Uniform bounds on the 1-norm of the inverse of lower triangular Toeplitz matrices*

X Liu[†] S McKee[‡] J Y Yuan[§] X Y Yuan[¶]

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Abstract

A uniform bound of the 1−norm is given for the inverse of lower triangular Toeplitz matrices with nonnegative monotonically decreasing entries whose limit is zero. The new bound is sharp under certain specified constraints. This result is then employed to throw light upon a long standing open problem posed by Brunner concerning the convergence of the one-point collocation method for the Abel's equation. In addition, the recent conjecture of Gauthier et al. is proved.

Keywords: 1−norm uniform upper bound, inverse of lower triangular Toeplitz matrix, Brunner's conjecture, Abel's equation.

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†LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, AMSS, CAS, Beijin 100190, China, liuxin@lsec.cc.ac.cn

‡Department of Mathematics, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XH, UK

§Departamento de Matemática - UFPR. Centro Politénico, CP: 19.081, 81531-980, Curitiba, Brazil, jin@mat.ufpr.br

¶Institute of Computational Mathematics and Scientific Computing Engineering, Academy of Mathematics and System Sciences, Chinese Academy, 10080, Beijing, China, yyx@slec.cc.ac.cn

1 Introduction

Consider the Abel equation

$$\int_0^t (t-s)^{-\alpha} y(s)ds = g(t), \quad t \in I := [0,T], \tag{1.1}$$

with $g(0)=0$, and $\alpha \in [0,1)$. Its unique solution $y \in C(I)$ is given by

$$y(t) = \frac{1}{\gamma_\alpha} \int_0^t (t-s)^{\alpha-1} g'(s)ds, \quad t \in I$$

with $\gamma_\alpha := \Gamma(\alpha)\Gamma(1-\alpha)$. Consider its associated difference equation

$$a_0(c;\alpha)y_{i+1} + \sum_{j=0}^{i-1} a_{i-j}(c;\alpha)y_{j+1} = b_i(c;\alpha), \quad i = 0,1,\dots,n-1, \tag{1.2}$$

where the parameter c lies in $(0,1]$ and where the coefficients are given by

$$\begin{array}{llcl}
(a) & a_0(c;\alpha) & = & \int_0^c (c-s)^{-\alpha} ds \\
(b) & a_{i-j}(c;\alpha) & = & \int_0^1 (i+c-j-s)^{-\alpha} ds \quad (j<i) \\
(c) & b_i(c;\alpha) & = & h^{\alpha-1} g(t_i + ch).
\end{array} \tag{1.3}$$

The points t_i belong to a uniform mesh $I_i = \{t_i = ih \; : \; i = 0,1,\dots,n; \quad t_n = T\}$.

This is a collocation method with a single collocation point t_{i+c} per sub-interval $[t_i, t_{i+1}]$.

Note that the case $c=1$ gives rise to the product Euler method (see e.g. Weiss and Anderssen [15], Eggermont [9]).

In [5] Brunner posed the problem:

Problem 1.1. *Given $\alpha \in [0,1)$, for which values of $c = c(\alpha) \in (0,1]$ do the solutions y_{i+1} of the difference equation (1.2) remain uniformly bounded as $n \to \infty$, $h \to 0$ with $nh = T$?*

By evaluating the integrals in (a) and (b) of (1.3) the totality of the difference equations (1.2) may be written as

$$T_n \mathbf{y} = \begin{pmatrix}
c^{1-\alpha} & & & O & \\
(1+c)^{1-\alpha} - c^{1-\alpha} & c^{1-\alpha} & & & \\
(2+c)^{1-\alpha} - (1+c)^{1-\alpha} & (1+c)^{1-\alpha} - c^{1-\alpha} & c^{1-\alpha} & & \\
\vdots & & & \ddots & \\
(n+c)^{1-\alpha} - (n-1+c)^{1-\alpha} & \dots & & \dots & c^{1-\alpha}
\end{pmatrix} \mathbf{y} = \mathbf{d} \tag{1.4}$$

where

$$\mathbf{y} = (y_1, \dots, y_{n+1})^T, \quad \mathbf{d} = (d_1, \dots, d_{n+1})^T.$$

We note that T_n is a Toeplitz matrix.

It is perhaps useful at this point to make clear what we mean by the L_1 and L_∞ norms of an $n \times n$ matrix A_n:

Definition

$$\begin{aligned} \|A_n\|_1 &= \max_i \sum_{j=1}^{n} |a_{ij}| \\ \|A_n\|_\infty &= \max_j \sum_{i=1}^{n} |a_{ij}|. \end{aligned}$$

In the case of the class of lower triangular Toeplitz matrices T_n we note that

$$\|T_n\|_1 = \|T_n\|_\infty$$

since

$$\max_i \sum_{j=1}^{n} |t_{ij}| = \max_j \sum_{i=1}^{n} |t_{ij}|.$$

It is easy to see that $\|T_n^{-1}\|_1$ $(= \|T_n^{-1}\|_\infty)$ being uniformly bounded is a necessary condition for $\mathbf{y}$ to remain uniformly bounded. We also note that if $\alpha = 0$ in (1.1), then the complete answer to Problem 1.1 is known: the solutions of the difference equation (1.2) remain uniformly bounded if, and only if, $c \geq \frac{1}{2}$ (c.f. Brunner [5, 6, 7]). For $0 < \alpha < 1$ a sufficient condition for uniform boundedness is $c \geq c^*(\alpha) := \left(\frac{1}{2}\right) [\alpha(1-\alpha)\gamma_\alpha]^{1/(1-\alpha)}$ (see [5]). For example, $c^*(\alpha) \simeq 0.3084$ when $\alpha = \frac{1}{2}$.

Consider also the following conjecture.

Conjecture A

If $T_n \in \mathbb{R}^{(n+1)\times(n+1)}$ is the lower triangular Toeplitz matrix whose first column is

$$\frac{1}{\sqrt{2}}(1, \sqrt{3}-1, \sqrt{5}-\sqrt{3}, \ldots, \sqrt{2n+1}-\sqrt{2n-1})^T \tag{1.5}$$

then for all n, $\|T_n^{-1}\|_1 < 3$.

Recently, Gauthier et al. [10] proved that an algorithm due to Chen and Mangasarian [8] for solving a mixed linear complementarity problem arising from the discretization of a special system of singular Volterra integral equations would converge if the above conjecture were true.

These and related problems have a long history dating back to the work of Holyhead [11], Weiss and Anderssen [15] and others; yet they appear to have evaded resolution.

Essentially the problem may be regarded as requiring that the 1-norm of the inverse of the lower triangular Toeplitz matrix with specific constraints on its elements be uniformly bounded with respect to its order. Consider the lower triangular $(n+1)\times(n+1)$ Toeplitz matrix

$$T_n = \begin{pmatrix} b_0 & & & & \\ b_1 & b_0 & & & \\ b_2 & b_1 & b_0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ b_n & \dots & \dots & b_1 & b_0 \end{pmatrix} \tag{1.6}$$

which may be characterized by its first column $(b_0,\ b_1,\dots,b_n)^T$ where $b_0 \geq b_1 \geq \dots \geq b_n \geq b \geq 0$.

The upper bounds for $\|T_n^{-1}\|_\infty$ given in [1, 13] (see Section 2) are not uniform with respect to n in the case where $\lim_{n\to\infty} b_n = b = 0$. In this paper we shall provide a sharp uniform upper bound for $\|T_n^{-1}\|_\infty$ in this case, subject to specified constraints on the elements of T_n. It will be observed that conjecture A is a special case of this result. Furthermore, it will provide sharp necessary conditions for Brunner's collocation problem.

It is worth mentioning that there is a fairly extensive history of work related to log-convexity of sequences, in particular, regarding the connections between Kaluza sequences, renewal sequences and power series expansions, e.g. [2], [12]. Although, in this paper, we do not treat block lower-triangular matrices that, for instance, arise in collocation based on higher-degree piecewise polynomials it may be that the approach adopted in this paper does not generalise and that the ideas in the book by Böttcher and Grudsky [4] may be amenable to treat this problem

The outline of this paper is as follows. In Section 2 a uniform bound is given for the 1−norm of the inverse of the lower triangular Toeplitz matrix (1.6) subject to certain constraints on its elements. Under these restrictions it is shown that this new bound is sharp. Conjecture A is proved in Section 3 and is verified by numerical experimentation. In the last section, Brunner's one-point collocation problem is also partially answered. Furthermore, it is proved that the L_1-norm of Brunner's associated T_n matrix is not uniformly bounded when $\alpha = 0$.

2 Uniform upper bound

Interesting results have already been obtained for matrices of the type defined by (1.6): the main result is given below.

Theorem [1, 13] (See, also the more recent papers [3], [14]) *An upper bound on* $\|T_n^{-1}\|_\infty$ *is given by*

$$\|T_n^{-1}\|_\infty \le \begin{cases} \frac{2}{b}[1-(1-\frac{b}{b_0})^{[\frac{n}{2}]+1}], & \text{if} \quad b>0 \\ \frac{2}{b_0}([\frac{n}{2}]+1), & \text{if} \quad b=0. \end{cases} \tag{2.1}$$

In particular if $b > 0$,

$$\|T_n^{-1}\|_\infty \le \frac{2}{b}$$

independently of n *and* b_0.

Note from (2.1) that the upper bound is dependent on n when $b = 0$. Conjecture A and Brunner's problem both involve the case of $b = 0$. However, numerical tests clearly show that $\|T_n^{-1}\|_\infty$ is bounded independently of n. Thus, this paper will deal with obtaining a uniform bound for T_n^{-1} in the 1-norm (or ∞-norm) when $b = 0$ subject to specified constraints on the elements of T_n.

Lemma 2.1 *Let* b_i, $i = 0, 1, 2, ...,$ *be a positive sequence such that*

$$\frac{b_{i+1}}{b_i}, \qquad i = 0, 1, 2, \ldots, \tag{2.2}$$

is non-decreasing, and let a_i, $i = 0, 1, 2, \ldots,$ *be a non-negative sequence such that*

$$a_{i+1} \ge \frac{b_{i+1}}{b_i} a_i\,, \quad i = 0, 1, 2, \ldots\ . \tag{2.3}$$

Then all the coefficients of the Taylor expansion of

$$\frac{\sum_{i=0}^{\infty} a_i x^i}{\sum_{i=0}^{\infty} b_i x^i} \tag{2.4}$$

are non-negative. Furthermore, these coefficients of the Taylor expansion are all positive if (2.3) holds as a strict inequality for all $i = 0, 1, 2, ...$

Proof.

We first define the sequence $\{a_i^{(n)}\}$ where

$$a_i^{(n)} = a_{i+1}^{(n-1)} - \frac{b_{i+1}}{b_0}\, a_0^{(n-1)}, \quad a_i^{(0)} = a_i. \tag{2.5}$$

Now consider the following identity:

$$\frac{\displaystyle\sum_{i=0}^{\infty} a_i x^i}{\displaystyle\sum_{i=0}^{\infty} b_i x^i} = \frac{a_0}{b_0} + \frac{\displaystyle\sum_{i=0}^{\infty} \left(a_i - \frac{a_0}{b_0}\, b_i\right) x^i}{\displaystyle\sum_{i=0}^{\infty} b_i x^i}\,. \tag{2.6}$$

Simple manipulation leads to

$$\frac{\sum_{i=0}^{\infty} a_i x^i}{\sum_{i=0}^{\infty} b_i x^i} = \frac{a_0}{b_0} + \sum_{i=1}^{\infty} \left(a_i - \frac{a_0}{b_0}\, b_i \right) x^i \Big/ \sum_{i=0}^{\infty} b_i x^i$$

$$= \frac{a_0}{b_0} + x \sum_{i=0}^{\infty} \left(a_{i+1} - \frac{a_0}{b_0}\, b_{i+1} \right) x^i \Big/ \sum_{i=0}^{\infty} b_i x^i$$

$$= \frac{a_0}{b_0} + x \sum_{i=0}^{\infty} a_i^{(1)} x^i \Big/ \sum_{i=0}^{\infty} b_i x^i$$

using (2.5).

In a similar manner, we may consider the identity:

$$\frac{\sum_{i=0}^{\infty} a_i^{(1)} x^i}{\sum_{i=0}^{\infty} b_i x^i} = \frac{a_0^{(1)}}{b_0} + \frac{\sum_{i=0}^{\infty} \left(a_i^{(1)} - \frac{a_0^{(1)}}{b_0}\, b_i \right) x^i}{\sum_{i=0}^{\infty} b_i x^i} .$$

The same argument as above yields

$$\frac{\sum_{i=0}^{\infty} a_i^{(1)} x^i}{\sum_{i=0}^{\infty} b_i x^i} = \frac{a_0^{(1)}}{b_0} + x \sum_{i=0}^{\infty} a_i^{(2)} x^i \Big/ \sum_{i=0}^{\infty} b_i x^i$$

where use has been made of $a_i^{(2)} = a_{i+1}^{(1)} - \frac{b_{i+1}}{b_0}\, a_0^{(1)}$.

Thus, (2.6) becomes

$$\frac{\sum_{i=0}^{\infty} a_i x^i}{\sum_{i=0}^{\infty} b_i x^i} = \frac{a_0}{b_0} + \frac{a_0^{(1)}}{b_0} x + x^2 \left(\sum_{i=0}^{\infty} a_i^{(2)} x^i \Big/ \sum_{i=0}^{\infty} b_i x^i \right) .$$

Continuing this argument N times leads to

$$\frac{\sum_{i=0}^{\infty} a_i x^i}{\sum_{i=0}^{\infty} b_i x^i} = \sum_{n=0}^{N} \frac{a_0^{(n)}}{b_0}\, x^n + x^{N+1} \left(\sum_{i=0}^{\infty} a_i^{(N+1)} x^i \Big/ \sum_{i=0}^{\infty} b_i x^i \right)$$

and, finally, by letting $N \to \infty$,

$$\frac{\sum_{i=0}^{\infty} a_i x^i}{\sum_{i=0}^{\infty} b_i x^i} = \sum_{n=0}^{\infty} \frac{a_0^{(n)} x^n}{b_0} . \tag{2.7}$$

Clearly, all that has to be done is demonstrate that $a_0^{(n)}$ is non-negative (since $b_0 > 0$). However, before we can do this, it is necessary to show that the following inequality holds for all $i = 0, 1, \ldots$:

$$a_{i+1}^{(n)} \geq \frac{b_{i+1}}{b_i}\, a_i^{(n)}. \tag{2.8}$$

This will be proved by induction on n.

Suppose that (2.8) holds for all $i = 0, 1, \ldots,$ and some n.

We have, by making use of (2.5),

$$\begin{aligned} a_{i+1}^{(n+1)} - \frac{b_{i+1}}{b_i}\, a_i^{(n+1)} &= a_{i+2}^{(n)} - \frac{b_{i+2}}{b_0}\, a_0^{(n)} - \frac{b_{i+1}}{b_i}\left(a_{i+1}^{(n)} - \frac{b_{i+1}}{b_0}\, a_0^{(n)}\right) \\ &\geq \frac{b_{i+2}}{b_{i+1}}\, a_{i+1}^{(n)} - \frac{b_{i+2}}{b_0}\, a_0^{(n)} - \frac{b_{i+1}}{b_i}\left(a_{i+1}^{(n)} - \frac{b_{i+1}}{b_0}\, a_0^{(n)}\right) \end{aligned} \tag{2.9}$$

by using (2.8) to replace the first term. The inequality (2.9) may be expressed in the form:

$$\begin{aligned} a_{i+1}^{(n+1)} - \frac{b_{i+1}}{b_i}\, a_i^{(n+1)} &\geq \frac{b_{i+2}}{b_{i+1}}\left(a_{i+1}^{(n)} - \frac{b_{i+1}}{b_0}\, a_0^{(n)}\right) - \frac{b_{i+1}}{b_i}\left(a_{i+1}^{(n)} - \frac{b_{i+1}}{b_0}\, a_0^{(n)}\right) \\ &= \left(a_{i+1}^{(n)} - \frac{b_{i+1}}{b_0}\, a_0^{(n)}\right)\left(\frac{b_{i+2}}{b_{i+1}} - \frac{b_{i+1}}{b_i}\right) \end{aligned}$$

after factorisation. However,the first bracket is non-negative by the induction hypothesis and the second one is non-negative since, by assumption, the sequence $\{b_{i+1}/b_i\}$ is non-decreasing. Hence

$$a_{i+1}^{(n+1)} \geq \frac{b_{i+1}}{b_i}\, a_i^{(n+1)}.$$

Clearly $a_{i+1}^{(0)} \geq \frac{b_{i+1}}{b_i}\, a_i^{(0)}$ using (2.3). Thus, by induction on n, (2.8) is true.

It now remains to demonstrate that the coefficients of the Taylor expansion of $\sum_{i=0}^{\infty} a_i x^i \Big/ \sum_{i=0}^{\infty} b_i x^i$ are non-negative. From (2.7) we know these are given by $a_0^{(n)}/b_0,\ n = 0, 1, \ldots$. We shall now prove by induction on n that this is so.

First, suppose that $a_i^{(n)} \geq 0$ for all $i = 0, 1, \ldots,$ and some n.

Then

$$\begin{aligned} a_i^{(n+1)} &= a_{i+1}^{(n)} - \frac{b_{i+1}}{b_0}\, a_0^{(n)} \\ &\geq \frac{b_{i+1}}{b_i}\, a_i^{(n)} - \frac{b_{i+1}}{b_0}\, a_0^{(n)} \end{aligned}$$

using (2.8). We may recast the inequality as

$$a_i^{(n+1)} \geq \frac{b_{i+1}}{b_i} \left(a_i^{(n)} - \frac{b_i}{b_0} a_0^{(n)} \right).$$

Eliminating $a_i^{(n)}$ in the same manner results in

$$a_i^{(n+1)} \geq \left(\frac{b_{i+1}}{b_i} \right) \left(\frac{b_i}{b_{i-1}} \right) \left(a_{i-1}^{(n)} - \frac{b_{i-1}}{b_0} a_0^{(n)} \right).$$

Continuing in this fashion we obtain

$$\begin{aligned} a_i^{(n+1)} &\geq \prod_{k=1}^{i} \left(\frac{b_{k+1}}{b_k} \right) \left(a_1^{(n)} - \frac{b_1}{b_0} a_0^{(n)} \right) \\ &= \frac{b_{i+1}}{b_1} \left(a_1^{(n)} - \frac{b_1}{b_0} a_0^{(n)} \right) \geq 0 \end{aligned}$$

since $b_i > 0$ and $a_1^{(n)} \geq \frac{b_1}{b_0} a_0^{(n)}$ by (2.8).

Clearly $a_1^{(0)} - \frac{b_1}{b_0} a_0^{(0)} = a_1 - \frac{b_1}{b_0} a_0 \geq 0$ by (2.3) and so the induction is complete.

In a similar way we can show that these coefficients are strictly positive if (2.3) holds as a strict inequality for all $i = 0, 1, \ldots$.

Corollary 2.1 *Let b_i, $i = 0, 1, 2, \ldots$, be a positive sequence such that $\frac{b_{i+1}}{b_i}$, $i = 0, 1, 2, \ldots.$, is non-decreasing, then all the coefficients of the Taylor expansion of*

$$\frac{1}{\sum_{i=0}^{\infty} b_i x^i} \tag{2.10}$$

are non-positive except the constant term. Furthermore, all these coefficients except the constant term are negative if $\frac{b_{i+1}}{b_i}$, $i = 0, 1, 2, \ldots.$, is strictly increasing for all $i = 0, 1, 2, \ldots$.

Proof. It is easy to see that

$$\frac{1}{\sum_{i=0}^{\infty} b_i x^i} = \frac{1}{b_0} - \frac{x}{b_0} \frac{\sum_{i=0}^{\infty} b_{i+1} x^i}{\sum_{i=0}^{\infty} b_i x^i}. \tag{2.11}$$

This relation and the previous lemma with $a_i = b_{i+1}$ imply that the corollary is true.

Now we come to a more general case. □

Corollary 2.2 *Let b_i, $i = 0, 1, 2, \ldots$, be a positive sequence such that $\frac{b_{i+1}}{b_i}$, $i = 2, 3, \ldots$, is non-decreasing and that relations*

$$\frac{b_2}{b_1} < \frac{b_1}{b_0} \leq \frac{b_3}{b_2} \tag{2.12}$$

and

$$b_4 - 2b_3\frac{b_1}{b_0} + 3b_2\left(\frac{b_1}{b_0}\right)^2 - b_1\left(\frac{b_1}{b_0}\right)^3 - b_0\left(\frac{b_2}{b_0}\right)^2 \geq 0 \tag{2.13}$$

hold. Then all the coefficients of the Taylor expansion of (2.10) are non-positive except the first three terms.

Proof. Again the following identity holds:

$$\frac{1}{\sum_{i=0}^{\infty} b_i x^i} = \frac{1}{b_0}\left[1 - \alpha_1 x - \alpha_2 x^2 - x^3 \frac{\sum_{i=0}^{\infty} a_i x^i}{\sum_{i=0}^{\infty} b_i x^i}\right] \tag{2.14}$$

where

$$\alpha_1 = \frac{b_1}{b_0}, \quad \alpha_2 = \alpha_1\left[\frac{b_2}{b_1} - \frac{b_1}{b_0}\right], \tag{2.15}$$

and

$$a_i = b_{i+3} - \alpha_1 b_{i+2} - \alpha_2 b_{i+1} \quad , \tag{2.16}$$

for all $i = 0, 1, 2, \ldots$. Assumption (2.12) implies that $\alpha_2 < 0$. Consequently, it follows from relation (2.16) that

$$a_i \geq b_{i+3} - \alpha_1 b_{i+2} = b_{i+2}\left[\frac{b_{i+3}}{b_{i+2}} - \frac{b_1}{b_0}\right] \geq b_{i+2}\left[\frac{b_3}{b_2} - \frac{b_1}{b_0}\right] \geq 0 \quad , \tag{2.17}$$

for all $i = 0, 1, 2, \ldots$. Moreover, we have that

$$\begin{aligned} a_{i+1} &= b_{i+4} - \alpha_1 b_{i+3} - \alpha_2 b_{i+2} \\ &= b_{i+3}\frac{b_{i+4}}{b_{i+3}} - \alpha_1 b_{i+3} - \alpha_2 b_{i+2} \\ &\geq \frac{b_{i+3}}{b_{i+2}}[b_{i+3} - \alpha_1 b_{i+2}] - \alpha_2 b_{i+2} \\ &\geq \frac{b_{i+2}}{b_{i+1}}[b_{i+3} - \alpha_1 b_{i+2}] - \alpha_2 b_{i+2} \\ &= \frac{b_{i+2}}{b_{i+1}}[b_{i+3} - \alpha_1 b_{i+2} - \alpha_2 b_{i+1}] \\ &= \frac{b_{i+2}}{b_{i+1}}\, a_i \\ &\geq \frac{b_{i+1}}{b_i}\, a_i \end{aligned} \tag{2.18}$$

for all $i = 0, 1, 2, \ldots$.

We have demonstrated that, in (2.14),

$$a_i \geq 0, \quad i = 0, 1, 2, \ldots \tag{2.19}$$

and

$$a_{i+1} \geq \frac{b_{i+1}}{b_i}\, a_i, \quad i = 0, 1, 2, \ldots \ . \tag{2.20}$$

Thus, we may appeal to lemma 2.1 to show that the coefficients of the Taylor expansion of

$$\frac{\sum a_i x^i}{\sum b_i x^i}$$

in (2.14) are all non-negative. □

We now impose an additional condition on the sequence $\{b_i\}$ and provide an estimate of the sum of the coefficients of the Taylor expansion of (2.10).

Lemma 2.2 *Suppose that the sequence $b_i, i = 0, 1, 2, \ldots$, satisfies the conditions of Corollary 2.2 and suppose that*

$$\frac{b_{i+1}}{b_i} \leq 1, \quad i = 0, 1, 2, \ldots \ . \tag{2.21}$$

Then,

$$\phi(x) = \frac{1}{\sum_{i=0}^{\infty} b_i x^i} = \frac{1}{b_0}\left[1 - \alpha_1 x - \alpha_2 x^2\right] - x^3 \sum_{i=0}^{\infty} \theta_i x^i, \tag{2.22}$$

where α_1 and α_2 are given by (2.15), and $\theta_i \geq 0$ for all $i = 0, 1, 2, \ldots$. Furthermore, we have that

$$\frac{1}{b_0}\left[1 - \alpha_1 - \alpha_2\right] - \beta = \sum_{i=0}^{\infty} \theta_i, \tag{2.23}$$

where

$$\beta = \begin{cases} 0, & \text{if } \sum_{i=0}^{\infty} b_i = \infty; \\ \frac{1}{\sum_{i=0}^{\infty} b_i}, & \text{otherwise.} \end{cases} \tag{2.24}$$

Proof. From Corollary 2.2 we have that the Taylor expansion of $\phi(x)$ is given by (2.22). Due to the assumption (2.21), it follows that $b_i \leq b_0$ for all $i \geq 0$. Thus, for any given $\delta \in (0, 1)$, the sequence $\sum_{i=0}^{\infty} b_i x^i$ is uniformly convergent on the interval $|x| \leq \delta$. Hence, we see that the relation (2.22) holds for all $|x| < 1$. Now (2.23) follows from (2.22) by letting $x \to 1_-$. This completes the proof. □

Now, we apply the above results to provide an estimate of the 1-norm of lower triangular Toeplitz matrices. First, we can write the matrix T_n given in (1.6) in the following form:

$$T_n = b_0 I + \sum_{i=1}^{n} b_i J^i = b_0 I + \sum_{i=1}^{\infty} b_i J^i, \tag{2.25}$$

where J is the Jordan matrix

$$J = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}, \tag{2.26}$$

since $J^k = O$ for $k \geq n+1$.

Thus, we have that

$$T_n^{-1} = \phi(J) \tag{2.27}$$

where $\phi(x)$ is defined in (2.22).

The following result follows directly from Corollary 2.1 and Lemma 2.2.

Theorem 2.1 *Suppose that T_n is defined by (1.6) with $b_i > 0$ for $i \geq 0$. If b_{i+1}/b_i is non-decreasing and bounded above by 1 for $i = 0, 1, 2, \ldots$, we have the bound:*

$$\|T_n^{-1}\|_1 \leq \frac{2}{b_0} - \beta, \tag{2.28}$$

where β is defined in (2.24). If b_i, $i = 0, 1, 2, \ldots$, is a positive sequence and if b_{i+1}/b_i $(i \geq 2)$ is non-decreasing and bounded above by 1 and if conditions (2.12) and (2.13) hold then

$$\|T_n^{-1}\|_1 \leq \frac{2}{b_0}\left[1 - \frac{b_2}{b_0} + \left(\frac{b_1}{b_0}\right)^2\right] - \beta. \tag{2.29}$$

Proof. If b_{i+1}/b_i is non-decreasing and bounded above by 1 for $i = 0, 1, 2, \ldots$, we see that the Taylor expansion of $\phi(x)$ given by (2.22) holds for all $|x| < 1$. Corollary 2.2 implies that $\theta_i \geq 0$ for all $i \geq 0$. Thus by letting $x \to 1_-$ we have that

$$\beta = \frac{1}{b_0}\left[1 - \alpha_1 - \alpha_2\right] - \sum_{i=0}^{\infty} \theta_i. \tag{2.30}$$

Therefore

$$\begin{aligned} \|T_n^{-1}\|_1 &= \frac{1}{b_0}\left[1 + |\alpha_1| + |\alpha_2|\right] + \sum_{i=0}^{\infty} |\theta_i| \\ &= \frac{1}{b_0}\left[1 + \alpha_1 + \alpha_2\right] + \sum_{i=0}^{\infty} \theta_i = \frac{2}{b_0} - \beta. \end{aligned} \tag{2.31}$$

This proves (2.28).

Now we assume that $b_{i+1}/b_i (i \geq 2)$ is non-decreasing (and bounded above by 1) and additionally conditions (2.12) and (2.13) are satisfied. In this case, we also have relation (2.29). The assumptions on the sequence $\{b_i, i = 0, 1, 2, \ldots\}$ and Corollary 2.2 imply that $\alpha_1 \geq 0$, $\alpha_2 \leq 0$ and $\theta_i \geq 0$ for all $i \geq 0$. Therefore, it follows that

$$\begin{aligned} \|T_n^{-1}\|_1 &= \frac{1}{b_0}\left[1 + |\alpha_1| + |\alpha_2|\right] + \sum_{i=0}^{\infty} |\theta_i| \\ &= \frac{1}{b_0}\left[1 + \alpha_1 + |\alpha_2|\right] + \sum_{i=0}^{\infty} \theta_i = \frac{2}{b_0}\left(1 + |\alpha_2|\right) - \beta. \end{aligned} \tag{2.32}$$

Since it is clear that $\alpha_2 < 0$, the above equality and (2.15) yield (2.29).

$\square$

Note that the bound given in Theorem 2.1 depends on b_0, b_1, b_2, b_3 and b_4 but not on n: consequently it is a uniform bound with respect to n.

3 Conjecture A

Since Conjecture A is relatively straightforward to demonstrate, we shall deal with it first.

Using the results given in the previous section, we can easily prove that Conjecture A is true.

Theorem 3.1 *Let $T_n \in \mathbb{R}^{(n+1)\times(n+1)}$ be the lower triangular Toeplitz matrix whose first column is given by (1.5). Then inequality*

$$\|T_n^{-1}\|_1 \le 2\sqrt{2}(5-\sqrt{3}-\sqrt{5}) = 2.91860082 < 3 \tag{3.1}$$

holds for all $n = 1, 2, \ldots$.

Proof. If $n = 1$, we have that

$$\begin{aligned} \|T_1^{-1}\|_1 &= \left\| \sqrt{2} \begin{pmatrix} 1 & 0 \\ -(\sqrt{3}-1) & 1 \end{pmatrix} \right\|_1 \\ &= \sqrt{2}[(\sqrt{3}-1)+1] < 2\sqrt{2}(5-\sqrt{3}-\sqrt{5}), \end{aligned} \tag{3.2}$$

which shows that (3.1) holds for $n = 1$. Similarly, direct calculations then provide

$$\begin{aligned} \|T_2^{-1}\|_1 &= \left\| \sqrt{2} \begin{pmatrix} 1 & 0 & 0 \\ -(\sqrt{3}-1) & 1 & 0 \\ 4-\sqrt{3}-\sqrt{5} & -(\sqrt{3}-1) & 1 \end{pmatrix} \right\|_1 \\ &= \sqrt{2}[1+(\sqrt{3}-1)+(4-\sqrt{3}-\sqrt{5})] \\ &= \sqrt{2}(4-\sqrt{5}) < 2\sqrt{2}(5-\sqrt{3}-\sqrt{5}), \end{aligned} \tag{3.3}$$

which demonstrates that (3.1) holds for $n = 2$ as well.

Now we consider the case when $n \ge 3$. Define the sequence b_i, $i = 0, 1, 2, \ldots$, as follows:

$$b_0 = \frac{1}{\sqrt{2}}, \quad b_i = \frac{\sqrt{2i+1}-\sqrt{2i-1}}{\sqrt{2}}, \quad i = 1, 2, \ldots . \tag{3.4}$$

We can easily check that all the conditions for b_i given in Lemma 2.2 are satisfied. It follows from (2.15) and (3.4) that

$$\alpha_2 = \frac{b_2}{b_0} - \left(\frac{b_1}{b_0}\right)^2 = \sqrt{5}-\sqrt{3}-(\sqrt{3}-1)^2 = \sqrt{5}+\sqrt{3}-4. \tag{3.5}$$

Moreover, (3.4) also gives $\beta = 0$. Thus, from Theorem 2.1, we have that

$$\|T_n^{-1}\|_1 \le 2\sqrt{2}(1 + |\sqrt{5} + \sqrt{3} - 4|) = 2\sqrt{2}(5 - \sqrt{3} - \sqrt{5}). \tag{3.6}$$

This shows that the theorem is true. □

4 Brunner's one-point collocation problem

Consider first the lower triangular Toeplitz matrix T_n whose first column is

$$(1,\ 2^{1-\alpha} - 1,\ 3^{1-\alpha} - 2^{1-\alpha},\ \ldots, (n+1)^{1-\alpha} - n^{1-\alpha})^T, \tag{4.1}$$

where $\alpha \in (0,\ 1)$. This corresponds to the case $c = 1$, which is the case of the implicit Euler product-integration method applied to the Abel's equation. Define the corresponding sequence

$$b_0 = 1, \quad b_i = (i+1)^{1-\alpha} - i^{1-\alpha},\ i = 1, 2, \ldots\ . \tag{4.2}$$

It follows from (2.28) of Theorem 2.1 that $\|T_n^{-1}\|_1 \le 2$. Matlab 7.0 with Laptop Sony Vaio VGN TZ370 was used to test the Abel's matrices in this case. The numerical results are displayed in Table 1.

n	$\alpha = 0$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.7$
50	2	1.9512	1.9091	1.7331
100	2	1.9682	1.9360	1.7838
1000	2	1.9920	1.9799	1.8912
2000	2	1.9947	1.9858	1.9122
2500	2	1.9954	1.9873	1.9179
3500	2	1.9962	1.9892	1.9258
5000	2	1.9970	1.9910	1.9333

Table 1: 1- Norm of the inverse of the Abel matrices

We now turn our attention to Brunner's problem where the Toeplitz matrix T_n is given in (1.4). We define $T_n = c^{1-\alpha}\tilde{T}_n$ so that the elements of $\tilde{T}_n$ become

$$b_0 = 1, \quad b_i = \left(\frac{c+i}{c}\right)^{1-\alpha} - \left(\frac{c+i-1}{c}\right)^{1-\alpha}, \quad i = 1, 2, \ldots\ . \tag{4.3}$$

First, we consider $\alpha \in (0, 1)$. It is clear that $b_i > 0$ for all $i \ge 0$. Moreover,

$$\frac{b_{i+1}}{b_i} = \frac{(1 + \frac{1}{c+i})^{1-\alpha} - 1}{1 - (1 - \frac{1}{c+i})^{1-\alpha}}, \quad i \ge 1. \tag{4.4}$$

Define the function

$$\psi(x) = \frac{(1+x)^{1-\alpha} - 1}{1-(1-x)^{1-\alpha}}\,, \tag{4.5}$$

A simple calculation yields

$$\psi'(x) = (1-\alpha)\,\frac{(1-x)^{\alpha} + (1+x)^{\alpha} - 2}{[1-(1-x)^{1-\alpha}]^2(1-x^2)^{\alpha}}\,. \tag{4.6}$$

It is easy to verify that $\psi'(x) < 0$ for all $\alpha \in (0,1)$ and $x \in (0,1)$. Therefore, if $\alpha \in (0,1)$ and $c > 0$, we have that b_{i+1}/b_i for $i \geq 2$ is monotonically increasing and converges to 1. In order to apply Theorem 2.1, we have to consider two separate cases.

Case A) $b_2/b_1 \geq b_1/b_0$. This inequality reduces to

$$\left(\frac{2+c}{c}\right)^{1-\alpha} - \left(\frac{1+c}{c}\right)^{1-\alpha} - \left[\left(\frac{1+c}{c}\right)^{1-\alpha} - 1\right]^2 \geq 0. \tag{4.7}$$

Since $1 > \dfrac{c}{1+c} > 0$, we multiply (4.7) by $\left(\dfrac{c}{1+c}\right)^{2(1-\alpha)}$ to obtain the equivalent but stronger inequality

$$\psi_1(\alpha, c) = \left(\frac{c}{1+c}\right)^{1-\alpha}\left[\left(1+\frac{1}{1+c}\right)^{1-\alpha} + 1\right] - 1 - \left(\frac{c}{1+c}\right)^{2(1-\alpha)} \geq 0. \tag{4.8}$$

It follows from (2.28) of Theorem 2.1 that $\|\tilde{T}_n^{-1}\|_1 \leq 2$ so that

$$\|T_n^{-1}\|_1 \leq \frac{2}{c^{1-\alpha}}\,. \tag{4.9}$$

Case B) In this case (2.12) and (2.13) hold. Thus, from Theorem 2.1, we have the following bound

$$\|T_n^{-1}\|_1 \leq 2(1+b_1^2-b_2)c^{\alpha-1} = \frac{2}{c^{1-\alpha}}\left[2 + \left(\frac{1+c}{c}\right)^{2(1-\alpha)} - \left(\frac{1+c}{c}\right)^{1-\alpha} - \left(\frac{2+c}{c}\right)^{1-\alpha}\right]. \tag{4.10}$$

The inequality (2.12) implies that (4.8) does not hold and $b_3/b_2 \geq b_1/b_0$. The latter is the following inequality

$$\frac{(3+c)^{1-\alpha} - (2+c)^{1-\alpha}}{(2+c)^{1-\alpha} - (1+c)^{1-\alpha}} \geq \frac{(1+c)^{1-\alpha} - c^{1-\alpha}}{c^{1-\alpha}}\,. \tag{4.11}$$

The above inequality can be rewritten equivalently as

$$\psi_2(\alpha, c) = \left[\left(\frac{3+c}{1+c}\right)^{1-\alpha} - 1\right]\left(\frac{c}{1+c}\right)^{1-\alpha} - \left(\frac{2+c}{1+c}\right)^{1-\alpha} + 1 \geq 0. \tag{4.12}$$

by rearranging and multiplying the resultant expression by $(1+c)^{2(\alpha-1)}$.

On the other hand condition (2.13) may be expressed as

$$\begin{aligned}\psi_3(\alpha,c) &= [(4+c)^{1-\alpha}-(3+c)^{1-\alpha}]c^{3(1-\alpha)} \\ &\quad -2c^{2(1-\alpha)}((3+c)^{1-\alpha}-(2+c)^{1-\alpha})((1+c)^{1-\alpha}-c^{1-\alpha}) \\ &\quad +3c^{1-\alpha}((2+c)^{1-\alpha}-(1+c)^{1-\alpha})((1+c)^{1-\alpha}-c^{1-\alpha})^2 \\ &\quad -((1+c)^{1-\alpha}-c^{1-\alpha})^4-c^{2(1-\alpha)}((2+c)^{1-\alpha}-(1+c)^{1-\alpha})^2 \\ &\geq 0. \end{aligned} \tag{4.13}$$

where in this case (2.13) has been multiplied by $c^{4(1-\alpha)}$.

Therefore, from Theorem 2.1, we have the following result.

Theorem 4.1 *Let T_n be defined in (1.4). Then we have that*

$$\|T_n^{-1}\|_1 \leq \frac{2}{c^{1-\alpha}} \tag{4.14}$$

if (4.8) holds. If (4.8) is not satisfied (i.e. $\psi_2(\alpha,c)<0$), we have that

$$\begin{aligned}\|T_n^{-1}\|_1 &\leq \frac{2}{c^{1-\alpha}}\left[2+\left(\frac{1+c}{c}\right)^{2(1-\alpha)}-\left(\frac{1+c}{c}\right)^{1-\alpha}-\left(\frac{2+c}{c}\right)^{1-\alpha}\right] \\ &= \Delta(\alpha,c), \end{aligned} \tag{4.15}$$

provided that α and c satisfy both (4.12) and (4.13).

Proof. If (4.8) holds, it follows from (2.28) in Theorem 2.1 that (4.14) is satisfied.

If, however, (4.8) does not hold and if the inequalities (4.12) and (4.13) are not valid, it follows from Theorem 2.1 that (2.29) is satisfied. The bound (2.29) and the form of the elements (4.3) then imply (4.15). □

Thus, by plotting the curves $\psi_i(\alpha,c)=0$, $i=1,2,3$, we can easily see the region of $\{(\alpha,c)\}$ for which $\|T_n^{-1}\|_1$ is uniformly bounded. Indeed, $\|T_n^{-1}\|_1$ is bounded uniformly for all n, if (α,c) is contained within the following two regions:

$$S_1 = \{(\alpha,c)|\psi_1(\alpha,c)\geq 0,\ \alpha\in(0,1),\ c\in(0,1].\} \tag{4.16}$$

$$\begin{aligned} S_2 = \{(\alpha,c)|&\psi_1(\alpha,c)<0,\ \psi_2(\alpha,c)\geq 0, \\ &\psi_3(\alpha,c)\geq 0,\ \alpha\in(0,1),\ c\in(0,1].\}\end{aligned} \tag{4.17}$$

For any point in S_1 and S_2, we can show that $\|T_n^{-1}\|_1$ is bounded uniformly with respect to n; therefore the solutions of the difference equation (1.2) remain uniformly bounded as $n\to\infty,\ h\to 0$ with $nh=T$.

It is easy to show that $\psi_1(\alpha,c)$ is an increasing function of c for any fixed $\alpha\in(0,1)$ with $\psi_1(\alpha,0)\leq 0$ and $\psi_1(\alpha,1)>0$. Thus, we can let $\bar{c}_1(\alpha)$ be the root of $\psi_1(\alpha,c)=0$ in the interval (0, 1). Consequently (4.14) holds for all $c \geq \bar{c}_1(\alpha)$. Some of the values of $\bar{c}_1(\alpha)$ are given in Table 2.

α	0.2	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99	0.999
$\bar{c}_1(\alpha)$	0.8225	0.6334	0.5332	0.4285	0.3182	0.2017	8.069×10^{-2}	2.429×10^{-2}	2.398×10^{-4}	4.768×10^{-7}

Table 2

From Table 2, for $\alpha = 0.2$ and $\bar{c}_1(0.2) = 0.8225$, we have that

$$\|T_n^{-1}\|_1 \leq 2/(0.8225)^{0.8} \approx 2.3384 \tag{4.18}$$

Another example is $\alpha = 0.9$ and $\bar{c}_1(0.9) = 8.069 \times 10^{-2}$ for which we have $\|T_n^{-1}\|_1 \leq 2/(0.08069)^{0.1} \simeq 2.0434$.

Now we consider the case when $c < \bar{c}_1(\alpha)$. For example, for the case when $c = 0.2$ and $\alpha = 0.75$ we can show that $\psi_2(0.75, 0.2) > 0$ and $\psi_3(0.75, 0.2) > 0$. Therefore we have that

$$\|T_n^{-1}\|_1 \leq \Delta(0.75, 0.2) \approx 3.1798. \tag{4.19}$$

In fact, we can define $\bar{c}_2(\alpha) \in (0, 1)$ and $\bar{c}_3(\alpha) \in (0, 1)$ by

$$\psi_2(\alpha, \bar{c}_2(\alpha)) = 0, \qquad \psi_3(\alpha, \bar{c}_3(\alpha)) = 0. \tag{4.20}$$

Now let

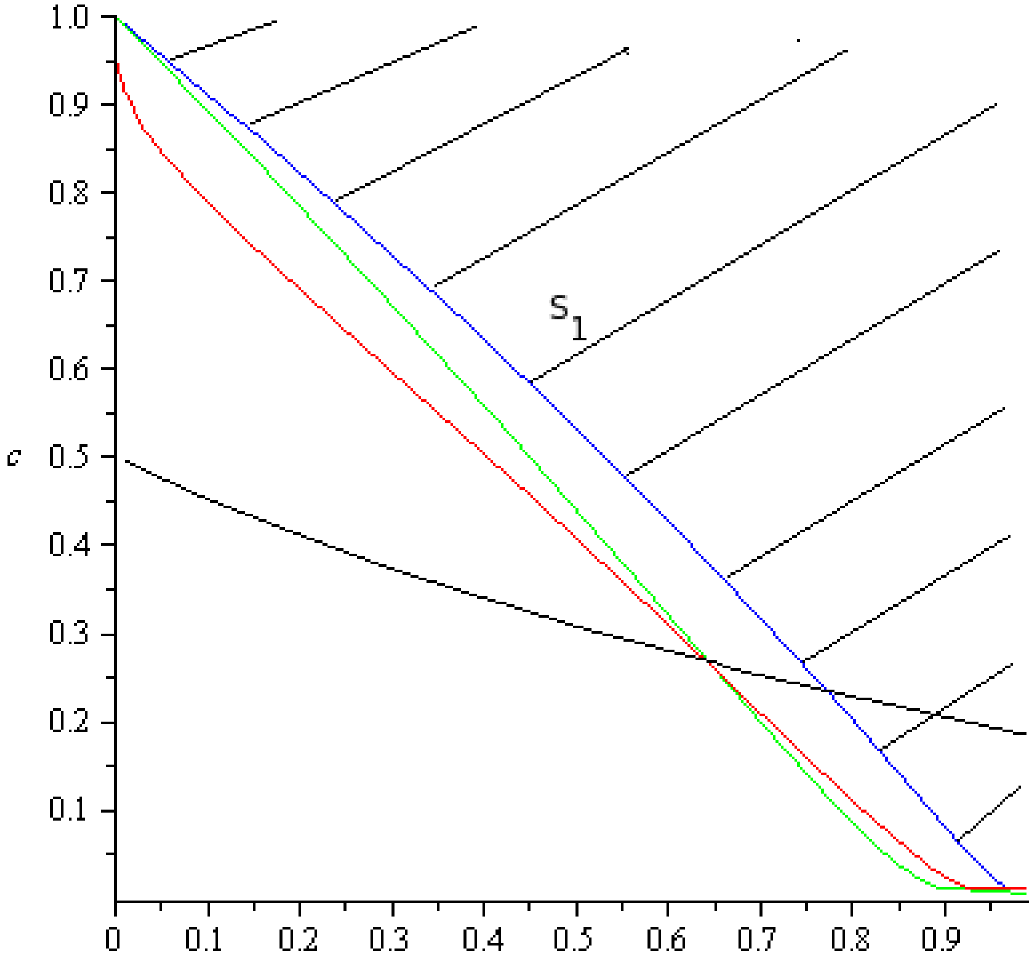
$$\bar{c}^*(\alpha) = \min[\bar{c}_1(\alpha), \max(\bar{c}_2(\alpha),\ \bar{c}_3(\alpha))]. \tag{4.21}$$

Theorem 4.1 implies that $\|T_n^{-1}\|_1$ is uniformly bounded for $c \geq \bar{c}^*(\alpha)$. Some of the values of $\bar{c}^*(\alpha)$ are given in Table 3: (Brunner's sufficient values $c^*(\alpha) = \frac{1}{2}(\alpha(1-\alpha)\gamma_\alpha)^{1/(1-\alpha)}$ for the boundedness of the solutions (1.2) are also given).

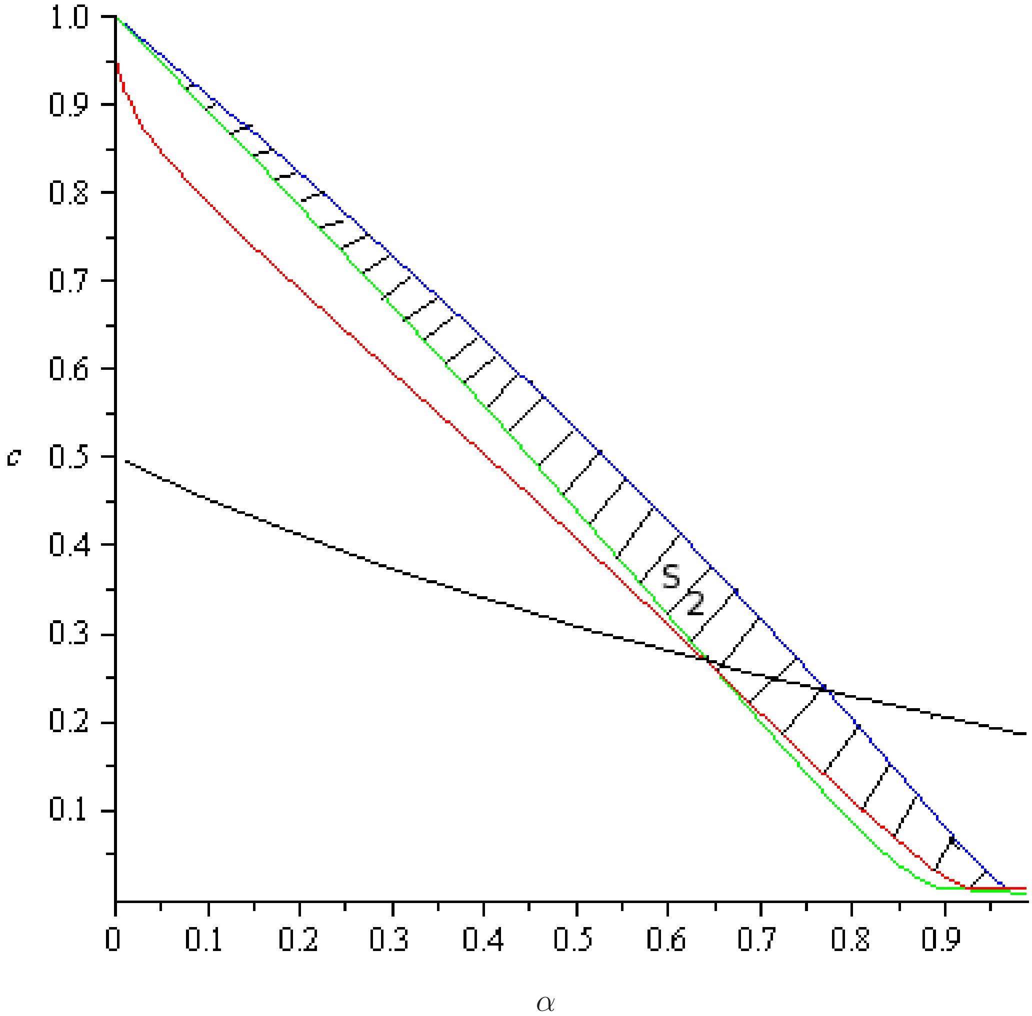
α	0.5	0.6	0.7	0.8	0.9	0.95	0.99	0.999
$\bar{c}^*(\alpha)$	0.5332	0.3196	0.2102	0.1107	0.2473×10^{-2}	2.2242×10^{-2}	4.685×10^{-7}	4.477×10^{-16}
$c^*(\alpha)$	0.3084	0.2798	0.2533	0.2287	0.2056	0.1946	0.1861	0.1842

Table 3

Below are displayed two graphs. Graph 1 corresponds to case A and the second (Graph 2) to case B: the regions S_1 and S_2 are shaded. The black curve corresponds to Brunner's condition $c_1^*(\alpha)$, while the blue, green and red curves correspond to the curves $\bar{c}_1^*(\alpha)$, $\bar{c}_2^*(\alpha)$ and $\bar{c}_3^*(\alpha)$, respectively.



Graph 1: Case A displaying the region S_1 in the (c, α)-plane in which $\|T_n^{-1}\|_1$ is uniformly bounded. (Brunner $c_1^*(\alpha)$-black, $\bar{c}_1^*(\alpha)$-blue, $c_2^*(\alpha)$-green, $\bar{c}_3^*$-red).



Graph 2: Case B displaying the region S_2 in the (c, α)-plane in which $\|T_n^{-1}\|_1$ is uniformly bounded. (Brunner $c_1^*(\alpha)$-black, $\bar{c}_1^*(\alpha)$-blue, $\bar{c}_2^*(\alpha)$-green, $\bar{c}_3^*(\alpha)$-red).

We note that in the case $\alpha = 0$, the matrix does not satisfy the conditions of this paper. However, if $\alpha = 0$, we have that

$$T_n = cI + J + J^2 + \cdots + J^n. \tag{4.22}$$

Direct calculations show that

$$T_n^{-1} = \frac{1}{c}\left[I - \frac{1}{c}\sum_{i=1}^{n}\left(1 - \frac{1}{c}\right)^{i-1} J^i\right]. \tag{4.23}$$

Thus,

$$\begin{aligned} \|T_n^{-1}\|_1 &= \frac{1}{c}\left[1 + \frac{1}{c}\sum_{i=1}^{n}\left|\left(1 - \frac{1}{c}\right)^{i-1}\right|\right] \\ &= \left\{\begin{array}{ll} 2(2n+1), & \text{if } c = 1/2; \\ \dfrac{\left(\frac{1}{c} - 1\right)^n - 2c}{c(1-2c)}, & \text{otherwise} \end{array}\right\} \end{aligned} \tag{4.24}$$

We note that $\|T_n^{-1}\|$ is uniformly bounded with respect to n if $c > \dfrac{1}{2}$; it is not uniformly bounded when $c = \dfrac{1}{2}$.

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Replies to the referees

Referee 1

We are grateful to this referee for drawing our attention to these papers. We have inserted a short paragraph near the end of Section 1 introducing two of them, while the other two are referenced in association with the first theorem of Section 2.

Referee 2

1. The referee is correct; the english and style are poor in places. All his points have been well taken and appropriately amended.
2. "Lemma 2.1 is unintelligible". We have rewritten this theorem to make it much clearer.
3. We would like to retain the colours in graph 1 and 2.

   Most readers, nowadays, download articles directly to colour printers rather than reading articles from library shelves so the fact that it will appear as black and white in printed form is almost irrelevant. However, if the Editor/Referee feels strongly about this point then we would be happy to produce an appropriate black and white version.
4. We have removed the concluding remarks.

Referee 3

1. The reference to Baker has been removed and reference to Weiss and Anderssen and Eggermont appropriately included. The reference to Holyhead's thesis has, however, been retained as it contains a great deal of relevant unpublished (elsewhere) material.
2. I am afraid we think the expression (2.5) is obvious, but we have added a short phrase of explanation. Extending the definition of $\phi(x)$ to $\phi(J)$ (J a real matrix) is standard in linear algebra (cf. Cayley-Hamilton theorem).
3. Yes, you are correct: it should be the implicit Euler method.
4. This is a good point and a rather subtle one. The second part of the last sentence is true; in fact $\|T_n^{-1}\|$ grows like n or $1/h$ where h is the mesh spacing. However, this does NOT imply that the (implicit) midpoint method for non-singular first kind Volterra integral equations yield unbounded solutions as $h \to 0$. This can be deduced from the general analysis of the (possibly little read) paper of Holyhead and McKee (1976)

   P.A.W. Holyhead and S. McKee, Stability and convergence of multistep methods for linear Volterra integral equations of the first kind, SIAM J. Numer. Anal. **13**, No.2 (1976) 269–711.