]\ M\cap K = \{0\}$ we have that F_P is bounded (hence compact), and so the set S_P is nonempty and compact; hence $v_P = v_D$ by Proposition 1 applied to F defined in (5). Similarly, for $M^\perp \cap K^+ = \{0\}$ the set S_D is nonempty and compact, and again $v_P = v_D$. In particular, if M is not proper we have that $v_P = v_D$. Assume that

$$M\cap K \neq \{0\}, \quad M^\perp\cap K^+ \neq \{0\}. \tag{6}$$

Suppose now that $\dim M = 1$, that is, $M = \mathbb{R}\overline{u}$ for some $\overline{u}\in X\setminus\{0\}$. From the preceding assumption we may assume that $\overline{u}\in K$, and so $\overline{x}+\mathbb{R}_+\overline{u} \subset (\overline{x}+M)\cap K = F_P \subset \overline{x}+\mathbb{R}\overline{u}$. If S_P is not a singleton then $\overline{u}\in\ker\overline{x}^* := \{x\in X \mid \langle x,\overline{x}^*\rangle = 0\}$, and so $\overline{x}^*\in M^\perp$. It follows that $\overline{x}^*$ is a feasible solution of (D), and so $v_P = \langle\overline{x},\overline{x}^*\rangle \leq v_D$. Therefore, $v_P = v_D$. If S_P is a singleton then clearly $v_P = v_D$ (by Proposition 1).

If $\dim M = \dim X - 1$ then $\dim M^\perp = 1$, and the conclusion follows changing the roles of (P) and (D).

We are interested now by the case in which (4) and (6) hold and, moreover,

$$2 \leq \dim M \leq \dim X - 2;$$

in particular, $\dim X \geq 4$.

Clearly, from the definition of F in (5), $\mathrm{Pr}_Y(\mathrm{dom}\, F) = K + M - \overline{x}$. By a known fact (see e.g. [4, Th. 2.7.1 (viii)]) we have that $\inf_{x\in X} F(x,0) = \max_{y^*\in Y^*}(-F^*(0,y^*))$ if $0 \in {}^i\,\mathrm{Pr}_Y(\mathrm{dom}\, F)$, that is, $\overline{x} \in {}^i(M+K) = M + {}^iK$; by ${}^iA$ we denote the algebraic interior (or intrinsic core) of the subset A of a real linear space E. Hence, in this case we have that $v_P = v_D$ and (D) has optimal solutions whenever (D) is feasible (this is the case because $\overline{x}^* \in K^+$). Because the dual problem of (D) is (P) and $\mathrm{Pr}_{X^*}(\mathrm{dom}\, F^*) = \overline{x}^* + K^- + M^\perp$, we have that $v_P = v_D$ and (P) has optimal solutions when $\overline{x}^* \in M^\perp + {}^i(K^+) = {}^i\left(M^\perp + K^+\right)$.

Assume that $\overline{x} \in \mathrm{rbd}(M+K) := \mathrm{cl}(M+K) \setminus {}^i(M+K)$, or, equivalently, $(\overline{x}+M) \cap {}^iK = \emptyset$. Then there exists $\overline{u}^* \in X^*$ such that $\langle \overline{x} + v, \overline{u}^* \rangle < \langle x, \overline{u}^* \rangle$ for all $v \in M$ and $x \in {}^iK$. Hence $\overline{u}^* \in M^\perp \cap K^+ \setminus \{0\}$. Since $\overline{x} \in K$ we obtain that $\langle \overline{x}, \overline{u}^* \rangle = 0$, and so $\overline{x} + M \subset \ker \overline{u}^*$. It follows that

$$(\overline{x}+M) \cap K = (\overline{x}+M) \cap (K \cap \ker \overline{u}^*)\,.$$

Setting

$$K_{\overline{u}^*} := K \cap \ker \overline{u}^*,$$

this shows that the problem (P) is equivalent to

(PR) $\min \langle x, \overline{x}^* \rangle$ s.t. $x \in \overline{x} + M$, $x \in K_{\overline{u}^*}$.

Assume that $K_{\overline{u}^*}$ is polyhedral. Then $S_P = S_{PR} \neq \emptyset$ and

$$v_P = v_{PR} = v_{DR} \geq v_D \geq 0,$$

where

(DR) $\max \langle \overline{x}, x^* \rangle$ s.t. $x^* \in M^\perp$, $\overline{x}^* - x^* \in (K_{\overline{u}^*})^+ = \mathrm{cl}\,(K^+ + \mathbb{R}\overline{u}^*)$.

If $\dim K_{u^*} := \dim(K_{u^*} - K_{u^*}) = 1$ we get $S_P \subset K_{\overline{u}^*} \subset \mathbb{R}\overline{x}$. Because $\overline{x}^*$ is constant on S_P and $\langle \overline{x}, \overline{x}^* \rangle > 0$, we have necessarily that S_P is a singleton. Therefore, $v_P = v_D$. The next result summerizes the above discussion.

Proposition 2 *Let $\overline{x} \in K$ and $\overline{x}^* \in K^+$. Then $v_P = v_D$ if one of the following condition is satisfied:*
(a) $\dim M \leq 1$ *or* $\dim M^\perp \leq 1$;
(b) $\overline{x} \in M + {}^iK$ *or* $\overline{x}^* \in M^\perp + {}^i(K^+)$;
(c) $\dim K_{u^*} = 1$ *for every* $u^* \in M^\perp \cap K^+ \setminus \{0\}$. □

If $\dim X \leq 3$ then (a) holds, and so we get [3, Prop. 4]. From (c) we get [3, Prop. 3] because in this case every nontrivial face of K has dimension one (see below the definition of a face of a convex set). The conclusion of Proposition 2 is well known when (b) holds.

As mentioned by Prof. A. Shapiro in a discussion, "the conjecture is that property (B) (in [3]) holds iff every face of cone K is polyhedral". In our framework, this conjecture translates as:

Conjecture 3 *Let $K \subset X$ be a closed convex cone. Then $v_P = v_D$ for all $\overline{x} \in K$, $\overline{x}^* \in K^+$ and all linear spaces $M \subset X$ if and only if all nontrivial faces of K are polyhedral.*

In [3] it is given an example of cone $K \subset \mathbb{R}^4$ for which the problems (P) and (D) have a finite duality gap; however, the cone K has a nontrivial face which is not polyhedral. Moreover, an example of nonpolyhedral cone K having only polyhedral nontrivial faces, one of which having dimension greater than one, for which Conjecture 3 is true is given in [3].

3 Relations between the faces of cones and their bases

Recall that a face of a convex set $C \subset E$, where E is a real linear space, is a nonempty convex subset $F \subset C$ with the property: $x, y \in C$, $\lambda \in (0,1)$, $\lambda x + (1-\lambda)y \in F \Rightarrow x, y \in F$. Note that the convexity of F is essential. For example, taking $C = [0,1] \subset \mathbb{R}$ and $F := \{0,1\}$, we have that $x, y \in C$, $\lambda \in (0,1)$, $\lambda x + (1-\lambda)y \in F \Rightarrow x, y \in F$; however, F is not a face of C.

The next two results are probably known; we give their proofs for readers convenience. They will be used in the next section.

Proposition 4 *Let $P \subset E$ be a convex cone having the base $B \subset E$, that is, B is a nonempty convex subset of P such that any $x \in P$ has a unique representation $x = \alpha u$ with $\alpha \geq 0$ and $u \in B$. Then F is a face of P iff $F = \{0\}$ or there exists a face D of B such that $F = \mathbb{R}_+ D$.*

Proof. Clearly, $0 \notin B$ and P is a pointed convex cone. Assume that D is a face of B and take $x, y \in P$ with $x \neq y$, and $\lambda \in (0,1)$ such that $z := \lambda x + (1-\lambda)y \in F = \mathbb{R}_+ D$. Then $x = \alpha a$, $y = \beta b$, $z = \gamma c$ with $\alpha, \beta, \gamma \geq 0$, $a, b \in B$ and $c \in D$. Because P is pointed, $z \neq 0$, whence $\gamma > 0$ and $\alpha + \beta > 0$. It follows that

$$z = \gamma c = (\lambda\alpha + (1-\lambda)\beta)\left(\frac{\lambda\alpha}{\lambda\alpha + (1-\lambda)\beta}a + \frac{(1-\lambda)\beta}{\lambda\alpha + (1-\lambda)\beta}b\right),$$

and so $\gamma = \lambda\alpha + (1-\lambda)\beta$. Hence

$$c = \frac{\lambda\alpha}{\lambda\alpha + (1-\lambda)\beta}a + \frac{(1-\lambda)\beta}{\lambda\alpha + (1-\lambda)\beta}b.$$

Since D is a face of B we get $a, b \in D$, whence $x, y \in F$. Hence F is a face of P. Let now $\{0\} \neq F \subset P$ be a face of P. Then F is a cone. Indeed, take $z \in F \setminus \{0\}$ and $\alpha \geq 0$. If $\alpha > 1$ then $z = \alpha^{-1}(\alpha z) + (1-\alpha^{-1})0$ with $\alpha z, 0 \in P$. It follows that $\alpha z, 0 \in F$; in particular $0 = 0z \in F$. If $\alpha \in (0,1)$ there exists $\beta > \alpha$ and $\lambda \in (0,1)$ such that $1 = \lambda\alpha + (1-\lambda)\beta$, and so $z = \lambda\alpha z + (1-\lambda)\beta z$. Since $\alpha z, \beta z \in P$ we get $\alpha z \in F$. Hence F is a (convex) cone. Set $D := \{a \in B \mid \exists \alpha > 0,\, x \in F : x = \alpha a\}$; clearly D is nonempty because $\emptyset \neq F \neq \{0\}$ and B is a base of P, and $F = \mathbb{R}_+ D$ because F is a cone. The set D is convex because for $a, b \in D$ and $\lambda \in (0,1)$ we have $a = \alpha^{-1}x$, $b = \beta^{-1}y$ with $\alpha, \beta > 0$ and $x, y \in F$, whence $B \ni \lambda a + (1-\lambda)b = 1 \cdot \left(\lambda\alpha^{-1}x + (1-\lambda)\beta^{-1}y\right)$, and so $\lambda a + (1-\lambda)b$ (taking into account the convexity of the cone F). Moreover, assume that $a, b \in B$, $\lambda \in (0,1)$ are such that $a \neq b$ and $\lambda a + (1-\lambda)b \in D \subset F$. Since F is a face of P we obtain that $a, b \in F$, and so $a, b \in D$ by the definition of D. Hence D is a face of B. □

Proposition 5 *Let $P \subset E$ be a convex cone having the base $B \subset E$. Then P is polyhedral iff P is algebraically closed and B is polyhedral.*

Proof. It is well known that there exists $\varphi_0 \in E'$, that is, $\varphi_0 : E \to \mathbb{R}$ is a linear functional, such that $B = \{x \in P \mid \varphi_0(x) = 1\}$. Assume first that P is polyhedral; clearly, P is algebraically closed (that is, the intersection of P with any line is closed in the line identified with $\mathbb{R}$). Then there exist $\varphi_1, \ldots, \varphi_k \in E'$ such that $P = \{x \in E \mid \varphi_i(x) \geq 0 \ \forall i \in \overline{1,k}\}$. Then $B = P \cap \{x \in E \mid \varphi_0(x) = 1\}$, and so B is polyhedral.

Assume that B is polyhedral and P is algebraically closed; hence $B = \{x \in E \mid \varphi_i(x) \geq \gamma_i \ \forall i \in \overline{1,k}\}$ with $\varphi_i \in E'$ and $\gamma_i \in \mathbb{R}$ for $i \in \overline{1,k}$. Because B is nonempty, take $\overline{x} \in B$; hence $\varphi_0(\overline{x}) = 1$ and $\varphi_i(\overline{x}) \geq \gamma_i$ for $i \in \overline{1,k}$. Then

$$P = \{x \in E \mid \varphi_0(x) \geq 0,\ \varphi_i(x) - \gamma_i \varphi_0(x) \geq 0 \ \forall i \in \overline{0,k}\}. \tag{7}$$

Indeed, let $x \in P \setminus \{0\}$. Then $\alpha := \varphi_0(x) > 0$ and $x' := \alpha^{-1}x \in B$. It follows that $\gamma_i \leq \varphi_i(x') = \alpha^{-1}\varphi_i(x)$, whence $\varphi_i(x) - \gamma_i\varphi_0(x) \geq 0$ for every $i \in \overline{1,k}$. Hence the inclusion $\subset$ holds in (7).

Take now $x \in E$ with $\varphi_0(x) \geq 0$, $\varphi_i(x) - \gamma_i\varphi_0(x) \geq 0$ for all $i \in \overline{0,k}$. Assume first that $\alpha := \varphi_0(x) \neq 0$. Then $\alpha > 0$ and setting $x' := \alpha^{-1}x$ we obtain that $\varphi_0(x') = 1$ and $\varphi_i(x') - \gamma_i\varphi_0(x') = \varphi_i(x') - \gamma_i \geq 0$ for all $i \in \overline{1,k}$. Hence $x' \in B$, and so $x \in \mathbb{R}_+B = P$. Assume now that $\alpha = 0$. Then, for $\lambda > 0$ we have $\varphi_0(x + \lambda\overline{x}) = \lambda > 0$ and $\varphi_i(x + \lambda\overline{x}) - \gamma_i\varphi_0(x + \lambda\overline{x}) = \varphi_i(x) - \gamma_i\varphi_0(x) + \lambda\left[\varphi_i(\overline{x}) - \gamma_i\right] \geq 0$ for $i \in \overline{1,k}$. By the previous situation we get $x + \lambda\overline{x} \in P$. Since $\lambda > 0$ is arbitrary and P is algebraically closed we get $x \in P$. □

4 Counter-examples to Conjecture 3

In the sequel we give examples showing that both implications in Conjecture 3 are not true. For constructing the examples we take into account the discussion in Section 2.

We consider first the set $A := A_1 \cup A_2 \subset \mathbb{R}^3$ with

$$A_1 := \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1,\ 0 \leq z \leq (1 - x^2 - y^2)^{1/2}\right\},$$
$$A_2 := \left\{(x, y, z) \in \mathbb{R}^3 \mid 0 \geq z \geq (x^2 + y^2)^{1/2} - 1\right\}.$$

The sets A_1 and A_2 are compact convex sets with nonempty interior.

Proposition 6 *The set A is compact, convex and $0 \in \operatorname{int} A$. Moreover, all the nontrivial faces of A are polyhedral .*

Proof. The analytical proof of the statement is quite involved. However the picture in Figure 1 (a) is self-explanatory. □

Let us set

$$K := \mathbb{R}_+ (A \times \{1\}) \subset \mathbb{R}^4. \tag{8}$$

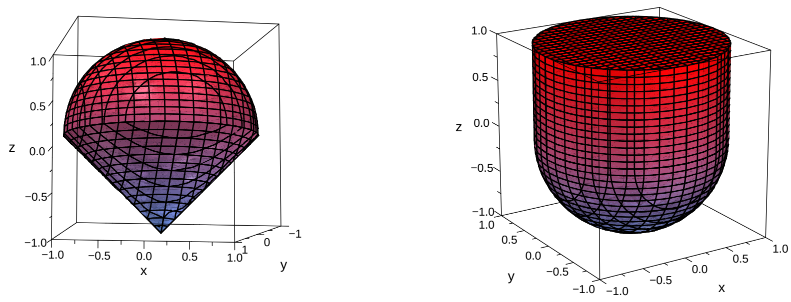
We have that K is a pointed closed convex cone with nonempty interior. Using Proposition 5 we obtain that K has only polyhedral nontrivial faces (of dimension 1 and 2).

After some computation we obtain that

$$K^+ = \mathbb{R}_+ (B \times \{1\}) \subset \mathbb{R}^4, \tag{9}$$

where

$$B := \left\{(a, b, c) \in \mathbb{R}^3 \mid c \in [0, 1],\ a^2 + b^2 \leq 1\right\} \cup \left\{(a, b, c) \in \mathbb{R}^3 \mid c \leq 0,\ a^2 + b^2 + c^2 \leq 1\right\}$$



(a) The set A. (b) The set B.

Figure 1: The sets A and B.

(see Figure 1 (b)).

We take $\overline{x} := (0, \frac{1}{2}, -\frac{1}{2}, 1)$; because $(0, \frac{1}{2}, -\frac{1}{2}) \in A_2$ we have that $\overline{x} \in K$. We take also $\overline{x}^* := (0, -1, \frac{1}{2}, 1)$; clearly $\overline{x}^* \in K^+$. Then $\langle \overline{x}, \overline{x}^* \rangle = \frac{1}{4} > 0$. Let $\overline{u}^* := (0, -1, 1, 1) \in K^+$; we have that $\langle \overline{x}, \overline{u}^* \rangle = 0$. Then

$$K_{\overline{u}^*} = K \cap \ker \overline{u}^* = \mathbb{R}_+ \{(0, z+1, z, 1) \mid z \in [-1, 0]\} = \{(0, \alpha + \beta, \alpha, \beta) \mid \beta \geq -\alpha \geq 0\}.$$

Having in view the discussion in Section 2, we have to take $M \subset \ker \overline{u}^*$ with $\dim M = 2$; so let

$$M := \{(\alpha, \beta - 2\alpha, -2\alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\}.$$

Then

$$M^\perp = \{(2\alpha, -\beta, \alpha + \beta, \beta) \mid \alpha, \beta \in \mathbb{R}\}.$$

The feasible set for problem (P) is

$$F_P = (\overline{x} + M) \cap K = (\overline{x} + M) \cap K_{\overline{u}^*} = \left\{(0, \beta, -\tfrac{1}{2}, \beta + \tfrac{1}{2}) \mid \beta \geq 0\right\}.$$

So $\langle x, \overline{x}^* \rangle = \frac{1}{4}$ for every $x \in F_P := (\overline{x} + M) \cap K$. Hence $v_P = \frac{1}{4}$ and the solution set of (P) is $S_P = F_P$. Let us evaluate v_D. The feasible set of (D) is $F_D = M^\perp \cap (\overline{x}^* - K^+)$. Take $x^* \in F_D$; hence $x^* = (2\alpha, -\beta, \alpha + \beta, \beta)$ with $\alpha, \beta \in \mathbb{R}$. Then $(-2\alpha, \beta - 1, \frac{1}{2} - \alpha - \beta, 1 - \beta) \in K^+$. Hence $1 - \beta \geq \frac{1}{2} - \alpha - \beta \geq 0$ and $1 - \beta \geq \left[4\alpha^2 + (\beta - 1)^2\right]^{1/2}$, in which case $\alpha = 0$ and $\beta \leq \frac{1}{2}$, or, else $\frac{1}{2} - \alpha - \beta \leq 0$ and $1 - \beta \geq \left[4\alpha^2 + (\beta - 1)^2 + (\frac{1}{2} - \alpha - \beta)^2\right]^{1/2}$, in which case $\alpha = 0$ and $\beta = \frac{1}{2}$. Hence

$$F_D = \{(0, -\beta, \beta, \beta) \in \mathbb{R}^4 \mid \beta \leq \tfrac{1}{2}\} = (-\infty, \tfrac{1}{2}] \cdot \overline{u}^*.$$

We have that $\langle \overline{x}, x^* \rangle = 0$ for every $x^* \in F_D$, and so $v_D = 0 < \frac{1}{4} = v_P$. Note that $(K_{\overline{u}^*})^+ = \left\{(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4 \mid \delta \geq \max\{\gamma, -\beta\}\right\}$, and so

$$F_{DR} = \{(2\alpha, -\beta, \alpha + \beta, \beta) \mid \alpha, \beta \in \mathbb{R},\ \alpha \geq -\tfrac{1}{2}\}.$$

It follows that $v_{DR} = \sup\{-\frac{1}{2}\alpha \mid \alpha \geq -\frac{1}{2}\} = \frac{1}{4} = v_P = v_{PR}$.

The previous example shows that the "if" part of Conjecture 3 is not true.

Note that assuming that the Shapiro's conjecture was true we would have that all nontrivial faces of a closed cone K are polyhedral if and only if all nontrivial faces of K^+ are polyhedral. The cone K defined by (8) has all the nontrivial faces polyhedral, while K^+ has a nontrivial face which is not polyhedral; this is $F := \mathbb{R}_+\{(x,y,1,1) \mid x,y \in \mathbb{R},\ x^2+y^2 \le 1\}$.

Having in view the previous example, one can ask if the following statement is true: *Let $K \subset X$ be a closed convex cone. Then $v_P = v_D$ for all $\overline{x} \in K$, $\overline{x}^* \in K^+$ and all linear spaces $M \subset X$ if and only if all nontrivial faces of K and K^+ are polyhedral.*

If this statement is true then one has the following consequence: *If all nontrivial faces of K have dimension one than all nontrivial faces of K^+ are polyhedral.*

In the following example we show that this statement is not true. Consider

$$D := \big\{(x,y,z) \in \mathbb{R}^3 \mid N(x,y,z) \le 1\big\}$$

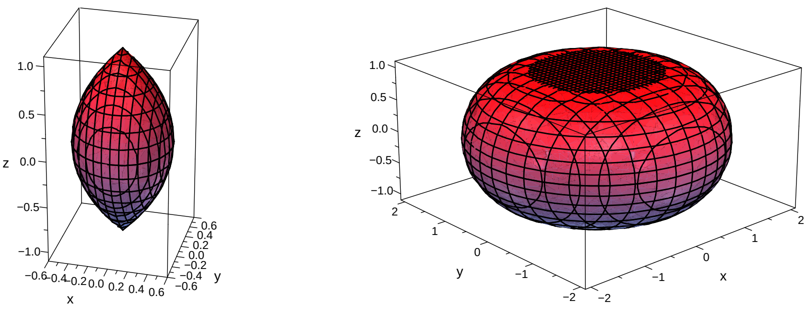
(see Figure 2 (a)), where $N(x,y,z) := \sqrt{x^2+y^2+z^2} + \sqrt{x^2+y^2}$. Clearly N is a norm on $\mathbb{R}^3$ which is strictly convex. It follows that D is strictly convex, and so any face of D is a singleton. Hence all the nontrivial faces of the cone $K_D := \mathbb{R}_+(D \times \{1\}) \subset \mathbb{R}^4$ have dimension 1. It follows that $K_D^+ = \mathbb{R}_+(D_* \times \{1\})$ with $D_* = \{u \in \mathbb{R}^3 \mid N_*(u) \le 1\}$, where N_* is the dual norm of N. But

$$N_*(a,b,c) = \sup\{ax+by+cz \mid N(x,y,z) \le 1\} = \begin{cases} |c| & \text{if } \sqrt{a^2+b^2} \le |c|, \\ \frac{a^2+b^2+c^2}{2\sqrt{a^2+b^2}} & \text{if } \sqrt{a^2+b^2} > |c|. \end{cases}$$

Hence (see Figure 2 (b))

$$D_* = \big\{(a,b,c) \mid a^2+b^2 \le 1,\ |c| \le 1\big\} \cup \Big\{(a,b,c) \mid 1 \le \sqrt{a^2+b^2} \le 1 + \sqrt{1-c^2}\Big\}.$$

One observes that $F := \{(a,b,1) \mid a^2+b^2 \le 1\}$ is a non polyhedral face of D_*, and so K^+ has non polyhedral nontrivial faces.



(a) The set D. (b) The set D_*.

Figure 2: The sets D and D_*.

Using Proposition 2 we obtain that property (B) holds for the cone K in the previous example, and so property (B) also holds for the cone K replaced by the cone K^+. This shows that the "only if" part of Conjecture 3 is not true.

As mentioned also by Prof. A. Shapiro, one can formulate the following problem.

Problem 7 Let $K \subset X$ be a closed convex cone such that all nontrivial faces of K and K^+ are polyhedral. Is it true that $v_P = v_D$ for all $\overline{x} \in K$, $\overline{x}^* \in K^+$ and all linear spaces $M \subset X$?

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