Max-min optimizations on the rank and inertia of a linear Hermitian matrix expression subject to range, rank and definiteness restrictions

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Abstract. The inertia of a Hermitian matrix is defined to be a triplet composed by the numbers of the positive, negative and zero eigenvalues of the matrix counted with multiplicities, respectively. In this paper, we give various closed-form formulas for the maximal and minimal values for the rank and inertia of the Hermitian expression $A + X$, where A is a given Hermitian matrix and X is a variable Hermitian matrix satisfying the range and rank restrictions $\mathrm{range}(X) \subseteq \mathrm{range}(B)$ and $\mathrm{rank}(X) \leqslant k$. Some expressions of the Hermitian matrix X such that $A + X$ attains the extremal ranks and inertias are also presented.

Keywords: Hermitian matrix expression; perturbation; rank; inertia; maximization; minimization; Moore–Penrose inverse; equality; inequality

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1. Introduction

Consider a Hermitian matrix perturbation problem

$$A + X, \tag{1.1}$$

where A is a given Hermitian matrix of order m, and X is a Hermitian perturbation matrix (also called a Hermitian modification or update matrix). The perturbation matrix X is not totally free, but is often assumed to take values from some matrix sets or satisfy some restriction on its range, rank, norm, definiteness, etc. In this paper, we assume that the Hermitian perturbation matrix X in (1.1) satisfies the following range and rank restrictions

$$\mathrm{range}(X) \subseteq \mathrm{range}(B) \ \text{ and } \ \mathrm{rank}(X) \leqslant k, \tag{1.2}$$

or satisfies the following range, rank and definiteness restrictions

$$\mathrm{range}(X) \subseteq \mathrm{range}(B), \ \ \mathrm{rank}(X) \leqslant k \ \text{ and } \ \pm X \geqslant 0, \tag{1.3}$$

in which B is a given $m \times n$ matrix with $\mathrm{rank}(B) \geqslant k$, and $X \geqslant 0$ means that X is nonnegative definite. The range, rank and definiteness restrictions in (1.2) and (1.3) are often used for demonstrating algebraic properties of the Hermitian perturbation matrix X. If k is fairly small in comparison with the order m or the rank of B, (1.2) and (1.3) are often called low-rank perturbations (or modifications) to the matrix A; see, e.g., [1, 7, 8, 12, 23, 36].

Note that the set inclusion $\mathrm{range}(X) \subseteq \mathrm{range}(B)$ for a Hermitian matrix X can be alternatively expressed as $X = BYB^*$ for some Hermitian matrix Y, where B^* denotes the conjugate transpose of B. Hence, the Hermitian perturbation in (1.1) subject to (1.2) or (1.3) can equivalently be formulated as the following Hermitian matrix expression

$$p(X) = A + BXB^* \ \text{ subject to } \ \mathrm{rank}(X) \leqslant k, \tag{1.4}$$

or

$$p(X) = A + BXB^* \ \text{ subject to } \ \mathrm{rank}(X) \leqslant k \text{ and } \pm X \geqslant 0, \tag{1.5}$$

where $k \leqslant \mathrm{rank}(B)$.

The rank and inertia of a Hermitian matrix are two basic concepts in matrix theory for describing the dimension of the row/column vector space and the sign distribution of the eigenvalues of the matrix, which are well understood and are easy to compute by the well-known elementary or congruent matrix operations. These two quantities play an essential role in characterizing algebraic properties of Hermitian matrices. In recent years, the present author showed many new formulas for ranks and inertias for Hermitian matrices

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and their operations, and presented various applications of these formulas in matrix theory and applications; see [37, 38, 39, 40, 41]. Just like the problems of maximizing/minimizing determinants, traces and norms of matrices, the problem of maximizing/minimizing the rank and inertia of a matrix expression could be regarded as a special topic in mathematical optimization theory, although it was not classified clearly in the literature.

In this paper, we consider the problems of maximizing/minimizing the rank and inertia of (1.1) subject to (1.2) and (1.3), respectively. The max-min optimization problems consist of determining the maximal and minimal ranks and inertias of a matrix expression, and finding the variable matrices such that the matrix expression attains the extremal ranks and inertias. In fact, maximizing/minimizing the rank/inertia of a matrix expression has been a challenging task that arises in studying many optimization problems and their applications, such as, machine learning, system and control theory, and discrete geometry. In particular, the problems on maximizing/minimizing the inertia of a matrix with low rank Hermitian perturbations was considered in the literature, see, e.g., [15, 18]. Just like the classic optimization problems on determinants, traces and norms of matrices, the problem of maximizing/minimizing the rank/inertia of a matrix expression could be regarded as a special topic in mathematical optimization theory, although it was not classified clearly in the literature. Since the variable entries in a matrix expression are often taken as continuous variables from some constraint sets, while the objective expressions—the rank and inertia of the matrix expression take values from a finite set of nonnegative integers. Hence, maximizing/minimizing the rank/inertia of a matrix expression can be regarded as a mixed continuous-discrete optimization problem. This kind of continuous-discrete optimization problems cannot be solved by various optimization methods for continuous or discrete cases, and there is no rigorous mathematical theory for studying a general rank/inertia optimization problem. In fact, optimization and completion problems on rank/inertia of a general matrix expression were regarded as NP-hard except some special cases that can be solved by pure algebraical methods; see, e.g., [13, 14, 19, 20, 17, 24, 28, 30, 33, 34].

Linear matrix expressions, and their special cases linear matrix equations, have been a class of fundamental object of study in matrix theory and applications. The linear matrix expression $A + BXB^*$ is one of the basic Hermitian matrix expressions that includes an arbitrary matrix. This matrix expression was considered by some authors in matrix theory and applications. In an earlier book [35], $A + BXB^*$ was studied and the inequality $A + BXB^* > 0$ was solved by making use of the SVD and generalized inverses of matrices. In two recent papers [37] and [41], certain expansion formulas for the rank/inertia of $A - BXB^*$ (see (2.36) and (2.37) below) were established by making use of generalized inverses of matrices and some Algebraic operations of block matrices, and a group of closed-form (symbolic) formulas for the extremal rank and inertia of $A - BXB^*$ with respect to a variable Hermitian matrix X were derived. The closed-form formulas obtained enable us to derive some valuable consequences on the nonsingularity and definiteness of the matrix expression $A - BXB^*$, least-rank Hermitian solution of the matrix equation $BXB^* = A$; as well as the existence of Hermitian solution of the matrix equation $BXB^* = A$. In addition, the closed-form formulas can be used to derive upper and lower bounds for the ranks and inertias of more complicated Hermitian matrix expressions, as well as to solve some rank/inertia completion problems of partial Hermitian matrices. As a continuation, we study in this paper the problems of maximizing and minimizing the rank and inertia of $A + BXB^*$ in (1.4) and (1.5).

Throughout this paper, $\mathbb{C}^{m\times n}$ and $\mathbb{C}^m_{\mathrm{H}}$ stand for the sets of all $m \times n$ complex matrices and all $m \times m$ complex Hermitian matrices, respectively; in particular, let

$$\mathbb{C}^m_{\mathrm{H},k} = \{\, X \in \mathbb{C}^m_{\mathrm{H}} \mid \mathrm{rank}(X) \leqslant k \leqslant m \,\}.$$

When $k = m$, $\mathbb{C}^m_{\mathrm{H},k} = \mathbb{C}^m_{\mathrm{H}}$. If $k = 1$ or 2, then $\mathbb{C}^m_{\mathrm{H},k}$ denotes the collection of all Hermitian matrices with rank less than 1 or 2. The symbols A^*, $r(A)$ and $\mathscr{R}(A)$ and stand for the conjugate transpose, rank and range (column space) of a matrix $A \in \mathbb{C}^{m\times n}$, respectively; I_m denotes the identity matrix of order m; $[\,A,\, B\,]$ denotes a row block matrix consisting of A and B. Two Hermitian matrices A and B of the same size are said to be congruent if there is an invertible matrix S such that $SAS^* = B$. We write $A > 0$ (or $A \geqslant 0$) if A is Hermitian positive (or nonnegative) definite. Two Hermitian matrices A and B of the same size are said to satisfy the inequality $A > B$ (or $A \geqslant B$) in the Löwner partial ordering if $A - B$ is positive (or nonnegative) definite. The Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is defined to be the unique solution X of the four matrix equations

$$\text{(i) } AXA = A, \quad \text{(ii) } XAX = X, \quad \text{(iii) } (AX)^* = AX, \quad \text{(iv) } (XA)^* = XA.$$

Further, the symbols E_A and F_A stand for the two orthogonal projectors $E_A = I_m - AA^{\dagger}$ and $F_A = I_n - A^{\dagger}A$ onto the null spaces A^* and A, respectively. A well-known property of the Moore–Penrose inverse is $(A^{\dagger})^* = (A^*)^{\dagger}$. In addition, $AA^{\dagger} = A^{\dagger}A$ if $A = A^*$. We shall repeatedly use them in the latter part of this paper. Results on the Moore–Penrose inverse can be found, e.g., in [3, 4, 21].

When considering a Hermitian matrix, we are usually concerned with distributions of the eigenvalues of the matrix, as well as its definiteness. Recall that the eigenvalues of a Hermitian matrix $A \in \mathbb{C}^m_{\mathrm{H}}$ are all real, and the inertia of A is defined to be the triplet

$$\operatorname{In}(A) = \{ i_+(A),\, i_-(A),\, i_0(A) \},$$

where $i_+(A)$, $i_-(A)$ and $i_0(A)$ are the numbers of the positive, negative and zero eigenvalues of A counted with multiplicities, respectively. $i_+(A)$ and $i_-(A)$ are are called the positive and negative index of inertia, respectively, and both of which are usually called the partial inertia (see, e.g., [2]). For a Hermitian matrix A, we have $r(A) = i_+(A) + i_-(A)$.

Note that the inertia of a Hermitian matrix describes the sign distribution of the real eigenvalues of the matrix. Hence, it can be used to characterize definiteness of the matrix. The following results are obvious from the definitions of the rank and inertia of a matrix.

Lemma 1.1. *Let $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, and $C \in \mathbb{C}_{\mathrm{H}}^{m}$. Then,*

(a) *A is nonsingular if and only if $r(A) = m$.*
(b) *$B = 0$ if and only if $r(B) = 0$.*
(c) *$C > 0$ ($C < 0$) if and only if $i_+(C) = m$ ($i_-(C) = m$).*
(d) *$C \geqslant 0$ ($C \leqslant 0$) if and only if $i_-(C) = 0$ ($i_+(C) = 0$).*

Lemma 1.2. *Let $\mathcal{S}$ be a set consisting of (square) matrices over $\mathbb{C}^{m\times n}$, and let $\mathcal{H}$ be a set consisting of Hermitian matrices over $\mathbb{C}_{\mathrm{H}}^{m}$. Then,*

(a) *$\mathcal{S}$ has a nonsingular matrix if and only if $\max\limits_{X\in\mathcal{S}} r(X) = m$.*
(b) *Any $X \in \mathcal{S}$ is nonsingular if and only if $\min\limits_{X\in\mathcal{S}} r(X) = m$.*
(c) *$0 \in \mathcal{S}$ if and only if $\min\limits_{X\in\mathcal{S}} r(X) = 0$.*
(d) *$\mathcal{S} = \{0\}$ if and only if $\max\limits_{X\in\mathcal{S}} r(X) = 0$.*
(e) *$\mathcal{H}$ has a matrix $X > 0$ ($X < 0$) if and only if $\max\limits_{X\in\mathcal{H}} i_+(X) = m$ $\left(\max\limits_{X\in\mathcal{H}} i_-(X) = m\right)$.*
(f) *Any $X \in \mathcal{H}$ satisfies $X > 0$ ($X < 0$) if and only if $\min\limits_{X\in\mathcal{H}} i_+(X) = m$ $\left(\min\limits_{X\in\mathcal{H}} i_-(X) = m\right)$.*
(g) *$\mathcal{H}$ has a matrix $X \geqslant 0$ ($X \leqslant 0$) if and only if $\min\limits_{X\in\mathcal{H}} i_-(X) = 0$ $\left(\min\limits_{X\in\mathcal{H}} i_+(X) = 0\right)$.*
(h) *Any $X \in \mathcal{H}$ satisfies $X \geqslant 0$ ($X \leqslant 0$) if and only if $\max\limits_{X\in\mathcal{H}} i_-(X) = 0$ $\left(\max\limits_{X\in\mathcal{H}} i_+(X) = 0\right)$.*

These two lemmas show that once certain formulas for the exact bounds of rank and partial inertia of a Hermitian matrix are derived, we can use them to characterize equalities and inequalities for the Hermitian matrix. This basic algebraic method, referred to as the matrix rank/inertia method, is available, as demonstrated below, for studying various Hermitian matrix expressions that involve generalized inverses of matrices and variable matrices. It has been realized that computing the rank/inertia of a matrix a hard problem in linear algebra and no method is known to get the inertia of a matrix exactly in general; see, e.g., [19, 20]. However, it is not hard to establish closed-form formulas for the extremal ranks/inertias of Hermitian matrices under some simple cases; see the author's recent paper [25, 26, 37, 38, 39, 41] and the results in Sections 2 and 3.

Some well-known equalities for ranks and partial inertias of matrices are given in the following lemmas.

Lemma 1.3 ([29]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$ be given. Then*

$$r[\, A,\, B\,] = r(A) + r(E_AB) = r(B) + r(E_BA), \tag{1.6}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C). \tag{1.7}$$

Lemma 1.4. *Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$, $B \in \mathbb{C}_{\mathrm{H}}^{n}$, $Q \in \mathbb{C}^{m\times n}$, and assume that $P \in \mathbb{C}^{m\times m}$ is nonsingular. Then*

$$i_{\pm}(PAP^*) = i_{\pm}(A), \tag{1.8}$$

$$i_{\pm}(\lambda A) = \begin{cases} i_{\pm}(A) & if\ \lambda > 0 \\ i_{\mp}(A) & if\ \lambda < 0 \end{cases}. \tag{1.9}$$

Eq. (1.8) is the well-known Sylvester's law of inertia. Eq. (1.9) is obvious from the definition of inertia.

Lemma 1.5. *Let $A \in \mathbb{C}_{\rm H}^{m}$, $B \in \mathbb{C}_{\rm H}^{n}$, and $P,\, Q \in \mathbb{C}^{m\times n}$. Then,*

$$i_{\pm}(P^{*}AP) \leqslant i_{\pm}(A). \tag{1.10}$$

In particular,

(a) *$r(P^{*}AP) = r(A)$ if and only if $i_{+}(P^{*}AP) = i_{+}(A)$ and $i_{-}(P^{*}AP) = i_{-}(A)$.*
(b) *If $P^{*}AP = B$ and $QBQ^{*} = A$, then $i_{\pm}(A) = i_{\pm}(B)$ and $r(A) = r(B)$.*

The two inequalities in (1.10) were given in [31]. Lemma 1.5(a) and (b) were given in [37, Lemma 1.6].

Lemma 1.6. *Let $A,\, B \in \mathbb{C}_{\rm H}^{m}$. Then*

$$r(\,A + B\,) \leqslant r(A) + r(B), \tag{1.11}$$
$$i_{\pm}(\,A + B\,) \leqslant i_{\pm}(A) + i_{\pm}(B), \tag{1.12}$$
$$r(\,A + B\,) \geqslant r(A) - r(B), \tag{1.13}$$
$$i_{\pm}(\,A + B\,) \geqslant i_{\pm}(A) - i_{\mp}(B). \tag{1.14}$$

In particular, if $B \geqslant 0$,

$$r(\,A + B\,) \geqslant i_{+}(\,A + B\,) \geq i_{+}(A), \tag{1.15}$$
$$r(\,A - B\,) \geqslant i_{-}(\,A - B\,) \geq i_{-}(A). \tag{1.16}$$

Eqs. (1.11) is a well-known rank inequality in elementary linear algebra. Eq. (1.12) was given in [15, 32]. Applying (1.9), (1.11) and (1.12) to $A = (\,A + B\,) + (-B)$ yields (1.13) and (1.14). Eqs. (1.15) and (1.16) follow from (1.9) and (1.14). More inequalities for the inertias of sums of Hermitian matrices can be found in [11], [37].

2. Extremal ranks and inertias of Hermitian expressions with rank restrictions

We first solve the rank and inertia optimization problem on (1.1) subject to $r(X) \leqslant k$.

Lemma 2.1. *Let $A \in \mathbb{C}_{\rm H}^{m}$ be given, $X \in \mathbb{C}_{{\rm H},k}^{m}$ a variable matrix. Then,*

$$\max_{X \in \mathbb{C}_{{\rm H},k}^{m}} r(\,A + X\,) = \min\{\,m, \quad r(A) + k\,\}, \tag{2.1}$$
$$\min_{X \in \mathbb{C}_{{\rm H},k}^{m}} r(\,A + X\,) = \max\{\,0, \quad r(A) - k\,\}, \tag{2.2}$$
$$\max_{X \in \mathbb{C}_{{\rm H},k}^{m}} i_{+}(\,A + X\,) = \min\{\,m, \quad i_{+}(A) + k\,\}, \tag{2.3}$$
$$\min_{X \in \mathbb{C}_{{\rm H},k}^{m}} i_{+}(\,A + X\,) = \max\{\,0, \quad i_{+}(A) - k\,\}, \tag{2.4}$$
$$\max_{X \in \mathbb{C}_{{\rm H},k}^{m}} i_{-}(\,A + X\,) = \min\{\,m, \quad i_{-}(A) + k\,\}, \tag{2.5}$$
$$\min_{X \in \mathbb{C}_{{\rm H},k}^{m}} i_{-}(\,A + X\,) = \max\{\,0, \quad i_{-}(A) - k\,\}. \tag{2.6}$$

The Hermitian matrices Xs satisfying these equalities can be formulated from the canonical form of certain operations A and B under the Hermitian congruence. Further,

(a) *For any integer t_1 between the two quantities on the right-hand sides of (2.1) and (2.2), there exists an $X \in \mathbb{C}_{\rm H}^{m\times m}$ with $r(X) \leqslant k$ such that $r(\,A + X\,) = t_1$.*
(b) *For any integer t_2 between the two quantities on the right-hand sides of (2.3) and (2.4), there exists an $X \in \mathbb{C}_{\rm H}^{m\times m}$ with $r(X) \leqslant k$ such that $i_{+}(\,A + X\,) = t_2$.*
(c) *For any integer t_3 between the two quantities on the right-hand sides of (2.5) and (2.6), there exists an $X \in \mathbb{C}_{\rm H}^{m\times m}$ with $r(X) \leqslant k$ such that $i_{-}(\,A + X\,) = t_3$.*

Proof Note that $r(\,A + X\,) \leqslant m$ always holds. Hence, it is easily derived from (1.9) and (1.12) that

$$\begin{aligned} r(\,A + X\,) &\leqslant \min\{\,m, \quad r(A) + r(X)\,\} \leqslant \min\{\,m, \quad r(A) + k\,\}, \\ i_{+}(\,A + X\,) &\leqslant \min\{\,m, \quad i_{+}(A) + i_{+}(X)\,\} \leqslant \min\{\,m, \quad i_{+}(A) + k\,\}, \\ i_{-}(\,A + X\,) &\leqslant \min\{\,m, \quad i_{-}(A) + i_{+}(X)\,\} \leqslant \min\{\,m, \quad i_{-}(A) + k\,\}. \end{aligned}$$

Hence, the right-hands of (2.1), (2.3) and (2.5) are upper bounds for the rank and partial inertia of $A+X$. Note from (1.13)–(1.16) that

$$\begin{aligned} r(A+X) &= r[A-(-X)] \geqslant \max\{0,\ \ r(A)-r(-X)\} \geqslant \max\{0,\ \ r(A)-k\},\\ i_{+}(A+X) &= i_{+}[A-(-X)] \geqslant \max\{0,\ \ i_{+}(A)-i_{+}(-X)\} \geqslant \max\{0,\ \ i_{+}(A)-k\},\\ i_{-}(A+X) &= i_{-}[A-(-X)] \geqslant \max\{0,\ \ i_{-}(A)-i_{-}(-X)\} \geqslant \max\{0,\ \ i_{-}(A)-k\}. \end{aligned}$$

Hence, the right-hands of (2.2), (2.4) and (2.6) are lower bounds for the ranks and partial inertias of $A+X$. We next show that there exist Xs such that the rank/inertia of $A+X$ attain the upper/lower bounds on the right-hands of (2.1)–(2.6), so that the upper/lower bounds are exactly the extremal rank and inertia of $A+X$. Assume that the canonical form of the Hermitian A under the Hermitian congruence is

$$A = UDU^{*}, \quad D = \mathrm{diag}\{I_p,\ -I_q,\ 0\}, \tag{2.7}$$

where $p = i_{+}(A)$, $q = i_{-}(A)$, and $U \in \mathbb{C}^{m\times m}$ is a nonsingular matrix. Consequently,

$$A + X = UDU^{*} + X = U\left[D + U^{-1}X(U^{-1})^{*}\right]U^{*}. \tag{2.8}$$

Applying (1.8) to (2.8) gives

$$r(A+X) = r(D+Y), \quad i_{+}(A+X) = i_{+}(D+Y), \quad i_{-}(A+X) = i_{-}(D+Y), \tag{2.9}$$

where $Y = U^{-1}X(U^{-1})^{*} \in \mathbb{C}^{m\times m}$ satisfying $r(Y) \leqslant k$. Without loss of generality, we assume that Y is a real diagonal matrix. Hence, the sum $D+Y$ is a real diagonal matrix as well.

Set $Y = \mathrm{diag}\{0,\ 2I_k\}$. Then, the first and second equalities in (2.9) become

$$r(A+X) = r\,(D+\mathrm{diag}\{0,\ 2I_k\}) = \min\{m,\ \ p+q+k\} = \min\{m,\ \ r(A)+k\}, \tag{2.10}$$

$$i_{+}(A+X) = r\,(D+\mathrm{diag}\{0,\ 2I_k\}) = \min\{m,\ \ p+k\} = \min\{m,\ \ i_{+}(A)+k\}, \tag{2.11}$$

establishing (2.1) and (2.3).

Set

$$Y = \begin{cases} \mathrm{diag}\{-I_k,\ 0\} & \text{for } k \leqslant p\\ \mathrm{diag}\{-I_p,\ I_{k-p},\ 0\} & \text{for } p < k \leqslant p+q\\ -D & \text{for } k > p+q. \end{cases}$$

Then, the first equality in (2.9) becomes

$$\begin{aligned} r(A+X) = r(D+Y) &= \begin{cases} p+q-k & \text{for } k \leqslant p+q\\ 0 & \text{for } k > p+q \end{cases}\\ &= \max\{0,\ p+q-k\} = \max\{0,\ r(A)-k\}, \end{aligned} \tag{2.12}$$

establishing (2.2). For any integer t_1 between the two quantities on the right-hand sides of (2.10) and (2.12), we can definitely choose the real diagonal matrix Y with $r(Y) \leqslant k$ such that the rank of the diagonal matrix $D+Y$ is $r(D+Y) = t_1$, as asserted in (a).

Set $Y = \mathrm{diag}\{-I_k,\ 0\}$. Then, the second equalities in (2.9) become

$$i_{+}(A+X) = i_{+}(D+Y) = \begin{cases} p-k & \text{for } k \leqslant p\\ 0 & \text{for } k > p \end{cases} = \max\{0,\ p-k\} = \min\{0,\ i_{+}(A)-k\}, \tag{2.13}$$

establishing (2.4). For any integer t_2 between the two quantities on the right-hand sides of (2.11) and (2.13), we can definitely choose the real diagonal matrix Y with $r(Y) \leqslant k$ such that $i_{+}(D+Y) = t_2$, as asserted in (b).

Set

$$Y = \begin{cases} \mathrm{diag}\{-2I_k,\ 0\} & \text{for } k \leqslant p\\ \mathrm{diag}\{-2I_p,\ 0,\ I_{k-p}\} & \text{for } k > p. \end{cases}$$

Then, the third equality in (2.9) becomes

$$i_{-}(A+X) = i_{-}(D+Y) = \begin{cases} q+k & \text{for } k \leqslant m-q\\ m & \text{for } k > m-q \end{cases} = \max\{m,\ q+k\} = \min\{m,\ i_{-}(A)+k\}, \tag{2.14}$$

establishing (2.5).

Set $Y = \mathrm{diag}\{\,0,\, I_k,\, 0\,\}$. Then, the third equality in (2.9) becomes

$$i_{-}(\,A + X\,) = i_{-}(\,D + Y\,) = \begin{cases} q - k & \text{for } k \leqslant q \\ 0 & \text{for } k > q \end{cases} = \max\{\,0,\;\; q - k\,\} = \min\{\,0,\;\; i_{-}(A) - k\,\}, \tag{2.15}$$

establishing (2.6). For any integer t_3 between the two quantities on the right-hand sides of (2.14) and (2.15), we can definitely choose the real diagonal matrix Y with $r(Y) \leqslant k$ such that $i_{+}(\,D + Y\,) = t_3$, as asserted in (c). □

A useful consequence of Lemma 2.1 is given below, which will be used to derive the extremal ranks and inertias of (1.4).

Corollary 2.2. *Let $A \in \mathbb{C}_{\mathrm{H}}^{n}$ and $B \in \mathbb{C}^{n\times m}$ be given with $\mathscr{R}(A) \subseteq \mathscr{R}(B)$, and let $X \in \mathbb{C}_{\mathrm{H},k}^{m}$ be a variable matrix with $k \leqslant r(B)$. Then,*

$$\max_{X\in\mathbb{C}_{\mathrm{H},k}^{m}} r(\,A + BXB^{*}\,) = \min\{\,r(B),\;\; r(A) + k\,\}, \tag{2.16}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H},k}^{m}} r(\,A + BXB^{*}\,) = \max\{\,0,\;\; r(A) - k\}\,, \tag{2.17}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H},k}^{m}} i_{+}(\,A + BXB^{*}\,) = \min\{\,r(B),\;\; i_{+}(A) + k\,\}\,, \tag{2.18}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H},k}^{m}} i_{+}(\,A + BXB^{*}\,) = \max\{\,0,\;\; i_{+}(A) - k\}\,, \tag{2.19}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H},k}^{m}} i_{-}(\,A + BXB^{*}\,) = \min\{\,r(B),\;\; i_{-}(A) + k\}\,, \tag{2.20}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H},k}^{m}} i_{-}(\,A + BXB^{*}\,) = \max\{\,0,\;\; i_{-}(A) - k\,\}. \tag{2.21}$$

In particular, if $k \geqslant r(B)$, then

$$\max_{X\in\mathbb{C}_{\mathrm{H},k}^{m}} r(\,A + BXB^{*}\,) = r(B), \tag{2.22}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H},k}^{m}} r(\,A + BXB^{*}\,) = 0, \tag{2.23}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H},k}^{m}} i_{+}(\,A + BXB^{*}\,) = r(B), \tag{2.24}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H},k}^{m}} i_{+}(\,A + BXB^{*}\,) = 0, \tag{2.25}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H},k}^{m}} i_{-}(\,A + BXB^{*}\,) = r(B), \tag{2.26}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H},k}^{m}} i_{-}(\,A + BXB^{*}\,) = 0. \tag{2.27}$$

The Hermitian matrices Xs satisfying these equalities can be formulated from the canonical form of certain operations A and B under the Hermitian congruence. Further,

- (a) *For any integer t_1 between the two quantities on the right-hand sides of (2.16) and (2.17), there exists an $X \in \mathbb{C}_{\mathrm{H},k}^{m}$ such that $r(\,A + BXB^{*}\,) = t_1$.*
- (b) *For any integer t_2 between the two quantities on the right-hand sides of (2.18) and (2.19), there exists an $X \in \mathbb{C}_{\mathrm{H},k}^{m}$ such that $i_{+}(\,A + BXB^{*}\,) = t_2$.*
- (c) *For any integer t_3 between the two quantities on the right-hand sides of (2.20) and (2.21), there exists an $X \in \mathbb{C}_{\mathrm{H},k}^{m}$ such that $i_{-}(\,A + BXB^{*}\,) = t_3$.*

Proof Note that $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ is equivalent to $A = BGB^{*}$ for some matrix $G \in \mathbb{C}_{\mathrm{H}}^{m}$. Also, assume that the rank decomposition of B is $B = PQ$, where $P \in \mathbb{C}^{n\times s}$, $Q \in \mathbb{C}^{s\times m}$ and $r(B) = r(P) = r(Q) = s$. Consequently,

$$A + BXB^{*} = BGB^{*} + BXB^{*} = PQGQ^{*}P^{*} + PQXQ^{*}P^{*} = P(\,QGQ^{*} + Y\,)P^{*}, \tag{2.28}$$

where $Y = QXQ^{*} \in \mathbb{C}_{\mathrm{H},k}^{s}$ with $r(Y) = r(QXQ^{*}) \leqslant k \leqslant s$. Applying Lemma 1.5 to (2.28) gives

$$r(\,A + BXB^{*}\,) = r(\,QGQ^{*} + Y\,), \;\;\; i_{\pm}(\,A + BXB^{*}\,) = i_{\pm}(\,QGQ^{*} + Y\,). \tag{2.29}$$

In this case, applying Lemma 2.1 to the right-hand sides of the equalities in (2.29) gives

$$\max_{Y\in\mathbb{C}^s_{\mathrm{H},k}} r(\,QGQ^* + Y\,) = \min\{\,s,\ \ r(QGQ^*) + k\,\}, \tag{2.30}$$

$$\min_{Y\in\mathbb{C}^s_{\mathrm{H},k}} r(\,QGQ^* + Y\,) = \max\{\,0,\ \ r(QGQ^*) - k\}\,, \tag{2.31}$$

$$\max_{Y\in\mathbb{C}^s_{\mathrm{H},k}} i_+(\,QGQ^* + Y\,) = \min\{\,s,\ \ i_+(QGQ^*) + k\,\}\,, \tag{2.32}$$

$$\min_{Y\in\mathbb{C}^s_{\mathrm{H},k}} i_+(\,QGQ^* + Y\,) = \max\{\,0,\ \ i_+(QGQ^*) - k\}\,, \tag{2.33}$$

$$\max_{Y\in\mathbb{C}^s_{\mathrm{H},k}} i_-(\,QGQ^* + Y\,) = \max\{\,s,\ \ i_-(QGQ^*) + k\}\,, \tag{2.34}$$

$$\min_{Y\in\mathbb{C}^s_{\mathrm{H},k}} i_-(\,QGQ^* + Y\,) = \max\{\,0,\ \ i_-(QGQ^*) - k\,\}. \tag{2.35}$$

Substituting $s = r(B)$, $r(QGQ^*) = r(A)$ and $i_\pm(QGQ^*) = i_\pm(A)$ into (2.30)–(2.35) yields (2.16)–(2.21). Set $k = r(B)$ in (2.16)–(2.21) yields (2.22)–(2.27). Results (a)–(c) follow from (2.30)–(2.35) and Lemma 2.1(a)–(c). □

The proof of Corollary 2.2 also gives the construction of the Hermitian matrices Xs that satisfy (2.16)–(2.21). It was shown in [37, 41] that for any Hermitian matrix expression $A + BXB^*$, the following equalities hold

$$r(\,A + BXB^*\,) = 2r[\,A,\,B\,] - r(M) + r[\,F_{S_1}(\,S^*M^\dagger S - X\,)F_{S_1}\,], \tag{2.36}$$

$$i_\pm(\,A + BXB^*\,) = r[\,A,\,B\,] - i_\mp(M) + i_\mp[\,F_{S_1}(\,S^*M^\dagger S - X\,)F_{S_1}\,], \tag{2.37}$$

where

$$M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ I_n \end{bmatrix}, \quad S_1 = S - MM^\dagger S, \tag{2.38}$$

where the matrix expression $F_{S_1}(\,S^*M^\dagger S - X\,)F_{S_1}$ on the right-hand sides of (2.36) and (2.37) is Hermitian. These expansion formulas for rank and inertia show that the extremal ranks and inertias of $A + BXB^*$ can be determined through those of the Hermitian matrix expression $F_{S_1}(\,S^*M^\dagger S - X\,)F_{S_1}$. Applying Corollary 2.2 to the right-hand sides of (2.36) and (2.37), we obtain the main result of this section.

Theorem 2.3. *Let $A \in \mathbb{C}^m_{\mathrm{H}}$ and $B \in \mathbb{C}^{m\times n}$ be given, M be as given in* (2.38), *and let $X \in \mathbb{C}^n_{\mathrm{H},k}$ be a variable matrix.*

(a) *If $k \leqslant r(M) - r[\,A,\,B\,]$, then,*

$$\max_{X\in\mathbb{C}^n_{\mathrm{H},k}} r(\,A + BXB^*\,) = \min\{\,r[\,A,\,B\,],\ \ r(A) + k\,\}, \tag{2.39}$$

$$\min_{X\in\mathbb{C}^n_{\mathrm{H},k}} r(\,A + BXB^*\,) = \max\{\,2r[\,A,\,B\,] - r(M),\ \ r(A) - k\}\,, \tag{2.40}$$

$$\max_{X\in\mathbb{C}^n_{\mathrm{H},k}} i_+(\,A + BXB^*\,) = \min\{\,i_+(M),\ \ i_+(A) + k\,\}\,, \tag{2.41}$$

$$\min_{X\in\mathbb{C}^n_{\mathrm{H},k}} i_+(\,A + BXB^*\,) = \max\{\,r[\,A,\,B\,] - i_-(M),\ \ i_+(A) - k\}\,, \tag{2.42}$$

$$\max_{X\in\mathbb{C}^n_{\mathrm{H},k}} i_-(\,A + BXB^*\,) = \min\{\,i_-(M),\ \ i_-(A) + k\,\}\,, \tag{2.43}$$

$$\min_{X\in\mathbb{C}^n_{\mathrm{H},k}} i_-(\,A + BXB^*\,) = \max\{\,r[\,A,\,B\,] - i_+(M),\ \ i_-(A) - k\,\}. \tag{2.44}$$

The Hermitian matrices Xs satisfying these equalities can be formulated from the canonical form of of certain operations A and B under the Hermitian congruence. Further,

(i) *For any integer t_1 between the two quantities on the right-hand sides of* (2.39) *and* (2.40), *there exists an $X \in \mathbb{C}^n_{\mathrm{H},k}$ such that $r(\,A + BXB^*\,) = t_1$.*

(ii) *For any integer t_2 between the two quantities on the right-hand sides of* (2.41) *and* (2.42), *there exists an $X \in \mathbb{C}^n_{\mathrm{H},k}$ such that $i_+(\,A + BXB^*\,) = t_2$.*

(iii) *For any integer t_3 between the two quantities on the right-hand sides of* (2.43) *and* (2.44), *there exists an $X \in \mathbb{C}^n_{\mathrm{H},k}$ such that $i_-(\,A + BXB^*\,) = t_3$.*

In particular,

(iv) *There exists an* $X \in \mathbb{C}^n_{\mathrm{H},k}$ *such that* $A + BXB^*$ *is nonsingular if and only if* $r[\,A,\,B\,] = m$ *and* $r(A) \geqslant m - k$.

(v) *There exists an* $X \in \mathbb{C}^n_{\mathrm{H},k}$ *such that* $A + BXB^* = 0$ *if and only if* $r(A) \leqslant k$ *and* $\mathscr{R}(A) \subseteq \mathscr{R}(B)$.

(vi) *There exists an* $X \in \mathbb{C}^n_{\mathrm{H},k}$ *such that* $A + BXB^* > 0$ *if and only if* $i_+(M) \geqslant m$ *and* $i_+(A) \geqslant m - k$.

(vii) *There exists an* $X \in \mathbb{C}^n_{\mathrm{H},k}$ *such that* $A + BXB^* \geqslant 0$ *if and only if* $i_-(M) \geqslant r[\,A,\,B\,]$ *and* $i_+(A) \leqslant k$.

(viii) *There exists an* $X \in \mathbb{C}^n_{\mathrm{H},k}$ *such that* $A + BXB^* < 0$ *if and only if* $i_-(M) \geqslant m$ *and* $i_-(A) \geqslant m - k$.

(ix) *There exists an* $X \in \mathbb{C}^n_{\mathrm{H},k}$ *such that* $A + BXB^* \leqslant 0$ *if and only if* $i_+(M) \geqslant r[\,A,\,B\,]$ *and* $i_-(A) \leqslant k$.

(b) *If* $k \geqslant r(M) - r[\,A,\,B\,]$, *then*

$$\max_{X \in \mathbb{C}^n_{\mathrm{H},k}} r(\,A + BXB^*\,) = r[\,A,\,B\,], \tag{2.45}$$

$$\min_{X \in \mathbb{C}^n_{\mathrm{H},k}} r(\,A + BXB^*\,) = 2r[\,A,\,B\,] - r(M), \tag{2.46}$$

$$\max_{X \in \mathbb{C}^n_{\mathrm{H},k}} i_+(\,A + BXB^*\,) = i_+(M), \tag{2.47}$$

$$\min_{X \in \mathbb{C}^n_{\mathrm{H},k}} i_+(\,A + BXB^*\,) = r[\,A,\,B\,] - i_-(M), \tag{2.48}$$

$$\max_{X \in \mathbb{C}^n_{\mathrm{H},k}} i_-(\,A + BXB^*\,) = i_-(M), \tag{2.49}$$

$$\min_{X \in \mathbb{C}^n_{\mathrm{H},k}} i_-(\,A + BXB^*\,) = r[\,A,\,B\,] - i_+(M). \tag{2.50}$$

Proof Eqs. (2.36) and (2.37) imply that

$$\max_{X \in \mathbb{C}^n_{\mathrm{H},k}} r(\,A + BXB^*\,) = 2r[\,A,\,B\,] - r(M) + \max_{X \in \mathbb{C}^n_{\mathrm{H},k}} r[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,], \tag{2.51}$$

$$\min_{X \in \mathbb{C}^n_{\mathrm{H},k}} r(\,A + BXB^*\,) = 2r[\,A,\,B\,] - r(M) + \min_{X \in \mathbb{C}^n_{\mathrm{H},k}} r[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,], \tag{2.52}$$

$$\max_{X \in \mathbb{C}^n_{\mathrm{H},k}} i_{\pm}(\,A + BXB^*\,) = r[\,A,\,B\,] - i_{\mp}(M) + \max_{X \in \mathbb{C}^n_{\mathrm{H},k}} i_{\mp}[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,], \tag{2.53}$$

$$\min_{X \in \mathbb{C}^n_{\mathrm{H},k}} i_{\pm}(\,A + BXB^*\,) = r[\,A,\,B\,] - i_{\mp}(M) + \min_{X \in \mathbb{C}^n_{\mathrm{H},k}} i_{\mp}[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,]. \tag{2.54}$$

It follows from (1.6) and (1.7) that

$$r(F_{S_1}) = n - r(E_M S) = n + r(M) - r[\,M,\,S\,] = r(M) - r[\,A,\,B\,]. \tag{2.55}$$

Under $k \leqslant r(F_{S_1}) = r(M) - r[\,A,\,B\,]$, applying Corollary 2.2 to the right-hand sides of (2.51)–(2.54) yields the following results

$$\max_{X \in \mathbb{C}^n_{\mathrm{H},k}} r[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = \min\{\,r(F_{S_1}), \quad r(F_{S_1}S^*M^{\dagger}SF_{S_1}) + k\,\}, \tag{2.56}$$

$$\min_{X \in \mathbb{C}^n_{\mathrm{H},k}} r[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = \max\left\{\,0, \quad r(F_{S_1}S^*M^{\dagger}SF_{S_1}) - k\right\}, \tag{2.57}$$

$$\max_{X \in \mathbb{C}^n_{\mathrm{H},k}} i_-[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = \min\left\{\,r(F_{S_1}), \quad i_-(F_{S_1}S^*M^{\dagger}SF_{S_1}) + k\,\right\}, \tag{2.58}$$

$$\min_{X \in \mathbb{C}^n_{\mathrm{H},k}} i_-[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = \max\left\{\,0, \quad i_-(F_{S_1}S^*M^{\dagger}SF_{S_1}) - k\,\right\}, \tag{2.59}$$

$$\max_{X \in \mathbb{C}^n_{\mathrm{H},k}} i_+[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = \min\left\{\,r(F_{S_1}), \quad i_+(F_{S_1}S^*M^{\dagger}SF_{S_1}) + k\,\right\}, \tag{2.60}$$

$$\min_{X \in \mathbb{C}^n_{\mathrm{H},k}} i_+[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = \max\{\,0,\; i_+(F_{S_1}S^*M^{\dagger}SF_{S_1}) - k\,\}. \tag{2.61}$$

Setting $X = 0$ in (2.36) and (2.37) yields

$$r[\,F_{S_1}(S^*M^{\dagger}S)F_{S_1}\,] = r(A) - 2r[\,A,\,B\,] + r(M), \tag{2.62}$$

$$i_{\pm}[\,F_{S_1}(S^*M^{\dagger}S)F_{S_1}\,] = i_{\mp}(A) - r[\,A,\,B\,] + i_{\pm}(M). \tag{2.63}$$

Substituting (2.55)–(2.63) into (2.51)–(2.54) and simplifying yields (2.39)–(2.44). Results (i)–(iii) in (a) follow from (2.56)–(2.61) and Lemma 2.1(a)–(c). Results (iv)–(ix) in (a) follow from (2.39)–(2.44) and Lemma 1.2.

Under $k \geqslant r(F_{S_1}) = r(M) - r[\,A,\, B\,]$, applying (2.22)–(2.27) to the right-hand sides of (2.51)–(2.54) yields the following results

$$\max_{X\in\mathbb{C}_{\mathrm{H},k}^{n}} r[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = r(F_{S_1}), \qquad \min_{X\in\mathbb{C}_{\mathrm{H},k}^{n}} r[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = 0, \tag{2.64}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H},k}^{n}} i_{-}[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = r(F_{S_1}), \qquad \min_{X\in\mathbb{C}_{\mathrm{H},k}^{n}} i_{-}[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = 0, \tag{2.65}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H},k}^{n}} i_{+}[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = r(F_{S_1}), \qquad \min_{X\in\mathbb{C}_{\mathrm{H},k}^{n}} i_{+}[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = 0. \tag{2.66}$$

Substituting (2.64)–(2.66) into (2.51)–(2.54) and simplifying, we obtain (2.45)–(2.50). □

Theorem 2.3 gives a group of explicit closed-form formulas for the extremal ranks and inertias of the Hermitian matrix expression in (1.4). As demonstrated in (i)–(vi) of Theorem 2.3(a), the closed-form formulas can be used to derive some basic algebraic properties of the matrix expressions, such as, necessary and sufficient conditions for $A + BXB^*$ to be nonsingular or definite, as well as necessary and sufficient conditions for the matrix equation $BXB^* = -A$ to have positive (nonnegative definite) solutions.

For convenience, we call the matrix $X \in \mathbb{C}_{\mathrm{H},k}^{n}$ satisfying (2.40) the rank-constrained least-rank solution of the matrix equation $BXB^* = -A$; the matrix $X \in \mathbb{C}_{\mathrm{H},k}^{n}$ satisfying (2.42) the rank-constrained least-positive-inertia solution of the matrix equation $BXB^* = -A$; the matrix $X \in \mathbb{C}_{\mathrm{H},k}^{n}$ satisfying (2.44) the rank-constrained least-negative-inertia solution of the matrix equation $BXB^* = -A$. These solutions are not necessarily unique. Therefore, it would be of interest to further choose such solution X with minimal norm.

From Theorem 2.3, we obtain the following results on the extremal ranks and inertias of (1.1) subject to (1.2).

Theorem 2.4. *Let $A \in \mathbb{C}_{\mathrm{H}}^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$ be given, and let M be as given in* (2.38). *Also, denote*

$$\mathcal{S} = \{\, X \in \mathbb{C}_{\mathrm{H},k}^{n} \mid \mathscr{R}(X) \subseteq \mathscr{R}(B) \quad \text{and} \quad k \leqslant r(B)\}. \tag{2.67}$$

Then,

$$\max_{X\in\mathcal{S}} r(\,A + X\,) = \min\{\, r[\,A,\, B\,], \quad r(A) + k\,\}, \tag{2.68}$$

$$\min_{X\in\mathcal{S}} r(\,A + X\,) = \max\{\, 2r[\,A,\, B\,] - r(M),\ r(A) - k\}\,, \tag{2.69}$$

$$\max_{X\in\mathcal{S}} i_{+}(\,A + X\,) = \min\{\, i_{+}(M), \quad i_{+}(A) + k\,\}\,, \tag{2.70}$$

$$\min_{X\in\mathcal{S}} i_{+}(\,A + X\,) = \max\{\, r[\,A,\, B\,] - i_{-}(M),\ i_{+}(A) - k\}\,, \tag{2.71}$$

$$\max_{X\in\mathcal{S}} i_{-}(\,A + X\,) = \min\{\, i_{-}(M), \quad i_{-}(A) + k\,\}\,, \tag{2.72}$$

$$\min_{X\in\mathcal{S}} i_{-}(\,A + X\,) = \max\{\, r[\,A,\, B\,] - i_{+}(M), \quad i_{-}(A) - k\,\}. \tag{2.73}$$

The Hermitian matrices Xs satisfying these equalities can be formulated from the canonical form of certain operations A and B under the Hermitian congruence. Further,

(i) *For any integer t_1 between the two quantities on the right-hand sides of* (2.68) *and* (2.69), *there exists an $X \in \mathbb{C}_{\mathrm{H},k}^{m}$ such that $r(\,A + X\,) = t_1$.*

(ii) *For any integer t_2 between the two quantities on the right-hand sides of* (2.70) *and* (2.71), *there exists an $X \in \mathbb{C}_{\mathrm{H},k}^{m}$ such that $i_{+}(\,A + X\,) = t_2$.*

(iii) *For any integer t_3 between the two quantities on the right-hand sides of* (2.72) *and* (2.73), *there exists an $X \in \mathbb{C}_{\mathrm{H},k}^{m}$ such that $i_{-}(\,A + X\,) = t_3$.*

Theorem 2.5. *Let $A \in \mathbb{C}_{\mathrm{H}}^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$ be given, and let M and $\mathcal{S}$ be as given in* (2.38) *and* (2.67). *Then,*

$$\max\{\, 2r[\,A,\, B\,] - r(M) - r(A), \quad -k\} \leqslant r(\,A + X\,) - r(A) \leqslant \min\{\, r[\,A,\, B\,] - r(A), \quad k\,\}, \tag{2.74}$$

$$\max\{\, r[\,A,\, B\,] - i_{-}(M) - i_{+}(A), \quad -k\} \leqslant i_{+}(\,A + X\,) - i_{+}(A) \leqslant \min\{\, i_{+}(M) - i_{+}(A), \quad k\,\}\,, \tag{2.75}$$

$$\max\{\, r[\,A,\, B\,] - i_{+}(M) - i_{-}(A), \quad -k\,\} \leqslant i_{-}(\,A + X\,) - i_{-}(A) \leqslant \min\{\, i_{-}(M) - i_{-}(A), \quad k\,\} \tag{2.76}$$

hold for any $X \in \mathcal{S}$. In particular, the equalities in (2.74)–(2.76) *hold for some $X \in \mathcal{S}$.*

3. Extremal ranks and inertias of Hermitian expressions with rank and definiteness restrictions

As a special case of (1.3), we first solve rank and inertia optimization problems on $A \pm X$ subject to $r(X) \leqslant k$ and $X \geqslant 0$.

Lemma 3.1. *Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$ be given, and let $X \in \mathbb{C}_{\mathrm{H},k}^{m}$ be a variable matrix. Then,*

$$\max_{0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}} r(\, A + X\,) = \min\{\, m, \quad r(A) + k\,\}, \tag{3.1}$$

$$\min_{0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}} r(\, A + X\,) = \max\{\, i_{+}(A), \quad r(A) - k\}\,, \tag{3.2}$$

$$\max_{0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}} i_{+}(\, A + X\,) = \min\{\, m, \quad i_{+}(A) + k\,\}\,, \tag{3.3}$$

$$\min_{0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}} i_{+}(\, A + X\,) = i_{+}(A), \tag{3.4}$$

$$\max_{0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}} i_{-}(\, A + X\,) = i_{-}(A), \tag{3.5}$$

$$\min_{0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}} i_{-}(\, A + X\,) = \max\{\, 0, \quad i_{-}(A) - k\,\}, \tag{3.6}$$

and

$$\max_{0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}} r(\, A - X\,) = \min\{\, m, \quad r(A) + k\,\}, \tag{3.7}$$

$$\min_{0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}} r(\, A - X\,) = \max\{\, i_{-}(A), \quad r(A) - k\}\,, \tag{3.8}$$

$$\max_{0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}} i_{+}(\, A - X\,) = i_{+}(A), \tag{3.9}$$

$$\min_{0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}} i_{+}(\, A - X\,) = \max\{\, 0, \; i_{+}(A) - k\,\}, \tag{3.10}$$

$$\max_{0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}} i_{-}(\, A - X\,) = \min\{\, m, \quad i_{-}(A) + k\,\}\,, \tag{3.11}$$

$$\min_{0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}} i_{-}(\, A - X\,) = i_{-}(A). \tag{3.12}$$

The Hermitian matrices Xs satisfying these equalities can be formulated from the canonical form of A under the Hermitian congruence. Further,

(a) *For any integer t_1 between the two quantities on the right-hand sides of* (3.1) *and* (3.2), *there exists a* $0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}$ *such that* $r(\, A + X\,) = t_1$.
(b) *For any integer t_2 between the two quantities on the right-hand sides of* (3.3) *and* (3.4), *there exists a* $0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}$ *such that* $i_{+}(\, A + X\,) = t_2$.
(c) *For any integer t_3 between the two quantities on the right-hand sides of* (3.5) *and* (3.6), *there exists a* $0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}$ *such that* $i_{-}(\, A + X\,) = t_3$.
(d) *For any integer t_4 between the two quantities on the right-hand sides of* (3.7) *and* (3.8), *there exists a* $0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}$ *such that* $r(\, A + X\,) = t_4$.
(e) *For any integer t_5 between the two quantities on the right-hand sides of* (3.9) *and* (3.10), *there exists a* $0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}$ *such that* $i_{+}(\, A + X\,) = t_5$.
(f) *For any integer t_6 between the two quantities on the right-hand sides of* (3.11) *and* (3.12), *there exists an* $0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}$ *such that* $i_{-}(\, A + X\,) = t_6$.

Proof Note that for any $0 \leqslant X \in \mathbb{C}_{\mathrm{H},k}^{m}$,

$$r(\, A \pm X\,) \leqslant m, \quad r(\pm X) \leqslant k, \quad i_{+}(X) = r(X), \quad i_{-}(X) = 0, \quad i_{+}(-X) = 0, \quad i_{-}(-X) = r(X) \tag{3.13}$$

always hold. Hence, it is easily derived from (1.11) and (1.12) that for any $0 \leqslant X \in \mathbb{C}_{\mathrm{H,k}}^{m}$,

$$\begin{aligned}
r(\, A + X\,) &\leqslant \min\{\, m, \quad r(A) + r(X)\,\} \leqslant \min\{\, m, \quad r(A) + k\,\},\\
i_{+}(\, A + X\,) &\leqslant \min\{\, m, \quad i_{+}(A) + i_{+}(X)\,\} \leqslant \min\{\, m, \quad i_{+}(A) + k\,\}\,,\\
i_{-}(\, A + X\,) &\leqslant i_{-}(A) + i_{-}(X) = i_{-}(A),\\
r(\, A - X\,) &\leqslant \min\{\, m, \quad r(A) + r(-X)\,\} \leqslant \min\{\, m, \quad r(A) + k\,\},\\
i_{+}(\, A - X\,) &\leqslant i_{+}(A) + i_{+}(-X) = i_{+}(A),\\
i_{-}(\, A - X\,) &\leqslant \min\{\, m, \quad i_{-}(A) + i_{-}(-X)\,\} \leqslant \min\{\, m, \quad i_{-}(A) + k\,\}\,.
\end{aligned}$$

Hence, the right-hand sides of (3.1), (3.3), (3.5), (3.7), (3.9) and (3.11) are upper bounds for the ranks and partial inertias of the matrix expressions on the left-hand sides. Note from (1.13)–(1.16) and (3.13) that

$$\begin{aligned} r(\,A+X\,) &\geqslant \max\{\,i_{+}(A), \;\; r(A)-r(X)\} \geqslant \max\{\,r(A)-k, \;\; i_{+}(A)\}\,, \\ i_{+}(\,A+X\,) &\geqslant i_{+}(A), \\ i_{-}(\,A+X\,) &\geqslant \max\{\,0, \;\; i_{-}(A)-i_{-}(-X)\,\} \geqslant \max\{\,0, \;\; i_{-}(A)-k\,\}, \\ r(\,A-X\,) &\geqslant \max\{\,r(A)-r(X), \;\; i_{-}(A)\} \geqslant \max\{\,i_{-}(A), \;\; r(A)-k\,\}, \\ i_{+}(\,A-X\,) &\geqslant \max\{\,0,\; i_{+}(A)-i_{+}(X)\,\} \geqslant \max\{\,0,\; i_{+}(A)-k\,\}, \\ i_{-}(\,A-X\,) &\geqslant i_{-}(A). \end{aligned}$$

Hence, the right-hand sides of (3.2), (3.4), (3.6), (3.8), (3.10) and (3.12) are lower bounds for the ranks and partial inertias of the matrix expressions on the left-hand sides. We next show that there exist Xs such that the ranks and partial inertias of the corresponding matrix expressions $A \pm X$ attain the upper and lower bounds on the right-hand sides of (3.1)–(3.12), so that the upper and lower bounds are exactly the extremal ranks and inertias of $A \pm X$.

Assume that the canonical form of the Hermitian A under the Hermitian congruence is

$$A = UDU^{*}, \quad D = \mathrm{diag}\{\,I_p, \; -I_q, \; 0\,\}, \tag{3.14}$$

where $p = i_{+}(A)$, $q = i_{-}(A)$, and $U \in \mathbb{C}^{m\times m}$ is a nonsingular matrix. Consequently, for any $0 \leqslant X \in \mathbb{C}^{m}_{\mathrm{H},k}$

$$A \pm X = UDU^{*} \pm X = U\left[\, D \pm U^{-1}X(U^{-1})^{*} \,\right] U^{*}, \tag{3.15}$$

where $0 \leqslant Y = U^{-1}X(U^{-1})^{*} \in \mathbb{C}^{m}_{\mathrm{H},k}$. Without loss of generality, we assume that Y is a real diagonal matrix. Hence, the sum $D+Y$ is a real diagonal matrix as well. Applying (1.8) to (3.15) gives

$$r(\,A+X\,) = r(\,D+Y\,), \quad r(\,A-X\,) = r(\,D-Y\,), \tag{3.16}$$

$$i_{+}(\,A+X\,) = i_{+}(\,D+Y\,), \quad i_{-}(\,A+X\,) = i_{-}(\,D+Y\,), \tag{3.17}$$

$$i_{+}(\,A-X\,) = i_{+}(\,D-Y\,), \quad i_{-}(\,A-X\,) = i_{-}(\,D-Y\,). \tag{3.18}$$

Set $Y = \mathrm{diag}\{\,0, \, 2I_k\,\}$. Then, $Y \geqslant 0$, and the first equalities in (3.16) and (3.17) become

$$r(\,A+X\,) = r(D+\mathrm{diag}\{\,0, \, 2I_k\,\}) = \min\{\,m, \;\, p+q+k\,\} = \min\{\,m, \;\, r(A)+k\,\}, \tag{3.19}$$

$$i_{+}(\,A+X\,) = i_{+}(D+\mathrm{diag}\{\,0, \, 2I_k\,\}) = \min\{\,m, \;\, p+k\,\} = \min\{\,m, \;\, i_{+}(A)+k\,\}, \tag{3.20}$$

establishing (3.1) and (3.3).

Set

$$Y = \begin{cases} \mathrm{diag}\{\,0, \, I_q, \, 0\,\} & \text{for } k \geqslant q \\ \mathrm{diag}\{\,0, \, I_k, \, 0\,\} & \text{for } k < q. \end{cases}$$

Then, $Y \geqslant 0$, and the first equality in (3.16) and the second equality in (3.17) become

$$\begin{aligned} r(\,A+X\,) = r(\,D+Y\,) = \begin{cases} p & \text{for } k \geqslant q \\ p+q-k & \text{for } k < q \end{cases} &= \max\{\,p, \;\, p+q-k\,\} \\ &= \max\{\,i_{+}(A), \;\, r(A)-k\,\}, \end{aligned} \tag{3.21}$$

$$\begin{aligned} i_{-}(\,A+X\,) = i_{-}(\,D+Y\,) = \begin{cases} 0 & \text{for } k \geqslant q \\ q-k & \text{for } k < q \end{cases} &= \max\{\,0, \;\, q-k\,\} \\ &= \max\{\,0, \;\, i_{-}(A)-k\,\}, \end{aligned} \tag{3.22}$$

establishing (3.2) and (3.6).

Set $Y = 0$ in (3.17) yields (3.4) and (3.5).

Set $Y = \mathrm{diag}\{\,0, \, 2I_k\,\}$. Then, $Y \geqslant 0$ and the second equality in (3.16) becomes

$$r(\,A-X\,) = r\,(D-\mathrm{diag}\{\,0, \, 2I_k\,\}) = \min\{\,m, \;\;\, p+q+k\,\} = \min\{\,m, \;\;\, r(A)+k\,\},$$

establishing (3.7).

Set

$$Y = \begin{cases} \mathrm{diag}\{\,I_p, \, 0\,\} & \text{for } k \geqslant p \\ \mathrm{diag}\{\,I_k, \, 0\,\} & \text{for } k < p. \end{cases}$$

Then, $Y \geqslant 0$, and the second equality in (3.16) and the first equality in (3.18) become

$$r(\,A - X\,) = r(\,D - Y\,) = \begin{cases} q & \text{for } k \geqslant p \\ q + p - k & \text{for } k < p \end{cases} = \max\{\,q, \quad p + q - k\,\}$$
$$= \max\{\,i_{-}(A), \quad r(A) - k\,\}, \tag{3.23}$$
$$i_{+}(\,A - X\,) = r(\,D - Y\,) = \begin{cases} 0 & \text{for } p \leqslant k \\ p - k & \text{for } p > k \end{cases} = \max\{\,0, \quad p - k\,\}$$
$$= \min\{\,0, \quad i_{+}(A) - k\,\}, \tag{3.24}$$

establishing (3.8) and (3.10).

Set

$$Y = \begin{cases} \mathrm{diag}\{\,2I_p,\, 0,\, I_{k-p}\,\} & \text{for } k \geqslant p \\ \mathrm{diag}\{\,2I_k,\, 0,\, I_{p-k}\,\} & \text{for } k < p. \end{cases}$$

Then, $Y \geqslant 0$, and the second equality in (3.18) becomes

$$i_{-}(\,A - X\,) = i_{-}(\,D - Y\,) = \begin{cases} m & \text{for } k \geqslant p \\ q + k & \text{for } k < p \end{cases} = \min\{\,m, \quad q + k\,\} = \min\{\,m, \quad i_{-}(A) + k\,\}, \tag{3.25}$$

establishing (3.11). Set $Y = 0$ in (3.18) yields (3.9) and (3.12). Results (a)–(g) follow from the similar arguments for establishing Lemma 2.1(a)–(c). $\square$

A few special cased of Lemma 3.1 are worth mentioning. For instance, setting $k = 1$ in Lemma 3.1 yields the following consequence.

Corollary 3.2. *Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$ be given, and $\alpha \in \mathbb{C}^{m\times 1}$ be a variable vector. Then,*

$$\max_{\alpha\in\mathbb{C}^{m\times 1}} r(\,A + \alpha\alpha^{*}\,) = \min\{\,m, \quad r(A) + 1\,\}, \tag{3.26}$$
$$\min_{\alpha\in\mathbb{C}^{m\times 1}} r(\,A + \alpha\alpha^{*}\,) = \max\{\,r(A) - 1, \; i_{+}(A)\}\,, \tag{3.27}$$
$$\max_{\alpha\in\mathbb{C}^{m\times 1}} i_{+}(\,A + \alpha\alpha^{*}\,) = \min\{\,m, \quad i_{+}(A) + 1\,\}\,, \tag{3.28}$$
$$\min_{\alpha\in\mathbb{C}^{m\times 1}} i_{+}(\,A + \alpha\alpha^{*}\,) = i_{+}(A), \tag{3.29}$$
$$\max_{\alpha\in\mathbb{C}^{m\times 1}} i_{-}(\,A + \alpha\alpha^{*}\,) = i_{-}(A), \tag{3.30}$$
$$\min_{\alpha\in\mathbb{C}^{m\times 1}} i_{-}(\,A + \alpha\alpha^{*}\,) = \max\{\,0, \quad i_{-}(A) - 1\,\}, \tag{3.31}$$

and

$$\max_{\alpha\in\mathbb{C}^{m\times 1}} r(\,A - \alpha\alpha^{*}\,) = \min\{\,m, \quad r(A) + 1\,\}, \tag{3.32}$$
$$\min_{\alpha\in\mathbb{C}^{m\times 1}} r(\,A - \alpha\alpha^{*}\,) = \max\{\,i_{-}(A), \quad r(A) - 1\}\,, \tag{3.33}$$
$$\max_{\alpha\in\mathbb{C}^{m\times 1}} i_{+}(\,A - \alpha\alpha^{*}\,) = i_{+}(A), \tag{3.34}$$
$$\min_{\alpha\in\mathbb{C}^{m\times 1}} i_{+}(\,A - \alpha\alpha^{*}\,) = \max\{\,0, \; i_{+}(A) - 1\,\}, \tag{3.35}$$
$$\max_{\alpha\in\mathbb{C}^{m\times 1}} i_{-}(\,A - \alpha\alpha^{*}\,) = \min\{\,m, \quad i_{-}(A) + 1\,\}\,, \tag{3.36}$$
$$\min_{\alpha\in\mathbb{C}^{m\times 1}} i_{-}(\,A - \alpha\alpha^{*}\,) = i_{-}(A). \tag{3.37}$$

Some previous work on the inertia of $A + \alpha\alpha^{*}$ can be found in [15]. Another useful consequence of Lemma 3.1 is given below, which will be used to derive the extremal ranks and inertias of Hermitian expressions in (1.5).

Corollary 3.3. *Let $A \in \mathbb{C}_{\mathrm{H}}^{n}$, and $B \in \mathbb{C}^{n\times m}$ be given given with $\mathscr{R}(A) \subseteq \mathscr{R}(B)$, and let $X \in \mathbb{C}_{\mathrm{H},k}^{m}$ be a variable* *matrix and* $k \leqslant r(B)$. *Then,*

$$\max_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} r(\, A + BXB^* \,) = \min\{\, r(B), \quad r(A) + k \,\}, \tag{3.38}$$

$$\min_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} r(\, A + BXB^* \,) = \max\{\, i_{+}(A), \quad r(A) - k \,\}, \tag{3.39}$$

$$\max_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{+}(\, A + BXB^* \,) = \min\{\, r(B), \quad i_{+}(A) + k \,\}, \tag{3.40}$$

$$\min_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{+}(\, A + BXB^* \,) = i_{+}(A), \tag{3.41}$$

$$\max_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{-}(\, A + BXB^* \,) = i_{-}(A), \tag{3.42}$$

$$\min_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{-}(\, A + BXB^* \,) = \max\{\, 0, \quad i_{-}(A) - k \,\}, \tag{3.43}$$

and

$$\max_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} r(\, A - BXB^* \,) = \min\{\, r(B), \quad r(A) + k \,\}, \tag{3.44}$$

$$\min_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} r(\, A - BXB^* \,) = \max\{\, i_{-}(A), \quad r(A) - k \,\}, \tag{3.45}$$

$$\max_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{+}(\, A - BXB^* \,) = i_{+}(A), \tag{3.46}$$

$$\min_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{+}(\, A - BXB^* \,) = \max\{\, 0, \; i_{+}(A) - k \,\}, \tag{3.47}$$

$$\max_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{-}(\, A - BXB^* \,) = \min\{\, r(B), \quad i_{-}(A) + k \,\}, \tag{3.48}$$

$$\min_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{-}(\, A - BXB^* \,) = i_{-}(A). \tag{3.49}$$

If $k \geqslant r(B)$, *then,*

$$\max_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} r(\, A + BXB^* \,) = r(B), \tag{3.50}$$

$$\min_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} r(\, A + BXB^* \,) = i_{+}(A), \tag{3.51}$$

$$\max_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{+}(\, A + BXB^* \,) = r(B), \tag{3.52}$$

$$\min_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{+}(\, A + BXB^* \,) = i_{+}(A), \tag{3.53}$$

$$\max_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{-}(\, A + BXB^* \,) = i_{-}(A), \tag{3.54}$$

$$\min_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{-}(\, A + BXB^* \,) = 0, \tag{3.55}$$

and

$$\max_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} r(\, A - BXB^* \,) = r(B), \tag{3.56}$$

$$\min_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} r(\, A - BXB^* \,) = i_{-}(A), \tag{3.57}$$

$$\max_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{+}(\, A - BXB^* \,) = i_{+}(A), \tag{3.58}$$

$$\min_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{+}(\, A - BXB^* \,) = 0, \tag{3.59}$$

$$\max_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{-}(\, A - BXB^* \,) = r(B), \tag{3.60}$$

$$\min_{0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}} i_{-}(\, A - BXB^* \,) = i_{-}(A). \tag{3.61}$$

Proof Note that $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ is equivalent to $A = BGB^*$ for some matrix $G \in \mathbb{C}^m_{\mathrm{H}}$. Also, assume that the rank decomposition of B is $B = PQ$, where $P \in \mathbb{C}^{n\times s}$, $Q \in \mathbb{C}^{s\times m}$ and $r(B) = r(P) = r(Q) = s$. Consequently, for any $0 \leqslant X \in \mathbb{C}^m_{\mathrm{H},k}$ and $r(X) \leqslant s$,

$$A \pm BXB^* = BGB^* \pm BXB^* = PQGQ^*P^* \pm PQXQ^*P^* = P(\, QGQ^* \pm Y \,)P^*, \tag{3.62}$$

where the matrix $Y = QXQ^* \geqslant 0$ and $r(Y) = r(QXQ^*) \leqslant k \leqslant s$. Applying Lemma 1.5 to (3.62) gives

$$
\begin{aligned}
r(\, A \pm BXB^* \,) &= r(\, QGQ^* \pm Y \,), & (3.63)\\
i_{\pm}(\, A + BXB^* \,) &= i_{\pm}(\, QGQ^* + Y \,), & (3.64)\\
i_{\pm}(\, A - BXB^* \,) &= i_{\pm}(\, QGQ^* - Y \,). & (3.65)
\end{aligned}
$$

In this case, applying Lemma 3.1 to the right-hand sides of (3.63)–(3.65) gives

$$
\begin{aligned}
\max_{0 \leqslant Y \in \mathbb{C}^m_{\mathrm{H},k}} r(\, G + Y \,) &= \min\{\, s, \quad r(G) + k \,\}, & (3.66)\\
\min_{0 \leqslant Y \in \mathbb{C}^m_{\mathrm{H},k}} r(\, G + Y \,) &= \max\{\, i_{+}(G), \quad r(G) - k \,\}, & (3.67)\\
\max_{0 \leqslant Y \in \mathbb{C}^m_{\mathrm{H},k}} i_{+}(\, G + Y \,) &= \min\{\, s, \quad i_{+}(G) + k \,\}, & (3.68)\\
\min_{0 \leqslant Y \in \mathbb{C}^m_{\mathrm{H},k}} i_{+}(\, G + Y \,) &= i_{+}(G), & (3.69)\\
\max_{0 \leqslant Y \in \mathbb{C}^m_{\mathrm{H},k}} i_{-}(\, G + Y \,) &= i_{-}(G), & (3.70)\\
\min_{0 \leqslant Y \in \mathbb{C}^m_{\mathrm{H},k}} i_{-}(\, G + Y \,) &= \max\{\, 0, \quad i_{-}(G) - k \,\}, & (3.71)
\end{aligned}
$$

and

$$
\begin{aligned}
\max_{0 \leqslant Y \in \mathbb{C}^m_{\mathrm{H},k}} r(\, G - Y \,) &= \min\{\, s, \quad r(G) + k \,\}, & (3.72)\\
\min_{0 \leqslant Y \in \mathbb{C}^m_{\mathrm{H},k}} r(\, G - Y \,) &= \max\{\, i_{-}(G), \quad r(G) - k \,\}, & (3.73)\\
\max_{0 \leqslant Y \in \mathbb{C}^m_{\mathrm{H},k}} i_{+}(\, G - Y \,) &= i_{+}(G), & (3.74)\\
\min_{0 \leqslant Y \in \mathbb{C}^m_{\mathrm{H},k}} i_{+}(\, G - Y \,) &= \max\{\, 0, \; i_{+}(G) - k \,\}, & (3.75)\\
\max_{0 \leqslant Y \in \mathbb{C}^m_{\mathrm{H},k}} i_{-}(\, G - Y \,) &= \min\{\, s, \quad i_{-}(G) + k \,\}, & (3.76)\\
\min_{0 \leqslant Y \in \mathbb{C}^m_{\mathrm{H},k}} i_{-}(\, G - Y \,) &= i_{-}(G). & (3.77)
\end{aligned}
$$

Substituting $s = r(B)$, $r(G) = r(A)$ and $i_{\pm}(G) = i_{\pm}(A)$ into (3.66)–(3.77) yields (3.38)–(3.49). When $k \geqslant r(B)$, (3.38)–(3.49) reduce to (3.50)–(3.61). $\square$

The proof of Corollary 3.3 also gives the construction of the Hermitian matrices Xs that satisfy (3.38)–(3.61). Applying Corollary 3.3 to the right-hand sides of (2.36) and (2.37), we obtain the main result of this section.

Theorem 3.4. *Let $A \in \mathbb{C}^m_{\mathrm{H}}$ and $B \in \mathbb{C}^{m \times n}$ be given, M be as given in* (2.38), *and let $X \in \mathbb{C}^n_{\mathrm{H},k}$ be a variable matrix.*

(a) *If $k \leqslant r(M) - r[\, A, \, B \,]$, then*

$$
\begin{aligned}
\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r(\, A + BXB^* \,) &= \min\{\, r[\, A, \, B \,], \quad r(A) + k \,\}, & (3.78)\\
\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r(\, A + BXB^* \,) &= \max\{\, i_{+}(A) + r[\, A, \, B \,] - i_{+}(M), \quad r(A) - k \,\}, & (3.79)\\
\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_{+}(\, A + BXB^* \,) &= \min\{\, i_{+}(M), \quad i_{+}(A) + k \,\}, & (3.80)\\
\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} (\, A + BXB^* \,) &= i_{+}(A), & (3.81)\\
\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_{-}(\, A + BXB^* \,) &= i_{-}(A), & (3.82)\\
\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_{-}(\, A + BXB^* \,) &= \max\{\, r[\, A, \, B \,] - i_{+}(M), \quad i_{-}(A) - k \,\}, & (3.83)
\end{aligned}
$$

and

$$\max_{0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}} r(\,A-BXB^*\,)=\min\{\,r[\,A,\,B\,],\quad r(A)+k\,\},\tag{3.84}$$

$$\min_{0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}} r(\,A-BXB^*\,)=\max\{\,i_-(A)+r[\,A,\,B\,]-i_-(M),\quad r(A)-k\,\}\,,\tag{3.85}$$

$$\max_{0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}} i_+(\,A-BXB^*\,)=i_+(A),\tag{3.86}$$

$$\min_{0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}} i_+(\,A-BXB^*\,)=\max\{\,r[\,A,\,B\,]-i_-(M),\quad i_+(A)-k\,\},\tag{3.87}$$

$$\max_{0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}} i_-(\,A-BXB^*\,)=\min\{\,i_-(M),\quad i_-(A)+k\,\}\,,\tag{3.88}$$

$$\min_{0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}} i_-(\,A-BXB^*\,)=i_-(A).\tag{3.89}$$

The Hermitian matrices Xs satisfying these equalities can be formulated from the canonical form of certain operations A and B under the Hermitian congruence. Further,

(i) *For any integer t_1 between the two quantities on the right-hand sides of* (3.78) *and* (3.79), *there exists a $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $r(\,A+X\,)=t_1$.*
(ii) *For any integer t_2 between the two quantities on the right-hand sides of* (3.80) *and* (3.81), *there exists a $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $i_+(\,A+X\,)=t_2$.*
(iii) *For any integer t_3 between the two quantities on the right-hand sides of* (3.82) *and* (3.83), *there exists a $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $i_-(\,A+X\,)=t_3$.*
(iv) *For any integer t_4 between the two quantities on the right-hand sides of* (3.84) *and* (3.85), *there exists a $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $r(\,A+X\,)=t_4$.*
(vi) *For any integer t_5 between the two quantities on the right-hand sides of* (3.86) *and* (3.87), *there exists a $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $i_+(\,A+X\,)=t_5$.*
(vi) *For any integer t_6 between the two quantities on the right-hand sides of* (3.88) *and* (3.89), *there exists a $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $i_-(\,A+X\,)=t_6$.*

In particular,

(vii) *There exists a $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $A+BXB^*$ is nonsingular if and only if $r[\,A,\,B\,]=m$ and $r(A)\geqslant m-k$.*
(viii) *$A+BXB^*$ is nonsingular for any $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ if and only if $r(A)=m$ and $BA^{-1}B^*\geqslant 0$.*
(ix) *There exists a $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $A+BXB^*=0$ if and only if $r(A)\leqslant k$, $A\leqslant 0$ and $\mathscr{R}(A)\subseteq\mathscr{R}(B)$.*
(x) *$A+BXB^*=0$ for any $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ if and only if $A=0$ and $B=0$.*
(xi) *There exists a $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $A+BXB^*>0$ if and only if $i_+(M)\geqslant m$ and $i_+(A)\geqslant m-k$.*
(xii) *$A+BXB^*>0$ for any $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ if and only if $A>0$.*
(xiii) *There exists a $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $A+BXB^*\geqslant 0$ if and only if $i_-(A)\leqslant k$ and $i_+(M)\geqslant r[\,A,\,B\,]$.*
(xiv) *$A+BXB^*\geqslant 0$ for any $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ if and only if $A\geqslant 0$.*
(xv) *There exists a $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $A+BXB^*<0$ if and only if $A<0$.*
(xvi) *$A+BXB^*<0$ for any $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ if and only if $A<0$ and $B=0$.*
(xvii) *There exists an $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $A+BXB^*\leqslant 0$ if and only if $A\leqslant 0$.*
(xviii) *$A+BXB^*\leqslant 0$ for any $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ if and only if $A\leqslant 0$ and $B=0$.*
(xix) *There exists a $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $A-BXB^*$ is nonsingular if and only if $r[\,A,\,B\,]=m$ and $r(A)\geqslant m-k$.*
(xx) *$A-BXB^*$ is nonsingular for any $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ if and only if if and only if $r(A)=m$ and $BA^{-1}B^*\leqslant 0$.*
(xxi) *There exists a $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $BXB^*=A$ if and only if $r(A)\leqslant k$, $A\geqslant 0$ and $\mathscr{R}(A)\subseteq\mathscr{R}(B)$.*
(xxii) *$BXB^*=A$ for any $X0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ if and only if $A=0$ and $B=0$.*
(xxiii) *There exists a $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $A-BXB^*>0$ if and only if $A>0$.*
(xxiv) *$A-BXB^*>0$ for any $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ if and only if $A>0$ and $B=0$.*
(xxv) *There exists an X with $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $A-BXB^*\geqslant 0$ if and only if $A\geqslant 0$.*
(xxvi) *$A-BXB^*\geqslant 0$ for any $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ if and only if $A\geqslant 0$ and $B=0$.*
(xxvii) *There exists an X with $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $A-BXB^*<0$ if and only if $i_-(M)\geqslant m$ and $i_-(A)\geqslant m-k$.*
(xxviii) *$A-BXB^*<0$ for any $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ if and only if $A<0$.*
(xxix) *There exists a $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ such that $A-BXB^*\leqslant 0$ if and only if $i_+(A)\leqslant k$ and $i_-(M)\geqslant r[\,A,\,B\,]$.*
(xxx) *$A-BXB^*\leqslant 0$ for any $0\leqslant X\in\mathbb{C}^n_{\mathrm{H},k}$ if and only if $(A)\leqslant 0$.*

(b) *If $k \geqslant r(M) - r[\, A,\, B\,]$, then*

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r(\, A + BXB^* \,) = r[\, A,\, B\,], \tag{3.90}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r(\, A + BXB^* \,) = i_+(A) + r[\, A,\, B\,] - i_+(M), \tag{3.91}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_+(\, A + BXB^* \,) = i_+(M), \tag{3.92}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_+(\, A + BXB^* \,) = i_+(A), \tag{3.93}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_-(\, A + BXB^* \,) = i_-(A), \tag{3.94}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_-(\, A + BXB^* \,) = r[\, A,\, B\,] - i_+(M), \tag{3.95}$$

and

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r(\, A - BXB^* \,) = r[\, A,\, B\,], \tag{3.96}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r(\, A - BXB^* \,) = i_-(A) + r[\, A,\, B\,] - i_-(M), \tag{3.97}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_+(\, A - BXB^* \,) = i_+(A), \tag{3.98}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_+(\, A - BXB^* \,) = r[\, A,\, B\,] - i_-(M), \tag{3.99}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_-(\, A - BXB^* \,) = i_-(M), \tag{3.100}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_-(\, A - BXB^* \,) = i_-(A). \tag{3.101}$$

Proof It follows from (2.36) and (2.37) imply that

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r(\, A \pm BXB^* \,) = 2r[\, A,\, B\,] - r(M) + \max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r[\, F_{S_1}(\, S^*M^{\dagger}S \mp X\,)F_{S_1}\,], \tag{3.102}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r(\, A \pm BXB^* \,) = 2r[\, A,\, B\,] - r(M) + \min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r[\, F_{S_1}(\, S^*M^{\dagger}S \mp X\,)F_{S_1}\,], \tag{3.103}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_{\pm}(\, A + BXB^* \,) = r[\, A,\, B\,] - i_{\mp}(M) + \max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_{\mp}[\, F_{S_1}(\, S^*M^{\dagger}S - X\,)F_{S_1}\,], \tag{3.104}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_{\pm}(\, A + BXB^* \,) = r[\, A,\, B\,] - i_{\mp}(M) + \min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_{\mp}[\, F_{S_1}(\, S^*M^{\dagger}S - X\,)F_{S_1}\,], \tag{3.105}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_{\pm}(\, A - BXB^* \,) = r[\, A,\, B\,] - i_{\mp}(M) + \max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_{\mp}[\, F_{S_1}(\, S^*M^{\dagger}S + X\,)F_{S_1}\,], \tag{3.106}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_{\pm}(\, A - BXB^* \,) = r[\, A,\, B\,] - i_{\mp}(M) + \min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_{\mp}[\, F_{S_1}(\, S^*M^{\dagger}S + X\,)F_{S_1}\,]. \tag{3.107}$$

If $k \leqslant r(F_{S_1}) = r(M) - r[\,A,\, B\,]$, then applying Corollary 3.3 to the right-hand sides of the right-hand sides of (3.102)–(3.107) yields

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = \min\{\,r(F_{S_1}),\;\; r(F_{S_1}S^*M^{\dagger}SF_{S_1}) + k\,\}, \tag{3.108}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = \max\left\{\,i_-(F_{S_1}S^*M^{\dagger}SF_{S_1}),\;\; r(F_{S_1}S^*M^{\dagger}SF_{S_1}) - k\right\}, \tag{3.109}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_-[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = \min\left\{\,r(F_{S_1}),\;\; i_-(F_{S_1}S^*M^{\dagger}SF_{S_1}) + k\,\right\}, \tag{3.110}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_-[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = i_-(F_{S_1}S^*M^{\dagger}SF_{S_1}), \tag{3.111}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_+[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = i_+(F_{S_1}S^*M^{\dagger}SF_{S_1}), \tag{3.112}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_+[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = \max\{\,0,\; i_+(F_{S_1}S^*M^{\dagger}SF_{S_1}) - k\,\}, \tag{3.113}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r[\,F_{S_1}(\,S^*M^{\dagger}S + X\,)F_{S_1}\,] = \min\{\,r(F_{S_1}),\;\; r(F_{S_1}S^*M^{\dagger}SF_{S_1}) + k\,\}, \tag{3.114}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r[\,F_{S_1}(\,S^*M^{\dagger}S + X\,)F_{S_1}\,] = \max\left\{\,i_+(F_{S_1}S^*M^{\dagger}SF_{S_1}),\;\; r(F_{S_1}S^*M^{\dagger}SF_{S_1}) - k\right\}, \tag{3.115}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_-[\,F_{S_1}(\,S^*M^{\dagger}S + X\,)F_{S_1}\,] = i_-(F_{S_1}S^*M^{\dagger}SF_{S_1}), \tag{3.116}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_-[\,F_{S_1}(\,S^*M^{\dagger}S + X\,)F_{S_1}\,] = \max\{\,0,\; i_-(F_{S_1}S^*M^{\dagger}SF_{S_1}) - k\,\}, \tag{3.117}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_+[\,F_{S_1}(\,S^*M^{\dagger}S + X\,)F_{S_1}\,] = \min\left\{\,r(F_{S_1}),\;\; i_+(F_{S_1}S^*M^{\dagger}SF_{S_1}) + k\,\right\}, \tag{3.118}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_+[\,F_{S_1}(\,S^*M^{\dagger}S + X\,)F_{S_1}\,] = i_+(F_{S_1}S^*M^{\dagger}SF_{S_1}). \tag{3.119}$$

Substituting (3.108)–(3.119) into (3.102)–(3.107) and simplifying, we obtain (3.78)–(3.89). Results (a)–(x) follow from (3.78)–(3.89) and Lemma 1.2.

If $k \geqslant r(F_{S_1}) = r(M) - r[\,A,\, B\,]$, then applying Corollary 3.3 to the right-hand sides of (3.102)–(3.107) yields

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = r(F_{S_1}), \tag{3.120}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = i_-(F_{S_1}S^*M^{\dagger}SF_{S_1}), \tag{3.121}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_-[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = r(F_{S_1}), \tag{3.122}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_-[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = i_-(F_{S_1}S^*M^{\dagger}SF_{S_1}), \tag{3.123}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_+[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = i_+(F_{S_1}S^*M^{\dagger}SF_{S_1}), \tag{3.124}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_+[\,F_{S_1}(\,S^*M^{\dagger}S - X\,)F_{S_1}\,] = 0, \tag{3.125}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r[\,F_{S_1}(\,S^*M^{\dagger}S + X\,)F_{S_1}\,] = r(F_{S_1}), \tag{3.126}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} r[\,F_{S_1}(\,S^*M^{\dagger}S + X\,)F_{S_1}\,] = i_+(F_{S_1}S^*M^{\dagger}SF_{S_1}), \tag{3.127}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_-[\,F_{S_1}(\,S^*M^{\dagger}S + X\,)F_{S_1}\,] = i_-(F_{S_1}S^*M^{\dagger}SF_{S_1}), \tag{3.128}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_-[\,F_{S_1}(\,S^*M^{\dagger}S + X\,)F_{S_1}\,] = i_-(F_{S_1}S^*M^{\dagger}SF_{S_1}), \tag{3.129}$$

$$\max_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_+[\,F_{S_1}(\,S^*M^{\dagger}S + X\,)F_{S_1}\,] = \min\left\{\,r(F_{S_1}),\;\; i_+(F_{S_1}S^*M^{\dagger}SF_{S_1}) + k\,\right\}, \tag{3.130}$$

$$\min_{0 \leqslant X \in \mathbb{C}^n_{\mathrm{H},k}} i_+[\,F_{S_1}(\,S^*M^{\dagger}S + X\,)F_{S_1}\,] = i_+(F_{S_1}S^*M^{\dagger}SF_{S_1}). \tag{3.131}$$

Substituting (3.120)–(3.131) into (3.102)–(3.107) and simplifying , we obtain (3.90)–(3.101). Results (i)–(vi) in (a) follow from the similar arguments for establishing Lemma 2.1(a)–(c). $\square$

From Theorem 3.4, we obtain the following results on the extremal ranks and inertias of (1.1) subject to (1.3).

Theorem 3.5. *Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$ and $B \in \mathbb{C}^{m\times n}$ be given, and let M be as given in* (2.38)*. Also, denote*

$$\mathcal{S} = \{\, X \in \mathbb{C}_{\mathrm{H},k}^{m} \mid \mathscr{R}(X) \subseteq \mathscr{R}(B) \text{ and } X \geqslant 0\}. \tag{3.132}$$

(a) *If $k \leqslant r(M) - r[\,A,\, B\,]$, then*

$$\max_{X\in\mathcal{S}} r(\,A + X\,) = \min\{\, r[\,A,\, B\,], \quad r(A) + k\,\}, \tag{3.133}$$
$$\min_{X\in\mathcal{S}} r(\,A + X\,) = \max\{\, i_{+}(A) + r[\,A,\, B\,] - i_{+}(M), \quad r(A) - k\,\}, \tag{3.134}$$
$$\max_{X\in\mathcal{S}} i_{+}(\,A + X\,) = \min\{\, i_{+}(M), \quad i_{+}(A) + k\,\}, \tag{3.135}$$
$$\min_{X\in\mathcal{S}} i_{+}(\,A + X\,) = i_{+}(A), \tag{3.136}$$
$$\max_{X\in\mathcal{S}} i_{-}(\,A + X\,) = i_{-}(A), \tag{3.137}$$
$$\min_{X\in\mathcal{S}} i_{-}(\,A + X\,) = \max\{\, r[\,A,\, B\,] - i_{+}(M), \quad i_{-}(A) - k\,\}, \tag{3.138}$$

and

$$\max_{X\in\mathcal{S}} r(\,A - X\,) = \min\{\, r[\,A,\, B\,], \quad r(A) + k\,\}, \tag{3.139}$$
$$\min_{X\in\mathcal{S}} r(\,A - X\,) = \max\{\, i_{-}(A) + r[\,A,\, B\,] - i_{-}(M), \quad r(A) - k\,\}, \tag{3.140}$$
$$\max_{X\in\mathcal{S}} i_{+}(\,A - X\,) = i_{+}(A), \tag{3.141}$$
$$\min_{X\in\mathcal{S}} i_{+}(\,A - X\,) = \max\{\, r[\,A,\, B\,] - i_{-}(M), \quad i_{+}(A) - k\,\}, \tag{3.142}$$
$$\max_{X\in\mathcal{S}} i_{-}(\,A - X\,) = \min\{\, i_{-}(M), \quad i_{-}(A) + k\,\}, \tag{3.143}$$
$$\min_{X\in\mathcal{S}} i_{-}(\,A - X\,) = i_{-}(A). \tag{3.144}$$

The Hermitian matrices Xs satisfying these equalities can be formulated from the canonical form of certain operations A and B under the Hermitian congruence.

(b) *If $r(M) - r[\,A,\, B\,] < k \leqslant r(B)$, then*

$$\max_{X\in\mathcal{S}} r(\,A + X\,) = r[\,A,\, B\,], \tag{3.145}$$
$$\min_{X\in\mathcal{S}} r(\,A + X\,) = i_{+}(A) + r[\,A,\, B\,] - i_{+}(M), \tag{3.146}$$
$$\max_{X\in\mathcal{S}} i_{+}(\,A + X\,) = i_{+}(M), \tag{3.147}$$
$$\min_{X\in\mathcal{S}} i_{+}(\,A + X\,) = i_{+}(A), \tag{3.148}$$
$$\max_{X\in\mathcal{S}} i_{-}(\,A + X\,) = i_{-}(A), \tag{3.149}$$
$$\min_{X\in\mathcal{S}} i_{-}(\,A + X\,) = r[\,A,\, B\,] - i_{+}(M), \tag{3.150}$$

and

$$\max_{X\in\mathcal{S}} r(\,A - X\,) = r[\,A,\, B\,], \tag{3.151}$$
$$\min_{X\in\mathcal{S}} r(\,A - X\,) = i_{-}(A) + r[\,A,\, B\,] - i_{-}(M), \tag{3.152}$$
$$\max_{X\in\mathcal{S}} i_{+}(\,A - X\,) = i_{+}(A), \tag{3.153}$$
$$\min_{X\in\mathcal{S}} i_{+}(\,A - X\,) = r[\,A,\, B\,] - i_{-}(M), \tag{3.154}$$
$$\max_{X\in\mathcal{S}} i_{-}(\,A - X\,) = i_{-}(M), \tag{3.155}$$
$$\min_{X\in\mathcal{S}} i_{-}(\,A - X\,) = i_{-}(A). \tag{3.156}$$

Theorem 3.6. *Let $A \in \mathbb{C}_{\rm H}^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$ be given, and let M and $\mathcal{S}$ be as given in (2.38) and (3.132). Then, for any $X \in \mathcal{S}$, the following rank and inertia inequalities hold*

$$\max\{\, r[\,A,\ B\,] - i_{+}(M) - i_{-}(A), \quad -k\} \leqslant r(\,A + X\,) - r(A) \leqslant \min\{\, r[\,A,\ B\,] - r(A), \quad k\,\}, \tag{3.157}$$

$$0 \leqslant i_{+}(\,A + X\,) - i_{+}(A) \leqslant \min\{\, i_{+}(M) - i_{+}(A), \quad k\,\}, \tag{3.158}$$

$$\max\{\, r[\,A,\ B\,] - i_{+}(M) - i_{-}(A), \quad -k\,\} \leqslant i_{-}(\,A + X\,) - i_{-}(A) \leqslant 0, \tag{3.159}$$

$$\max\{\, r[\,A,\ B\,] - i_{-}(M) - i_{+}(A), \quad -k\} \leqslant r(\,A - X\,) - r(A) \leqslant \min\{\, r[\,A,\ B\,] - r(A), \quad k\,\}, \tag{3.160}$$

$$\max\{\, r[\,A,\ B\,] - i_{-}(M) - i_{+}(A), \quad -k\,\} \leqslant i_{+}(\,A - X\,) - i_{+}(A) \leqslant 0, \tag{3.161}$$

$$0 \leqslant i_{-}(\,A - X\,) - i_{-}(A) \leqslant \min\{\, i_{-}(M) - i_{-}(A), \quad k\,\}. \tag{3.162}$$

4. Concluding remarks

The results obtained in the previous sections can be used to solve rank and inertia optimization problems on some partially specified Hermitian matrices. For instance,

$$M(X) = \begin{bmatrix} A & B \\ B^{*} & X \end{bmatrix},$$

where $A \in \mathbb{C}_{\rm H}^{m}$ and $B \in \mathbb{C}^{m\times n}$ are given, and $X \in \mathbb{C}_{\rm H}^{n}$ is a variable matrix satisfying some range, rank and definiteness restrictions, as given in (1.2) and (1.3). Note that the $M(X)$ can be rewritten as

$$M(X) = \begin{bmatrix} A & B \\ B^{*} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_n \end{bmatrix} X[\,0,\ I_n\,],$$

which can be regarded as a special case of the matrix expression in (1.4). Hence, applying the results in the previous sections to the partial block matrix $M(X)$ may yield a group of consequences on the maximal/minimal values of the rank/inertia of $M(X)$. Inverse and definiteness completion problems on the partial block matrix $M(X)$ can also be solved. Some previous and recent work on rank/inertia completions of partially specified Hermitian matrices can be found, e.g., in [5, 6, 9, 10, 39].

Note that any matrix A can be decomposed as

$$A = (\,A - X\,) + X, \tag{4.1}$$

where X is any matrix. If both A and X are Hermitian and

$$r(A) = r(\,A - X\,) + r(X), \quad i_{\pm}(A) = i_{\pm}(\,A - X\,) + i_{\pm}(X), \tag{4.2}$$

then both $A - X$ and X are said to satisfy the rank/inertia additivity conditions, or equivalently, both A and X are often said in the literature to satisfy the rank/inertia subtractivity conditions. In the investigation of Hermitian matrix perturbations, the perturbation matrix X is naturally assumed to satisfy the restrictions in (4.2). In such a case, it would be of interest to consider the problems of maximizing and minimizing the rank and inertia of

$$A - X \ \text{ subject to } \ r(\,A - X\,) = r(A) - r(X), \ \ \mathscr{R}(X) \subseteq \mathscr{R}(B) \text{ and } r(X) \leqslant k, \tag{4.3}$$

or

$$A - X \ \text{ subject to } r(\,A - X\,) = r(A) - r(X), \ \mathscr{R}(X) \subseteq \mathscr{R}(B), \ r(X) \leqslant k \text{ and } \ X \geqslant 0\,(X \leqslant 0). \tag{4.4}$$

A more challenging task is to solve the problems of maximizing and minimizing the rank and inertia of

$$A + BXB^{*} \ \text{ subject to } \ r(X) = k. \tag{4.5}$$

In this case, the ranks and inertias of BXB^{*} and $A + BXB^{*}$ may vary with respect to the choice of X.

Since many basic or classic problems in matrix theory, like solvability of matrix equations and matrix inequalities, can be converted to some max-min optimization problems on ranks and inertias of matrices. So that these problems can be approached through the matrix rank/inertia methods. It has been realized that the matrix/rank methods can serve as effective tools to deal with matrices and their operations. The new techniques for solving max-min optimization problems on rank/inertia of matrices enable us to develop new extensions of classic theory on matrix equations and matrix inequalities, which allowed us to analyze algebraic properties of a wide variety of Hermitian matrix expression that could not be handled before. We expect that more optimization problems on maximizing/minimizing ranks/inertias of matrix expressions can be proposed reasonably and solved analytically, while the matrix rank/inertia methods will play more important roles in matrix theory and applications.

References

[1] P. Arbenz and G.H. Golub, On the spectral decomposition of Hermitian matrices modified by low rank perturbations with applications, SIAM. J. Matrix Anal. Appl. 9(1988), 40–58.
[2] W. Barrett, H.T. Hall and R. Loewy, The inverse inertia problem for graphs: cut vertices, trees, and a counterexample, Linear Algebra Appl. (2009), 431(2009), 1147–1191.
[3] A. Ben-Israel and T.N.E. Greville, Generalized Inverses: Theory and Applications, Second ed., Springer, New York, 2003.
[4] D.S. Bernstein, Matrix Mathematics: Theory, Facts and Formulas, Second ed., Princeton University Press, Princeton, 2009.
[5] N. Cohen and J. Dancis. Maximal rank Hermitian completions of partially specified Hermitian matrices, Linear Algebra Appl., 244(1996), 265–276.
[6] N. Cohen and J. Dancis. Inertias of block band matrix completions. SIAM J. Matrix Anal. Appl. 19(1998), 583–612.
[7] M. Dana and Kh. D. Ikramov, On low-rank perturbations of Hermitian and unitary matrices, Zh. Vychisl. Mat. Mat. Fiz. 44(2004), 1259–1273.
[8] M. Dana and Kh. D. Ikramov, On rank-one corrections of complex symmetric matrices, J. Math. Sci. 141(2007), 1614–1617.
[9] J. Dancis, The possible inertias for a Hermitian matrix and its principal submatrices, Linear Algebra Appl. 85(1987), 121–151. 1987.
[10] J. Dancis, Positive semidefinite completions of partial Hermitian matrices. Linear Algebra Appl. 175(1992), 97–114.
[11] E.M. de Sá, On the inertia of sums of Hermitian matrices, Linear Algebra Appl. 37(1981), 143–159.
[12] C. Eckart and G. Young, The approximation of one matrix by another of lower rank, Psychometrika 1(1936), 211–218.
[13] M. Fazel, H. Hindi and S. Boyd, Rank minimization and applications in system Theory, In: Proceedings of the 2004 American Control Conference, 2004, pp. 3273–3278.
[14] J.F. Geelen, Maximum rank matrix completion, Linear Algebra Appl. 288(1999), 211–217.
[15] D.A. Gregory, B. Heyink and K.N. Vander Meulen, Inertia and biclique decompositions of joins of graphs, J. Combin. Theory Ser. B 88(2003), 135–151.
[16] D.A. Gregory, B.L. Shader and V.L. Watts, Biclique decompositions and Hermitian rank, Linear Algebra Appl. 292(1999), 267–280.
[17] N.J.A. Harvey, D.R. Karger and S. Yekhanin, The complexity of matrix completion, In: Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithm, Association for Computing Machinery, New York, pp. 1103–1111, 2006.
[18] N.J. Higham and S.H. Cheng, Modifying the inertia of matrices arising in optimization, Linear Algebra Appl. 275/276(1998), 261–279.
[19] T.M. Hoang and T. Thierauf, The complexity of the inertia, Lecture Notes in Computer Science Vol. 2556(2002), 206–217.
[20] T.M. Hoang and T. Thierauf, The complexity of the inertia and some closure properties of GapL, In: Proceedings of Twentieth Annual IEEE Conference on Computational Complexity, 2005, pp. 28–37.
[21] L. Hogben, Handbook of Linear Algebra, Chapman & Hall/CRC, 2007.
[22] C.R. Johnson and R. Loewy, Possible inertias for the Hermitian part of $A + BX$, Linear and Multilinear Algebra 28(1991), 225–227.
[23] M.G. Kamalvand and Kh. D. Ikramov, Low-rank perturbations of symmetric matrices and their condensed forms under unitary congruences, J. Comput. Math. Math. Phys. 49(2009), 573–578.
[24] M. Laurent, Matrix completion problems, In: The Encyclopedia of Optimization (C. Floudas and P. Pardalos, eds.), Vol. III, Kluwer, 2001, pp. 221–229.
[25] Y. Liu and Y. Tian, More on extremal ranks of the matrix expressions $A - BX \pm X^*B^*$ with statistical applications, Numer. Linear Algebra Appl. 15(2008), 307–325.
[26] Y. Liu and Y. Tian, A simultaneous decomposition of a matrix triplet with applications, Numer. Linear Algebra Appl., DOI:10.1002/nla.701.
[27] Y. Liu and Y. Tian, Max-min problems on the ranks and inertias of the matrix expressions $A - BXC \pm (BXC)^*$ with applications, J. Optim. Theory Appl., accepted.
[28] M. Mahajan and J. Sarma, On the complexity of matrix rank and rigidity, Lecture Notes in Computer Science, Vol. 4649, Springer, 2007, pp. 269–280.
[29] G. Marsaglia and G.P.H. Styan, Equalities and inequalities for ranks of matrices, Linear and Multilinear Algebra 2(1974), 269–292.
[30] B.K. Natarajan, Sparse approximate solutions to linear systems, SIAM J. Comput. 24(1995), 227–234.
[31] A. Ostrowski, A quantitative formulation of Sylvester's law of inertia II, Proc. Nat. Acad. Sci. USA 46(1960), 859–862.
[32] D.V. Ouellette, Schur complements and statistics, Linear Algebra Appl. 36(1981), 187–295.
[33] B. Recht, M. Fazel, P.A. Parrilo, Guaranteed minimum rank Solutions to linear matrix equations via nuclear norm minimization, SIAM Review 52 (2010) 471–501.
[34] B. Recht, W. Xu and B. Hassibi, Null space conditions and thresholds for rank minimization, Math. Progam., DOI: 10.1007/s10107-010-0422-2.
[35] R.E. Skelton, T. Iwasaki and K.M. Grigoriadis, A unified Algebraic Approach to Linear Control Design, Taylor & Francis, London, 1997.
[36] P. Tarazaga, Eigenvalue estimates for symmetric matrices, Linear Algebra Appl. 135(1990), 171–179.
[37] Y. Tian, Equalities and inequalities for inertias of Hermitian matrices with applications, Linear Algebra Appl. 433(2010), 263–296.

[38] Y. Tian, Rank and inertia of submatrices of the Moore–Penrose inverse of a Hermitian matrix, Electron. J. Linear Algebra 20(2010), 226–240.
[39] Y. Tian, Completing block Hermitian matrices with maximal and minimal ranks and inertias, Electron. Linear Algebra Appl. 21(2010), 141–158.
[40] Y. Tian, Expansion formulas for the inertias of Hermitian matrix polynomials and matrix pencils of orthogonal projectors, J. Math. Anal. Appl., doi:10.1016/j.jmaa.2010.09.038.
[41] Y. Tian and Y. Liu, Extremal ranks of some symmetric matrix expressions with applications, SIAM J. Matrix Anal. Appl. 28(2006), 890–905.