The Split Variational Inequality Problem

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Abstract

We propose a new variational problem which we call the Split Variational Inequality Problem (SVIP). It entails finding a solution of one Variational Inequality Problem (VIP), the image of which under a given bounded linear transformation is a solution of another VIP. We construct iterative algorithms that solve such problems, under reasonable conditions, in Hilbert space and then discuss special cases, some of which are new even in Euclidean space.

1 Introduction

In this paper we introduce a new problem, which we call the *Split Variational Inequality Problem* (SVIP). Let H_1 and H_2 be two real Hilbert spaces. Given operators $f : H_1 \to H_1$ and $g : H_2 \to H_2$, a bounded linear operator $A : H_1 \to H_2$, and nonempty, closed and convex subsets $C \subseteq H_1$ and $Q \subseteq H_2$, the SVIP is formulated as follows:

$$\text{find a point } x^* \in C \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C \tag{1.1}$$

and such that

$$\text{the point } y^* = Ax^* \in Q \text{ solves } \langle g(y^*), y - y^* \rangle \geq 0 \text{ for all } y \in Q. \tag{1.2}$$

When Looked at separately, (1.1) is the classical *Variational Inequality Problem* (VIP) and we denote its solution set by $SOL(C, f)$. The SVIP constitutes a pair of VIPs, which have to be solved so that the image $y^* = Ax^*$, under a given bounded linear operator A, of the solution x^* of the VIP in H_1, is a solution of another VIP in another space H_2.

SVIP is quite general and should enable split minimization between two spaces so that the image of a solution point of one minimization problem, under a given bounded linear operator, is a solution point of another minimization problem. Another special case of the SVIP is the *Split Feasibility Problem* (SFP) which had already been studied and used in practice as a model in intensity-modulated radiation therapy (IMRT) treatment planning; see [8, 10].

We consider two approaches to the solution of the SVIP. The first approach is to look at the product space $H_1 \times H_2$ and transform the SVIP (1.1)–(1.2) into an equivalent *Constrained VIP* (CVIP) in the product space. We study this CVIP and devise an iterative algorithm for its solution, which becomes applicable to the original SVIP via the equivalence between the problems. Our new iterative algorithm for the CVIP, thus for the SVIP, is inspired by an extension of the extragradient method of Korpelevich [22]. In the second approach we present a method that does not require the translation to a product space. This algorithm is inspired by the work of Censor and Segal [14] and Moudafi [26].

Our paper is organized as follows. In Section 2 we present some preliminaries. In Section 3 the algorithm for the constrained VIP is presented. In Section 4 we analyze the SVIP and present its equivalence with the CVIP in the product space. In Section 5 we first present our method for solving the SVIP, which does not rely on any product space formulation, and then prove convergence. In Section 6 we present some applications of the SVIP. It turns out that in addition to helping us solve the SVIP, the CVIP unifies and improves several existing problems and methods where a VIP has to be solved with some additional constraints. Relations of our results to some previously published work are discussed in detail after Theorems 3.5 and 5.3.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let D be a nonempty, closed and convex subset of H. We write $x^k \rightharpoonup x$ to indicate that the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ converges weakly to x, and $x^k \to x$ to indicate that the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in D, denoted by $P_D(x)$. This point satisfies

$$\|x - P_D(x)\| \leq \|x - y\| \text{ for all } y \in D. \tag{2.1}$$

The mapping P_D is called the metric projection of H onto D. We know that P_D is a nonexpansive operator of H onto D, i.e.,

$$\|P_D(x) - P_D(y)\| \leq \|x - y\| \text{ for all } x, y \in H. \tag{2.2}$$

The metric projection P_D is characterized by the fact that $P_D(x) \in D$ and

$$\langle x - P_D(x), P_D(x) - y\rangle \geq 0 \text{ for all } x \in H,\ y \in D, \tag{2.3}$$

and has the property

$$\|x - y\|^2 \geq \|x - P_D(x)\|^2 + \|y - P_D(x)\|^2 \text{ for all } x \in H,\ y \in D. \tag{2.4}$$

It is known that in a Hilbert space H,

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \tag{2.5}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

The following lemma was proved in [32, Lemma 3.2].

Lemma 2.1 *Let H be a Hilbert space and let D be a nonempty, closed and convex subset of H. If the sequence $\left\{x^k\right\}_{k=0}^{\infty} \subset H$ is* `Fejér-monotone` *with respect to D, i.e., for every $u \in D$,*

$$\|x^{k+1} - u\| \leq \|x^k - u\| \textit{ for all } k \geq 0, \tag{2.6}$$

then $\left\{P_D\left(x^k\right)\right\}_{k=0}^{\infty}$ converges strongly to some $z \in D$.

The next lemma is also known (see, e.g., [27, Lemma 3.1]).

Lemma 2.2 *Let H be a Hilbert space, $\{\alpha_k\}_{k=0}^{\infty}$ be a real sequence satisfying $0 < a \leq \alpha_k \leq b < 1$ for all $k \geq 0$, and let $\left\{v^k\right\}_{k=0}^{\infty}$ and $\left\{w^k\right\}_{k=0}^{\infty}$ be two sequences in H such that for some $\sigma \geq 0$,*

$$\limsup_{k\to\infty} \|v^k\| \leq \sigma, \text{ and } \limsup_{k\to\infty} \|w^k\| \leq \sigma. \tag{2.7}$$

If

$$\lim_{k\to\infty} \|\alpha_k v^k + (1-\alpha_k)w^k\| = \sigma, \tag{2.8}$$

then

$$\lim_{k\to\infty} \|v^k - w^k\| = 0. \tag{2.9}$$

Definition 2.3 *Let H be a Hilbert space, D a closed and convex subset of H, and let $M : D \to H$ be an operator. Then M is said to be* `demiclosed` *at $y \in H$ if for any sequence $\left\{x^k\right\}_{k=0}^{\infty}$ in D such that $x^k \rightharpoonup \overline{x} \in D$ and $M(x^k) \to y$, we have $M(\overline{x}) = y$.*

Our next lemma is the well-known Demiclosedness Principle [4].

Lemma 2.4 *Let H be a Hilbert space, D a closed and convex subset of H, and $N : D \to H$ a nonexpansive operator. Then $I - N$ (I is the identity operator on H) is* `demiclosed` *at $y \in H$.*

For instance, the orthogonal projection P onto a closed and convex set is a demiclosed operator everywhere because $I - P$ is nonexpansive [21, page 17].

The next property is known as the *Opial condition* [28, Lemma 1]. It characterizes the weak limit of a weakly convergent sequence in Hilbert space.

Condition 2.5 ***(Opial)*** *For any sequence $\left\{x^k\right\}_{k=0}^{\infty}$ in H that converges weakly to x,*

$$\liminf_{k\to\infty} \|x^k - x\| < \liminf_{k\to\infty} \|x^k - y\| \text{ for all } y \neq x. \tag{2.10}$$

Definition 2.6 *Let $h : H \to H$ be an operator and let $D \subseteq H$.*

(i) h is called `inverse strongly monotone (ISM)` *with constant α on $D \subseteq H$ if*

$$\langle h(x) - h(y), x - y\rangle \geq \alpha\|h(x) - h(y)\|^2 \text{ for all } x, y \in D. \tag{2.11}$$

(ii) h is called `monotone` *on $D \subseteq H$ if*

$$\langle h(x) - h(y), x - y\rangle \geq 0 \text{ for all } x, y \in D. \tag{2.12}$$

Definition 2.7 *An operator $h : H \to H$ is called* ***Lipschitz continuous*** *on $D \subseteq H$ with constant $\kappa > 0$ if*

$$\|h(x) - h(y)\| \leq \kappa \|x - y\| \text{ for all } x, y \in D. \tag{2.13}$$

Definition 2.8 *Let $S : H \rightrightarrows 2^H$ be a point-to-set operator defined on a real Hilbert space H. S is called a* ***maximal monotone operator*** *if S is* ***monotone****, i.e.,*

$$\langle u - v, x - y \rangle \geq 0, \text{ for all } u \in S(x) \text{ and for all } v \in S(y), \tag{2.14}$$

and the graph $G(S)$ of S,

$$G(S) := \{(x, u) \in H \times H \mid u \in S(x)\}, \tag{2.15}$$

is not properly contained in the graph of any other monotone operator.

It is clear that a monotone operator S is maximal if and only if, for each $(x, u) \in H \times H$, $\langle u - v, x - y \rangle \geq 0$ for all $(v, y) \in G(S)$ implies that $u \in S(x)$.

Definition 2.9 *Let D be a nonempty, closed and convex subset of H. The* ***normal cone*** *of D at the point $w \in D$ is defined by*

$$N_D(w) := \{d \in H \mid \langle d, y - w \rangle \leq 0 \text{ for all } y \in D\}. \tag{2.16}$$

Let h be an α-ISM operator on $D \subseteq H$, let $N_D(w)$ be the normal cone of D at a point $w \in D$, and define the following point-to-set operator:

$$S(w) := \begin{cases} h(w) + N_D(w), & w \in C, \\ \emptyset, & w \notin C. \end{cases} \tag{2.17}$$

In these circumstances, it follows from [30, Theorem 3] that S is maximal monotone. In addition, $0 \in S(w)$ if and only if $w \in SOL(D, h)$.

For $T : H \to H$, denote by $\text{Fix}(T)$ the fixed point set of T, i.e.,

$$\text{Fix}(T) := \{x \in H \mid T(x) = x\}. \tag{2.18}$$

It is well-known that

$$x^* \in SOL(C, f) \Leftrightarrow x^* = P_C(x^* - \lambda f(x^*)), \tag{2.19}$$

i.e., $x^* \in \mathrm{Fix}(P_C(I - \lambda f))$. It is also known that every nonexpansive operator $T : H \to H$ satisfies, for all $(x, y) \in H \times H$, the inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x) \rangle \leq (1/2) \|(T(x) - x) - (T(y) - y)\|^2 \tag{2.20}$$

and therefore we get, for all $(x, y) \in H \times \mathrm{Fix}(T)$,

$$\langle x - T(x), y - T(x) \rangle \leq (1/2) \|T(x) - x\|^2; \tag{2.21}$$

see, e.g., [20, Theorem 3] and [19, Theorem 1].

In the next lemma we collect several important properties that will be needed in the sequel.

Lemma 2.10 *Let $D \subseteq H$ be a nonempty, closed and convex subset and let $h : H \to H$ be an α-ISM operator on H. If $\lambda \in [0, 2\alpha]$, then*

(i) the operator $P_D(I - \lambda h)$ is nonexpansive on D.

If, in addition, for all $x^* \in SOL(D, h)$,

$$\langle h(x), P_D(I - \lambda h)(x) - x^* \rangle \geq 0 \text{ for all } x \in H, \tag{2.22}$$

then the following inequalities hold:

(ii) for all $x \in H$ and $q \in \mathrm{Fix}(P_D(I - \lambda h))$,

$$\langle P_D(I - \lambda h)(x) - x, P_D(I - \lambda h)(x) - q \rangle \leq 0; \tag{2.23}$$

(iii) for all $x \in H$ and $q \in \mathrm{Fix}(P_D(I - \lambda h))$,

$$\|P_D(I - \lambda h)(x) - q\|^2 \leq \|x - q\|^2 - \|P_D(I - \lambda h)(x) - x\|^2. \tag{2.24}$$

Proof. (i) Let $x, y \in H$. Then

$$\begin{aligned}
\|P_D(I - \lambda h)(x) - P_D(I - \lambda h)(y)\|^2 &= \|P_D(x - \lambda h(x)) - P_D(y - \lambda h(y))\|^2 \\
&\leq \|x - \lambda h(x) - (y - \lambda h(y))\|^2 \\
&= \|(x - y) - \lambda(h(x) - h(y))\|^2 \\
&= \|x - y\|^2 - 2\lambda \langle x - y, h(x) - h(y) \rangle \\
&+ \lambda^2 \|h(x) - h(y)\|^2 \\
&\leq \|x - y\|^2 - 2\lambda\alpha \|h(x) - h(y)\|^2 \\
&+ \lambda^2 \|h(x) - h(y)\|^2 \\
&= \|x - y\|^2 + \lambda(\lambda - 2\alpha) \|h(x) - h(y)\|^2 \\
&\leq \|x - y\|^2.
\end{aligned} \tag{2.25}$$

(ii) Let $x \in H$ and $q \in \mathrm{Fix}(P_D(I - \lambda h))$. Then

$$\begin{aligned} &\langle P_D(x - \lambda h(x)) - x, P_D(x - \lambda h(x)) - q\rangle \\ &= \langle P_D(x - \lambda h(x)) - x + \lambda h(x) - \lambda h(x), P_D(x - \lambda h(x)) - q\rangle \\ &= \langle P_D(x - \lambda h(x)) - (x - \lambda h(x)), P_D(x - \lambda h(x)) - q\rangle \\ &- \lambda\langle h(x), P_D(x - \lambda h(x)) - q\rangle. \end{aligned} \tag{2.26}$$

By (2.3), (2.19) and (2.22), we get

$$\langle P_D(x - \lambda h(x)) - x, P_D(x - \lambda h(x)) - q\rangle \leq 0. \tag{2.27}$$

(iii) Let $x \in H$ and $q \in \mathrm{Fix}(P_D(I - \lambda h))$. Then

$$\begin{aligned} \|q - x\|^2 &= \|(P_D(I - \lambda h)(x) - x) - (P_D(I - \lambda h)(x) - q)\|^2 \\ &= \|P_D(I - \lambda h)(x) - x\|^2 + \|P_D(I - \lambda h)(x) - q\|^2 \\ &- 2\langle P_D(I - \lambda h)(x) - x, P_D(I - \lambda h)(x) - q\rangle. \end{aligned} \tag{2.28}$$

By (ii), we get

$$-2\langle P_D(I - \lambda h)(x) - x, P_D(I - \lambda h)(x) - q\rangle \geq 0. \tag{2.29}$$

Thus,

$$\|q - x\|^2 \geq \|P_D(I - \lambda h)(x) - x\|^2 + \|P_D(I - \lambda h)(x) - q\|^2 \tag{2.30}$$

or

$$\|P_D(I - \lambda h)x - q\|^2 \leq \|q - x\|^2 - \|P_D(I - \lambda h)x - x\|^2, \tag{2.31}$$

as asserted. ■

Equation (2.23) means that the operator $P_D(I - \lambda h)$ belongs to the class of operators called the $\mathcal{T}$-class. This class $\mathcal{T}$ of operators was introduced and investigated by Bauschke and Combettes in [2, Definition 2.2] and by Combettes in [18]. Operators in this class were named *directed operators* by Zaknoon [34] and further studied under this name by Segal [31] and by Censor and Segal [15, 14, 16]. Cegielski [6, Def. 2.1] studied these operators under the name *separating operators*. Since both *directed* and *separating* are key words of other, widely-used, mathematical entities, Cegielski and Censor have recently introduced the term *cutter operators* [7]. This class coincides with the class $\mathcal{F}^\nu$ for $\nu = 1$ [19] and with the class $\mathrm{DC}_{\boldsymbol{p}}$ for $\boldsymbol{p} = -1$ [24]. The term *firmly quasi-nonexpansive* (FQNE) for $\mathcal{T}$-class operators was used by Yamada and Ogura [33] because every *firmly nonexpansive* (FNE) mapping [21, page 42] is obviously FQNE.

3 An algorithm for solving the constrained variational inequality problem

Let $f : H \to H$, and let C and Ω be nonempty, closed and convex subsets of H. The *Constrained Variational Inequality Problem* (CVIP) is:

$$\text{find } x^* \in C \cap \Omega \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C. \tag{3.1}$$

The iterative algorithm for this CVIP, presented next, is inspired by our earlier work [11, 12] in which we modified the extragradient method of Korpelevich [22]. The following conditions are needed for the convergence theorem.

Condition 3.1 *f is monotone on C.*

Condition 3.2 *f is Lipschitz continuous on H with constant $\kappa > 0$.*

Condition 3.3 $\Omega \cap SOL(C, f) \neq \emptyset$.

Let $\{\lambda_k\}_{k=0}^{\infty} \subset [a, b]$ for some $a, b \in (0, 1/\kappa)$, and let $\{\alpha_k\}_{k=0}^{\infty} \subset [c, d]$ for some $c, d \in (0, 1)$. Then the following algorithm generates two sequences that converge to a point $z \in \Omega \cap \mathrm{SOL}(C, f)$, as the convergence theorem that follows shows.

Algorithm 3.4

Initialization: *Select an arbitrary starting point $x^0 \in H$.*

Iterative step: *Given the current iterate x^k, compute*

$$y^k = P_C(x^k - \lambda_k f(x^k)), \tag{3.2}$$

construct the half-space T_k the bounding hyperplane of which supports C at y^k,

$$T_k := \{w \in H \mid \left\langle \left(x^k - \lambda_k f(x^k)\right) - y^k, w - y^k \right\rangle \leq 0\}, \tag{3.3}$$

and then calculate the next iterate by

$$x^{k+1} = \alpha_k x^k + (1 - \alpha_k) P_\Omega \left(P_{T_k}(x^k - \lambda_k f(y^k)) \right). \tag{3.4}$$

Theorem 3.5 *Let $f : H \to H$, and let C and Ω be nonempty, closed and convex subsets of H. Assume that Conditions 3.1–3.3 hold, and let $\left\{x^k\right\}_{k=0}^{\infty}$ and $\left\{y^k\right\}_{k=0}^{\infty}$ be any two sequences generated by Algorithm 3.4 with $\{\lambda_k\}_{k=0}^{\infty} \subset [a,b]$ for some $a,b \in (0, 1/\kappa)$ and $\{\alpha_k\}_{k=0}^{\infty} \subset [c,d]$ for some $c,d \in (0,1)$. Then $\left\{x^k\right\}_{k=0}^{\infty}$ and $\left\{y^k\right\}_{k=0}^{\infty}$ converge to the same point $z \in \Omega \cap SOL(C,f)$ and*

$$z = \lim_{k\to\infty} P_{\Omega\cap SOL(C,f)}(x^k). \tag{3.5}$$

Proof. For the special case of fixed $\lambda_k = \tau$ for all $k \geq 0$ this theorem is a direct consequence of our [12, Theorem 7.1] with the choice of the nonexpansive operator S there to be P_Ω. However, a careful inspection of the proof of [12, Theorem 7.1] reveals that it also applies to a variable sequence $\{\lambda_k\}_{k=0}^{\infty}$ as used here. ■

To relate our results to some previously published works we mention two lines of research related to our notion of the CVIP. Takahashi and Nadezhkina [27] proposed an algorithm for finding a point $x^* \in \mathrm{Fix}(N) \cap \mathrm{SOL}(C,f)$, where $N : C \to C$ is a nonexpansive operator. The iterative step of their algorithm is as follows. Given the current iterate x^k, compute

$$y^k = P_C(x^k - \lambda_k f(x^k)) \tag{3.6}$$

and then

$$x^{k+1} = \alpha_k x^k + (1-\alpha_k) N\left(P_C(x^k - \lambda_k f(y^k))\right). \tag{3.7}$$

The restriction $P_\Omega|_C$ of our P_Ω in (3.4) is, of course, nonexpansive, and so it is a special case of N in [27]. But a significant advantage of our Algorithm 3.4 lies in the fact that we compute P_{T_k} onto a half-space in (3.4) whereas the authors of [27] need to project onto the convex set C.

Bertsekas and Tsitsiklis [3, Page 288] consider the following problem in Euclidean space: given $f : R^n \to R^n$, polyhedral sets $C_1 \subset R^n$ and $C_2 \subset R^m$, and an $m \times n$ matrix A, find a point $x^* \in C_1$ such that $Ax^* \in C_2$ and

$$\langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C_1 \cap \{y \mid Ay \in C_2\}. \tag{3.8}$$

Denoting $\Omega = A^{-1}(C_2)$, we see that this problem becomes similar to, but not identical with a CVIP. While the authors of [3] seek a solution in $\mathrm{SOL}(C_1 \cap \Omega, f)$, we aim in our CVIP at $\Omega \cap \mathrm{SOL}(C,f)$. They propose to solve their problem by the method of multipliers, which is a different approach than ours, and they need to assume that either C_1 is bounded or $A^T A$ is invertible, where A^T is the transpose of A.

4 The split variational inequality problem as a constrained variational inequality problem in a product space

Our first approach to the solution of the SVIP (1.1)–(1.2) is to look at the product space $\boldsymbol{H} = H_1 \times H_2$ and introduce in it the product set $\boldsymbol{D} := C \times Q$ and the set

$$\boldsymbol{V} := \{\mathbf{x} = (x, y) \in \boldsymbol{H} \mid Ax = y\}. \tag{4.1}$$

We adopt the notational convention that objects in the product space are represented in boldface type. We transform the SVIP (1.1)–(1.2) into the following equivalent CVIP in the product space:

$$\begin{aligned} &\text{Find a point } \boldsymbol{x}^* \in \boldsymbol{D} \cap \boldsymbol{V}, \text{ such that } \langle \boldsymbol{h}(\boldsymbol{x}^*), \boldsymbol{x} - \boldsymbol{x}^* \rangle \geq 0 \\ &\qquad \text{for all } \boldsymbol{x} = (x, y) \in \boldsymbol{D}, \end{aligned} \tag{4.2}$$

where $\boldsymbol{h} : \boldsymbol{H} \to \boldsymbol{H}$ is defined by

$$\boldsymbol{h}(x, y) = (f(x), g(y)). \tag{4.3}$$

A simple adaptation of the decomposition lemma [3, Proposition 5.7, page 275] shows that problems (1.1)–(1.2) and (4.2) are equivalent, and, therefore, we can apply Algorithm 3.4 to the solution of (4.2).

Lemma 4.1 *A point $\boldsymbol{x}^* = (x^*, y^*)$ solves (4.2) if and only if x^* and y^* solve (1.1)–(1.2).*

Proof. If (x^*, y^*) solves (1.1)–(1.2), then it is clear that (x^*, y^*) solves (4.2). To prove the other direction, suppose that (x^*, y^*) solves (4.2). Since (4.2) holds for all $(x, y) \in \boldsymbol{D}$, we may take $(x^*, y) \in \boldsymbol{D}$ and deduce that

$$\langle g(Ax^*), y - Ax^* \rangle \geq 0 \text{ for all } y \in Q. \tag{4.4}$$

Using a similar argument with $(x, y^*) \in \boldsymbol{D}$, we get

$$\langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C, \tag{4.5}$$

which means that (x^*, y^*) solves (1.1)–(1.2). ■

Using this equivalence, we can now employ Algorithm 3.4 in order to solve the SVIP. The following conditions are needed for the convergence theorem.

Condition 4.2 *f is monotone on C and g is monotone on Q.*

Condition 4.3 *f is Lipschitz continuous on H_1 with constant $\kappa_1 > 0$ and g is Lipschitz continuous on H_2 with constant $\kappa_2 > 0$.*

Condition 4.4 $\boldsymbol{V} \cap SOL(\boldsymbol{D}, \boldsymbol{h}) \neq \emptyset$.

Let $\{\lambda_k\}_{k=0}^{\infty} \subset [a, b]$ for some $a, b \in (0, 1/\kappa)$, where $\kappa = \min\{\kappa_1, \kappa_2\}$, and let $\{\alpha_k\}_{k=0}^{\infty} \subset [c, d]$ for some $c, d \in (0, 1)$. Then the following algorithm generates two sequences that converge to a point $\boldsymbol{z} \in \boldsymbol{V} \cap SOL(\boldsymbol{D}, \boldsymbol{h})$, as the convergence theorem given below shows.

Algorithm 4.5

Initialization: *Select an arbitrary starting point $\boldsymbol{x}^0 \in \boldsymbol{H}$.*

Iterative step: *Given the current iterate $\boldsymbol{x}^k$, compute*

$$\boldsymbol{y}^k = \boldsymbol{P}_{\boldsymbol{D}}(\boldsymbol{x}^k - \lambda_k \boldsymbol{h}(\boldsymbol{x}^k)), \tag{4.6}$$

construct the half-space $\boldsymbol{T}_k$ the bounding hyperplane of which supports $\boldsymbol{D}$ at $\boldsymbol{y}^k$,

$$\boldsymbol{T}_k := \{\boldsymbol{w} \in \boldsymbol{H} \mid \left\langle \left(\boldsymbol{x}^k - \lambda_k \boldsymbol{h}(\boldsymbol{x}^k)\right) - \boldsymbol{y}^k, \boldsymbol{w} - \boldsymbol{y}^k \right\rangle \leq 0\}, \tag{4.7}$$

and then calculate

$$\boldsymbol{x}^{k+1} = \alpha_k \boldsymbol{x}^k + (1 - \alpha_k) \boldsymbol{P}_{\boldsymbol{V}} \left(\boldsymbol{P}_{\boldsymbol{T}_k}(\boldsymbol{x}^k - \lambda_k \boldsymbol{h}(\boldsymbol{y}^k)) \right). \tag{4.8}$$

Our convergence theorem for Algorithm 4.5 follows from Theorem 3.5.

Theorem 4.6 *Consider $f : H_1 \to H_1$ and $g : H_2 \to H_2$, a bounded linear operator $A : H_1 \to H_2$, and nonempty, closed and convex subsets $C \subseteq H_1$ and $Q \subseteq H_2$. Assume that Conditions 4.2–4.4 hold, and let $\left\{\boldsymbol{x}^k\right\}_{k=0}^{\infty}$ and $\left\{\boldsymbol{y}^k\right\}_{k=0}^{\infty}$ be any two sequences generated by Algorithm 4.5 with $\{\lambda_k\}_{k=0}^{\infty} \subset [a, b]$ for some $a, b \in (0, 1/\kappa)$, where $\kappa = \min\{\kappa_1, \kappa_2\}$, and let $\{\alpha_k\}_{k=0}^{\infty} \subset [c, d]$ for some $c, d \in (0, 1)$. Then $\left\{\boldsymbol{x}^k\right\}_{k=0}^{\infty}$ and $\left\{\boldsymbol{y}^k\right\}_{k=0}^{\infty}$ converge to the same point $\boldsymbol{z} \in \boldsymbol{V} \cap SOL(\boldsymbol{D}, \boldsymbol{h})$ and*

$$\boldsymbol{z} = \lim_{k \to \infty} \boldsymbol{P}_{\boldsymbol{V} \cap SOL(\boldsymbol{D}, \boldsymbol{h})}(\boldsymbol{x}^k). \tag{4.9}$$

The value of the product space approach, described above, depends on the ability to "translate" Algorithm 4.5 back to the original spaces H_1 and H_2. Observe that due to [29, Lemma 1.1] for $\boldsymbol{x}=(x,y) \in \boldsymbol{D},$ we have $\boldsymbol{P_D}(\boldsymbol{x}) = (P_C(x), P_Q(y))$ and a similar formula holds for $\boldsymbol{P_{T_k}}$. The potential difficulty lies in $\boldsymbol{P_V}$ of (4.8). In the finite-dimensional case, since $\boldsymbol{V}$ is a subspace, the projection onto it is easily computable by using an orthogonal basis. For example, if U is a k-dimensional subspace of R^n with the basis $\{u_1, u_2, ..., u_k\}$, then for $x \in R^n$, we have

$$P_U(x) = \sum_{i=1}^{k} \frac{\langle x, u_i \rangle}{\|u_i\|^2} u_i. \tag{4.10}$$

5 Solving the split variational inequality problem without a product space

In this section we present a method for solving the SVIP, which does not need a product space formulation as in the previous section. Recalling that $SOL(C, f)$ and $SOL(Q, g)$ are the solution sets of (1.1) and (1.2), respectively, we see that the solution set of the SVIP is

$$\Gamma := \Gamma(C, Q, f, g, A) := \{z \in SOL(C, f) \mid Az \in SOL(Q, g)\}. \tag{5.1}$$

Using the abbreviations $T := P_Q(I - \lambda g)$ and $U := P_C(I - \lambda f)$, we propose the following algorithm.

Algorithm 5.1

Initialization: *Let $\lambda > 0$ and select an arbitrary starting point $x^0 \in H_1$.*

Iterative step: *Given the current iterate x^k, compute*

$$x^{k+1} = U(x^k + \gamma A^*(T - I)(Ax^k)), \tag{5.2}$$

where $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A, and A^* is the adjoint of A.

The following lemma, which asserts Fejér-monotonicity, is crucial for the convergence theorem.

Lemma 5.2 *Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Let $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be α_1-ISM and α_2-ISM operators on H_1 and H_2, respectively, and set $\alpha := \min\{\alpha_1, \alpha_2\}$. Assume that $\Gamma \neq \emptyset$ and that $\gamma \in (0, 1/L)$. Consider the operators $U = P_C(I - \lambda f)$ and $T = P_Q(I - \lambda g)$ with $\lambda \in [0, 2\alpha]$. Then any sequence $\left\{x^k\right\}_{k=0}^{\infty}$, generated by Algorithm 5.1, is Fejér-monotone with respect to the solution set Γ.*

Proof. Let $z \in \Gamma$. Then $z \in SOL(C, f)$ and, therefore, by (2.19) and Lemma 2.10(i), we get

$$\begin{aligned}\left\|x^{k+1} - z\right\|^2 &= \left\|U\left(x^k + \gamma A^*(T - I)(Ax^k)\right) - z\right\|^2 \\ &= \left\|U\left(x^k + \gamma A^*(T - I)(Ax^k)\right) - U(z)\right\|^2 \\ &\le \left\|x^k + \gamma A^*(T - I)(Ax^k) - z\right\|^2 \\ &= \left\|x^k - z\right\|^2 + \gamma^2\left\|A^*(T - I)(Ax^k)\right\|^2 \\ &+ 2\gamma\left\langle x^k - z, A^*(T - I)(Ax^k)\right\rangle. \end{aligned} \tag{5.3}$$

Thus

$$\begin{aligned}\left\|x^{k+1} - z\right\|^2 \le \left\|x^k - z\right\|^2 &+ \gamma^2\left\langle (T - I)(Ax^k), AA^*(T - I)(Ax^k)\right\rangle \\ &+ 2\gamma\left\langle x^k - z, A^*(T - I)(Ax^k)\right\rangle. \end{aligned} \tag{5.4}$$

From the definition of L it follows, by standard manipulations, that

$$\begin{aligned}\gamma^2\left\langle (T - I)(Ax^k), AA^*(T - I)(Ax^k)\right\rangle &\le L\gamma^2\left\langle (T - I)(Ax^k), (T - I)(Ax^k)\right\rangle \\ &= L\gamma^2\left\|(T - I)(Ax^k)\right\|^2. \end{aligned} \tag{5.5}$$

Denoting $\Theta := 2\gamma\left\langle x^k - z, A^*(T - I)(Ax^k)\right\rangle$ and using (2.21), we obtain

$$\begin{aligned}\Theta &= 2\gamma\left\langle A(x^k - z), (T - I)(Ax^k)\right\rangle \\ &= 2\gamma\left\langle A(x^k - z) + (T - I)(Ax^k) - (T - I)(Ax^k), (T - I)(Ax^k)\right\rangle \\ &= 2\gamma\left(\left\langle T(Ax^k) - Az, (T - I)(Ax^k)\right\rangle - \left\|(T - I)(Ax^k)\right\|^2\right) \\ &\le 2\gamma\left((1/2)\left\|(T - I)(Ax^k)\right\|^2 - \left\|(T - I)(Ax^k)\right\|^2\right) \\ &\le -\gamma\left\|(T - I)(Ax^k)\right\|^2. \end{aligned} \tag{5.6}$$

Applying (5.5) and (5.6) to (5.4), we see that

$$\left\|x^{k+1} - z\right\|^2 \le \left\|x^k - z\right\|^2 + \gamma(L\gamma - 1)\left\|(T - I)(Ax^k)\right\|^2. \tag{5.7}$$

From the definition of γ, we get

$$\left\|x^{k+1} - z\right\|^2 \le \left\|x^k - z\right\|^2, \tag{5.8}$$

which completes the proof. ■

Now we present our convergence result for Algorithm 5.1.

Theorem 5.3 *Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Let $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be α_1-ISM and α_2-ISM operators on H_1 and H_2, respectively, and set $\alpha := \min\{\alpha_1, \alpha_2\}$. Assume that $\gamma \in (0, 1/L)$. Consider the operators $U = P_C(I - \lambda f)$ and $T = P_Q(I - \lambda g)$ with $\lambda \in [0, 2\alpha]$. Assume further that $\Gamma \neq \emptyset$ and that, for all $x^* \in SOL(C, f)$,*

$$\langle f(x), P_C(I - \lambda f)(x) - x^* \rangle \ge 0 \text{ for all } x \in H. \tag{5.9}$$

Then any sequence $\left\{x^k\right\}_{k=0}^{\infty}$, generated by Algorithm 5.1, converges weakly to a solution point $x^* \in \Gamma$.

Proof. Let $z \in \Gamma$. It follows from (5.8) that the sequence $\left\{\left\|x^k - z\right\|\right\}_{k=0}^{\infty}$ is monotonically decreasing and therefore convergent, which shows, by (5.7), that,

$$\lim_{k\to\infty} \left\|(T - I)(Ax^k)\right\| = 0. \tag{5.10}$$

Fejér-monotonicity implies that $\left\{x^k\right\}_{k=0}^{\infty}$ is bounded, so it has a weakly convergent subsequence $\left\{x^{k_j}\right\}_{j=0}^{\infty}$ such that $x^{k_j} \rightharpoonup x^*$. By the assumptions on λ and g, we get from Lemma 2.10(i) that T is nonexpansive. Applying the demiclosedness of $T - I$ at 0 to (5.10), we obtain

$$T(Ax^*) = Ax^*, \tag{5.11}$$

which means that $Ax^* \in SOL(Q, g)$. Denote

$$u^k := x^k + \gamma A^*(T - I)(Ax^k). \tag{5.12}$$

Then

$$u^{k_j} = x^{k_j} + \gamma A^*(T - I)(Ax^{k_j}). \tag{5.13}$$

Since $x^{k_j} \rightharpoonup x^*$, (5.10) implies that $u^{k_j} \rightharpoonup x^*$ too. It remians to be shown that $x^* \in SOL(C, f)$. Assume, by negation, that $x^* \notin SOL(C, f)$, i.e., $Ux^* \neq x^*$. By the assumptions on λ and f, we get from Lemma 2.10(i) that U is nonexpansive and, therefore, $U - I$ is demiclosed at 0. So, the negation assumption must lead to

$$\lim_{j\to\infty} \left\| U(u^{k_j}) - u^{k_j} \right\| \neq 0. \tag{5.14}$$

Therefore there exists an $\varepsilon > 0$ and a subsequence $\left\{u^{k_{j_s}}\right\}_{s=0}^{\infty}$ of $\left\{u^{k_j}\right\}_{j=0}^{\infty}$ such that

$$\left\| U(u^{k_{j_s}}) - u^{k_{j_s}} \right\| > \varepsilon \text{ for all } s \geq 0. \tag{5.15}$$

Inequality (2.24) now yields, for all $s \geq 0$,

$$\begin{aligned} \left\| U(u^{k_{j_s}}) - U(z) \right\|^2 = \left\| U(u^{k_{j_s}}) - z \right\|^2 &\leq \left\| u^{k_{j_s}} - z \right\|^2 - \left\| U(u^{k_{j_s}}) - u^{k_{j_s}} \right\|^2 \\ &< \left\| u^{k_{j_s}} - z \right\|^2 - \varepsilon^2. \end{aligned} \tag{5.16}$$

By arguments similar to those in the proof of Lemma 5.2, we have

$$\left\| u^k - z \right\| = \left\| \left(x^k + \gamma A^*(T - I)(Ax^k)\right) - z \right\| \leq \left\| x^k - z \right\|. \tag{5.17}$$

Since U is nonexpansive,

$$\left\| x^{k+1} - z \right\| = \left\| U(u^k) - z \right\| \leq \left\| u^k - z \right\|. \tag{5.18}$$

Combining (5.17) and (5.18), we get

$$\left\| x^{k+1} - z \right\| \leq \left\| u^k - z \right\| \leq \left\| x^k - z \right\|, \tag{5.19}$$

which means that the sequence $\{x^1, u^1, x^2, u^2, \ldots\}$ is Fejér-monotone with respect to Γ. Since $x^{k_{j_s}+1} = U(u^{k_{j_s}})$, we obtain

$$\left\| u^{k_{j_{s+1}}} - z \right\|^2 \leq \left\| u^{k_{j_s}} - z \right\|^2. \tag{5.20}$$

Hence $\left\{u^{k_{j_s}}\right\}_{s=0}^{\infty}$ is also Fejér-monotone with respect to Γ. Now, (5.16) and (5.19) imply that

$$\left\| u^{k_{j_{s+1}}} - z \right\|^2 < \left\| u^{k_{j_s}} - z \right\|^2 - \varepsilon^2 \text{ for all } s \geq 0, \tag{5.21}$$

which leads to a contradiction. Therefore $x^* \in SOL(C, f)$ and finally, $x^* \in \Gamma$. Since the subsequence $\left\{x^{k_j}\right\}_{j=0}^{\infty}$ was arbitrary, we get that $x^k \rightharpoonup x^*$. ■

Relations of our results to some previously published works are as follows. In [14] an algorithm for the Split Common Fixed Point Problem (SCFPP) in Euclidean spaces was studied. Later Moudafi [26] presented a similar result for Hilbert spaces. In this connection, see also [25].

To formulate the SCFPP, let H_1 and H_2 be two real Hilbert spaces. Given operators $U_i : H_1 \to H_1$, $i = 1, 2, \ldots, p$, and $T_j : H_2 \to H_2$, $j = 1, 2, \ldots, r$, with nonempty fixed point sets C_i, $i = 1, 2, \ldots, p$ and Q_j, $j = 1, 2, \ldots, r$, respectively, and a bounded linear operator $A : H_1 \to H_2$, the SCFPP is formulated as follows:

$$\text{find a point } x^* \in C := \cap_{i=1}^{p} C_i \text{ such that } Ax^* \in Q := \cap_{j=1}^{r} Q_j. \tag{5.22}$$

Our result differs from those in [14] and [26] in several ways. Firstly, the spaces in which the problems are formulated. Secondly, the operators U and T in [14] are assumed to be firmly quasi-nonexpansive (FQNE; see the comments after Lemma 2.10 above), where in our case here only U is FQNE, while T is just nonexpansive. Lastly, Moudafi [26] obtains weak convergence for a wider class of operators, called demicontractive. The iterative step of his algorithm is

$$x^{k+1} = (1 - \alpha_k)u^k + \alpha_k U(u^k), \tag{5.23}$$

where $u^k := x^k + \gamma A^*(T - I)(Ax^k)$ for $\alpha_k \in (0, 1)$. If $\alpha_k = 1$, which is not allowed there, were possible, then the iterative step of [26] would coincide with that of [14].

5.1 A parallel algorithm for solving the multiple set split variational inequality problem

We extend the SVIP to the *Multiple Set Split Variational Inequality Problem* (MSSVIP), which is formulated as follows. Let H_1 and H_2 be two real Hilbert spaces. Given a bounded linear operator $A : H_1 \to H_2$, functions $f_i : H_1 \to H_1$, $i = 1, 2, \ldots, p$, and $g_j : H_2 \to H_2$, $j = 1, 2, \ldots, r$, and nonempty, closed and convex subsets $C_i \subseteq H_1$, $Q_j \subseteq H_2$ for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r$, respectively, the Multiple Set Split Variational Inequality Problem (MSSVIP) is formulated as follows:

$$\begin{cases} \text{find a point } x^* \in C := \cap_{i=1}^{p} C_i \text{ such that } \langle f_i(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C_i \\ \text{and for all } i = 1, 2, \ldots, p, \text{ and such that} \\ \text{the point } y^* = Ax^* \in Q := \cap_{i=1}^{r} Q_j \text{ solves } \langle g_j(y^*), y - y^* \rangle \geq 0 \text{ for all } y \in Q_j \\ \text{and for all } j = 1, 2, \ldots, r. \end{cases} \tag{5.24}$$

For the MSSVIP we do not yet have a solution approach which does not use a product space formalism. Therefore we present a simultaneous algorithm for the MSSVIP the analysis of which is carried out via a certain product space. Let Ψ be the solution set of the MSSVIP:

$$\Psi := \left\{ z \in \cap_{i=1}^{p} SOL(C_i, f_i) \mid Az \in \cap_{i=1}^{r} SOL(Q_j, g_j) \right\}. \tag{5.25}$$

We introduce the spaces $\boldsymbol{W}_1 : =H_1$ and $\boldsymbol{W}_2 := H_1^p \times H_2^r$, where r and p are the indices in (5.24). Let $\{\alpha_i\}_{i=1}^{p}$ and $\{\beta_j\}_{j=1}^{r}$ be positive real numbers. Define the following sets in their respective spaces:

$$\boldsymbol{C} := H_1 \text{ and} \tag{5.26}$$

$$\boldsymbol{Q} := \left(\prod_{i=1}^{p} \sqrt{\alpha_i} C_i \right) \times \left(\prod_{j=1}^{r} \sqrt{\beta_j} Q_j \right), \tag{5.27}$$

and the operator

$$\boldsymbol{A} := \left(\sqrt{\alpha_1} I, \ldots, \sqrt{\alpha_p} I, \sqrt{\beta_1} A^*, \ldots, \sqrt{\beta_r} A^* \right)^*, \tag{5.28}$$

where A^* stands for adjoint of A. Denote $U_i := P_{C_i}(I - \lambda f_i)$ and $T_j := P_{Q_j}(I - \lambda g_j)$ for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r$, respectively. Define the operator $\boldsymbol{T} : \boldsymbol{W}_2 \to \boldsymbol{W}_2$ by

$$\begin{aligned} \boldsymbol{T}(\boldsymbol{y}) &= \boldsymbol{T} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{p+r} \end{pmatrix} \\ &= \left(\left(U_1\left(y_1\right)\right)^*, \ldots, \left(U_p\left(y_p\right)\right)^*, \left(T_1\left(y_{p+1}\right)\right)^*, \ldots, \left(T_r(y_{p+r})\right)^* \right)^*, \end{aligned} \tag{5.29}$$

where $y_1, y_2, ..., y_p \in H_1$ and $y_{p+1}, y_{p+2}, ..., y_{p+r} \in H_2$.

This leads to an SVIP with just two operators $\boldsymbol{F}$ and $\boldsymbol{G}$ and two sets $\boldsymbol{C}$ and $\boldsymbol{Q}$, respectively, in the product space, when we take $\boldsymbol{C}$=H_1, $\boldsymbol{F} \equiv \boldsymbol{0}$,

$\boldsymbol{Q}\subseteq\boldsymbol{W}_2$, $\boldsymbol{G}(\boldsymbol{y}) = (f_1(y_1), f_2(y_2)\ldots, f_p(y_p), g_1(y_{p+1}), g_2(y_{p+2}), \ldots, g_r(y_{p+r}))$, and the operator $\boldsymbol{A} : H_1\rightarrow\boldsymbol{W}_2$. It is easy to verify that the following equivalence holds:

$$x \in \Psi \text{ if and only if } \boldsymbol{A}x \in \boldsymbol{Q}. \tag{5.30}$$

Therefore we may apply Algorithm 5.1,

$$x^{k+1} = x^k + \gamma\boldsymbol{A}^*(\boldsymbol{T} - \boldsymbol{I})(\boldsymbol{A}x^k) \text{ for all } k \geq 0, \tag{5.31}$$

to the problem (5.26)–(5.29) in order to obtain a solution of the original MSSVIP. We translate the iterative step (5.31) to the original spaces H_1 and H_2 using the relation

$$\boldsymbol{T}(\boldsymbol{A}x) = \left(\sqrt{\alpha_1}U_1(x), \ldots, \sqrt{\alpha_p}U_p(x), \sqrt{\beta_1}AT_1(x), \ldots, \sqrt{\beta_r}AT_r(x)\right)^* \tag{5.32}$$

and obtain the following algorithm.

Algorithm 5.4

Initialization: *Select an arbitrary starting point* $x^0 \in H_1$.

Iterative step: *Given the current iterate* x^k*, compute*

$$x^{k+1} = x^k + \gamma\left(\sum_{i=1}^{p}\alpha_i(U_i - I)(x^k) + \sum_{j=1}^{r}\beta_j A^*(T_j - I)(Ax^k)\right), \tag{5.33}$$

where $\gamma \in (0, 1/L)$*, with* $L = \sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{r}\beta_j\|A\|^2$.

The following convergence result follows from Theorem 5.3.

Theorem 5.5 *Let H_1 and H_2 be two real Hilbert spaces and let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Let $f_i : H_1 \rightarrow H_1$, $i = 1, 2, \ldots, p$, and $g_j : H_2 \rightarrow H_2$, $j = 1, 2, \ldots, r$, be α-ISM operators on nonempty, closed and convex subsets $C_i \subseteq H_1$, $Q_j \subseteq H_2$ for $i = 1, 2, \ldots, p$, and $j = 1, 2, \ldots, r$, respectively. Assume that $\gamma \in (0, 1/L)$ and $\Psi \neq \emptyset$. Set $U_i := P_{C_i}(I - \lambda f_i)$ and $T_j := P_{Q_j}(I - \lambda g_j)$ for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r$, respectively, with $\lambda \in [0, 2\alpha]$. If, in addition, for each $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r$ we have*

$$\langle f_i(x), P_{C_i}(I - \lambda f_i)(x) - x^*\rangle \geq 0 \textit{ for all } x \in H \tag{5.34}$$

for all $x^* \in SOL(C_i, f_i)$ and

$$\langle g_j(x), P_{Q_j}(I - \lambda g_j)(x) - x^*\rangle \geq 0 \textit{ for all } x \in H, \tag{5.35}$$

for all $x^* \in SOL(C_i, f_i)$, then any sequence $\left\{x^k\right\}_{k=0}^{\infty}$, generated by Algorithm 5.1, converges weakly to a solution point $x^* \in \Psi$.

Proof. Apply Theorem 5.3 to the two-operator SVIP in the product space setting with $U = \boldsymbol{I} : H_1 \to H_1$, $\operatorname{Fix} U = \boldsymbol{C}$, $T = \boldsymbol{T} : \boldsymbol{W} \to \boldsymbol{W}$, and $\operatorname{Fix} T = \boldsymbol{Q}$. ■

Remark 5.6 *Observe that conditions (5.34) and (5.35) imposed on U_i and T_j for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r$, respectively, in Theorem 5.5, which are necessary for our treatment of the problem in a product space, ensure that these operators are firmly quasi-nonexpansive (FQNE). Therefore, the SVIP under these conditions may be considered a* `Split Common Fixed Point Problem` *(SCFPP), first introduced in [14], with $\boldsymbol{C}$, $\boldsymbol{Q}$, $\boldsymbol{A}$ and $\boldsymbol{T} : \boldsymbol{W}_2 \to \boldsymbol{W}_2$ as above, and the identity operator $\boldsymbol{I} : \boldsymbol{C} \to \boldsymbol{C}$. Therefore, we could also apply [14, Algorithm 4.1]. If, however, we drop these conditions, then the operators are nonexpansive, by Lemma 2.10(i), and the result of [26] would apply.*

6 Applications

The following problems are special cases of the SVIP. They are listed here because their analysis can benefit from our algorithms for the SVIP and because known algorithms for their solution may be generalized in the future to cover the more general SVIP. The list includes known problems such as the Split Feasibility Problem (SFP) and the Convex Feasibility Problem (CFP). In addition, we introduce two new "split" problems that have, to the best of our knowledge, never been studied before. These are the Common Variational Inequality Point Problem (CVIPP) and the Split Zeros Problem (SZP).

6.1 The split feasibility and convex feasibility problems

The Split Feasibility Problem (SFP) in Euclidean space is formulated as follows:

$$\text{find a point } x^* \text{ such that } x^* \in C \subseteq R^n \text{ and } Ax^* \in Q \subseteq R^m, \tag{6.1}$$

where $C \subseteq R^n$, $Q \subseteq R^m$ are nonempty, closed and convex sets, and $A : R^n \to R^m$ is given. Originally introduced in Censor and Elfving [9], it was later used in the area of intensity-modulated radiation therapy (IMRT) treatment planning; see [10, 8]. Obviously, it is formally a special case of the SVIP obtained from (1.1)–(1.2) by setting $f \equiv g \equiv 0$. The Convex Feasibility Problem (CFP) in a Euclidean space is:

$$\text{find a point } x^* \text{ such that } x^* \in \cap_{i=1}^m C_i \neq \emptyset, \tag{6.2}$$

where C_i, $i = 1, 2, \ldots, m$, are nonempty, closed and convex sets in R^n. This, in its turn, becomes a special case of the SFP by taking in (6.1) $n = m$, $A = I$ and $C = \cap_{i=1}^m C_i$. Many algorithms for solving the CFP have been developed; see, e.g., [1, 17]. Byrne [5] established an algorithm for solving the SFP, called the CQ-Algorithm, with the following iterative step:

$$x^{k+1} = P_C\left(x^k + \gamma A^t (P_Q - I) A x^k\right), \tag{6.3}$$

which does not require calculation of the inverse of the operator A, as in [9], but needs only its transpose A^t. A recent excellent paper on the multiple-sets SFP which contains many references that reflect the state-of-the-art in this area is [23].

It is of interest to note that looking at the SFP from the point of view of the SVIP enables us to find the minimum-norm solution of the SFP, i.e., a solution of the form

$$x^* = \operatorname{argmin}\{\|x\| \mid x \text{ solves the SFP (6.1)}\}. \tag{6.4}$$

This is done, and easily verified, by solving (1.1)–(1.2) with $f = I$ and $g \equiv 0$.

6.2 The common variational inequality point problem

The Common Variational Inequality Point Problem (CVIPP), newly introduced here, is defined in Euclidean space as follows. Let $\{f_i\}_{i=1}^m$ be a family of functions from R^n into itself and let $\{C_i\}_{i=1}^m$ be nonempty, closed and convex subsets of R^n with $\cap_{i=1}^m C_i \neq \emptyset$. The CVIPP is formulatd as follows:

$$\begin{aligned} &\text{find a point } x^* \in \cap_{i=1}^m C_i \text{ such that } \langle f_i(x^*), x - x^* \rangle \geq 0 \\ &\qquad \text{for all } x \in C_i,\ i = 1, 2, \ldots, m. \end{aligned} \tag{6.5}$$

This problem can be transformed into a CVIP in an appropriate product space (different from the one in Section 4). Let R^{mn} be the product space and define $\boldsymbol{F} : R^{mn} \to R^{mn}$ by

$$\boldsymbol{F}\left((x_1, x_2, ..., x_m)^t\right) = \left((f_1(x_1), ..., f_m(x_m))^t\right), \tag{6.6}$$

where $x_i \in R^n$ for all $i = 1, 2, \ldots, m$. Let the diagonal set in R^{mn} be

$$\boldsymbol{\Delta} := \{\boldsymbol{x} \in R^{mn} \mid \boldsymbol{x}{=}(a, a, ..., a),\ a \in R^n\} \tag{6.7}$$

and define the product set

$$\boldsymbol{C} := \Pi_{i=1}^m C_i. \tag{6.8}$$

The CVIPP in R^n is equivalent to the following CVIP in R^{mn}:

$$\text{find a point } \boldsymbol{x}^* \in \boldsymbol{C} \cap \boldsymbol{\Delta} \text{ such that } \langle \boldsymbol{F}(\boldsymbol{x}^*), \boldsymbol{x} - \boldsymbol{x}^* \rangle \geq 0$$
$$\text{for all } \boldsymbol{x} = (x_1, x_2, ..., x_m) \in \boldsymbol{C}. \tag{6.9}$$

So, this problem can be solved by using Algorithm 3.4 with $\Omega = \boldsymbol{\Delta}$. A new algorithm specifically designed for the CVIPP appears in [13].

6.3 The split minimization and the split zeros problems

From optimality conditions for convex optimization (see, e.g., Bertsekas and Tsitsiklis [3, Proposition 3.1, page 210]) it is well-known that if $F : R^n \to R^n$ is a continuously differentiable convex function on a closed and convex subset $X \subseteq R^n$, then $x^* \in X$ minimizes F over X if and only if

$$\langle \nabla F(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in X, \tag{6.10}$$

where ∇F is the gradient of F. Since (6.10) is a VIP, we make the following observation. If $F : R^n \to R^n$ and $G : R^m \to R^m$ are continuously differentiable convex functions on closed and convex subsets $C \subseteq R^n$ and $Q \subseteq R^m$, respectively, and if in the SVIP we take $f = \nabla F$ and $g = \nabla G$, then we obtain the following *Split Minimization Problem* (SMP):

$$\text{find a point } x^* \in C \text{ such that } x^* = \text{argmin}\{f(x) \mid x \in C\} \tag{6.11}$$
$$\text{and such that}$$
$$\text{the point } y^* = Ax^* \in Q \text{ solves } y^* = \text{argmin}\{g(y) \mid y \in Q\}. \tag{6.12}$$

The *Split Zeros Problem* (SZP), newly introduced here, is defined as follows. Let H_1 and H_2 be two Hilbert spaces. Given operators $B_1 : H_1 \to H_1$

and $B_2 : H_2 \to H_2$, and a bounded linear operator $A : H_1 \to H_2$, the SZP is formulated as follows:

$$\text{find a point } x^* \in H_1 \text{ such that } B_1(x^*) = 0 \text{ and } B_2(Ax^*) = 0. \tag{6.13}$$

This problem is a special case of the SVIP if A is a surjective operator. To see this, take in (1.1)–(1.2) $C = H_1$, $Q = H_2$, $f = B_1$ and $g = B_2$, and choose $x := x^* - B_1(x^*) \in H_1$ in (1.1) and $x \in H_1$ such that $Ax := Ax^* - B_2(Ax^*) \in H_2$ in (1.2).

The next lemma shows when the only solution of an SVIP is a solution of an SZP. It extends a similar result concerning the relationship between the (un-split) zero finding problem and the VIP.

Lemma 6.1 *Let H_1 and H_2 be real Hilbert spaces, and $C \subseteq H_1$ and $Q \subseteq H_2$ nonempty, closed and convex subsets. Let $B_1 : H_1 \to H_1$ and $B_2 : H_2 \to H_2$ be α-ISM operators and let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $C \cap \{x \in H_1 \mid B_1(x) = 0\} \neq \emptyset$ and that $Q \cap \{y \in H_2 \mid B_2(y) = 0\} \neq \emptyset$, and denote*

$$\Gamma := \Gamma(C, Q, B_1, B_2, A) := \{z \in SOL(C, B_1) \mid Az \in SOL(Q, B_2)\}. \tag{6.14}$$

Then, for any $x^* \in C$ with $Ax^* \in Q$, x^* solves (6.13) if and only if $x^* \in \Gamma$.

Proof. First assume that $x^* \in C$ with $Ax^* \in Q$ and that x^* solves (6.13). Then it is clear that $x^* \in \Gamma$. In the other direction, assume that $x^* \in C$ with $Ax^* \in Q$ and that $x^* \in \Gamma$. Applying (2.4) with C as D there, $(I - \lambda B_1)(x^*) \in H_1$, for any $\lambda \in (0, 2\alpha]$, as x there, and $q_1 \in C \cap \operatorname{Fix}(I - \lambda B_1)$, with the same λ, as y there, we get

$$\begin{aligned} &\|q_1 - P_C(I - \lambda B_1)(x^*)\|^2 + \|(I - \lambda B_1)(x^*) - P_C(I - \lambda B_1)(x^*)\|^2 \\ &\leq \|(I - \lambda B_1)(x^*) - q_1\|^2, \end{aligned} \tag{6.15}$$

and, similarly, applying (2.4) again, we obtain

$$\begin{aligned} &\|q_2 - P_Q(I - \lambda B_2)(Ax^*)\|^2 + \|(I - \lambda B_2)(Ax^*) - P_Q(I - \lambda B_2)(Ax^*)\|^2 \\ &\leq \|(I - \lambda B_2)(Ax^*) - q_2\|^2. \end{aligned} \tag{6.16}$$

Using the characterization of (2.19), we get

$$\|q_1 - x^*\|^2 + \|(I - \lambda B_1)(x^*) - x^*\|^2 \leq \|(I - \lambda B_1)(x^*) - q_1\|^2 \tag{6.17}$$

and

$$\|q_2 - Ax^*\|^2 + \|(I - \lambda B_2)(Ax^*) - x^*\|^2 \leq \|(I - \lambda B_2)(Ax^*) - q_2\|^2 . \quad (6.18)$$

It can be seen from the proof of Lemma 2.10(i) that the operators $I - \lambda B_1$ and $I - \lambda B_2$ are nonexpansive for every $\lambda \in [0, 2\alpha]$, so with $q_1 \in C \cap \mathrm{Fix}(I - \lambda B_1)$ and $q_2 \in Q \cap \mathrm{Fix}(I - \lambda B_2)$,

$$\|(I - \lambda B_1)(x^*) - q_1\|^2 \leq \|x^* - q_1\|^2 \quad (6.19)$$

and

$$\|(I - \lambda B_2)(Ax^*) - q_2\|^2 \leq \|Ax^* - q_2\|^2 . \quad (6.20)$$

Combining the above inequalities, we obtain

$$\|q_1 - x^*\|^2 + \|(I - \lambda B_1)(x^*) - x^*\|^2 \leq \|x^* - q_1\|^2 \quad (6.21)$$

and

$$\|q_2 - Ax^*\|^2 + \|(I - \lambda B_2)(Ax^*) - x^*\|^2 \leq \|Ax^* - q_2\|^2 . \quad (6.22)$$

Hence, $\|(I - \lambda B_1)(x^*) - x^*\|^2 = 0$ and $\|(I - \lambda B_2)(Ax^*) - Ax^*\|^2 = 0$. Since $\lambda > 0$, we get that $B_1(x^*) = 0$ and $B_2(Ax^*) = 0$, as claimed ■

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