Facets of the minimum-adjacency vertex coloring polytope

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1 Introduction

In this work we are interested in a combinatorial optimization problem arising from frequency assignment problems in wireless communication networks, that was motivated by the types of interference generated in GSM mobile phone networks [4].

A wireless network employs some portion of the electromagnetic spectrum to establish communications between the transmitter/receiver network antennas, called TRXs. A certain part of the electromagnetic spectrum is licensed to the company operating the network and is divided into discrete channels. Each TRX must operate through one channel, although whenever two TRXs overlap their coverage areas *co-channel interference* occurs if both are using the same channel, and communications cannot be established within the common area. Moreover, if these conflicting TRXs are assigned to adjacent channels, then the so-called *adjacent-channel interference* occurs, generating in this case a minor interference only. In a typical scenario, a good channel assignment *must* avoid co-channel interference and *should* avoid adjacent-channel interference.
Several other constraints arise in practical settings as, e.g., blocked channels and separation constraints (see, e.g., [4, 5, 6, 10]) but in this work we focus on the basic model as stated in the previous paragraph. We are interested in the polyhedral structure generated by such a combinatorial optimization problem, which includes a graph coloring structure with additional considerations on adjacent channels/colors. Based on these observations, we introduce in this work the *minimum-adjacency vertex coloring problem* and present an initial polyhedral study for it.

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This paper is organized as follows. In Section 2 we formally state the minimum-adjacency vertex coloring problem and provide an integer programming model for this problem. Section 3 presents two families of facet-inducing valid inequalities.

2 Problem formulation and integer programming model

We first introduce the *interference graph* $G = (V, E)$ associated with an instance, in such a way that V represents the set of TRXs in the network and, whenever the coverage areas of two TRXs overlap, an edge in E joins the corresponding vertices. Throughout this work we shall use the notation $n = |V|$ and $m = |E|$. Let $C = \{1, \ldots, t\}$ be a set of consecutive colors representing the t available channels. A *C-coloring* of G is a function $c : V \to C$ such that $c(i) \neq c(j)$ for every $ij \in E$. Clearly, a C-coloring of G corresponds to a feasible frequency assignment avoiding co-channel interference. Finally, for every $vw \in E$, we define $\psi(vw) \in [0, 1]$ to be the level of interference generated when the vertices v and w are assigned adjacent colors/channels.

Minimum-adjacency vertex coloring problem. *Given an inteference graph $G = (V, E)$, a set of consecutive colors $C = \{1, \ldots, t\}$, and an interference function $\psi : E \to [0, 1]$, find a C-coloring of G minimizing the total adjacent-channel interference, i.e.,*

$$\min_{y \in \mathcal{C}(G,C)} \sum \{\psi(vw) : vw \in E \text{ and } |y(v) - y(w)| = 1\},$$

where $\mathcal{C}(G, C)$ represents the set of all C-colorings of G.

In the minimum-adjacency vertex coloring problem we ask for a frequency assignment with no co-channel interference (i.e., a C-coloring of G) minimizing the less critical adjacent-channel interference. This provides a good compromise between forbidding both kinds of interference and allowing them with a penalty term in the objective function. This problem is clearly $\mathcal{NP}$-hard since it generalizes the classical vertex coloring problem, thus motivating the integer programming approach started in this work.

Throughout this work we shall consider $\psi(vw) = 1$ for every $vw \in E$, so the minimum-adjacency vertex coloring problem reduces to finding a C-coloring of G which minimizes the number of edges receiving adjacent colors. Note that the adjacencies within the color set C are not circular, i.e., the colors 1 and t are not adjacent if $t \geq 3$.

In order to state an integer programming formulation for the minimum-adjacency vertex coloring problem, for every $v \in V$ and every $c \in C$ we introduce the binary *assignment variable* x_{vc} representing whether the color c is assigned to the vertex v or not. We also introduce, for every $vw \in E$, $v < w$, the binary *adjacency variable* z_{vw} asserting whether the vertices v and w receive adjacent colors or not. With these definitions, a model for the minimum-adjacency vertex coloring problem is given by:

$$\min \sum_{vw \in E} \psi(vw) \quad z_{vw}$$

$$\sum_{c \in C} x_{vc} = 1 \qquad \forall v \in V \tag{1}$$

$$x_{vc} + x_{wc} \leq 1 \qquad \forall vw \in E, v < w, \quad \forall c \in C \tag{2}$$

$$x_{vc_1} + x_{wc_2} \leq 1 + z_{vw} \qquad \forall vw \in E, v < w, \quad \forall c_1, c_2 \in C, |c_1 - c_2| = 1 \tag{3}$$

$$x_{vc} \in \{0, 1\} \qquad \forall v \in V, \quad \forall c \in C \tag{4}$$

$$z_{vw} \in \{0, 1\} \qquad \forall vw \in E, v < w. \tag{5}$$

Constraints (1) ensure that every vertex is assigned exactly one color from C and constraints (2) forbid adjacent vertices to be assigned the same color. Constraints (3) force adjacent vertices to be assigned non-adjacent colors unless the corresponding adjacency variable takes the value 1. Finally, constraints (4) and (5) force the variables to be binary. We call this formulation the *stable-set model*, which is a straightforward adaptation of the formulation presented in [12, 13] for the classical vertex coloring problem.

Note that the constraints allow feasible solutions to have active adjacency variables even when the corresponding vertices are not receiving adjacent colors, however, this is not the case in any optimal solution. We say that $y = (x, z)$ is an *undominated* solution if $z_{vw} = 0$ for every $vw \in E$ such that v are w recieving non-adjacent colors.

3 Polyhedral study

In this section we introduce the polytope associated with the stable-set formulation, we characterize the dimension of this polytope for $|C| > \chi(G)$, and we present some families of facet-inducing valid inequalities.

Definition 1 *Given a graph $G = (V, E)$ and a set of colors C, we define $PS(G, C)$ to be the convex hull of the incidence vectors $(x, z) \in \mathbb{R}^{nt+m}$ of feasible solutions to the stable-set model (1)-(5).*

Proposition 1 *If $|C| > \chi(G)$ and $E \neq \emptyset$, then $dim(PS(G, C)) = n(t-1) + m$, and a minimal equation system is defined by (1).*

Proof. Let $\lambda \in \mathbb{R}^{nt+m}$ and $\lambda_0 \in \mathbb{R}$ such that $\lambda y = \lambda_0$ for every feasible solution $y \in PS(G, C)$. It suffices to show that (λ, λ_0) is a linear combination of the model constraints (1).

Let $vw \in E$, and consider a feasible solution $y = (x, z) \in PS(G, C)$ such that the colors assigned to v and w are nonadjacent and, furthermore, $z_{vw} = 0$. Such a solution can always be constructed since $|C| > \chi(G) \geq 2$ (as $E \neq \emptyset$). Let $y' = (x, z') \in PS(G, C)$ be the feasible solution obtained from y by setting $z'_{vw} = 1$ and leaving the remaining variables unchanged. The solution y' is clearly feasible and only differs from y in the z_{vw}-variable, hence $\lambda_{z_{vw}} = 0$.
Let $v \in V$. Since $|C| > \chi(G)$, there exists a feasible solution $y = (x, z) \in PS(G, C)$ such that at least one color from C is not used by any vertex. Let $c \in C$ be the color assigned to the vertex v by y, and let $c' \in C$ be a color not used in y. Define $y' = (x', z') \in PS(G, C)$ to be the feasible solution obtained from y by setting $x'_{vc} = 0$, $x'_{vc'} = 1$, and leaving the remaining x-variables unchanged. The solution y' is feasible as the color c' is not used in y and, furthermore, x only differs from x' in the x'_{vc}- and $x'_{vc'}$-variables, and possibly some z-variables. Since $\lambda_z = 0$, we conclude $\lambda_{x_{vc}} = \lambda_{x_{vc'}}$. By arbitrarily renaming the colors, we conclude that $\lambda_{x_{vc}} = \lambda_{x_{vc'}}$ for any $c, c' \in C$.

By combining these observations, we conclude that λ is a linear combination of the coefficient vectors of the model constraints (1). Since the coefficient vectors of these constraints are linearly independent, we conclude that (1) is a minimal equation system for $PS(G, C)$, so $\dim(PS(G, C)) = n(t-1) + m$. □

3.1 Consecutive colors inner clique inequalities

In this and the following section we introduce two classes of facet-inducing inequalities for $PS(G, C)$ arising from a clique $K \subseteq V$ and a distinguished vertex $k \in K$. In these constructions, the clique structure is crucial both for validity and facetness, suggesting the importance of the cliques of the interference graph for the structure of $PS(G, C)$.

Let $K \subseteq V$ be a clique, $k \in K$ be some vertex of K, and $\{c_1, c_2, c_3\} \subseteq C$ be a set of three consecutive colors. If some vertex $v \in K \setminus \{k\}$ receives the color c_2 then an adjacency with k is generated if the latter receives either c_1 or c_3. This idea can be generalizaed for a set $Q = \{c_1, \ldots, c_q\} \subseteq C$ of an odd number of consecutive colors; if vertices from $K \setminus \{k\}$ are using every color $c_i \in Q$ with an odd index i, then any color $c_j \in Q$ with an even index j generates adjacencies when assigned to k. The following lifted family of valid inequalities arises from this observation.

Definition 2 (Consecutive colors inner clique inequality) *Let $K \subseteq V$ be a clique of G, and fix a vertex $k \in K$. Let $Q = \{c_1, ..., c_q\} \subseteq C$ with an odd q, be a set of consecutive colors such that $c_{i+1} = c_i + 1$ for $i = 1, ..., q-1$. Let $I = \{c_1, c_3, ..., c_q\}$ and $P = \{c_2, c_4, ..., c_{q-1}\}$. We define the* consecutive colors inner clique inequality *to be*

$$(x_{kc_1} + x_{kc_q} + \sum_{c_i \in I \setminus \{c_1, c_q\}} 2x_{kc_i} + \sum_{c_j \in P} x_{kc_j}) + \sum_{v \in K \setminus \{k\}} \sum_{c_j \in P} x_{vc_j} \quad \leq \quad \frac{q-1}{2} + \sum_{v \in K \setminus \{k\}} z_{vk}. \tag{6}$$

Note that if $Q = \{c_1, c_2, c_3\}$, $K = \{v, w\}$ and $k = v$ then the resulting consecutive colors inner clique inequality dominates the constraint (3) corresponding to the edge vw and the colors c_1 and c_2 (resp. the edge vw and the colors c_3 and c_2).

Proposition 2 *The consecutive colors inner clique inequalities (6) are valid for $PS(G, C)$.*

Proof. Let $y = (x, z) \in PS(G, C)$ be a feasible solution. Define $\delta_c \in \{0, 1\}$ as

$$\delta_c = \sum_{v \in K \setminus \{k\}} x_{vc},$$

i.e., $\delta_c = 1$ iff c is assigned to a vertex from $K \setminus \{k\}$. For any color $c_j \in P$, if $\delta_{c_j}(x_{kc_{j-1}} + x_{kc_{j+1}}) = 1$, then a color adjacency exists between k and some vertex from $K \setminus \{k\}$. So, we can bound the RHS of (6) as

$$\frac{q-1}{2} + \sum_{c_j \in P} \delta_{c_j}(x_{kc_{j-1}} + x_{kc_{j+1}}) \;\leq\; \frac{q-1}{2} + \sum_{v \in K \setminus \{k\}} z_{kv}. \tag{7}$$

Note that no color adjacency is counted more than once since $\sum_{c \in Q} x_{kc} \leq 1$. On the other hand, we can write the LHS of (6) as

$$\sum_{c_j \in P} (x_{kc_{j-1}} + x_{kc_j} + x_{kc_{j+1}}) + \sum_{c_j \in P} \delta_{c_j}. \tag{8}$$

Hence we can prove validity of (6) by showing that (8) is not greater than the LHS of (7). To this end, we can show, for every $c_j \in P$, that

$$(x_{kc_{j-1}} + x_{kc_j} + x_{kc_{j+1}}) + \delta_{c_j} \leq 1 + \delta_{c_j}(x_{kc_{j-1}} + x_{kc_{j+1}}) \tag{9}$$

and then, sum over P.

If $\delta_{c_j} = 0$, then (9) holds as $(x_{kc_{j-1}} + x_{kc_j} + x_{kc_{j+1}}) \leq 1$. Finally, if $\delta_{c_j} = 1$ the inequality holds only if $x_{kc_j} = 0$, and this is true as $\delta_{c_j} = 1$.

Since the solution y is arbitrary, the inequality (6) is valid for $PS(G, C)$. □

Theorem 1 *If $|C| > \chi(G)$ and $|C| \geq |K| + 5$, then the consecutive colors inner clique inequalities (6) are facet-defining for $PS(G, C)$.*

Proof. Let $\pi \in \mathbb{R}^{nt+m}$ be the coefficient vector of (6) and $\pi_0 \in \mathbb{R}$ the independent term of (6). The face of $PS(G, C)$ defined by this inequality is

$$F = \{y \in \mathbb{R}^{nt+m} : \pi y = \pi_0\} \cap PS(G, C).$$

Let $\lambda \in \mathbb{R}^{nt+m}$ and $\lambda_0 \in \mathbb{R}$ such that $\lambda y = \lambda_0, \forall y \in F$. We will prove that λ is a linear combination of π and the coefficient vectors of the equations (1) from the model, implying that F is a facet of $PS(G, C)$.

We first characterize the integer solutions $y = (x, z) \in F$ in four types:

Type (I) If the vertex k is not assigned to any color from Q then the only case in which (6) is satisfied by equality is when $\frac{q-1}{2}$ vertices from the clique are using colors from P, i.e., every color from P, and no other vertex from K is assigned to a color adjacent to the one assigned to k. That is,

- $x_{kc} = 0$, for all $c \in Q$,
- for all $c_j \in P$, there exists $v \in K \setminus \{k\}$ such that $x_{vc_j} = 1$, and
- $z_{kv} = 0$ for all $v \in K \setminus \{k\}$.

Type (II) The vertex k uses a color $c_j \in P$. In order to satisfy (6) with equality in this case, every other color from P must be assigned to vertices from K and the colors c_{j-1} and c_{j+1} cannot be assigned to any vertex from the clique, as it will generate color adjacencies with k without increasing the LHS of (6). Obviously, c_j cannot be assigned to vertices from K, since it is being used by k. This is,

- $x_{kc_j} = 1$ with $c_j \in P$,
- for every $c_i \in P \setminus \{c_j\}$, there exists $v \in K \setminus \{k\}$ such that $x_{vc_i} = 1$, and
- $x_{vc_{j-1}} = x_{vc_j} = x_{vc_{j+1}} = 0$ for all $v \in K \setminus \{k\}$.

Type (III) The vertex k uses c_1 (resp. c_q). In order to satisfy (6) with equality in this case, every color from $P \setminus \{c_2\}$ (resp. in $P \setminus \{c_{q-1}\}$) must be assigned to vertices from the clique and the color preceding c_1 (resp. following c_q) cannot be assigned to any other vertex from the clique, as it will generate color adjacencies with k without increasing the LHS of (6). The color c_2 (resp. c_{q-1}) may or may not be assigned to a vertex from K, but for every $v \in K \setminus \{k\}$, if v is not assigned to c_2 (resp. c_{q-1}), then $z_{kv} = 0$. We call type (IIIa) and (IIIb) to the solutions from this type using c_1 or c_q, respectively. This is,

Type (IIIa):

- $x_{kc_1} = 1$,
- for every $c_j \in P \setminus \{c_2\}$, there exists $v \in K \setminus \{k\}$ such that $x_{vc_j} = 1$,
- $x_{v,c_1-1} = 0$ for all $v \in K \setminus \{k\}$, and
- for every $v \in K \setminus \{k\}$, $z_{kv} = 1 \Leftrightarrow x_{vc_2} = 1$.

Type (IIIb):

- $x_{kc_q} = 1$,
- for every $c_j \in P \setminus \{c_{q-1}\}$, there exists $v \in K \setminus \{k\}$ such that $x_{vc_j} = 1$,
- $x_{v,c_q+1} = 0$ for all $v \in K \setminus \{k\}$, and
- for every $v \in K \setminus \{k\}$, $z_{kv} = 1 \Leftrightarrow x_{vc_{q-1}} = 1$.

Type (IV) The vertex k uses a color $c_i \in I \setminus \{c_1, c_q\}$. In order to satisfy the equality in this case, every color from $P \setminus \{c_{i-1}, c_{i+1}\}$ must be assigned to vertices from the clique. The colors c_{i-1} and c_{i+1} may or may not be assigned to vertices from the clique, but for every $v \in K \setminus \{k\}$, if v is not using any of these colors then $z_{kv} = 0$.

- $x_{kc_i} = 1$ with $c_i \in I \setminus \{c_1, c_q\}$,
- for all $c_j \in P \setminus \{c_{i-1}, c_{i+1}\}$, there exists $v \in K \setminus \{k\}$ such that $x_{vc_j} = 1$, and
- for every $v \in K \setminus \{k\}$, $z_{kv} = 1 \Leftrightarrow (x_{vc_{i-1}} = 1 \;\vee\; x_{vc_{i+1}} = 1)$.

The following illustrates the types of solutions described above. The white spaces in the table denote that there are no restrictions for the vertices receiving the corresponding colors and the symbol $\emptyset_K$ denotes that the color cannot be assigned to a vertex from the clique. The vertices $\hat{v}$ represent vertices from the clique.

		c		c_1	c_2	c_3	c_4	c_5	c_6	$\dots$	c_{q-1}	c_q	
$(I) \rightarrow$	$\emptyset_K$	k	$\emptyset_K$		$\hat{v}$		$\hat{v}$		$\hat{v}$		$\hat{v}$		
$(II) \rightarrow$					$\hat{v}$	$\emptyset_K$	k	$\emptyset_K$	$\hat{v}$		$\hat{v}$		
$(IIIa) \rightarrow$			$\emptyset_K$	k			$\hat{v}$		$\hat{v}$		$\hat{v}$		
$(IIIb) \rightarrow$					$\hat{v}$		$\hat{v}$		$\hat{v}$			k	$\emptyset_K$
$(IV) \rightarrow$					$\hat{v}$			k			$\hat{v}$		

To prove that λ is a linear combination of π and the coefficient vectors of the equations (1) from the model we will state several claims on the coefficients from λ.

Let $vw \in E$ such that $vw \notin \{kj : j \in K\}$, we will see that $\lambda_{z_{vw}} = 0$. Let $y = (x, z) \in PS(G, C)$ be a solution of type (II) in which vertices v and w receive nonadjacent colors and $z_{vw} = 0$. This solution can be obtained by permuting the color classes from any valid solution, in order to obtain the non-adjacency between v and w. We now build the solution $y' = (x', z') \in PS(G, C)$ from y but setting $z'_{vw} = 1$. This solution is still valid and since z_{vw} is not involved in (6), the equality holds. As both y and y' belong to F, we know that $\lambda y = \lambda_0$ and $\lambda y' = \lambda_0$, so $\lambda y = \lambda y'$. We also know that y equals y' in every coefficient but z_{vw}, and so $\lambda_{z_{vw}} z_{vw} = \lambda_{z_{vw}} z'_{vw}$. Hence,

$$\textit{[Claim 1]} \qquad \lambda_{z_{vw}} = 0 \quad \forall vw \in E \text{ such that } vw \notin \{kj : j \in K\}. \tag{10}$$

Take now a vertex $v \notin K$, we will see that $\lambda_{x_{vc}} = \lambda_{x_{vc'}}$ for every pair of colors $c, c' \in C$. As $|C| > \chi(G)$, it is possible to build a solution where the color c' is not assigned to any vertex. Assume first that $c' \notin \{c_i : i \in P\}$, we will show later what happens in that case. Consider a solution $y = (x, z) \in PS(G, C)$ of type (I) in which c' is not assigned to any vertex and in which v receives the color c (this can be achieved by permuting the color class of v in any solution, assuming that the color assigned to v does not belong to $\{c_i : i \in P\}$). We now build the solution $y' = (x', z')$ from y, in which the vertex v receives the color c'. We can visualize the difference between these solutions in the next diagram:

		c		c'	
$y \to$		v		$\emptyset$	
$y' \to$				v	

The solution y' is feasible as c is not being used by any vertex in y. Moreover, this solution belongs to the face F since it also represents a type (I) solution. As both y and y' belong to F, we can deduce again that $\lambda y = \lambda y'$. In this case y equals y' in every coefficient except for x_{vc}, $x_{vc'}$ and eventually some adjacency variables incidents to v, however (10) asserts that $\lambda_{z_{vw}} = 0, \forall vw \in E$, and so we obtain

$$\lambda_{x_{vc}} x_{vc} + \lambda_{x_{vc'}} x_{vc'} = \lambda_{x_{vc}} x'_{vc} + \lambda_{x_{vc'}} x'_{vc'},$$

and using the corresponding variable values we get $\lambda_{x_{vc}} = \lambda_{x_{vc'}}$.

If $c' \in \{c_i : i \in P\}$, there are no type (I) solutions in F with c' free of vertices. However, we can show that $\lambda_{x_{vc}} = \lambda_{x_{vc'}}$ by resorting to solutions of type (IV). Hence,

$$\textit{[Claim 2]} \qquad \lambda_{x_{vc}} = \lambda_{x_{vc'}} \quad \forall v \notin K \quad \forall c, c' \in C. \tag{11}$$

Next, consider a vertex $v \in K \setminus \{k\}$, we will see that for every $c_j \in P$, $\lambda_{x_{vc_j}} = \lambda_{x_{vc}} - \lambda_{z_{kv}}$ for all $c \in C \setminus P$. Let $y = (x, z) \in PS(G, C)$ be a solution of type (IV) where v is assigned to c_j and k to c_{j-1} (if $j = 2$, then the solution was type (IIIa)). Assume there is a color $c \in C \setminus (P \cup \{c_{j-2}\})$ free from vertices (this can be achieved since $|C| > \chi(G)$). We now build another solution $y' = (x', z')$ from y but assigning c to v.

		c		c_{j-2}	c_{j-1}	c_j	
$y \to$		$\emptyset$			k	v	
$y' \to$		v			k		

Note that y' is feasible as y assingns no vertex to color c. Moreover, y' represents a type (IV) solution (or a type (IIIa), if $j = 2$) so it belongs to F. Again, as in the previous cases, $\lambda y = \lambda y'$. Both solutions are identical for every coefficient but for x_{vc_j}, x_{vc}, z_{kv} and other adjacency variables for v. The latter, however, are void since (10) asserts that their coefficients in λ are null. Hence,

$$\lambda_{x_{vc_j}} + \lambda_{z_{kv}} = \lambda_{x_{vc}}.$$

It is worth mentioning that if $c_{j-2} \notin P$, then in order to prove the equality for $c = c_{j-2}$ we can use a type (IV) solution in which k is assigned to c_{j+1} instead of c_{j-1} (it will be a type (IIIb) solution if $j = q - 1$), following the same line. So, as c is an arbitrary color from $C \setminus P$, we have

$$\textit{[Claim 3]} \qquad \lambda_{x_{vc_j}} + \lambda_{z_{kv}} = \lambda_{x_{vc}} \quad \forall v \in K \setminus \{k\},\ \forall c_j \in P,\ \forall c \in C \setminus P. \tag{12}$$

Note also that (12) implies

$$\textit{[Claim 4]} \qquad \lambda_{x_{vc}} = \lambda_{x_{vc'}} \quad \forall v \in K \setminus \{k\} \quad \forall c, c' \in C \setminus P. \tag{13}$$

We will see next that $\lambda_{x_{kc_1}} = \lambda_{x_{kc_q}} = \lambda_{x_{kc}} - \lambda_{z_{kv}}$ for all $c \in C \setminus Q$ and $v \in K \setminus \{k\}$. First we will prove it for $\lambda_{x_{kc1}}$.
Let $y = (x, z) \in PS(G, C)$ be a type (IIIa) solution in which v is assigned the color c_2 and there is a color $c \in C \setminus Q$ free from vertices (again this can be achieved as $|C| > \chi(G)$). We also assume that no vertex from K is assigned to the colors adjacent to c; this is possible if $|C \setminus \{c\}| \geq |K| + 2$, which is implied by the hypothesis. We now build a solution $y' = (x', z')$ from y by assigning c to k.

			c			c_1	c_2	c_3	c_4	...
$y \rightarrow$		$\emptyset_K$	$\emptyset$	$\emptyset_K$		k	v		$\hat{v}$	
$y' \rightarrow$		$\emptyset_K$	k	$\emptyset_K$			v		$\hat{v}$	

Note that y' is feasible as y assingns no vertex to color c. Moreover, y' belongs to F as it represents a type (I) solution and therefore, as in the previous cases, $\lambda y = \lambda y'$. Here, y equals y' in every variable but x_{kc_1}, x_{kc}, z_{kv} and eventually some adjacency variables that can be ignored based on (10). We thus obtain

$$\lambda_{x_{kc_1}} + \lambda_{z_{kv}} = \lambda_{x_{kc}}. \tag{14}$$

Analogously, by taking a type (IIIb) solution instead of a type (IIIa) solution, we get

$$\lambda_{x_{kc_q}} + \lambda_{z_{kv}} = \lambda_{x_{kc}}. \tag{15}$$

Finally, from (14) and (15) we have

$$\textit{[Claim 5]} \quad \lambda_{x_{kc_1}} = \lambda_{x_{kc_q}} = \lambda_{x_{kc}} - \lambda_{z_{kv}}, \quad \forall v \in K \setminus \{k\}, \forall c \in C \setminus Q. \tag{16}$$

Note that v is an arbitrary vertex from $K \setminus \{k\}$, therefore (16) also implies

$$\textit{[Claim 6]} \quad \lambda_{z_{kv}} = \lambda_{z_{kw}} \quad \forall v, w \in K \setminus \{k\}. \tag{17}$$

Next we will see that for every $c_i \in I \setminus \{c_1, c_q\}$, we have $\lambda_{x_{kc_i}} = \lambda_{x_{kc}} - \lambda_{z_{kv}} - \lambda_{z_{kw}}$, for all $c \in C \setminus Q$ and $v, w \in K \setminus \{k\}$. Let $y = (x, z)$ be a type (IV) solution in which k is assigned to c_i, v to c_{i-1} and w to c_{i+1}. Assume also that c is free from vertices in y and that the colors adjacent to c are free from vertices from K. We now build a solution $y' = (x', z')$ from y by assigning c to k.

		c		c_1	c_2	$\dots$	c_{i-1}	c_i	c_{i+1}	$\dots$	c_{q-1}	c_q	
$y \rightarrow$	$\emptyset_K$	$\emptyset$	$\emptyset_K$		$\hat{v}$		v	k	w		$\hat{v}$		
$y' \rightarrow$	$\emptyset_K$	k	$\emptyset_K$		$\hat{v}$		v		w		$\hat{v}$		

The solution y' is feasible and satisfy (6) with equality as it is a type (I) solution. Again, we get $\lambda y = \lambda y'$ and in this case the only difference between y and y' is given by x_{kc_i}, x_{kc}, z_{kv}, z_{kw} and eventually some adjacency variables. Hence,

$$\text{[Claim 7]} \quad \lambda_{x_{kc_i}} = \lambda_{x_{kc}} - \lambda_{z_{kv}} - \lambda_{z_{kw}}, \qquad \forall v, w \in K \setminus \{k\},\ \forall i \in I \setminus \{1, q\}, \forall c \in C \setminus Q. \tag{18}$$

We now show $\lambda_{x_{kc_j}} = \lambda_{x_{kc}} - \lambda_{z_{kv}}$ for every $c_j \in P$ for all $c \in C \setminus Q$ and $v \in K \setminus \{k\}$. Let $y = (x, z)$ be a type (II) solution in which k is assigned to color c_j. Assume for a moment that $c_j \neq c_2$ and call v the vertex of the clique assigned to c_{j-2}. Assume also that the color c_{j-1} is free from vertices in y. We now build the solution $y' = (x', z')$ from y in which we move k to the color c_{j-1}.

	c_1	c_2	$\dots$	c_{j-2}	c_{j-1}	c_j	c_{j+1}	$\dots$	c_{q-1}	c_q	
$y \rightarrow$		$\hat{v}$		v	$\emptyset$	k	$\emptyset_K$		$\hat{v}$		
$y' \rightarrow$		$\hat{v}$		v	k		$\emptyset_K$		$\hat{v}$		

The resulting solution is valid and it satisfies (6) with equality as it is a type (IV) solution. Once again, $\lambda y = \lambda y'$ and the differences between y and y' are given by x_{kc_j}, $x_{kc_{j-1}}$, z_{kv} and eventually some adjacency variables. So we have

$$\lambda_{x_{kc_j}} = \lambda_{x_{kc_{j-1}}} + \lambda_{z_{kv}}.$$

If $c_j = c_2$, a similar argument can be given by moving k to the color c_{j+1}, defining v to be the vertex of K using c_{j+2}, and analogously deducing

$$\lambda_{x_{kc_j}} = \lambda_{x_{kc_{j+1}}} + \lambda_{z_{kv}}.$$

By (17) and (18) we conclude that

$$\text{[Claim 8]} \quad \lambda_{x_{kc_j}} = \lambda_{x_{kc}} - \lambda_{z_{kv}}, \quad \forall v \in K \setminus \{k\},\ \forall j \in P, \forall c \in C \setminus Q, \tag{19}$$

as both c_{j-1} and c_{j+1} belong to I.

Finally we show that $\lambda_{x_{kc}} = \lambda_{x_{kc'}}$ for all $c, c' \in C \setminus Q$. Let $y = (x, z)$ be a type (I) solution in which k is assigned to the color c and no vertex gets the color c'. In addition, assume that the colors adjacent to c and to c' are free from vertices from K; such a solution exists if $|C \setminus \{c\}| \geq |K| + 4$, which is implied by the hypothesis. We now build a solution $y' = (x', z')$ from y by assigning c' to k.

			c				c'		
$y \rightarrow$		$\emptyset_K$	k	$\emptyset_K$		$\emptyset_K$	$\emptyset$	$\emptyset_K$	
$y' \rightarrow$		$\emptyset_K$		$\emptyset_K$		$\emptyset_K$	k	$\emptyset_K$	

The solution y' is feasible as y assigns no vertex to c'. Moreover, y' belongs to F as it is a type (I) solution. In this case the difference between both solutions is given by x_{kc}, $x_{kc'}$ and eventually some adjacency variables, so we obtain

$$\textit{[Claim 9]} \quad \lambda_{x_{kc}} = \lambda_{x_{kc'}} \quad \forall c, c' \in C \setminus Q. \tag{20}$$

To conclude the proof, we now show that λ is a linear combination of π and the coefficient vectors of (1).

Claim 2 asserts that given a vertex $v \notin K$, the coefficient in λ for the variables x_{vc} is the same for any color c. Call β_v this value, so for $v \notin K$

$$\lambda_{x_{vc}} = \beta_v \quad \forall c \in C.$$

Claim 4 asserts that for a vertex $v \in K \setminus \{k\}$, the coefficient in λ for x_{vc} is the same for any color $c \in C \setminus P$. Call β_v this value, then for $v \in K \setminus \{k\}$ we have

$$\lambda_{x_{vc}} = \beta_v \quad \forall c \in C \setminus P.$$

Similarly, Claim 9 states that $\lambda_{x_{kc}}$ is the same for any color c outside Q. By defining β_k to be this value we have

$$\lambda_{x_{kc}} = \beta_k \quad \forall c \in C \setminus Q.$$

On the other hand, Claim 6 shows that $\lambda_{z_{kv}} = \lambda_{z_{kw}}$ for every pair of vertex $v, w \in K \setminus \{k\}$. Call α this value, so

$$\lambda_{z_{kv}} = \alpha \quad \forall v \in K \setminus \{k\}.$$

We can now compute the remaining coefficients of λ. Claim 3 implies

$$\lambda_{x_{vc_j}} = \beta_v - \alpha \quad \forall j \in P,\ \forall v \in K \setminus \{k\},$$

and from Claim 5 we have

$$\lambda_{x_{kc_1}} = \lambda_{x_{kc_q}} = \beta_k - \alpha.$$

Finally, from Claim 7 we deduce

$$\lambda_{x_{kc_i}} = \beta_k - 2\ \alpha, \quad \forall i \in I \setminus \{1, q\},$$

and from Claim 8 we obtain

$$\lambda_{x_{kc_j}} = \beta_k - \alpha, \quad \forall j \in P.$$

In this manner, using also Claim 1, we verify that λ can be obtained as the linear combination

$$\lambda = \sum_{v \in V} \beta_v e^{(v)} - \alpha \pi,$$

where $e^{(v)}$ represents the coefficient vector of the constraint (1) corresponding to the vertex v. Hence, the *consecutive colors inner clique inequalities* define facets of $PS(G, C)$. □

It is interesting to note that we can define the set $Q = \{c_1, \ldots, c_q\}$ even when c_q represents a color outside the limits of C, i.e., when c_{q-1} is the last color from C. The inequalities obtained by omitting the variables associated to c_q are still valid, and they also define facets of $PS(G, C)$. The same applies to c_1, when c_2 represents the first color of C.

To conclude this section, note that for any undominated integer solution $y = (x, z) \in PS(G, C)$ there are several consecutive colors inner clique inequalities satisfied with equality by y. To this end, take a vertex $k \in V$ and call c the color assigned to it. If $c \neq 1, t$, then by defining $Q = \{c-2, c-1, c, c+1, c+2\}$ and taking any clique K such that $k \in K$, the associated valid inequality is satisfied with equality by y. If c is the first color resp. the last color of C, by defining $Q = \{c, c+1, c+2\}$ resp. $Q = \{c-2, c-1, c\}$ we obtain a consecutive colors inner clique inequality satisfied with equality by y.

3.2 Consecutive colors subset clique inequalities

The class of valid inequalities introduced in this section arises from similar considerations as for the inner clique inequalities.

Definition 3 (Consecutive colors subset clique inequality) *Let $K \subseteq V$ be a clique of G, and fix a vertex $k \in K$. Let $Q = \{c_1, ..., c_q\} \subseteq C$ be a set of consecutive colors such that $c_{i+1} = c_i + 1$ for $i = 1, ..., q-1$. We define the* consecutive colors subset clique inequality *to be*

$$\big(x_{kc_1} + 2x_{kc_2} + \sum_{i=3}^{q-2} 3x_{kc_i} + 2x_{kc_{q-1}} + x_{kc_q}\big) + \sum_{v \in K \setminus \{k\}} \sum_{i=2}^{q-1} x_{vc_i} \leq (q-2) + \sum_{v \in K \setminus \{k\}} z_{vk}. \tag{21}$$

Proposition 3 *The consecutive colors subset clique inequalities (21) are valid for $PS(G, C)$.*

Proof. Let $y = (x, z) \in PS(G, C)$ be a feasible solution. Define $\delta_c \in \{0, 1\}$ as

$$\delta_c = \sum_{v \in K \setminus \{k\}} x_{vc},$$

i.e., $\delta_c = 1$ iff c is assigned to a vertex from $K \setminus \{k\}$. For every $i = 2, \dots, q-1$, if $\delta_{c_i}(x_{kc_{i-1}} + x_{kc_{i+1}}) = 1$, then a color adjacency exists between k and some vertex from $K \setminus \{k\}$. So, we can bound the RHS of (21) as

$$(q-2) \ + \sum_{i=2}^{q-1} \delta_{c_i}(x_{kc_{i-1}} + x_{kc_{i+1}}) \ \leq \ (q-2) \ + \sum_{v \in K \setminus \{k\}} z_{kv}. \qquad (22)$$

Note that no color adjacency is counted more than once since $\sum_{c \in Q} x_{kc} \leq 1$.

On the other hand, we can write the LHS of (21) as

$$\sum_{i=2}^{q-1}(x_{kc_{i-1}} + x_{kc_i} + x_{kc_{i+1}}) + \sum_{i=2}^{q-1} \delta_{c_i}. \qquad (23)$$

Hence, we can prove validity of (21) by showing that (23) is not greater than the LHS of (22). To this end, we can show, for every $i = 2, \dots, q-1$, that

$$(x_{kc_{i-1}} + x_{kc_i} + x_{kc_{i+1}}) + \delta_{c_i} \leq 1 + \delta_{c_i}(x_{kc_{i-1}} + x_{kc_{i+1}}) \qquad (24)$$

and then, sum over $\{2, \dots, q-1\}$.

If $\delta_{c_i} = 0$, then (24) holds as $(x_{kc_{i-1}} + x_{kc_i} + x_{kc_{i+1}}) \leq 1$. Finally, if $\delta_{c_i} = 1$ the inequality holds only if $x_{kc_i} = 0$, and this is true as $\delta_{c_i} = 1$.

Since the solution y is arbitrary, the inequality (21) is valid for $PS(G, C)$. □

Theorem 2 *If $|C| > \chi(G)$ and $|C| \geq |K| + 5$, the consecutive colors subset clique inequalities (21) are facet-defining for $PS(G, C)$.*

Proof. Let $Q' = Q \setminus \{c_1, c_q\}$ and $Q'' = Q' \setminus \{c_2, c_{q-1}\}$. Let $\pi \in \mathbb{R}^{nt+m}$ be the coefficient vector from (21) and let $\pi_0 \in \mathbb{R}$ the independent term of (21). The face of $PS(G, C)$ defined by this inequality is

$$F = \{y \in \mathbb{R}^{nt+m} : \pi y = \pi_0\} \cap PS(G, C).$$

Let $\lambda \in \mathbb{R}^{nt+m}$ and $\lambda_0 \in \mathbb{R}$ such that $\lambda y = \lambda_0, \forall y \in F$. We will prove that λ is a linear combination of π and the coefficient vectors of the equations (1) from de model, implying that F is a facet of $PS(G, C)$.

We first characterize the integer solutions $y = (x, z) \in F$ in four types:

Type (I) If the vertex k is not assigned to any color from Q, then the only way to satisfy (21) with equality is by assigning $q-2$ colors from Q' to vertices from K, i.e., every color from Q'. In addition, no vertex from K can use a color adjacent to the color assigned to k. This is,

- $x_{kc} = 0$, for all $c \in Q$,
- for every $c \in Q'$, there exists $v \in K \setminus \{k\}$ such that $x_{vc} = 1$, and
- $z_{kv} = 0$ for all $v \in K \setminus \{k\}$.

Type (II) The vertex k is assigned to c_1 (resp. c_q). In this case (21) holds with equality only if every color from $Q' \setminus \{c_2\}$ (resp. $Q' \setminus \{c_{q-1}\}$) is assigned to a vertex from K. Moreover, no vertex from the clique can use the color preceding c_1 (resp. following c_q) as that would generate a color adjacency with k without increasing the LHS of the inequality. The color c_2 (resp. c_{q-1}) may or may not be assigned to vertices from K but for every $v \in K \setminus \{k\}$, if that color is not assigned to v then $z_{kv} = 0$. We call type (IIa) and type (IIb) to the solutions of this type in which k is assigned to c_1 and c_q, respectively. This is,

Type (IIa):

- $x_{kc_1} = 1$,
- for every $c \in Q' \setminus \{c_2\}$, there exists $v \in K \setminus \{k\}$ such that $x_{vc} = 1$,
- $x_{v,c_1-1} = 0$ for all $v \in K \setminus \{k\}$, and
- for every $v \in K \setminus \{k\}$, $z_{kv} = 1 \Leftrightarrow x_{vc_2} = 1$.

Type (IIb):

- $x_{kc_q} = 1$,
- for every $c \in Q' \setminus \{c_{q-1}\}$, there exists $v \in K \setminus \{k\}$ such that $x_{vc} = 1$,
- $x_{v,c_q+1} = 0$ for all $v \in K \setminus \{k\}$, and
- for every $v \in K \setminus \{k\}$, $z_{kv} = 1 \Leftrightarrow x_{vc_2} = 1$.

Type (III) The vertex k is assigned to c_2 resp. c_{q-1}. In this case (21) holds with equality only if every color from $Q'' \setminus \{c_3\}$ (resp. $Q'' \setminus \{c_{q-2}\}$) is assigned to a vertex from K. Moreover, no vertex from the clique can use c_1 (resp. c_q) as that would generate a color adjacency with k without increasing the LHS of the inequality. The color c_3 (resp. c_{q-2}) may or may not be assigned to vertices from K but for every $v \in K \setminus \{k\}$, if that color is not assigned to v then $z_{kv} = 0$. We call type (IIIa) and type (IIIb) to the solutions of this type in which k is assigned to c_2 and c_{q-1}, respectively. This is,

Type (IIIa):

- $x_{kc_2} = 1$,
- for every $c \in Q'' \setminus \{c_3\}$, there exists $v \in K \setminus \{k\}$ such that $x_{vc} = 1$,
- $x_{v,c_1} = 0$ for all $v \in K \setminus \{k\}$, and
- for every $v \in K \setminus \{k\}$, $z_{kv} = 1 \Leftrightarrow x_{vc_3} = 1$.

Type (IIIb):

- $x_{kc_{q-1}} = 1$,
- for every $c \in Q'' \setminus \{c_{q-2}\}$, there exists $v \in K \setminus \{k\}$ such that $x_{vc} = 1$,
- $x_{v,c_q} = 0$ for all $v \in K \setminus \{k\}$, and
- for every $v \in K \setminus \{k\}$, $z_{kv} = 1 \Leftrightarrow x_{vc_{q-2}} = 1$.

Type (IV) The vertex k is assigned to a color $c \in Q''$. The inequality (21) holds with equality only if every color from $Q' \setminus \{c-1, c, c+1\}$ is assigned to a vertex from K. The colors $c-1$ and $c+1$ may or may not be assigned to vertices from K but for every $v \in K \setminus \{k\}$, if v does not receive either $c-1$ or $c+1$, then $z_{kv} = 0$. This is,

- $x_{kc} = 1$ for some $c \in Q''$,
- for every $c \in Q' \setminus \{c-1, c, c+1\}$, there exists $v \in K \setminus \{k\}$ such that $x_{vc} = 1$, and
- for every $v \in K \setminus \{k\}$, $z_{kv} = 1 \Leftrightarrow (x_{v,c-1} = 1 \vee x_{v,c+1} = 1)$.

The following illustrates the types of solutions described above. The white spaces in the table denote that there are no restrictions for the vertices receiving the corresponding colors and the symbol $\emptyset_K$ denotes that the color cannot be assigned to a vertex from K. The vertices $\hat{v}$ represent vertices from K.

		c		c_1	c_2	c_3	c_4	c_5	c_6	$\dots$	c_{q-2}	c_{q-1}	c_q	
$(I) \rightarrow$	$\emptyset_K$	k	$\emptyset_K$		$\hat{v}$	$\hat{v}$	$\hat{v}$	$\hat{v}$	$\hat{v}$	$\dots$	$\hat{v}$	$\hat{v}$		
$(IIa) \rightarrow$			$\emptyset_K$	k		$\hat{v}$	$\hat{v}$	$\hat{v}$	$\hat{v}$	$\dots$	$\hat{v}$	$\hat{v}$		
$(IIb) \rightarrow$					$\hat{v}$	$\hat{v}$	$\hat{v}$	$\hat{v}$	$\hat{v}$	$\dots$	$\hat{v}$		k	$\emptyset_K$
$(IIIa) \rightarrow$				$\emptyset_K$	k		$\hat{v}$	$\hat{v}$	$\hat{v}$	$\dots$	$\hat{v}$	$\hat{v}$		
$(IIIb) \rightarrow$					$\hat{v}$	$\hat{v}$	$\hat{v}$	$\hat{v}$	$\hat{v}$	$\dots$		k	$\emptyset_K$	
$(IV) \rightarrow$					$\hat{v}$		k		$\hat{v}$	$\dots$	$\hat{v}$	$\hat{v}$		

To show that λ is a linear combination of π and the coefficient vectors of the model constraints (1) we will prove several claims on the coefficients of λ.

Let $vw \in E$ such that $vw \notin \{kj : j \in K\}$. Let $y = (x, z) \in PS(G, C)$ be a solution of type (I) in which the vertices v and w receive nonadjacent colors and $z_{vw} = 0$. This solution can be obtained by permuting the color classes from any valid solution, in order to obtain the non-adjacency between v and w. We now build the solution $y' = (x', z') \in PS(G, C)$ from y by setting $z'_{vw} = 1$. This solution is still valid and since z_{vw} is not involved in (21), equality holds. As both y and y' belong to F, we know that $\lambda y = \lambda_0$ and $\lambda y' = \lambda_0$, so $\lambda y = \lambda y'$. We also know that y equals y' in every coefficient but z_{vw}, and so $\lambda_{z_{vw}} z_{vw} = \lambda_{z_{vw}} z'_{vw}$. Hence,

$$\textit{[Claim 1]} \qquad \lambda_{z_{vw}} = 0 \quad \forall vw \in E \text{ such that } vw \notin \{kj : j \in K\}. \tag{25}$$

Take now a vertex $v \notin K$, we will see that $\lambda_{x_{vc}} = \lambda_{x_{vc'}}$ for every pair of colors $c, c' \in C$. As $|C| > \chi(G)$, it is possible to build a solution where the color c'

is not assigned to any vertex. Consider a solution $y = (x, z) \in PS(G, C)$ of type (I) in which c' is not assigned to any vertex and in which v receives the color c (this can be achieved by permuting the color class corresponding to v in any solution). we now build the solution $y' = (x', z')$ from y, in which vertex v receives the color c'. We can visualize the difference between these solutions in the next diagram:

		c		c'	
$y \to$		v		$\emptyset$	
$y' \to$				v	

The solution y' is feasible $x_{vc} = 0$ for every $v \in V$. Moreover, $y' \in F$ since it is a type (I) solution. As both y and y' belong to F, we get $\lambda y = \lambda y'$. In this case y equals y' in every coefficient except for x_{vc}, $x_{vc'}$ and eventually some adjacency variables, however (25) states that $\lambda_{z_{vw}} = 0, \forall vw \in E$, and so obtain

$$\lambda_{x_{vc}} x_{vc} + \lambda_{x_{vc'}} x_{vc'} = \lambda_{x_{vc}} x'_{vc} + \lambda_{x_{vc'}} x'_{vc'},$$

and using the corresponding variable values we get $\lambda_{x_{vc}} = \lambda_{x_{vc'}}$. Since c and c' are arbitrary colors, we conclude

$$\textit{[Claim 2]} \qquad \lambda_{x_{vc}} = \lambda_{x_{vc'}} \quad \forall v \notin K \quad \forall c, c' \in C. \tag{26}$$

Next, consider a vertex $v \in K \setminus \{k\}$ and a color $c \in Q'$, we will see that $\lambda_{x_{vc}} = \lambda_{x_{vc'}} - \lambda_{z_{kv}}$ for all $c' \in C \setminus Q'$. Let $y = (x, z) \in PS(G, C)$ be a solution of type (IV) in which v is assigned to the color c and k is assigned to a color adjacent to c. Assume that c' is free from vertices (such a solution exists as $|C| > \chi(G)$). We now build another solution $y' = (x', z')$ from y by assigning c' to v.

		c'		c_1	$\dots$	c	$c+1$	$\dots$	c_q	
$y \to$		$\emptyset$			$\dots$	v	k	$\dots$		
$y' \to$		v			$\dots$		k	$\dots$		

The solution y' is feasible $x_{vc} = 0$ for every $v \in V$. Moreover, $y' \in F$ since it is a type (IV) solution, so it belongs to F and, therefore, $\lambda y = \lambda y'$. Both solutions are identical for every coefficient but x_{vc}, $x_{vc'}$, z_{kv} and some adjacency variables for v. The latter are void since (25) asserts that their coefficients in λ are null. Hence,

$$\lambda_{x_{vc}} + \lambda_{z_{kv}} = \lambda_{x_{vc'}}.$$

If there are no solutions of type (IV) in $PS(G, C)$, if $|Q| < 5$, the same conclusion can be achieved by resorting to type (IIa) or type (IIb) solutions. So, we conclude:

$$\textit{[Claim 3]} \qquad \lambda_{x_{vc}} = \lambda_{x_{vc'}} - \lambda_{z_{kv}} \quad \forall v \in K \setminus \{k\},\ \forall c \in Q', \forall c' \in C \setminus Q'. \tag{27}$$

Note also that (27) implies

$$\text{[Claim 4]} \qquad \lambda_{x_{vc}} = \lambda_{x_{vc'}} \quad \forall v \in K \setminus \{k\} \quad \forall c, c' \in C \setminus Q'. \tag{28}$$

We show next that $\lambda_{x_{kc_1}} = \lambda_{x_{kc_q}} = \lambda_{x_{kc}} - \lambda_{z_{kv}}$ for all $c \in C \setminus Q$ and $v \in K \setminus \{k\}$. Let $y = (x, z) \in PS(G, C)$ be a solution of type (IIa) in which a vertex $v \in K \setminus \{k\}$ uses the color c_2 and a color $c \in C \setminus Q$ is free from vertices (again this can be achieved as $|C| > \chi(G)$). We also assume that no vertex from K is assigned to a color adjacent to c; this is possible if $|C \setminus \{c\}| \geq |K| + 2$, which is implied by the hypothesis. We now build a solution $y' = (x', z')$ from y but assigning c to k.

		c			c_1	c_2	$\dots$
$y \rightarrow$	$\emptyset_K$	$\emptyset$	$\emptyset_K$	$\emptyset_K$	k	v	$\dots$
$y' \rightarrow$	$\emptyset_K$	k	$\emptyset_K$	$\emptyset_K$		v	$\dots$

The solution y' is feasible $x_{vc} = 0$ for every $v \in V$. Moreover, $y' \in F$ since it is a type (I) solution and, therefore, $\lambda y = \lambda y'$. Here, y equals y' in every variable but x_{kc_1}, x_{kc}, z_{kv} and eventually some adjacency variables that can be ignored based on (25). So we deduce

$$\lambda_{x_{kc_1}} + \lambda_{z_{kv}} = \lambda_{x_{kc}}. \tag{29}$$

Analogously, by taking a type (IIb) solution instead of a type (IIa) solution we get

$$\lambda_{x_{kc_q}} + \lambda_{z_{kv}} = \lambda_{x_{kc}}. \tag{30}$$

Finally, from (29) and (30) we have

$$\text{[Claim 5]} \quad \lambda_{x_{kc_1}} = \lambda_{x_{kc_q}} = \lambda_{x_{kc}} - \lambda_{z_{kv}}, \qquad \forall v \in K \setminus \{k\}, \forall c \in C \setminus Q. \tag{31}$$

Note that v is an arbitrary vertex from $K \setminus \{k\}$, therefore (31) also implies

$$\text{[Claim 6]} \quad \lambda_{z_{kv}} = \lambda_{z_{kw}} \quad \forall v, w \in K \setminus \{k\}. \tag{32}$$

Similarly as c is any color from $C \setminus Q$, we also obtain

$$\text{[Claim 7]} \quad \lambda_{x_{kc}} = \lambda_{x_{kc'}} \quad \forall c, c' \in C \setminus Q. \tag{33}$$

Let $y = (x, z) \in PS(G, C)$ be a type (IIIa) solution in which a vertex $v \in K \setminus \{k\}$ is assigned to c_3 and the color c_1 is free from vertices (such a solution exists as $|C| > \chi(G)$). We also assume that no vertex from K is assigned to a color preceding c_1; this is possible if $|C \setminus \{c\}| \geq |K| + 2$, which is implied by the hypothesis. We now build a solution $y' = (x', z')$ from y but assigning c_1 to k.

		c_1	c_2	c_3	$\dots$
$y \rightarrow$	$\emptyset_K$	$\emptyset$	k	v	$\dots$
$y' \rightarrow$	$\emptyset_K$	k		v	$\dots$

The solution y' is valid and it satisfies (21) with equality as it is a type (IIa) solution. Once again, $\lambda y = \lambda y'$ and the differences between y and y' are given by x_{kc_2}, x_{kc_1}, z_{kv} and eventually some adjacency variables. So we have

$$\lambda_{x_{kc_2}} + \lambda_{z_{kv}} = \lambda_{x_{kc_1}}. \tag{34}$$

Analogously, with a type (IIIb) instead of a type (IIIa) solution, we can show

$$\lambda_{x_{kc_{q-1}}} + \lambda_{z_{kv}} = \lambda_{x_{kc_q}}. \tag{35}$$

Finally, by combining (34), (35) and (31) we have

$$\text{[Claim 8]} \quad \lambda_{x_{kc_2}} = \lambda_{x_{kc_{q-1}}} = \lambda_{x_{kc}} - 2\lambda_{z_{kv}}, \quad \forall v \in K \setminus \{k\}, \forall c \in C \setminus Q. \tag{36}$$

Let $y = (x, z) \in PS(G, C)$ be a type (IV) solution in which k is assigned to a color $c \in Q''$, a vertex $v_1 \in K \setminus \{k\}$ is assigned to $c + 1$ and the color $c - 1$ is free from vertices (again, this can be achieved as $|C| > \chi(G)$). Assume for now that $c > c_3$, and call v_2 to the vertex from K assigned to $c - 2$ (such vertex exists as y is a type (IV) solution). We now build a solution $y' = (x', z')$ from y by moving k to the color $c - 1$.

	c_1	$\dots$	$c-2$	$c-1$	c	$c+1$	$\dots$	c_q
$y \rightarrow$		$\dots$	v_2	$\emptyset$	k	v_1	$\dots$	
$y' \rightarrow$		$\dots$	v_2	k		v_1	$\dots$	

The solution y' is valid as $c - 1$ is free from vertices in y. Moreover, $y' \in F$ as y' represents a type (IV) solution and, therefore, $\lambda y = \lambda y'$. The only difference between y and y' is given by x_{kc}, $x_{k,c-1}$, z_{kv_1}, z_{kv_2} and eventually some adjacency variables that can be ignored, so we obtain

$$\lambda_{x_{kc}} + \lambda_{z_{kv_1}} = \lambda_{x_{k,c-1}} + \lambda_{z_{kv_2}}$$

and using (32) we conclude

$$\lambda_{x_{kc}} = \lambda_{x_{k,c-1}}, \quad \forall c \in Q'' \setminus \{c_3\}. \tag{37}$$

Suppose now that $c = c_3$, then the color $c - 2 = c_1$ may be free from vertices from the clique. We keep in y the color $c - 1 = c_2$ free from vertices and build y' the same way than before, moving k to color $c - 1$, namely to c_2.

	c_1	c_2	c_3	c_4	$\dots$	c_q
$y \rightarrow$	$\emptyset_K$	$\emptyset$	k	v_1	$\dots$	
$y' \rightarrow$	$\emptyset_K$	k		v_1	$\dots$	

This solution is valid and it belongs to F as it is a type (IIIa) solution, hence $\lambda y = \lambda y'$. The difference between y and y' is given by x_{kc_3}, x_{kc_2}, z_{kv_1} and eventually some adjacency variables that can be ignored, so we obtain

$$\lambda_{x_{kc_3}} + \lambda_{z_{kv_1}} = \lambda_{x_{kc_2}}.$$

Combining this result with (37) and then using (36) we conclude that

$$\textit{[Claim 9]} \quad \lambda_{x_{kc}} = \lambda_{x_{kc'}} - 3\lambda_{z_{kv}}, \quad \forall c \in Q'', \forall v \in K \setminus \{k\}, \forall c' \in C \setminus Q. \quad (38)$$

We are now in position of proving that λ is in fact a linear combination of π and the coefficient vectors of (1).

Claim 2 asserts that given a vertex $v \notin K$, the coefficient in λ for the variables x_{vc} is the same for any color c. Define β_v to be this value, so for $v \notin K$

$$\lambda_{x_{vc}} = \beta_v \quad \forall c \in C.$$

Claim 4 asserts that for a vertex $v \in K \setminus \{k\}$, the coefficient in λ for x_{vc} is the same for any color $c \in C \setminus Q'$. Define β_v to be this value, then for $v \in K \setminus \{k\}$ we have

$$\lambda_{x_{vc}} = \beta_v \quad \forall c \in C \setminus Q'.$$

Similarly, Claim 7 asserts that $\lambda_{x_{kc}}$ is the same for any color c outside Q. By defining β_k to be this value, we have

$$\lambda_{x_{kc}} = \beta_k \quad \forall c \in C \setminus Q.$$

On the other hand, Claim 6 shows that $\lambda_{z_{kv}} = \lambda_{z_{kw}}$ for every pair of vertices $v, w \in K \setminus \{k\}$. By defining α to be this value, we get

$$\lambda_{z_{kv}} = \alpha \quad \forall v \in K \setminus \{k\}.$$

We can now compute the remaining coefficients of λ. Claim 3 implies

$$\lambda_{x_{vc}} = \beta_v - \alpha \quad \forall c \in Q',\ \forall v \in K \setminus \{k\},$$

and from Claim 5 we have

$$\lambda_{x_{kc_1}} = \lambda_{x_{kc_q}} = \beta_k - \alpha.$$

Finally, from Claim 8 we conclude

$$\lambda_{x_{kc_2}} = \lambda_{x_{kc_{q-1}}} = \beta_k - 2\ \alpha$$

and from Claim 9 we obtain

$$\lambda_{x_{kc}} = \beta_k - 3\ \alpha, \quad \forall c \in Q''.$$

Finally, by applying Claim 1, we can see that λ can be obtained as the linear combination

$$\lambda = \sum_{v \in V} \beta_v e^{(v)} - \alpha\pi,$$

where $e^{(v)}$ represents the coefficient vector of (1) corresponding to vertex v. Hence, the *consecutive colors subset clique inequalitites* define facets of $PS(G, C)$. $\square$

As in the consecutive colors inner clique inequalities, the set $Q = \{c_1, \ldots, c_q\}$ can be defined even when c_1, resp. c_q, represents a color outside the limits of C. The inequalities obtained by omitting the variables associated to these colors are still valid, and they also define facets of $PS(G, C)$.

Again, we conclude this section noting that for any undominated integer solution $y = (x, z) \in PS(G, C)$ there are several consecutive colors subset clique inequalities satisfied with equality by y. To this end, take a vertex $k \in V$ and call c the color assigned to it. If $c \neq 1, t$, then by defining $Q = \{c-2, c-1, c, c+1, c+2\}$ and taking any clique K such that $k \in K$, the associated valid inequality is satisfied with equality by y. If c is the first color resp. the last color of C, by defining $Q = \{c, c+1, c+2\}$ resp. $Q = \{c-2, c-1, c\}$ we obtain a consecutive colors subset clique inequality satisfied with equality by y (these are actually the same inequalities as in the *inner clique*).

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