Convex envelopes for quadratic and polynomial functions over polytopes

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Abstract

In this paper we consider the problem of computing the value and a supporting hyperplane of the convex envelope for quadratic and polynomial functions over polytopes with known vertex set. We show that for general quadratic functions the computation can be carried on through a copositive problem, but in some cases (including all the two-dimensional ones) we can solve a semidefinite problem. The result is also extended to two-dimensional polynomial functions satisfying certain conditions.

KEYWORDS: convex envelopes, semidefinite programming, copositive programming, quadratic problems, polynomial problems.

1 Introduction

The convex envelope of a nonconvex function f over some region X is the best (largest) possible convex underestimator of f over X, i.e.,

$$conv_{f,X}(\mathbf{x}) = \sup\{c(\mathbf{x}) \ : \ c(\mathbf{x}') \leq f(\mathbf{x}') \ \ \forall \ \mathbf{x}' \in X, \ \ c \text{ is convex}\}$$

An equivalent definition is

$$conv_{f,X}(\mathbf{x}) = \sup\{c(\mathbf{x}) \ : \ c(\mathbf{x}') \leq f(\mathbf{x}') \ \ \forall \ \mathbf{x}' \in X, \ \ c \text{ is affine}\} \qquad (1)$$

where the requirement "c is convex" is substituted by the milder one "c is affine". Finding the convex envelope for a general function f and region X is at least as difficult as minimizing f over X. It may be a hard task even for simple functions and/or regions. E.g., in [4] it has been proved that finding the convex envelope of a multilinear function over the unit hypercube is $\mathcal{NP}$-hard.

For some functions the convex envelope is a polyhedral one, i.e., it is the maximum of a *finite* number of affine underestimators. Results about polyhedral convex envelopes are available, e.g., in [10, 11, 16]. For non polyhedral

convex envelopes some results are available in [6, 13] when the region is a box.
Different results have been proved for some specific functions and regions. Relevant examples are the fractional function y/x over rectangles (see [14, 17]), and the bilinear function xy. For the latter different results are available over different regions:

- *rectangles*: this is the well known result by McCormick [9];
- *D-polytopes* : these are polytopes whose edges all lie along lines with nonpositive slope, for which it can be proved that the convex envelope of the bilinear function is polyhedral (see [12]);
- *triangles with a single edge lying along a line with positive slope*: this result has been derived in [7] and employed within a branch-and-bound scheme to define a new branching technique based on a subdivision into rectangles and/or triangles;
- *triangles*: in [1] a representation of the convex envelope over general triangles through doubly nonnegative matrices (i.e., matrices which are both semidefinite and nonnegative) is given.

In this paper we present some results for the computation of the value and of a supporting hyperplane of the convex envelope for some quadratic and polynomial functions f over polytopes P with known vertex set. In particular, these functions include the bilinear one xy (in fact, they include all two-dimensional quadratic functions). In that case we will show that the computation can be carried on, in a time polynomial with respect to the number of vertices of the polytope, through a semidefinite problem. We immediately remark that, with respect to the previously mentioned results about the bilinear term, here we are not claiming that we are able to return the *analytic form* of the convex envelope. We can only return the *value* of the convex envelope at some points of the polytope, together with a supporting hyperplane at the same points. However, knowledge of supporting hyperplanes is quite relevant for the definition of a polyhedral convex underestimator of the convex envelope (maximum of the supporting hyperplanes at a finite set of points in P). As remarked in [15] (see also a comment in [7]), the use of polyhedral underestimators of the convex envelope rather than the convex envelope itself is advisable in view of the higher speed and stability of linear programming solvers with respect to nonlinear ones.
The structure of the paper is the following. In Section 2 we recall a result proved in [8] for some bivariate functions over general two-dimensional

polytopes. In Section 3 we prove some results about the convex envelope of quadratic functions over polytopes, including one about bivariate quadratic functions which shows that the value of their convex envelope at some point of a polytope is returned by the solution of a semidefinite problem. Finally, in Section 4 we prove the same result for two-dimensional polynomial functions satisfying some conditions.

2 Convex envelopes of bivariate functions over polytopes

In [8] a technique has been proposed to compute the value and a supporting hyperplane for the convex envelope of a bivariate function f over a polytope P. We briefly recall it here. Function f was required to satisfy the following conditions.

Condition 1 the Hessian of f has always at least one negative eigenvalue in the interior of P;

Condition 2 the restriction of f along each edge of P is either concave or strictly convex;

Condition 3 $f \in \mathcal{C}^2$, i.e., f is twice-continuously differentiable.

It view of the definition (1) of the convex envelope, given some point $(x_0, y_0) \in P$, the value of the convex envelope of f over P is given by the solution of the following optimization problem in the three variables a, b and c with an infinite number of linear constraints

$$\begin{array}{rl} Conv_{f,P}(x_0, y_0) = & \max \ c \\ & f(x,y) - [a(x-x_0) + b(y-y_0) + c] \geq 0 \quad \forall\ (x,y) \in P. \end{array}$$

The optimal solution (a^*, b^*, c^*) of this problem defines the value of the convex envelope (i.e., c^*) and a supporting hyperplane at point (x_0, y_0) (i.e., $a^*(x - x_0) + b^*(y - y_0) + c^*$). The infinite number of constraints can be substituted by a single one involving, however, a further optimization problem

$$\begin{array}{rl} Conv_{f,P}(x_0, y_0) = & \max \ c \\ & \min_{(x,y)\in P} f(x,y) - [a(x-x_0) + b(y-y_0) + c] \geq 0. \end{array}$$

Condition 1 guarantees that, for each possible choice of a, b, c, the minimum of $f(x, y) - [a(x - x_0) + b(y - y_0) + c]$ can not be attained in the interior of

P, and is always attained at a vertex of P or along an edge of P such that the restriction of f along the edge is a strictly convex function. Therefore, in [8] it is observed that, under Conditions 1-3, we also have

$$\begin{array}{lll} Conv_{f,P}(x_0,y_0) = & \max\ c & \\ & f(x_i,y_i) - [a(x_i-x_0)+b(y_i-y_0)+c] \geq 0 & \forall\ (x_i,y_i) \in V'(P) \\ & \min_{(x,y)\in e_j} f(x,y) - [a(x-x_0)+b(y-y_0)+c] \geq 0 & \forall\ e_j \in E'(P) \end{array} \tag{2}$$

where:

- $E'(P)$ denotes the set of edges of P along which f is strictly convex;
- $V'(P)$ denotes the set of vertices of P which do not belong to edges in $E'(P)$.

The constraints related to the vertices in $V'(P)$ are simple linear ones with respect to the unknowns a, b and c. For what concerns the constraints related to the edges in $E'(P)$, it is shown that each of them can be split into three set of constraints: two sets made up by two linear constraints, and a third set made up by two linear constraints and a convex one. Therefore, if we denote by s the cardinality of $E'(P)$, it turns out that the computation of the value and of a supporting hyperplane at some point of a polytope P for a bivariate function satisfying Conditions 1-3, requires the solution of (at most) 3^s three-dimensional convex optimization problems. In the next section we would like to show that, for more specific functions, we are able to derive the value and a supporting hyperplane of the convex envelope over polytopes through the solution of some different problem.

3 Convex envelopes of quadratic forms over polytopes

Let us consider a quadratic form $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ and a polytope P. Similarly to what we have seen in Section 2, given a point $\mathbf{x}_0 \in P$, the value of the convex envelope of f over P at $\mathbf{x}_0$ is given by the optimal value of the following optimization problem with an infinite number of linear constraints

$$\begin{array}{ll} convenv_{f,P}(\mathbf{x}_0) = & \max\ \gamma \\ & \mathbf{x}^T\mathbf{A}\mathbf{x} - \boldsymbol{\alpha}^T(\mathbf{x}-\mathbf{x}_0) - \gamma \geq 0 \quad \forall\ \mathbf{x} \in P \end{array} \tag{3}$$

The optimal solution $(\boldsymbol{\alpha}^*, \gamma^*)$ also allows to define a supporting hyperplane of the convex envelope at $\mathbf{x}_0$. Now, let

$$V(P) = \{\mathbf{v}_i\ :\ i = 1, \ldots, t\},$$

denote the vertex set of P and let t be its cardinality. Then, we can represent P as follows

$$P = \{\mathbf{V}\boldsymbol{\lambda} \ : \ \boldsymbol{\lambda} \in \Delta_t\},$$

where $\mathbf{V}$ is the matrix whose columns are the vertices of P, while

$$\Delta_t = \{\boldsymbol{\lambda} \ : \ \sum_{i=1}^{t} \lambda_i = 1, \quad \lambda_i \geq 0, \ i = 1, \ldots, t\},$$

is the t-dimensional unit simplex. Next, let us the introduce copositive cone of order t:

$$\mathcal{C}_t = \{\mathbf{Q} \in \mathcal{S}_t \ : \ \mathbf{x}^T\mathbf{Q}\mathbf{x} \geq 0 \quad \forall \ \mathbf{x} \geq \mathbf{0}\},$$

($\mathcal{S}_t$ denotes the set of symmetric matrices of order t). Then, the following theorem shows that the value of the convex envelope of a quadratic function over a polytope can be computed by solving a copositive problem.

Theorem 3.1 *It holds that*

$$\begin{array}{rl} \mathit{convenv}_{f,P}(\mathbf{x}_0) = & \max \ \gamma \\ & \mathbf{V}^T\mathbf{A}\mathbf{V} - \frac{1}{2}\left[\boldsymbol{\alpha}\mathbf{V}\mathbf{e}^T + \mathbf{e}\mathbf{V}^T\boldsymbol{\alpha}^T\right] - (\gamma - \boldsymbol{\alpha}^T\mathbf{x}_0)\mathbf{E} \in \mathcal{C}_t \end{array} \tag{4}$$

Proof. Problem (3) can be rewritten as

$$\begin{array}{l} \mathit{convenv}_{f,P}(\mathbf{x}_0) = \max \ \gamma \\ \quad \boldsymbol{\lambda}^T\left\{\mathbf{V}^T\mathbf{A}\mathbf{V} - \frac{1}{2}\left[\boldsymbol{\alpha}\mathbf{V}\mathbf{e}^T + \mathbf{e}\mathbf{V}^T\boldsymbol{\alpha}^T\right] - (\gamma - \boldsymbol{\alpha}^T\mathbf{x}_0)\mathbf{E}\right\}\boldsymbol{\lambda} \geq 0 \quad \forall \ \boldsymbol{\lambda} \in \Delta_t \end{array} \tag{5}$$

where $\mathbf{e}$ is the vector whose components are all equal to ones and $\mathbf{E}$ the matrix whose entries are all equal to ones. Here we exploit the fact that any affine function $\mathbf{a}^T\boldsymbol{\lambda} + a_0$ over the unit simplex can be represented with the quadratic form

$$\boldsymbol{\lambda}^T\left[1/2(\mathbf{e}\mathbf{a}^T + \mathbf{a}\mathbf{e}^T) + a_0\mathbf{E}\right]\boldsymbol{\lambda}.$$

But

$$\boldsymbol{\lambda}^T\left\{\mathbf{V}^T\mathbf{A}\mathbf{V} - \frac{1}{2}\left[\boldsymbol{\alpha}\mathbf{V}\mathbf{e}^T + \mathbf{e}\mathbf{V}^T\boldsymbol{\alpha}^T\right] - (\gamma - \boldsymbol{\alpha}^T\mathbf{x}_0)\mathbf{E}\right\}\boldsymbol{\lambda} \geq 0 \quad \forall \ \boldsymbol{\lambda} \in \Delta_t$$

$$\Updownarrow$$

$$\mathbf{V}^T\mathbf{A}\mathbf{V} - \frac{1}{2}\left[\boldsymbol{\alpha}\mathbf{V}\mathbf{e}^T + \mathbf{e}\mathbf{V}^T\boldsymbol{\alpha}^T\right] - (\gamma - \boldsymbol{\alpha}^T\mathbf{x}_0)\mathbf{E} \in \mathcal{C}_t$$

from which the result of the theorem follows. □

The copositive problem (4) is in general difficult to solve because of the copositivity condition, unless t (which, we recall, is the cardinality of the vertex set of P) is not larger than 4 (which holds, e.g., for triangles and quadrilaterals in $\mathbb{R}^2$ or simplices in $\mathbb{R}^3$). Indeed, in these cases the copositive cone $\mathcal{C}_t$ can be replaced by cone $\mathcal{P}_t + \mathcal{N}_t$, i.e., the cone obtained as the sum of the cone of semidefinite matrices

$$\mathcal{P}_t = \{\mathbf{Q} \in \mathcal{S}_t \ : \ \mathbf{x}^T\mathbf{Q}\mathbf{x} \geq 0 \quad \forall\, \mathbf{x} \in \mathbb{R}^t\},$$

and the cone of nonnegative matrices

$$\mathcal{N}_t = \{\mathbf{Q} \in \mathcal{S}_t \ : \ \mathbf{Q} \geq \mathbf{O}\}.$$

The equality $\mathcal{C}_t = \mathcal{P}_t + \mathcal{N}_t$ does not hold for t larger than 4 (see, e.g., [5]). While problems over the copositive cone $\mathcal{C}_t$ are not solvable in polynomial time, those over the cone $\mathcal{P}_t + \mathcal{N}_t$ can be solved in polynomial time.

What we will prove now is that we are able to end up with a problem solvable in polynomial time under conditions more general than $|V(P)| \leq 4$. Let us denote by $\mathcal{F}$ the set of *maximal* faces of P over which f is strictly convex. For $F \in \mathcal{F}$, let

$$I_F = \{i \ : \ \mathbf{v}_i \in V(F)\}$$

be the index set of the vertices in $V(F)$. Given a matrix $\mathbf{Q}$ and an index set I, we denote by $\{\mathbf{Q}\}_I$ the restriction of $\mathbf{Q}$ to the rows and columns in the index set I. Then, we can prove the following theorem.

Theorem 3.2

$$\begin{array}{l} conv env_{f,P}(\mathbf{x}_0) = \max \ \gamma \\ \qquad \left\{\mathbf{V}^T\mathbf{A}\mathbf{V} - \frac{1}{2}\left[\boldsymbol{\alpha}\mathbf{V}\mathbf{e}^T + \mathbf{e}\mathbf{V}^T\boldsymbol{\alpha}^T\right] - (\gamma - \boldsymbol{\alpha}^T\mathbf{x}_0)\mathbf{E}\right\}_{I_F} \in \mathcal{C}_{|I_F|} \quad \forall\, F \in \mathcal{F}. \end{array} \tag{6}$$

Proof. We can rewrite (3) as follows

$$\begin{array}{rl} conv env_{f,P}(\mathbf{x}_0) = & \max \ \gamma \\ & \mathbf{x}^T\mathbf{A}\mathbf{x} - \boldsymbol{\alpha}^T(\mathbf{x} - \mathbf{x}_0) - \gamma \geq 0 \quad \forall\, \mathbf{x} \in F, \ \forall\, F \in \mathcal{F} \end{array} \tag{7}$$

Indeed, for each fixed $(\boldsymbol{\alpha}, \gamma)$, the minimum value of

$$\mathbf{x}^T\mathbf{A}\mathbf{x} - \boldsymbol{\alpha}^T(\mathbf{x} - \mathbf{x}_0) - \gamma$$

can never be attained only in the interior of a face where f is not strictly convex. From (7) and recalling the proof of Theorem 3.1, we end up with problem (6). □

A corollary, whose proof is immediate, is the following

Corollary 3.1 *Let us assume that*

$$|V(F)| \leq 4, \quad \forall\ F \in \mathcal{F}, \tag{8}$$

i.e., each convex face F has at most four vertices. Then,

$$\begin{aligned} & \mathit{convenv}_{f,P}(\mathbf{x}_0) = \max\ \gamma \\ & \quad \left\{\mathbf{V}^T\mathbf{A}\mathbf{V} - \tfrac{1}{2}\left[\boldsymbol{\alpha}\mathbf{V}\mathbf{e}^T + \mathbf{e}\mathbf{V}^T\boldsymbol{\alpha}^T\right] - (\gamma - \boldsymbol{\alpha}^T\mathbf{x}_0)\mathbf{E}\right\}_{I_F} \in \mathcal{P}_{|I_F|} + \mathcal{N}_{|I_F|} \quad \forall\ F \in \mathcal{F} \end{aligned} \tag{9}$$

The above corollary always applies to any nonconvex two-dimensional quadratic form (thus including also the bilinear term) over a generic two-dimensional polytope. Indeed, in this case, unless f is already a convex function (in which case it coincides with its convex envelope), the maximal faces in F are either vertices or edges of P. In both cases the cardinality of I_F is not larger than 2, so that (9) applies and the value of the convex envelope can be computed in a time polynomial with respect to the number of vertices in P.

4 Convex envelopes for two-dimensional polynomial functions

In this section we consider the case where f is a two-dimensional polynomial function of some degree d satisfying Conditions 1-3 (actually, Condition 3 is always satisfied). As an instance, one might consider the function x^2y^2 over the unit square $[0,1]^2$. For some interval $[a,b]$ we will denote by $\mathcal{Q}_k([a,b])$ the set of all one-dimensional polynomials of degree not larger than k, which are nonnegative over $[a,b]$. We will also denote by Σ_k the set of Sum-Of-Square (SOS) one-dimensional polynomials of degree at most k, i.e., polynomials f of degree not larger than k for which there exists a finite number of polynomials h_i, $i = 1,\ldots,r$ such that

$$f(x) = \sum_{i=1}^{r} [h_i(x)]^2.$$

The following theorem proves that the value of the convex envelope at some point of a polytope is returned by the solution of a semidefinite problem.

Theorem 4.1 *Let $P \subset \mathbb{R}^2$ be a polytope and f a two-dimensional polynomial function of degree d satisfying Conditions 1-3. Then, the value and a supporting hyperplane for the convex envelope of f over P is returned by the solution of a semidefinite problem.*

Proof. As shown in Section 2, the value and a supporting hyperplane for the convex envelope of f over P is returned by the solution of problem (2). Now, for a given edge $e_j \in E'(P)$, let $\mathbf{v}^j, \mathbf{w}^j$ be the two vertices of edge e_j. Then,

$$\begin{aligned} & g^j(\lambda; a, b, c) = \\ & = f(\lambda v_x^j + (1-\lambda) w_x^j, \lambda v_y^j + (1-\lambda)\ w_y^j) - \\ & -[a(\lambda v_x^j + (1-\lambda) w_x^j - x_0) + b(\lambda v_y^j + (1-\lambda)\ w_y^j - y_0) + c] \end{aligned}$$

is the one-dimensional polynomial function, of degree not larger than d, corresponding to the restriction of

$$f(x,y) - [a(x-x_0) + b(y-y_0) + c]$$

along the edge e_j. Then,

$$\min_{(x,y)\in e_j} f(x,y) - [a(x-x_0)+b(y-y_0)+c] \geq 0 \quad \Leftrightarrow \quad g^j(\lambda; a,b,c) \geq 0 \ \ \forall\, \lambda \in [0,1]$$

or, equivalently

$$\min_{(x,y)\in e_j} f(x,y) - [a(x-x_0)+b(y-y_0)+c] \geq 0 \quad \Leftrightarrow \quad g^j(\cdot; a,b,c) \in \mathcal{Q}_d([0,1]).$$

It is well known (see, e.g., [2]) that, if d is even, then

$$g^j(\cdot; a,b,c) \in \mathcal{Q}_d([0,1]) \quad \Leftrightarrow \quad g^j(\lambda; a,b,c) = p^j(\lambda) + q^j(\lambda)(1-\lambda)\lambda$$

for $p^j \in \Sigma_d$, $q^j \in \Sigma_{d-2}$, while if d is odd, then

$$g^j(\cdot; a,b,c) \in \mathcal{Q}_d([0,1]) \quad \Leftrightarrow \quad g^j(\lambda; a,b,c) = r^j(\lambda) + \lambda p^j(\lambda) + (1-\lambda) q^j(\lambda),$$

for $r^j, p^j, q^j \in \Sigma_{d-1}$. Therefore, problem (2) can be rewritten as

$$\begin{aligned} Conv_{f,P}(x_0,y_0) = \ & \max\ c \\ & f(x_i,y_i) - [a(x_i-x_0) + b(y_i-y_0) + c] \geq 0 \quad \forall\ (x_i,y_i) \in V'(P) \\ & g^j(\cdot; a,b,c) \in \mathcal{Q}_d([0,1]) \qquad \forall\ e_j \in E'(P) \end{aligned}$$

and since each SOS condition can be rewritten as a semidefinite one (see, e.g., [3]) we end up with a semidefinite problem. □

5 Conclusions and future research

In this paper we have presented some techniques for computing the value and a supporting hyperplane for the convex envelope of quadratic and polynomial functions over polytopes whose vertex set is known. For quadratic functions we have proved that the computation can be carried on through the solution of a copositive problem. It has also been shown that in some cases it is possible to solve a semidefinite problem. Such cases always include two-dimensional quadratic functions. The results have then been extended also to two-dimensional polynomial functions satisfying certain conditions. The theory of nonnegative polynomials shows that also in this case we end up with the solution of a semidefinite problem. A possible subject for future research is that of identifying more general conditions under which the computation of the value for the convex envelope can be done through the solution of a semidefinite problem.

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