The Gomory-Chvátal closure of a non-rational polytope is a rational polytope

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Abstract

The question as to whether the Gomory-Chvátal closure of a non-rational polytope is a polytope has been a longstanding open problem in integer programming. In this paper, we answer this question in the affirmative, by combining ideas from polyhedral theory and the geometry of numbers. [1]

1 Introduction

Cutting-plane methods, when combined with branch and bound, are among the most successful techniques for solving integer programming problems in practice; numerous types of cutting planes have been studied in the literature and several of them are used in commercial solvers (see, e.g., Cornuéjols (2008) and the references therein). In general, a cutting plane for a polyhedron P is an inequality that is satisfied by all integer points in P, and, when added to the polyhedron P, it typically yields a stronger relaxation of its integer hull. A Gomory-Chvátal cutting plane (Gomory 1958; Chvátal 1973) is an inequality of the form $cx \leq \lfloor \delta \rfloor$, where c is an integral vector and $cx \leq \delta$ is valid for P. The Gomory-Chvátal closure of P is the intersection of all half-spaces defined by such inequalities; it is typically denoted by P'. Even though the Gomory-Chvátal closure of a polyhedron P is defined as the intersection of an infinite number of half-spaces, Schrijver (1980) proved that, for rational polyhedra, a finite number of inequalities is sufficient to describe P'. In other words, the Gomory-Chvátal closure of a rational polyhedron is again a polyhedron. In contrast, it is well known that the integer hull (and,

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[1] After completing this work, it was pointed out to us that Dadush, Dey, and Vielma recently showed that the Gomory-Chvátal closure of any compact convex set is a polytope. The proof presented in this paper has been obtained independently, without any knowledge of the ideas and techniques used in Dadush, Dey, and Vielma (2010a).

therefore, the Gomory-Chvátal closure) of not necessarily rational polyhedra may not be a rational polyhedron. However, for bounded polyhedra, i.e., polytopes, the situation is different: While the integer hull of an arbitrary polytope is, obviously, a polytope, it was unknown whether the same is true for the Gomory-Chvátal closure. We show that this is indeed the case: The Gomory-Chvátal closure of a non-rational polytope is again a rational polytope, that is, it can be described by a finite set of rational inequalities.

Even though Gomory-Chvátal cuts were originally introduced for polyhedra, they have lately been applied to other convex sets as well. Of particular relevance is the work by Dey and Vielma (2010), who showed that the Gomory-Chvátal closure of a full-dimensional bounded ellipsoid described by rational data is a polytope. Dadush, Dey, and Vielma (2010b) recently extended this result to strictly convex bodies and to the intersection of strictly convex bodies with rational polyhedra. Since the original proof of Schrijver (1980) for rational polyhedra relies strongly on polyhedral properties, Dey and Vielma and Dadush, Dey, and Vielma had to develop a whole new proof technique, which can roughly be described as follows: One first shows that there exists a finite set of Gomory-Chvátal cuts that separate every non-integral point on the boundary of the strictly convex body. In a second step, one proves that, if the intersection of the boundary of a convex body with a finite set of Gomory-Chvátal cuts is contained in the Gomory-Chvátal closure, only a finite set of additional inequalities are needed to fully describe the Gomory-Chvátal closure of the body. Dadush, Dey, and Vielma (2010b) themselves point out that their way of proving the first step depends crucially on the fact that a strictly convex body does not have any flat faces of dimension greater than 0. Consequently, their proof does not apply to convex bodies with higher-dimensional flat faces; in particular, it does not apply to non-rational polytopes.

Our proof uses ideas from convex analysis, polyhedral theory, and the geometry of numbers as, for example, Diophantine approximations and reduced lattice bases. After introducing our notation in Section 2, we provide a sketch of our proof in Section 3. Section 4 covers some required background material, and Section 5 contains the main part of the proof.

2 Basics and Notations

For any closed convex set $K \subseteq \mathbb{R}^n$ and any vector $a \in \mathbb{Z}^n$, we define $a_K := \max\{ax \mid x \in K\}$. We use the notation $(ax = a_0)$ for the hyperplane $\{x \in \mathbb{R}^n \mid ax = a_0\}$ and, similarly, $(ax \leq a_0)$ denotes the half-space of all points satisfying the inequality $ax \leq a_0$. If y is a real number, $\lfloor y \rfloor$ denotes the largest integer less than or equal to y. The greatest common divisor of integer numbers $a_1, \ldots, a_n$ is denoted by $\gcd(a_1, \ldots, a_n) = \gcd(a)$, where $a = (a_1, \ldots, a_n)$.

For a subset U of $\mathbb{R}^n$, the interior of U is denoted by $\text{int}(U)$. The relative boundary and relative interior of U (that is, the boundary and interior of U considered as a subset of $\operatorname{aff}(U)$) are denoted by $\operatorname{rbd}(U)$ and $\operatorname{ri}(U)$, respectively. Moreover, $B(0,\varepsilon)$ denotes the full-dimensional ball in $\mathbb{R}^n$ around the origin with radius ε. For any set $S \subseteq \mathbb{Z}^n$, we define

$$C_S(P) := \bigcap_{a \in S} \left(ax \leq \lfloor a_P \rfloor \right) \ ,$$

to denote the intersection of all Gomory-Chvátal cuts with normal vector in S. The Gomory-Chvátal closure of the polytope P, that is, the intersection of all Gomory-Chvátal cuts $C_{\mathbb{Z}^n}(P)$, is denoted by P'.

3 General Proof Idea

In this section, we give an outline of our proof and provide some intuition. The proof is divided into 4 steps:

Step 1: Show that there exists a finite set $S_1 \subseteq \mathbb{Z}^n$ such that $C_{S_1}(P) \subseteq P$.

Step 2: Show that for any face F of P, $F' = P' \cap F$. In particular, if $F = P \cap (ax = a_P)$, then for every Gomory-Chvátal cut for F, there exists a Gomory-Chvátal cut for P that has the same impact on the maximal rational affine subspace of $(ax = a_P)$.

Step 3: Show that if there exists a finite set S such that $C_S(P) \subseteq P$ and $C_S(P) \cap \operatorname{rbd}(P) \subseteq P'$, then P' is a rational polytope.

Step 4: Prove that P' is a rational polytope by induction on the dimension of $P \subseteq \mathbb{R}^n$.

The general proof technique is inspired by the work of Dadush, Dey, and Vielma (2010b) who showed that the Gomory-Chvátal closure of a strictly convex body is a polytope. However, Steps 1 to 3 require significant modification. This is on the one hand due to the fact that a main argument in their proof relies on *strict* convexity, which is not given in the case of polytopes. On the other hand, we cannot assume that P is full-dimensional, since P might be contained in a non-rational affine subspace. Therefore, it is not possible to apply a unimodular transformation to reduce the problem for a non-full-dimensional polytope to the full-dimensional case.

Step 1 (Corollary 5.12) is the most challenging part of the proof and is shown in a series of lemmata. First, we establish the existence of a sequence of integral normal vectors (Lemma 5.2) satisfying a specific list of properties. These normal vectors give rise to Gomory-Chvátal cuts that separate the points of every non-rational face $F = P \cap (ax = a_P)$ of P that are not contained in the maximal rational affine subspace of $(ax = a_P)$. In particular, these sequences imply that for each non-rational inequality $ax \leq a_P$ defining P, there exists a finite set of integral vectors S_a such that $C_{S_a}(P) \subseteq (ax \leq a_P)$ (Lemma 5.7). The proof of Step 1 involves ideas from convex and polyhedral theory, as well as number theory. More specifically, we use modified Diophantine approximations to approximate the normal vectors of non-rational hyperplanes. Furthermore, integral lattices and reduced lattice bases play a crucial role in this part of the proof.

Step 2 states a property of the Gomory-Chvátal closure that is well-known for rational polytopes: If we apply the closure operator to a face of a polytope, the result is the same as if we intersect the closure of the polytope with the face. As it turns out, the same is true for non-rational polytopes (Lemma 5.14).

A statement similar to the one in Step 3 has been shown in Dadush, Dey, and Vielma (2010b) for the case of full-dimensional convex bodies. Since P can be contained in some non-rational affine subspace and, thus, a unimodular transformation of P to a full-dimensional polytope in a lower-dimensional space is not possible, we need the extension to lower-dimensional polytopes (Lemma 5.15). However, the basic observation for proving this part is the same as in Dadush, Dey, and Vielma (2010b): Every additional undominated Gomory-Chvátal cut has to separate a point that is contained in the relative interior of the polytope. Even though in the non-full-dimensional case there are infinitely many cuts with this property, we argue that only a finite number of them need to be considered.

Finally, in Step 4 of the proof (Theorem 5.16), we combine the observations of the first three steps and use induction over the dimension of the polytope to prove the main result.

Each of the four steps will be discussed in a separate subsection of Section 5.

4 Preliminaries

In this section, we state various results from the literature and derive some additional observations regarding Diophantine approximations and bases of lattices that are utilized in the subsequent sections to show that the Gomory-Chvátal closure of any polytope is a rational polytope.

The first lemma links the absolute value of the determinant of an integral non-singular square matrix to the number of integer points contained in the parallelepiped spanned by the columns of the matrix (see, e.g., Barvinok 2002).

Lemma 4.1 *Let $v_1, \ldots, v_n \in \mathbb{Z}^n$ be linearly independent vectors. Then the number of integer points* *in the semi-open parallelepiped*

$$\left\{ \sum_{i=1}^{n} \lambda_i v_i \,\middle|\, 0 \le \lambda_i < 1,\ i = 1, \ldots, n \right\}$$

is equal to the absolute value of the determinant of the matrix with columns $v_1, \ldots, v_n$.

For linearly independent vectors $b_1, \ldots, b_l$ in $\mathbb{R}^n$ and $B = (b_1, \ldots, b_l)$, the *lattice* generated by the basis B is the set

$$\Lambda(B) := \left\{ x \in \mathbb{R}^n \,\middle|\, x = \sum_{j=1}^{l} \lambda_j b_j,\ \lambda_j \in \mathbb{Z},\ 1 \le j \le l \right\} .$$

For any lattice $\Lambda \subset \mathbb{R}^n$, the volume of the fundamental parallelepiped of a basis of the lattice (the parallelepiped spanned by the basis vectors) does not depend on the basis itself. If we define $L_0 := \{0\}$ and $L_k = \text{span}(b_1, \ldots, b_k)$ for $k = 1, \ldots, l$, and if $\tilde{b}_k$ denotes the orthogonal projection of b_k onto $L_{k-1}^{\perp}$ for $k = 1, \ldots, l$, then

$$\det\bigl(\Lambda(B)\bigr) = \prod_{i=1}^{l} \left\| \tilde{b}_i \right\| .$$

A famous result due to Lenstra, Lenstra, and Lovász (1982) is that for every lattice in $\mathbb{R}^n$, there exists a basis whose vectors are *almost* orthogonal to each other. Such basis is referred to as *reduced basis* and its *orthogonality defect* can be bounded by a constant that only depends on the dimension n. We slightly modify their *lattice basis reduction algorithm* and obtain the following result.

Theorem 4.2 *Let $(ax = 0)$ be a hyperplane that is spanned by n integral points in $\mathbb{R}^n$. Let $U_0 \subseteq (ax = 0)$ be a k-dimensional linear vector space spanned by integral vectors $u_1, \ldots, u_k$ and assume that $u_1, \ldots, u_k$ form a basis of the lattice defined by the integer points in U_0. Let $l := n - k - 1$. If $k \ge 1$, assume that for any $v \in \bigl((ax = 0) \cap \mathbb{Z}^n\bigr) \setminus U_0$,*

$$\|v\|^2 \ge \frac{1}{2} \left(\sum_{p=1}^{k} \|u_p\| \right)^2 . \tag{1}$$

Then there exist vectors $v_1, \ldots, v_l$ that satisfy the following properties:

(1) The vectors $u_1, \ldots, u_k, v_1, \ldots, v_l$ form a basis of the lattice $(ax = 0) \cap \mathbb{Z}^n$.

(2) If we define $U_j := \text{span}(u_1, \ldots, u_k, v_1, \ldots, v_j)$ and if $\tilde{v}_j$ denotes the orthogonal projection of v_j onto $U_{j-1}^{\perp}$, for $j = 1, \ldots, l$, then there exists a positive constant c only depending on l such that

for $j = 1, \ldots, l$,

$$\|\tilde{v}_j\| \geq c \|v_j\| \quad .$$

In the following lemma, we show that if a point can be written as linear combination of an orthogonal basis $\tilde{w}_1, \ldots, \tilde{w}_l$ derived from vectors $w_1, \ldots, w_l$ with small multipliers and if the orthogonal projections are not too short, the point can also be written as a linear combination of $w_1, \ldots, w_l$ with multipliers that are not too large.

Lemma 4.3 *Let $R > 0$ and let $u_1, \ldots, u_k, w_1, \ldots, w_l$ be linearly independent vectors in $\mathbb{R}^n$ with $\|w_j\| = R$, for $j = 1, \ldots, l$. Furthermore, let us define the linear vector spaces $U_0 := \operatorname{span}(u_1, \ldots, u_k)$ and $U_j := \operatorname{span}(u_1, \ldots, u_k, w_1, \ldots, w_j)$, for $j = 1, \ldots, l$. Let $\tilde{w}_j$ denote the orthogonal projection of w_j onto $U_{j-1}^{\perp}$, for $j = 1, \ldots, l$. If there exists a constant $c > 0$ such that $\|\tilde{w}_j\| \geq cR$ for $j = 1, \ldots, l$, then there exists a constant c_1 only depending on l and c such that*

$$\left\{ u + \sum_{j=1}^{l} \tilde{\lambda}_j \tilde{w}_j \,\middle|\, u \in U;\ -1 \leq \tilde{\lambda}_j \leq 1,\ j = 1, \ldots, l \right\}$$

$$\subseteq \left\{ u + \sum_{j=1}^{l} \lambda_j w_j \,\middle|\, u \in U;\ -c_1 \leq \lambda_j \leq c_1,\ j = 1, \ldots, l \right\} .$$

Proof. The proof of the lemma is by induction on l. For $j = 1, \ldots, l$, the orthogonal projection $\tilde{w}_j$ of w_j onto $U_{j-1}^{\perp}$ has a unique representation:

$$\tilde{w}_j = w_j - \sum_{p=1}^{k} \alpha_{jp} u_p - \sum_{t=1}^{j-1} \alpha_{jt} \tilde{w}_t \ , \tag{2}$$

where $\alpha_{jp} \in \mathbb{R}$ for $p = 1, \ldots, k$, and

$$\alpha_{jt} = \frac{w_j \tilde{w}_t}{\|\tilde{w}_t\|^2}$$

for $t = 1, \ldots, j-1$. First, consider the case $l = 1$. Take an arbitrary $x = u + \tilde{\lambda}_1 \tilde{w}_1$, where $u \in U$ and $\tilde{\lambda}_1 \in [-1, 1]$. Then

$$x = u + \tilde{\lambda}_1 \tilde{w}_1 = u + \tilde{\lambda}_1 \left(w_1 - \sum_{p=1}^{k} \alpha_{1p} u_p \right) = \left(u - \tilde{\lambda}_1 \sum_{p=1}^{k} \alpha_{1p} u_p \right) + \tilde{\lambda}_1 w_1 \ ,$$

and $c_1 = 1$ satisfies the conditions of the lemma. Therefore, assume that the statement of the lemma is true for some $l \geq 1$ with constant $c_1 = c_1(l, c)$. Now take an $x = u + \sum_{j=1}^{l+1} \tilde{\lambda}_j \tilde{w}_j$, where $u \in U$ and $\tilde{\lambda}_j \in [-1, 1]$ for $j = 1, \ldots, l+1$. Using the induction assumption and (2), we get

$$\begin{aligned} x &= u + \sum_{j=1}^{l} \tilde{\lambda}_j \tilde{w}_j + \tilde{\lambda}_{l+1} \tilde{w}_{l+1} \\ &= u' + \sum_{j=1}^{l} \lambda_j w_j + \tilde{\lambda}_{l+1} \left(w_{l+1} - \sum_{p=1}^{k} \alpha_{l+1 p} u_p - \sum_{t=1}^{l} \frac{w_{l+1} \tilde{w}_t}{\|\tilde{w}_t\|^2} \, \tilde{w}_t \right) \\ &= u'' + \sum_{j=1}^{l} \lambda_j w_j + \tilde{\lambda}_{l+1} \left(w_{l+1} - \sum_{j=1}^{l} \frac{w_{l+1} \tilde{w}_j}{\|\tilde{w}_j\|^2} \, \tilde{w}_j \right) \quad , \end{aligned}$$

for some $u', u'' \in U$ and numbers λ_j satisfying $|\lambda_j| \leq c_1(l, c)$, for $j = 1, \ldots, l$. Let us define

$$y := \sum_{j=1}^{l} \frac{w_{l+1} \tilde{w}_j}{\|\tilde{w}_j\|^2} \, \tilde{w}_j = \sum_{j=1}^{l} \nu_j \tilde{w}_j \quad .$$

Then

$$|\nu_j| = \frac{|w_{l+1} \tilde{w}_j|}{\|\tilde{w}_j\|^2} \leq \frac{\|w_{l+1}\| \, \|\tilde{w}_j\|}{\|\tilde{w}_j\|^2} = \frac{\|w_{l+1}\|}{\|\tilde{w}_j\|} \leq \frac{R}{Rc} = \frac{1}{c} \quad .$$

By applying the induction assumption a second time, we get

$$\begin{aligned} y &\in \left\{ u + \sum_{j=1}^{l} \nu_j \tilde{w}_j \,\middle|\, u \in U; \; -\frac{1}{c} \leq \nu_j \leq \frac{1}{c}, \; j = 1, \ldots, l \right\} \\ &= \frac{1}{c} \left\{ u + \sum_{j=1}^{l} \nu_j \tilde{w}_j \,\middle|\, u \in U; \; -1 \leq \nu_j \leq 1, \; j = 1, \ldots, l \right\} \\ &\subseteq \frac{1}{c} \left\{ u + \sum_{j=1}^{l} \gamma_j w_j \,\middle|\, u \in U; \; -c_1 \leq \gamma_j \leq c_1, \; j = 1, \ldots, l \right\} \\ &= \left\{ u + \sum_{j=1}^{l} \gamma_j w_j \,\middle|\, u \in U; \; -\frac{c_1}{c} \leq \gamma_j \leq \frac{c_1}{c}, \; j = 1, \ldots, l \right\} \quad . \end{aligned}$$

In particular, there exists some $u''' \in U$ and numbers $\gamma_j \in [-c_1/c, c_1/c]$, for $j = 1, \ldots, l$, such that

$$y = u''' + \sum_{j=1}^{l} \gamma_j w_j \quad .$$

Hence, we obtain

$$\begin{aligned} x &= u'' + \sum_{j=1}^{l} \lambda_j w_j + \tilde{\lambda}_{l+1} \left(w_{l+1} - u''' - \sum_{j=1}^{l} \gamma_j w_j \right) \\ &= \hat{u} + \sum_{j=1}^{l} (\lambda_j - \tilde{\lambda}_{l+1}\gamma_j)\, w_j + \tilde{\lambda}_{l+1} w_{l+1} \ , \end{aligned}$$

where $\hat{u} \in U$ and

$$\left|\lambda_j - \tilde{\lambda}_{l+1}\gamma_j\right| \le |\lambda_j| + |\gamma_j| \le c_1(l,c) + \frac{c_1(l,c)}{c} \ .$$

Thus, $c_1(l+1,c) := c_1(l,c)\,(1+1/c)$ is the desired constant for $l+1$. □

Next, we review a famous result regarding simultaneous Diophantine approximations: a finite set of real numbers can be approximated by rational numbers with one common low denominator (see, e.g., Schrijver 1986).

Theorem 4.4 (Dirichlet) *For $a \in \mathbb{R}^n$ and $0 < \varepsilon < 1$, there exist integers $p_1, \ldots, p_n$ and $q > 0$ such that for $i = 1, \ldots, n$,*

$$\left| a_i - \frac{p_i}{q} \right| < \frac{\varepsilon}{q} \ .$$

We now extend the theorem to the case that there are rational linear dependencies between the components of the non-rational vector that also should be satisfied by its approximation.

Lemma 4.5 *Let $a \in \mathbb{R}^n$ and let $u_1, \ldots, u_k$, where $k \le n-1$, be linearly independent vectors in $\mathbb{Z}^n$ such that $au_j = 0$ for $j = 1, \ldots, k$. For any $0 < \varepsilon < 1$, there exists an integer vector $p = (p_1, \ldots, p_n)$ and an integer $q > 0$ such that $pu_j = 0$ for $j = 1, \ldots, k$, and such that for $i = 1, \ldots, n$,*

$$\left| a_i - \frac{p_i}{q} \right| < \frac{\varepsilon}{q} \ .$$

Proof. Let U denote the $k \times n$ matrix with rows $u_1, \ldots, u_k$, that is, $Ua = 0$. Since $\operatorname{rank}(U) = k$, there exists (after possibly reordering the indices) a rational $k \times (n-k)$ matrix $\tilde{U}$ such that the system of equalities $Ua = 0$ is equivalent to the system

$$\begin{bmatrix} a_{n-k+1} \\ \vdots \\ a_n \end{bmatrix} = \tilde{U} \begin{bmatrix} a_1 \\ \vdots \\ a_{n-k} \end{bmatrix} \ .$$

In particular, one can find a positive integer s and integers r_{ij}, for $n-k+1 \le i \le n$ and $j = 1, \dots, n-k$, such that for $i = n-k+1, \dots, n$,

$$a_i = \frac{1}{s} \sum_{j=1}^{n-k} r_{ij}\, a_j \ .$$

Let us define the constants

$$K_1 := \min\left\{ \frac{1}{s}, \frac{s}{(n-k)\max_{i,j}|r_{ij}|} \right\}$$

and $\varepsilon_1 := K_1\, \varepsilon$. Let $\tilde{p}_1, \dots, \tilde{p}_{n-k}$ and $\tilde{q}$ be integers according to Theorem 4.4 that satisfy

$$\left| a_i - \frac{\tilde{p}_i}{\tilde{q}} \right| < \frac{\varepsilon_1}{\tilde{q}}$$

for $i = 1, \dots, n-k$. We define

$$\begin{aligned} q &:= s\,\tilde{q} \\ p_i &:= s\,\tilde{p}_i && \text{for } i = 1, \dots, n-k \\ p_i &:= \sum_{j=1}^{n-k} r_{ij}\,\tilde{p}_j && \text{for } i = n-k+1, \dots, n \ . \end{aligned}$$

Note that

$$\begin{bmatrix} p_{n-k+1} \\ \vdots \\ p_n \end{bmatrix} = \tilde{U} \begin{bmatrix} p_1 \\ \vdots \\ p_{n-k} \end{bmatrix} \ ,$$

implying $pu_j = 0$ for $j = 1, \dots, k$. Furthermore, for $i = 1, \dots, n-k$, we have

$$\left| a_i - \frac{p_i}{q} \right| = \left| a_i - \frac{\tilde{p}_i}{\tilde{q}} \right| < \frac{\varepsilon_1}{q/s} \le \frac{\varepsilon}{q} \ .$$

Then we obtain for $i = n-k+1, \dots, n$,

$$\left| a_i - \frac{p_i}{q} \right| = \left| a_i - \frac{1}{s}\sum_{j=1}^{n-k} r_{ij}\, \frac{\tilde{p}_j}{\tilde{q}} \right| = \frac{1}{s} \left| \sum_{j=1}^{n-k} r_{ij}\, a_j - \sum_{j=1}^{n-k} r_{ij}\, \frac{\tilde{p}_j}{\tilde{q}} \right| \le \frac{1}{s} \sum_{j=1}^{n-k} |r_{ij}| \left| a_j - \frac{\tilde{p}_j}{\tilde{q}} \right| \le \frac{\varepsilon}{q} \ ,$$

and the lemma follows. □

From the last lemma, we obtain the following corollary.

Corollary 4.6 *Let $a \in \mathbb{R}^n$ and let $u_1, \dots, u_k$, where $k \le n-1$, be linearly independent vectors in $\mathbb{Z}^n$ such that $au_j = 0$ for $j = 1, \dots, k$. Then there exists a sequence $\{a^i\}_{i \in \mathbb{N}} \subseteq \mathbb{Z}^n$ such that $a^i \perp u_j$ for $j = 1, \dots, k$ and such that*

$$\left\| a^i \right\| \left\| \bar{a}^i - \bar{a} \right\| \longrightarrow 0 \ , \tag{3}$$

where $\bar{a} = a / \|a\|$ and $\bar{a}^i = a^i / \left\| a^i \right\|$.

5 The Main Proof

In this part, we prove the main result of the paper, proceeding through the sequence of four steps outlined in Section 3.

5.1 Step 1

As first step of our proof, we show that there exists a finite set of Gomory-Chvátal cuts that defines a subset of P. Such a set of cuts must separate all parts of the boundary of P that are not contained in some rational affine subspace. The basic idea for the proof is an observation made for rational polytopes, which is illustrated in Figure 1: Suppose $F \subseteq (ax = 0)$ is a rational polytope in $\mathbb{R}^n$, such that $(ax = 0)$ is spanned by integral vectors $u_1, \ldots, u_{n-2}, v$. Assume that these vectors are chosen such that the parallelepiped spanned by them does not contain any interior integral points. Let $U = \text{span}(u_1, \ldots, u_{n-2})$. If F is contained in the set $U + \{\lambda v \mid \lambda < 1\}$, we can find a Gomory-Chvátal cut that separates every point in $F \cap (U + \{\lambda v \mid \lambda > 0\})$. In other words, $F' \subseteq U + \{\lambda v \mid \lambda \leq 0\}$.

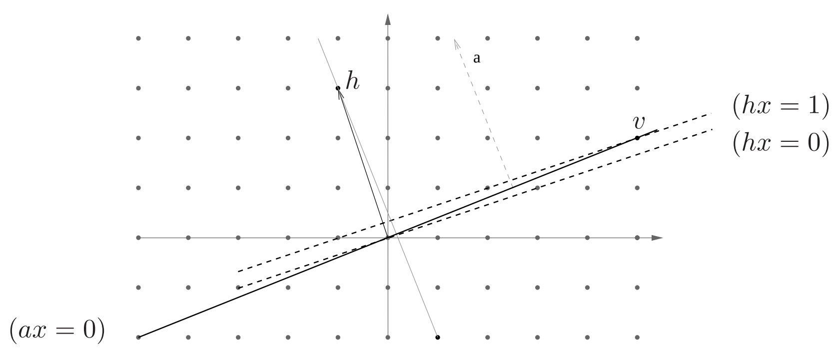

Figure 1: Illustration of the basic geometric idea for $n = 2$: The hyperplane $(ax = 0)$ with $a = (-2, 5)$ is spanned by $v = (5, 2)$. The line segment $[0, v]$ does not contain any interior integral points. This implies that there is a vector $h = (-1, 3)$ such that $hv = 1$, and the same is true for any $h + ka$ with $k \in \mathbb{Z}$. If $F \subset \{\lambda v \mid \lambda < 1\}$, then $\max\{hx \mid x \in F\} < hv = 1$, and $hx \leq 0$ is a Gomory-Chvátal cut for F that separates every point in $(0, v)$.

Now suppose that $(ax = 0)$ is a non-rational hyperplane and, for illustration only, that $u_1, \ldots, u_{n-2} \in \mathbb{Z}^n$ and $v \in \mathbb{R}^n \setminus \mathbb{Q}^n$. Then we can approximate $(ax = 0)$ by a sequence of rational hyperplanes $(a^i x = 0)$ spanned by vectors $u_1, \ldots, u_{n-2}$ and $v^i \in \mathbb{Z}^n$, such that the norm of the vector v^i

increases with the quality of the approximation. This will guarantee that for a reasonably good approximation the cut, that would separates every point in $U + \{\lambda v^i \mid \lambda > 0\}$ in the rational case, will give rise to a valid cut for the non-rational polytope that separates every point in $U + \{\lambda v \mid \lambda > 0\}$. Using a symmetry argument, we can find at most two cuts that remove the non-rational parts of $(ax = 0)$. This idea can be extended to arbitrary non-rational hyperplanes.

First, we show a lemma that formalizes the above intuition for the rational case.

Lemma 5.1 *Let $u_1, \ldots, u_{n-2}$ and v be linearly independent vectors in $\mathbb{Z}^n$ such that*

$$\left\{ \sum_{i=1}^{n-2} \gamma_i u_i + \lambda v \,\middle|\, \gamma_i \in \mathbb{R},\ i = 1, \ldots, n-2;\ 0 < \lambda < 1 \right\} \cap \mathbb{Z}^n = \emptyset \ . \tag{4}$$

Then there exists a vector $y \in \mathbb{Z}^n$ such that $u_i y = 0$, for $i = 1, \ldots, n-2$, and such that $vy = 1$.

Proof. First, let us assume that the semi-open parallelepiped spanned by the vectors $u_1, \ldots, u_{n-2}$ does not contain any integral points apart from 0, that is,

$$\left\{ \sum_{i=1}^{n-2} \gamma_i u_i \,\middle|\, 0 \le \gamma_i < 1,\ i = 1, \ldots, n-2 \right\} \cap \mathbb{Z}^n = \{0\} \ . \tag{5}$$

Together with (4), we have

$$\left\{ \sum_{i=1}^{n-2} \gamma_i u_i + \lambda v \,\middle|\, 0 \le \gamma_i < 1,\ i = 1, \ldots, n-2;\ 0 \le \lambda < 1 \right\} \cap \mathbb{Z}^n = \{0\} \ ,$$

that is, also the semi-open parallelepiped spanned by all $n-1$ vectors does not contain any integral points apart from 0. Now consider the system

$$Vy := \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ v \end{bmatrix} y = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} =: b \ .$$

Note that V has full row rank and column rank $n-1$. Therefore, there exists a unimodular matrix $U \in \mathbb{Z}^{n\times n}$, that is, $|\det(U)| = 1$, such that

$$VU = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ v \end{bmatrix} U = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_{n-2} \\ \tilde{v} \end{bmatrix} =: \left[\tilde{V} \,|\, 0\right],$$

where each $\tilde{u}_i = u_i U$ and $\tilde{v} = vU$ has its n-th component zero and where $\tilde{V}$ is a nonsingular integral $(n-1)\times(n-1)$ matrix. The semi-open parallelepiped spanned by the vectors $\tilde{u}_1, \ldots, \tilde{u}_{n-2}$, and $\tilde{v}$ in $(x_n = 0)$ does not contain any integral points apart from 0. Indeed, suppose there was an integral point $z = \gamma_1\tilde{u}_1 + \ldots + \gamma_{n-2}\tilde{u}_{n-2} + \lambda\tilde{v}$ with $0 \le \gamma_i < 1$, for $i = 1, \ldots, n-2$, and $0 \le \lambda < 1$, such that not all of these coefficients are zero. Then

$$zU^{-1} = \gamma_1\tilde{u}_1U^{-1} + \ldots + \gamma_{n-2}\tilde{u}_{n-2}U^{-1} + \lambda\tilde{v}U^{-1} = \gamma_1 u_1 + \ldots + \gamma_{n-2}u_{n-2} + \lambda v$$

is an integral point different from 0 in the semi-open parallelepiped spanned by $u_1, \ldots, u_{n-2}$, and v, which is a contradiction. Now observe that Lemma 4.1 implies $|\det(\tilde{V})| = 1$. Therefore, the system

$$\tilde{V}\tilde{y} = b$$

has an integral solution $\tilde{y} \in \mathbb{Z}^{n-1}$. The vector $\bar{y} = [\tilde{y}^T \; 0]^T$ satisfies $VU\bar{y} = b$ and, consequently, $y = U\bar{y}$ is an integral solution to $Vy = b$.

If assumption (5) is not satisfied, we can find a set of $n-2$ integral vectors $u'_1, \ldots, u'_{n-2}$ spanning the same linear vector space as $u_1, \ldots, u_{n-2}$ such that (5) holds. Consequently, there is a vector $y \in \mathbb{Z}^n$ such that $u'_i y = 0$ for $i = 1, \ldots, n-2$ and $vy = 1$. Since every u_i can be written as a linear combination of the u'_i, we have $u_i\, y = 0$ as well. □

The following Lemma 5.2 can be seen as the core of the proof of Step 1. Therein, we establish for every non-rational vector space $V = (ax = 0)$ the existence of sequences of vectors and numbers, which satisfy a distinct list of properties. The sequences are associated with integral approximations of the non-rational hyperplane V. The starting point in the construction of these sequences is a special type of Diophantine approximation $\{a^i\}$ of the non-rational normal vector a. If $u_1, \ldots, u_k$ denote a maximal set of integral and linearly independent vectors in V, then the normal vectors $a^i \in \mathbb{Z}^n$ are perpendicular to each of the vectors $u_1, \ldots, u_k$. As a result, the approximations $(a^i x = 0)$ of the hyperplane V contain the maximal rational subspace $V_R = \text{span}(u_1, \ldots, u_k)$ of V. In particular, $(ax = 0) \cap (a^i x = 0) = V_R$. Each integral hyperplane $(a^i x = 0)$ is spanned by the vectors $u_1, \ldots, u_k$ together with $l = n - 1 - k$

additional integral vectors, denoted by $v_1^i, \ldots, v_l^i$, which can be regarded as approximations of the non-rational directions of V. These vectors will be chosen very carefully among the infinite number of possible sets of vectors spanning $(a^i x = 0)$, as not all choices will guarantee the properties that we require for the other sequences and numbers derived from them. Most importantly, they will be *almost* orthogonal to one another. The vectors v_j^i give rise to non-rational vectors w_j^i that span the non-rational part of $(ax = 0)$. More precisely, each w_j^i is obtained as projection of the vector v_j^i onto $(ax = 0)$, scaled by a factor, so that all w_j^i have a same given length. As the quality of the approximations of V increases with the index i, the w_j^i's will, at some point, also be almost orthogonal to one another. This property of the w_j^i's will turn out to be material in the subsequent proof of Step 1. Apart from the mentioned sequences a^i, v_j^i, and w_j^i, which have very natural geometric interpretations, we also establish a sequence of integral vectors $h^i(\delta)$, for each $\delta \in \{-1, 1\}^l$, whose construction is more involved. They arise as integral linear combinations of the integral vectors found in Lemma 5.1, which were the basis for Gomory-Chvátal cuts separating points in rational facets between affine layers of integral points. Some of the properties that these vectors satisfy are as follows: Each $h^i(\delta)$ is perpendicular to the vectors $u_1, \ldots, u_k$ and, therefore, the hyperplane $(h^i x = 0)$ is parallel to V_R. Moreover, the scalar product of $h^i(\delta)$ with each non-rational vector $\delta_j w_j^i$ is strictly positive, but very small.

To understand the motivation behind these properties, let us consider the non-rational parallelepiped $Q(\delta)$ that is spanned by $u_1, \ldots, u_k$ and the non-rational vectors $\delta_1 w_1^i, \ldots, \delta_l w_l^i$. When maximizing $h^i(\delta)$ over $Q(\delta)$, the maximum is attained at $\bar{w}(\delta) = \delta_1 w_1^i + \ldots + \delta_l w_l^i$, or any other point in $Q(\delta)$ that can be written as $\bar{w}(\delta) + u$ for some $u \in V_R$. Moreover, the properties of $h^i(\delta)$ guarantee that $0 < h^i(\delta)\bar{w}(\delta) < 1$. As a consequence, $h^i(\delta)x \leq 0$ is a Gomory-Chvátal cut for $Q(\delta)$, and this cut implies that $\big(Q(\delta)\big)' \subseteq V_R$. Thus, for the special case that the non-rational polytope is the $(n-1)$-dimensional parallelepiped $Q(\delta)$ or, contained in it, the single integral vector $h^i(\delta)$ implies a finite set S with the properties that we are looking for in Step 1 of the proof.

For a general polytope P with a facet $F = P \cap (ax = 0)$, we can cover F by at most 2^l parallelepipeds in V. Then every vector $h^i(\delta)$ will give rise to a Gomory-Chvátal cut that separates all the points in corresponding parallelepiped that do not belong to V_R. Note that for this, we also need the property that, when $h^i(\delta)$ is maximized over P, the maximum is attained at a vertex in F. In other words, every vector $h^i(\delta)$ must have a non-positive scalar product with the directions of edges connecting a vertex in F and a vertex outside of F. Indeed, we construct the $h^i(\delta)$ in Lemma 5.2 with the requirement that for a given finite set of vectors $r_1, \ldots, r_m$, their scalar product with these vectors is nonpositive.

The proof of Lemma 5.2 strongly relies on properties of reduced bases of integral lattices.

Lemma 5.2 *Let $R > 0$ be a constant and let $V \subset \mathbb{R}^n$ be a non-rational vector space of dimension*

$n-1$, that is, $V = (ax = 0)$ for some $a \in \mathbb{R}^n \setminus \mathbb{Q}^n$. Let U be the maximal rational subspace of V and assume that U is spanned by vectors $u_1, \dots, u_k \in \mathbb{Z}^n$, that is, $\dim(U) = k$, where $0 \leq k \leq n-2$. Let $l := n-k-1$. Furthermore, let $r_1, \dots, r_m \in \mathbb{R}^n$ such that for $s = 1, \dots, m$

$$r_s\, a < 0 \ . \tag{6}$$

Then there exists a constant $c > 0$ only depending on l such that there exist sequences

$$\{a^i\} \in \mathbb{Z}^n, \quad \{v_1^i, \dots, v_l^i\} \in \mathbb{Z}^{n \times l}, \quad \{q_1^i, \dots, q_l^i\} \in \mathbb{R}^l, \quad \{w_1^i, \dots, w_l^i\} \in \mathbb{R}^{n \times l}$$

that satisfy the following properties:

(i) $\gcd(a^i) = 1$.

(ii) $r_s\, a^i \leq 0$ *for* $s = 1, \dots, m$.

(iii) $\left\|a^i\right\| \left\|\bar{a}^i - \bar{a}\right\| \longrightarrow 0$, *where* $\bar{a}^i = a^i / \left\|a^i\right\|$ *and* $\bar{a} = a / \|a\|$.

(iv) $(a^i x = 0) = \operatorname{span}(u_1, \dots, u_k, v_1^i, \dots, v_l^i)$.

(v) $\left\|v_j^i\right\| \longrightarrow \infty$.

(vi) $\left\|\frac{v_j^i}{q_j^i} - w_j^i\right\| \longrightarrow 0$.

(vii) $\left\|w_j^i\right\| = R$.

(viii) $V = \operatorname{span}(u_1, \dots, u_k, w_1^i, \dots, w_l^i)$.

(ix) $\left\|\tilde{w}_j^i\right\| \geq c\,R$, *where* $\tilde{w}_j^i$ *denotes the orthogonal projection of* w_j^i *onto* $\left(\operatorname{span}(u_1, \dots, u_k, w_1^i, \dots, w_{j-1}^i)\right)^{\perp}$.

(x) There exists a constant $C > 0$ such that for any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that for all $i \geq N(\varepsilon)$ and for all $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{R}_+^l$ with $\|\alpha\|_\infty \leq 1$, there exist vectors $\{h_\alpha^i(\delta)\} \subseteq \mathbb{Z}^n$ for all $\delta \in \{-1, 1\}^l$ such that

$$\begin{array}{rcll} h_\alpha^i(\delta) & \perp & u_p & \text{for} \quad p = 1, \dots, k \\ \alpha_j - \varepsilon & \leq & \delta_j w_j^i\, h_\alpha^i(\delta) \leq \alpha_j + \varepsilon & \text{for} \quad j = 1, \dots, l \\ \delta_j h_\alpha^i(\delta)\, v_j^i & = & \left\lfloor \alpha_j q_j^i \right\rfloor & \text{for} \quad j = 1, \dots, l \\ 0 & \geq & r_s\, h_\alpha^i(\delta) & \text{for} \quad s = 1, \dots, m \\ \left|h_\alpha^i(\delta)\, a^i\right| & \leq & C \left\|a^i\right\|^2 \ . & \end{array}$$

Proof. Let us assume w.l.o.g. that the vectors $u_1, \ldots, u_k$ form a basis of the lattice $U \cap \mathbb{Z}^n$. If this is not the case, we can replace the original vectors by another set of vectors in U that has this property. Let V_{IR} denote the set of points in V that are not contained in the maximal rational subspace of V, that is, $V_{IR} := V \setminus U$. Let $\{a^i\} \subseteq \mathbb{Z}^n$ be a sequence of vectors according to Corollary 4.6 such that for $p = 1, \ldots, k$,

$$a^i \perp u_p$$

and

$$\left\|a^i\right\| \left\|\bar{a}^i - \bar{a}\right\| \longrightarrow 0 \ . \tag{7}$$

We can assume w.l.o.g. that $\gcd(a^i) = 1$, since the same properties hold if we divide a^i by some positive integer. Thus, the sequence $\{a^i\}$ satisfies properties (i) and (iii). Furthermore,(7) implies for $s = 1, \ldots, m$,

$$\left|r_s \bar{a}^i - r_s \bar{a}\right| \longrightarrow 0 \ .$$

As $r_s \bar{a} < 0$ by assumption (6), there exists some constant $\beta > 0$ such that $r_s \bar{a}^i \leq -\beta$ for large enough i. Hence, noting that $\left\|a^i\right\| \longrightarrow \infty$ because of $a \in \mathbb{R}^n \setminus \mathbb{Q}^n$, it also holds that for $s = 1, \ldots, m$ and large enough i,

$$r_s a^i \leq -\beta \ . \tag{8}$$

In particular, property (ii) is guaranteed for large enough i.

Let $\Lambda^i = (a^i x = 0) \cap \mathbb{Z}^n$ denote the lattice defined by the integer points in the integral hyperplane $(a^i x = 0)$. In the following claim, we show that norm of the shortest vector in $\Lambda^i \setminus U$ grows with i.

Claim 5.3 *Let z^i denote a shortest vector in $\Lambda^i \setminus U$. Then $\left\|z^i\right\| \longrightarrow \infty$.*

Proof of claim. Suppose that there exists some positive constant K such that for all i, one can find a point $z^i \in \Lambda^i \setminus U$ with $\left\|z^i\right\| \leq K$. Let $\text{proj}(z^i)$ denote the projection of z^i onto the hyperplane $(ax = 0)$, that is, $\text{proj}(z^i) + \lambda a = z^i$, where $\lambda = (az^i)/\left\|a\right\|^2$. As $z^i \notin (ax = 0)$, we have $\left\|z^i - \text{proj}(z^i)\right\| > 0$. Furthermore, since the number of integer points in $B(0, K)$ is finite, there must exist some positive number D such that $\left\|z^i - \text{proj}(z^i)\right\| \geq D$ for every i. However, using $\bar{a}^i z^i = 0$ and (7), we get

$$\left\|z^i - \text{proj}(z^i)\right\| = |\lambda| \left\|a\right\| = \frac{\left|az^i\right|}{\left\|a\right\|} = \left|\bar{a} z^i\right| = \left|\bar{a} z^i - \bar{a}^i z^i\right| \leq \left\|\bar{a} - \bar{a}^i\right\| \ K \longrightarrow 0 \ ,$$

which is a contradiction. ☐

Claim 5.3 implies that, for sufficiently large i, we can assume for every $v \in \Lambda^i \setminus U$,

$$\left\|v\right\|^2 \geq \frac{1}{2} \left(\sum_{p=1}^{k} \left\|u_p\right\| \right)^2 \ .$$

Since $(a^i x = 0)$ is an integral hyperplane and $U \subseteq (a^i x = 0)$, we can find integral vectors $v_1^i, \ldots, v_l^i$ according to Theorem 4.2. That is,

$$(a^i x = 0) = \text{span}(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i) \ ,$$

and the vectors $u_1, \ldots, u_k, v_1^i, \ldots, v_l^i$ form a basis of the lattice Λ^i. Furthermore, let $\tilde{v}_1^i$ be the orthogonal projection of v_1^i onto $U^\perp$ and let $\tilde{v}_j^i$ denote the orthogonal projection of v_j^i onto $\big(\text{span}(u_1, \ldots, u_k, v_1^i, \ldots, v_{j-1}^i)\big)^\perp$, for $j = 2, \ldots, l$. Then it also holds by Theorem 4.2 that for $j = 1, \ldots, l$,

$$\left\|\tilde{v}_j^i\right\| \geq c_1 \left\|v_j^i\right\| \ , \tag{9}$$

where c_1 is a constant that only depends on l. Note that property (iv) of the lemma follows. Furthermore, observe that $v_j^i \in \Lambda^i \setminus U$ for $j = 1, \ldots, l$. Consequently, Claim 5.3 implies property (v).

Since $u_1, \ldots, u_k, v_1^i, \ldots, v_l^i$ form a basis of Λ^i, we have for every $s \in \{1, \ldots, l\}$,

$$\left\{ \sum_{p=1}^k \gamma_p u_p + \sum_{j=1}^l \lambda_j v_j^i \ \middle| \ \gamma_p \in \mathbb{R},\ p = 1, \ldots, k;\ \lambda_j \in \mathbb{R},\ j = 1, \ldots, l;\ 0 < \lambda_s < 1 \right\} \cap \mathbb{Z}^n = \emptyset \ . \tag{10}$$

Indeed, if this was not the case and there was a point $z \in \mathbb{Z}^n$ such that

$$z = \sum_{p=1}^k \gamma_p u_p + \sum_{j=1}^l \lambda_j v_j^i$$

and such that $0 < \lambda_s < 1$, then

$$z' = \sum_{p=1}^k \big(\gamma_p - \lfloor \gamma_p \rfloor\big) u_p + \sum_{j=1}^l \big(\lambda_j - \lfloor \lambda_j \rfloor\big) v_j^i \in \Big(\mathbb{Z}^n \cap \Pi(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i)\Big) \ .$$

Hence, z' is an integral point in the semi-open parallelepiped spanned by the basis vectors. Because of $0 < \lambda_s < 1$, it holds that $z' \neq 0$, and this cannot be true.

Now, let us define for $j = 1, \ldots, l$,

$$(w_j^i, q_j^i) := \arg\min \left\{ \left\| \frac{1}{q} v_j^i - w \right\| \ \middle| \ w \in V_{IR};\ \|w\| = R;\ q \in \mathbb{R}_+ \right\} \ . \tag{11}$$

Intuitively, w_j^i is the closest point in the intersection of V_{IR} with the ball $B(0, R)$ to the line spanned by v_j^i. The definition of w_j^i immediately implies property (vii) of the lemma. In the following claim, we show property (vi).

Claim 5.4 *For $j = 1, \ldots, l$, we have $q_j^i \longrightarrow \infty$ and*

$$\left\| \frac{v_j^i}{q_j^i} - w_j^i \right\| \longrightarrow 0 \ .$$

Proof of claim. We first show the second part. Let w denote the projection of the point $(R\,\bar{v}_j^i)$ onto the non-rational hyperplane $(ax = 0)$, where $\bar{v}_j^i = v_j^i / \left\| v_j^i \right\|$. We have $w = R\bar{v}_j^i - \lambda a$, where $\lambda = (aR\bar{v}_j^i)/\left\|a\right\|^2$. Furthermore, let $q = \left\| v_j^i \right\| / R > 0$. Note that for $\bar{w} = w/\left\|w\right\|$, it holds that $R\bar{w} \in V_{IR}$ and $\left\|R\bar{w}\right\| = R$. Therefore, $(R\bar{w}, q)$ is a feasible pair in the minimization (11) that defines (w_j^i, q_j^i). Consequently,

$$\begin{aligned}\left\| \frac{v_j^i}{q_j^i} - w_j^i \right\| &\leq \left\| \frac{v_j^i}{q} - R\,\bar{w} \right\| = \left\| \frac{v_j^i}{\left\| v_j^i \right\| / R} - R\,\bar{w} \right\| = \left\| R\bar{v}_j^i - R\bar{w} + (w - w) \right\| \\ &\leq \left\| R\bar{v}_j^i - w \right\| + \left\| w - R\bar{w} \right\| \ .\end{aligned}$$

We get, using $\bar{a}^i \bar{v}_j^i = 0$ and (7), that

$$\left\| R\bar{v}_j^i - w \right\| = |\lambda| \left\|a\right\| = \left| R\bar{a}\bar{v}_j^i \right| = R \left| \bar{a}\bar{v}_j^i - \bar{a}^i \bar{v}_j^i \right| \leq R \left\| \bar{a} - \bar{a}^i \right\| \left\| \bar{v}_j^i \right\| = R \left\| \bar{a} - \bar{a}^i \right\| \longrightarrow 0 \ .$$

This also implies that $\left\|w\right\| \longrightarrow R$ and, therefore, the second part of the claim holds. The first part, $q_j^i \longrightarrow \infty$, follows from $\left\| v_j^i \right\| \longrightarrow \infty$, $\left\| w_j^i \right\| = R$, and $\left\| v_j^i / q_j^i - w_j^i \right\| \longrightarrow 0$. □

Next, we prove property (ix). By (9), we have for $j = 1, \ldots, l$,

$$\frac{1}{q_j^i} \left\| \tilde{v}_j^i \right\| \geq \frac{1}{q_j^i} c_1 \left\| v_j^i \right\| \ . \tag{12}$$

Let $\tilde{w}_j^i$ denote the orthogonal projection of w_j^i onto $\left(\mathrm{span}(u_1, \ldots, u_k, w_1^i, \ldots, w_{j-1}^i)\right)^{\perp}$, for $j = 1, \ldots, l$. Because of Claim 5.4, there is for every $\tau > 0$ a number $N(\tau)$, such that for all $i \geq N(\tau)$,

$$\frac{\left\| v_j^i \right\|}{q_j^i} - \tau \leq \left\| w_j^i \right\| \leq \frac{\left\| v_j^i \right\|}{q_j^i} + \tau$$

and

$$\frac{\left\| \tilde{v}_j^i \right\|}{q_j^i} - \tau \leq \left\| \tilde{w}_j^i \right\| \leq \frac{\left\| \tilde{v}_j^i \right\|}{q_j^i} + \tau \ .$$

Now let γ be some small constant such that $c_1 > \gamma > 0$. By (12),

$$\frac{\left\| \tilde{v}_j^i \right\|}{q_j^i} - (c_1 - \gamma) \frac{\left\| v_j^i \right\|}{q_j^i} \geq \gamma \frac{\left\| v_j^i \right\|}{q_j^i} \ .$$

Using this observation and $R = \left\| w_j^i \right\|$, we obtain

$$\begin{aligned} \|\tilde{w}_j^i\| - (c_1 - \gamma)R &\geq \frac{\left\|\tilde{v}_j^i\right\|}{q_j^i} - \tau - (c_1 - \gamma)\left(\frac{\left\|v_j^i\right\|}{q_j^i} + \tau\right) \\ &\geq \gamma\frac{\left\|v_j^i\right\|}{q_j^i} - \tau - (c_1 - \gamma)\tau \\ &\geq \gamma(R - \tau) - \tau - (c_1 - \gamma)\tau \ . \end{aligned}$$

Note that we can choose τ small enough such that the last expression is nonnegative. Hence, $c = (c_1 - \delta) > 0$ is the desired constant for property (ix). Since c_1 only depends on l, the same is true for c.

Now observe that property (ix) implies that the vectors $u_1, \ldots, u_k, w_1^i, \ldots, w_l^i$ are linearly independent. This is because $\left\| \tilde{w}_j^i \right\| > 0$ for $j = 1, \ldots, l$ and

$$\text{span}(u_1, \ldots, u_k, w_1^i, \ldots, w_l^i) = \text{span}(\tilde{u}_1, \ldots, \tilde{u}_k, \tilde{w}_1^i, \ldots, \tilde{w}_l^i) \ ,$$

where $\tilde{u}_1 = u_1$, and where for $p = 2, \ldots, k$, the vector $\tilde{u}_p$ denotes the orthogonal projection of u_p onto $(\text{span}(u_1, \ldots, u_{p-1}))^{\perp}$. Consequently, property (viii) is satisfied.

In the remainder of the proof, we show property (x). Because of (10) and Lemma 5.1, there exists for each $s = 1, \ldots, l$ a vector $y_s^i \in \mathbb{Z}^n$ such that

$$y_s^i \in (u_1 x = 0) \cap \ldots \cap (u_k x = 0) \cap \bigcap_{j \neq s} (v_j^i x = 0) \cap (v_s^i x = 1) =: L_s^i \ . \tag{13}$$

Since L_s^i is the intersection of $n-1$ linearly independent hyperplanes in $\mathbb{R}^n$, it is a line. Because $a^i \perp u_j$ and $a^i \perp v_j^i$, the direction of the line is a^i. Let us assume w.l.o.g. that $a_1 \neq 0$, and therefore $a_1^i \neq 0$ for large enough i. Let $\bar{y}_s^i$ denote the intersection of L_s^i with the hyperplane $(x_1 = 0)$. Note that $\bar{y}_s^i \neq \pm\infty$

because of the assumption $a_1^i \neq 0$. That is, $\bar{y}_s^i$ is the unique solution to the system

$$\begin{bmatrix} e_1 \\ u_1 \\ \vdots \\ u_k \\ v_1^i \\ \vdots \\ v_{s-1}^i \\ v_s^i \\ v_{s+1}^i \\ \vdots \\ v_l^i \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

For convenience, we introduce some additional notation: Let U denote the matrix with rows u_p, $p = 1, \ldots, k$, and let V_{-s}^i denote the matrix with rows v_j^i for all $j = 1, \ldots, l$ such that $j \neq s$. Similarly, let W_{-s}^i denote the matrix with rows w_j^i for all $j = 1, \ldots, l$ with $j \neq s$. Finally, let V^i and W^i denote the matrices with rows v_j^i and w_j^i, for $j = 1, \ldots, l$, respectively. Then the above system becomes

$$\begin{bmatrix} e_1 \\ U \\ V_{-s}^i \\ v_s^i \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Claim 5.5 *For $s = 1, \ldots, l$, $\bar{y}_s^i q_s^i \longrightarrow \bar{x}_s^i$, where $\bar{x}_s^i$ denotes the unique solution to the linear system of equations*

$$\begin{bmatrix} e_1 \\ U \\ W_{-s}^i \\ w_s^i \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Proof of claim. Let v_j^i/q_j^i denote the vector obtained by dividing every component of v_j^i with the scalar q_j^i. Furthermore, let V_{-s}^i/q_{-s}^i be the matrix with rows v_j^i/q_j^i for all $j \neq s$. Then

$$\begin{bmatrix} e_1 \\ U \\ V_{-s}^i \\ v_s^i \end{bmatrix} \bar{y}_s^i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} e_1 \\ U \\ V_{-s}^i \\ v_s^i/q_s^i \end{bmatrix} \bar{y}_s^i q_s^i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} e_1 \\ U \\ V_{-s}^i/q_{-s}^i \\ v_s^i/q_s^i \end{bmatrix} \bar{y}_s^i q_s^i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

where Claim 5.4 implies

$$\begin{bmatrix} e_1 \\ U \\ V^i_{-s}/q^i_{-s} \\ v^i_s/q^i_s \end{bmatrix} \longrightarrow \begin{bmatrix} e_1 \\ U \\ W^i_{-s} \\ w^i_s \end{bmatrix} .$$

□

Now we will show that the entries of $\bar{x}^i_s$ cannot become arbitrarily large.

Claim 5.6 *There exists a constant $K_1 > 0$ such that for sufficiently large i,*

$$\|\bar{x}^i_s\|_\infty \leq K_1 .$$

Proof of claim. By definition,

$$\bar{x}^i_s = \begin{bmatrix} e_1 \\ U \\ W^i_{-s} \\ w^i_s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} .$$

Therefore, it suffices to show that the entries of the inverse matrix in the above equation cannot be arbitrarily large. We have

$$\left(A^i\right)^{-1} := \begin{bmatrix} e_1 \\ U \\ W^i \end{bmatrix}^{-1} = \frac{1}{\det(A^i)} \operatorname{adj}(A^i) ,$$

where $\operatorname{adj}(A^i)$ denotes the adjugate matrix of A^i. Since all entries of A^i are bounded (note that $\left\|w^i_j\right\| = R$), every entry of $\operatorname{adj}(A^i)$ is bounded as well. Hence, it is sufficient to show that $\left|\det(A^i)\right|$ can be bounded from below for large enough i. The absolute value of the determinant of A^i corresponds to the volume of the parallelepiped spanned by the vectors $u_1, \dots, u_k, w^i_1, \dots, w^i_l, e_1$. Therefore, it holds that

$$\left|\det(A^i)\right| = \|\tilde{u}_1\| \dots \|\tilde{u}_k\| \left\|\tilde{w}^i_1\right\| \dots \left\|\tilde{w}^i_l\right\| \|\tilde{e}_1\| .$$

Here, $\tilde{e}_1$ denotes the orthogonal projection of e_1 onto $(\operatorname{span}(u_1, \dots, u_k, w^i_1, \dots, w^i_l))^\perp$. Hence, by property (viii), the vector $\tilde{e}_1$ is the orthogonal projection of e_1 onto $V^\perp$. Because of the assumption $a_1 \neq 0$, it follows that $\|\tilde{e}_1\| > 0$. With property (ix), we obtain for sufficiently large i,

$$\left|\det(A^i)\right| \geq (cR)^l \ \|\tilde{u}_1\| \dots \|\tilde{u}_k\| \|\tilde{e}_1\| .$$

As the expression on the right in the last inequality is a strictly positive constant, the claim follows. □

Now let us define for any vector $M^i = (M^i_1, \ldots, M^i_l) \in \mathbb{N}^l$, the set

$$L^i(M^i) := (u_1 x = 0) \cap \ldots \cap (u_k x = 0) \cap (v^i_1 x = M^i_1) \cap \ldots \cap (v^i_l x = M^i_l) \ .$$

By virtue of (13), we have

$$\left(M^i_1 y^i_1 + \ldots + M^i_l y^i_l\right) \in L^i(M^i) \cap \mathbb{Z}^n$$

and therefore,

$$L^i(M^i) \cap \mathbb{Z}^n \neq \emptyset \ .$$

Moreover, $L^i(M^i)$ is a line with direction a^i that intersects the hyperplane $(x_1 = 0)$ in $\bar{y}^i(M^i) := M^i_1 \, \bar{y}^i_1 + \ldots + M^i_l \, \bar{y}^i_l$. Therefore, we can write

$$L^i(M^i) = \left\{ x \in \mathbb{R}^n \,\middle|\, x = \bar{y}^i(M^i) + \mu a^i, \ \mu \in \mathbb{R} \right\} \ .$$

Observe that every line segment of length $\|a^i\|$ of $L^i(M^i)$ must contain an integral point.

Now let $\varepsilon > 0$ be an arbitrary constant. In the remainder of the proof we show that there exists a constant C and an integer $N(\varepsilon)$ such that for all $i \geq N(\varepsilon)$ and for every $\alpha \in \mathbb{R}^l_+$ with $\|\alpha\|_\infty \leq 1$, there is a vector $M^i = M^i(\alpha) \in \mathbb{N}^l$ and a number $\mu^i_0 = \mu^i_0(\alpha) \in \mathbb{R}$ such that for each $\mu \in [\mu^i_0, \mu^i_0 + 1]$ and each $\delta \in \{-1, 1\}^l$ the vector

$$h^i(\delta M^i, \mu) := \bar{y}^i(\delta M^i) + \mu a^i = \delta_1 M^i_1 \, \bar{y}^i_1 + \ldots + \delta_l M^i_l \, \bar{y}^i_l + \mu \, a^i \tag{14}$$

satisfies

$$\begin{aligned}
h^i(\delta M^i, \mu) \quad &\perp \quad u_p && \text{for } p = 1, \ldots, k && (15)\\
\alpha_j - \varepsilon \quad &\leq \quad \delta_j w^i_j \, h^i(\delta M^i, \mu) \leq \alpha_j + \varepsilon && \text{for } j = 1, \ldots, l && (16)\\
\delta_j h^i(\delta M^i, \mu) \, v^i_j \quad &= \quad \lfloor \alpha_j q^i_j \rfloor && \text{for } j = 1, \ldots, l && (17)\\
0 \quad &\geq \quad r_s \, h^i(\delta M^i, \mu) && \text{for } s = 1, \ldots, m && (18)\\
\left| h^i(\delta M^i, \mu) \, a^i \right| \quad &\leq \quad C \left\| a^i \right\|^2 \ . && && (19)
\end{aligned}$$

Here, the notation δM^i means $(\delta_1 M^i_1, \ldots, \delta_l M^i_l)$. Since every line segment of length $\|a^i\|$ contains an integral point, there must exists some $\mu \in [\mu^i_0, \mu^i_0 + 1]$ such that $h^i(\delta M^i, \mu)$ is an integral vector. Consequently, this would imply property (x) of the lemma.

First, observe that condition (15) always holds, since any h of the form (14) is a linear combination of vectors that are perpendicular to the vectors u_p, $p = 1, \ldots, k$. Using definition (14) of $h^i(\delta M^i, \mu)$, condition (18) becomes

$$0 \geq \delta_1 M^i_1 \, r_s \, \bar{y}^i_1 + \ldots + \delta_l M^i_l \, r_s \, \bar{y}^i_l + \mu \, r_s a^i \ .$$

Now let $\beta > 0$ be the constant from (8), that is, $r_s\, a^i \leq -\beta$, for $s = 1, \ldots, m$. Then (18) becomes

$$\mu \geq \frac{\delta_1 M_1^i\, r_s\, \bar{y}_1^i + \ldots + \delta_l M_l^i\, r_s\, \bar{y}_l^i}{-r_s a^i} \ ,$$

and for

$$\mu_0^i := \max_{s=1,\ldots,m} \left\{ \frac{\delta_1 M_1^i\, r_s\, \bar{y}_1^i + \ldots + \delta_l M_l^i\, r_s\, \bar{y}_l^i}{\beta} \right\} \ ,$$

this condition is satisfied for all $\mu \geq \mu_0^i$. Let $r^i \in \{r_1, \ldots, r_m\}$ such that

$$\mu_0^i = \frac{\delta_1 M_1^i\, r^i\, \bar{y}_1^i + \ldots + \delta_l M_l^i\, r^i\, \bar{y}_l^i}{\beta} \ .$$

Then by (14),

$$h^i(\delta M^i, \mu_0^i) = \delta_1 M_1^i \left(\bar{y}_1^i + \frac{1}{\beta} r^i\, \bar{y}_1^i\, a^i \right) + \ldots + \delta_l M_l^i \left(\bar{y}_l^i + \frac{1}{\beta} r^i\, \bar{y}_l^i\, a^i \right) \ ,$$

and (16) becomes for $\mu = \mu_0^i$ and $j = s$,

$$\alpha_s - \varepsilon \leq \delta_s \left(\sum_{j=1}^{l} \delta_j M_j^i \left(w_s^i\, \bar{y}_j^i + \frac{1}{\beta}\, r^i\, \bar{y}_j^i\, w_s^i\, a^i \right) \right) \leq \alpha_s + \varepsilon \ . \tag{20}$$

Now let us define $M_j^i := \left\lfloor \alpha_j q_j^i \right\rfloor$. Note that $M^i \in \mathbb{N}^l$. In the following, we will show that this choice for M^i satisfies (20). For this, we consider the terms in (20) separately. We start with the terms $\delta_s \delta_j M_j^i\, w_s^i\, \bar{y}_j^i$. If $j = s$, we get with Claims 5.4 ($q_s^i \to \infty$) and 5.5 ($w_s^i \bar{x}_s^i = 1$),

$$\begin{aligned}
\left|\delta_s \delta_s M_s^i\, w_s^i\, \bar{y}_s^i - \alpha_s\right| &= \left|\left\lfloor \alpha_s q_s^i \right\rfloor w_s^i\, \bar{y}_s^i - \alpha_s\right| \leq \left|\alpha_s w_s^i\, \bar{y}_s^i q_s^i - \alpha_s\right| + \left|w_s^i\, \bar{y}_s^i\right| \\
&= \left|\alpha_s w_s^i\, \bar{y}_s^i q_s^i - \alpha_s w_s^i \bar{x}_s^i\right| + \frac{1}{q_s^i} \left|w_s^i\, \bar{y}_s^i q_s^i - w_s^i\, \bar{x}_s^i + w_s^i\, \bar{x}_s^i\right| \\
&\leq \left|w_s^i\, (\bar{y}_s^i q_s^i - \bar{x}_s^i)\right| + \frac{1}{q_s^i} \left|w_s^i\, (\bar{y}_s^i q_s^i - \bar{x}_s^i)\right| + \frac{1}{q_s^i} \left|w_s^i\, \bar{x}_s^i\right| \\
&\leq \left(1 + \frac{1}{q_s^i}\right) \left\|w_s^i\right\| \left\|\bar{y}_s^i q_s^i - \bar{x}_s^i\right\| + \frac{1}{q_s^i} \\
&= R \left(1 + \frac{1}{q_s^i}\right) \left\|\bar{y}_s^i q_s^i - \bar{x}_s^i\right\| + \frac{1}{q_s^i} \longrightarrow 0 \ .
\end{aligned}$$

Hence, for every $\varepsilon_1 > 0$ and for sufficiently large i,

$$\alpha_s - \varepsilon_1 \leq \delta_s \delta_s M_s^i\, w_s^i\, \bar{y}_s^i \leq \alpha_s + \varepsilon_1 \ . \tag{21}$$

For $j \neq s$, it similarly follows by Claims 5.4 ($q_j^i \to \infty$) and 5.5 ($w_s^i \bar{x}_j^i = 0$) that

$$\begin{aligned}
\left|\delta_s \delta_j M_j^i \, w_s^i \, \bar{y}_j^i\right| &= \left|\lfloor \alpha_j q_j^i \rfloor \, w_s^i \, \bar{y}_j^i\right| = \left|\alpha_j w_s^i \, \bar{y}_j^i q_j^i - \alpha_j w_s^i \bar{x}_j^i\right| + \left|w_s^i \, \bar{y}_j^i\right| \\
&\leq \left|w_s^i \left(\bar{y}_j^i q_j^i - \bar{x}_j^i\right)\right| + \frac{1}{q_j^i} \left|w_s^i \left(\bar{y}_j^i q_j^i - \bar{x}_j^i\right)\right| \\
&\leq R \left(1 + \frac{1}{q_j^i}\right) \left\|\bar{y}_j^i q_j^i - \bar{x}_j^i\right\| \longrightarrow 0 \ .
\end{aligned}$$

Thus, for sufficiently large i,

$$-\,\varepsilon_1 \leq \delta_s \delta_j M_j^i \, w_s^i \, \bar{y}_j^i \leq \varepsilon_1 \ . \tag{22}$$

Now consider the terms $\delta_s \delta_j M_j^i \frac{1}{\beta} r^i \, \bar{y}_j^i \, w_s^i \, a^i$. With Claim 5.5, we obtain

$$\begin{aligned}
\left|\delta_s \delta_j M_j^i \frac{1}{\beta} r^i \, \bar{y}_j^i \, w_s^i \, a^i\right| &= \frac{1}{\beta} \left|\lfloor \alpha_j q_j^i \rfloor \, (r^i \, \bar{y}_j^i) \, (w_s^i \, a^i)\right| \\
&\leq \frac{1}{\beta} \left|\alpha_j \, (r^i \, \bar{y}_j^i \, q_j^i) \, (w_s^i \, a^i)\right| + \frac{1}{\beta} \left|(r^i \, \bar{y}_j^i) \, (w_s^i \, a^i)\right| \\
&\leq \frac{1}{\beta} \left|r^i \, \bar{y}_j^i \, q_j^i\right| \left|w_s^i \, a^i\right| + \frac{1}{\beta} \left|r^i \, \bar{y}_j^i\right| \left|w_s^i \, a^i\right| \\
&= \frac{1}{\beta}\left(1 + \frac{1}{q_j^i}\right) \left|r^i \, \bar{y}_j^i \, q_j^i\right| \left|w_s^i \, a^i\right| \\
&= \frac{1}{\beta}\left(1 + \frac{1}{q_j^i}\right) \left|r^i \, \bar{y}_j^i \, q_j^i - r^i \bar{x}_j^i + r^i \bar{x}_j^i\right| \left|w_s^i \, a^i\right| \\
&\leq \frac{1}{\beta}\left(1 + \frac{1}{q_j^i}\right) \Big(\left|r^i \, (\bar{y}_j^i \, q_j^i - \bar{x}_j^i)\right| + \left|r^i \, \bar{x}_j^i\right| \Big) \left|w_s^i \, a^i\right| \\
&\leq \frac{1}{\beta}\left(1 + \frac{1}{q_j^i}\right) \left\|r^i\right\| \Big(\left\|\bar{y}_j^i q_j^i - \bar{x}_j^i\right\| + \left\|\bar{x}_j^i\right\| \Big) \left|w_s^i \, a^i\right| \ .
\end{aligned}$$

We can bound

$$\frac{1}{\beta}\left(1 + \frac{1}{q_j^i}\right) \left\|r^i\right\| \Big(\left\|\bar{y}_j^i q_j^i - \bar{x}_j^i\right\| + \left\|\bar{x}_j^i\right\| \Big)$$

from above by Claims 5.4, 5.5, and 5.6 for sufficiently large i. Furthermore, using (7) and $\bar{a} w_s^i = 0$, we get for each $s = 1, \ldots, l$,

$$\left|w_s^i a^i\right| = \left\|a^i\right\| \left|\bar{a}^i w_s^i\right| = \left\|a^i\right\| \left|\bar{a}^i w_s^i - \bar{a} w_s^i\right| \leq \left\|a^i\right\| \left\|\bar{a}^i - \bar{a}\right\| \left\|w_s^i\right\| = R \left\|a^i\right\| \left\|\bar{a}^i - \bar{a}\right\| \longrightarrow 0 \ . \tag{23}$$

It follows that

$$\left|\delta_s \delta_j M_j^i \,\frac{1}{\beta}\, r^i\, \bar{y}_j^i\, w_s^i\, a^i\right| \longrightarrow 0 \ .$$

Consequently, for large enough i,

$$-\,\varepsilon_1 \le \delta_s \delta_j M_j^i \,\frac{1}{\beta}\, r^i\, \bar{y}_j^i\, w_s^i\, a^i \le \varepsilon_1 \ . \tag{24}$$

Plugging (21),(22), and (24) into (20), we can bound $\delta_s w_s^i\, h^i(\delta M^i, \mu_0^i)$ for large enough i by

$$\alpha_s - 2\, l\, \varepsilon_1 \le \delta_s w_s^i\, h^i(\delta M^i, \mu_0^i) \le \alpha_s + 2\, l\, \varepsilon_1 \ .$$

Because of (23), we have $\left|w_s^i a^i\right| \le \varepsilon_1$ for large enough i. Therefore, for all $\mu \in [\mu_0^i, \mu_0^i + 1]$,

$$\alpha_s - (2l+1)\, \varepsilon_1 \le \delta_s w_s^i\, h^i(\delta M^i, \mu) \le \alpha_s + (2l+1)\, \varepsilon_1 \ .$$

Finally, by choosing $\varepsilon_1 < \frac{\varepsilon}{(2l+1)}$, we obtain that there exists some integer $N(\varepsilon)$ such that for all $i \ge N(\varepsilon)$ and for all $\mu \in [\mu_0^i, \mu_0^i + 1]$,

$$\alpha_s - \varepsilon \le \delta_s w_s^i\, h^i(\delta M^i, \mu) \le \alpha_s + \varepsilon \ .$$

In particular, there must exist an integral $h_\alpha^i(\delta)$ with this property. Note that $N(\varepsilon)$ does not depend on α. This proves condition (16). For condition (17), observe that $hv_j^i = M_j^i$ for every $h \in L^i(M^i)$ and every $j = 1, \ldots, l$. We thus get

$$\delta_j h^i(\delta M^i, \mu) v_j^i = \delta_j^2 M_j^i = \left\lfloor \alpha_j q_j^i \right\rfloor \ .$$

Finally, consider condition (19). For every $\mu \in [\mu_0^i, \mu_0^i + 1]$, we have

$$\begin{aligned}
\left|h(\delta M^i, \mu) a^i\right| &\le \left|h(\delta M^i, \mu_0^i)\, a^i\right| + \left|a^i a^i\right| \\
&= \left|\sum_{j=1}^{l} \delta_j \left\lfloor a_j q_j^i \right\rfloor \left(\bar{y}_j^i + \frac{1}{\beta}(r^i \bar{y}_j^i) a^i\right) a^i\right| + \left\|a^i\right\|^2 \\
&\le \sum_{j=1}^{l} \left(\left|\alpha_j q_j^i \left(\bar{y}_j^i\, a^i + \frac{1}{\beta}\, r^i\, \bar{y}_j^i\, a^i\, a^i\right)\right| + \left|\bar{y}_j^i\, a^i + \frac{1}{\beta}\, r^i\, \bar{y}_j^i\, a^i\, a^i\right| \right) + \left\|a^i\right\|^2 \\
&\le \sum_{j=1}^{l} \left(1 + \frac{1}{q_j^i}\right) \left(\left|\bar{y}_j^i q_j^i\, a^i\right| + \frac{1}{\beta}\, \left|r^i\, \bar{y}_j^i q_j^i\, a^i\, a^i\right| \right) + \left\|a^i\right\|^2 \\
&\le \sum_{j=1}^{l} \left(1 + \frac{1}{q_j^i}\right) \left\|\bar{y}_j^i q_j^i\right\| \left(\left\|a^i\right\| + \frac{1}{\beta} \left\|r^i\right\| \left\|a^i\right\|^2 \right) + \left\|a^i\right\|^2 \ .
\end{aligned}$$

Since $\left\|\bar{y}_j^i q_j^i\right\|$ is bounded because of Claims 5.5 and 5.6, since $q_j^i \to \infty$ by Claim 5.4, and since $\left\|r^i\right\|$ is bounded as well, condition (19) is satisfied for sufficiently large i. $\square$

With this, we are prepared to show that for every inequality $ax \leq a_0$ of the non-rational system $A^2x \leq b^2$, there exists a finite set S_a of integral vectors such that $C_{S_a}(P) \subseteq (ax \leq a_0)$. Observe that this immediately implies the existence of a finite set $S \subset \mathbb{Z}^n$ with $C_S(P) \subseteq P$.

Lemma 5.7 *Let $(a, a_P) \in \mathbb{R}^{n+1} \setminus \mathbb{Q}^{n+1}$ such that $(ax = a_P)$ is a non-rational hyperplane with $P \subseteq (ax \leq a_P)$ and $P \cap (ax = a_P) \neq \emptyset$. Then there exists a finite set $S \subseteq \mathbb{Z}^n$ such that $C_S(P) \subseteq (ax \leq a_P)$.*

Proof. There are three possible types of non-rational inequalities $ax \leq a_P$:

(a) $a \in \mathbb{Q}^n$ and $a_P \in \mathbb{R} \setminus \mathbb{Q}$.

(b) $a \in \mathbb{R}^n \setminus \mathbb{Q}^n$ and $(ax = a_P) \cap \mathbb{Q}^n \neq \emptyset$.

(c) $a \in \mathbb{R}^n \setminus \mathbb{Q}^n$ and $(ax = a_P) \cap \mathbb{Q}^n = \emptyset$.

We will consider each case separately:

Case (a): If $a \in \mathbb{Q}^n$, we can assume w.l.o.g. that $a \in \mathbb{Z}^n$ by scaling (a, a_P) by some rational number, if necessary. Consequently, $ax \leq \lfloor a_P \rfloor$ is a Gomory-Chvátal cut for P and $(ax \leq \lfloor a_P \rfloor) \subseteq (ax \leq a_P)$. Then $S = \{a\}$ has the desired property and we are done.

In the following, let us assume that $a \in \mathbb{R}^n \setminus \mathbb{Q}^n$ and that the same is true for every λa with $\lambda \in \mathbb{R}$. Let $F = P \cap (ax = a_P)$ and let $r_1, \ldots, r_m \in \mathbb{R}^n$ denote the set of edge directions emanating from the vertices of F to vertices of P that are not in F. Note that $r_s\, a < 0$, for $s = 1, \ldots, m$. First, consider case (b):

Case (b): Let V_R denote the maximal rational affine subspace contained in $(ax = a_P)$ and let $u_1, \ldots, u_k \in \mathbb{Z}^n$ and $x_0 \in (ax = a_P) \cap \mathbb{Q}^n$ such that $V_R = x_0 + \mathrm{span}(u_1, \ldots, u_k)$. Define $l := n - k - 1$ and $U := \mathrm{span}(u_1, \ldots, u_k)$. Note that $U = \{0\}$ is possible. Since P is bounded, there exists an $R_1 > 0$ such that for every $x \in F$ there is an $u \in U$ with

$$x \in x_0 + u + B(0, R_1) \ . \tag{25}$$

Let $p_0 \in \mathbb{Z}^n$ and let $q_0 \geq 1$ be an integer such that $x_0 = p_0/q_0$. Furthermore, let c be the constant from property (ix) in Lemma 5.2 and let c_1 be the constant from Lemma 4.3. Let us fix a constant R such that $R \geq R_1 c_1 / c$ and consider the sequences that exist according to Lemma 5.2 for $V = (ax = 0)$ and R.

First, observe that we can choose i large enough such that $a^i x \le \lfloor a^i x_0 \rfloor$ is a Gomory-Chvátal cut for P: Property (ii) in Lemma 5.2 implies

$$\max\left\{a^i x \mid x \in P\right\} = \max\left\{a^i x \mid x \in F\right\} = a^i x_0 + \max\left\{a^i (x - x_0) \mid x \in F\right\} \ ,$$

and by property (iii) and the boundedness of P, we have for all $x \in F$,

$$\begin{aligned} \left\|a^i(x - x_0)\right\| &= \left\|a^i\right\| \left\|\bar{a}^i(x - x_0)\right\| = \left\|a^i\right\| \left\|(\bar{a}^i - \bar{a})(x - x_0)\right\| \\ &\le \left\|a^i\right\| \left\|\bar{a}^i - \bar{a}\right\| \|x - x_0\| \longrightarrow 0 \ . \end{aligned}$$

Hence, we can choose i large enough such that

$$\max\left\{a^i x \mid x \in P\right\} \le a^i x_0 + \frac{1}{2q_0} \ ,$$

which implies that $a^i x \le \lfloor a^i x_0 \rfloor$ is a Gomory-Chvátal cut for P.

Now let $\alpha = \frac{1}{2q_0(l+1)}(1, \ldots, 1)$. Also by Lemma 5.2, there exists an index i such that the vectors $v_j := v^i_j$ and $w_j := w^i_j$, for $j = 1, \ldots, l$, and the integral vectors $h(\delta) := h^i_\alpha(\delta)$, for each $\delta \in \{-1, 1\}^l$, satisfy

$$h(\delta) \perp u_p \qquad \text{for } p = 1, \ldots, k \tag{26}$$

$$0 < \delta_j w_j \, h(\delta) \le (q_0(l+1))^{-1} \qquad \text{for } j = 1, \ldots, l \tag{27}$$

$$\delta_j h(\delta) \, v_j \ge 1 \qquad \text{for } j = 1, \ldots, l \tag{28}$$

$$0 \ge r_s \, h(\delta) \qquad \text{for } s = 1, \ldots, m \ . \tag{29}$$

Moreover, it holds that $\|\tilde{w}_j\| \ge cR$, where $\tilde{w}_j$ denotes the orthogonal projection of w_j onto $(\operatorname{span}(u_1, \ldots, u_k, w_1, \ldots, w_{j-1}))^\perp$. Observe that by (25), every point $x \in F$ can be written as

$$x = x_0 + u' + \sum_{j=1}^{l} \tilde{\lambda}_j \tilde{w}_j \ ,$$

where $u' \in U$ and $|\tilde{\lambda}_j| \le R_1/(cR)$, for $j = 1, \ldots, l$. Then it follows by Lemma 4.3 that every $x \in F$ can be expressed as

$$x = x_0 + u + \sum_{j=1}^{l} \lambda_j w_j \ , \tag{30}$$

where $u \in U$ and $|\lambda_j| \le c_1 R_1/(cR) \le 1$, for $j = 1, \ldots, l$. Now consider any $\delta \in \{-1, 1\}^l$. We obtain with (26)-(29) and (30),

$$\begin{aligned}\max\{h(\delta)\,x \mid x \in P\} &= \max\{h(\delta)\,x \mid x \in F\} \le h(\delta)\,x_0 + \sum_{j=1}^{l} \max_{\lambda_j \in [-1,1]} \{\lambda_j h(\delta)\,w_j\} \\ &= h(\delta)\,x_0 + \sum_{j=1}^{l} \delta_j h(\delta)\,w_j \le h(\delta)\,x_0 + l(q_0(l+1))^{-1} \\ &< h(\delta)\,x_0 + \frac{1}{q_0} \le \lfloor h(\delta)\,x_0 \rfloor + 1 \ .\end{aligned}$$

Hence, $h(\delta)\,x \le \lfloor h(\delta)\,x_0 \rfloor$ is a Gomory-Chvátal cut for P for every $\delta \in \{-1,1\}^l$. Now consider an arbitrary $x \in (ax = a_P) \setminus V_R$. By (30), there exists an $u \in U$ and $\lambda_j \in \mathbb{R}_+$ and $\delta_j \in \{-1,1\}$ for $j = 1,\ldots,l$ such that

$$x = x_0 + u + \sum_{j=1}^{l} \lambda_j \delta_j w_j \ .$$

Note that $\sum_{j=1}^{l} \lambda_j > 0$, as $x \notin V_R$. Consequently,

$$h(\delta)\,x = h(\delta)\,x_0 + \sum_{j=1}^{l} \lambda_j \delta_j h(\delta)\,w_j > h(\delta)\,x_0 \ge \lfloor h(\delta)\,x_0 \rfloor \ ,$$

that is, x violates the Gomory-Chvátal cut $h(\delta)\,x \le \lfloor h(\delta)\,x_0 \rfloor$. Now let H denote the polyhedron defined by the intersection of the 2^l half-spaces associated with the Gomory-Chvátal cuts $h(\delta)\,x \le \lfloor h(\delta)\,x_0 \rfloor$, with $\delta \in \{-1,1\}^l$. Then by the last observation,

$$\Big((ax = a_P) \cap H\Big) = \Big((ax = a_P) \cap \bigcap_{\delta \in \{-1,1\}^l} \big(h(\delta)\,x \le \lfloor h(\delta)\,x_0 \rfloor\big)\Big) \subseteq V_R \ . \tag{31}$$

Similarly, let us consider the integral hyperplane $(a^i x = a^i x_0)$. Any $x \in (a^i x = a^i x_0) \setminus V_R$ can be written as

$$x = x_0 + u + \sum_{j=1}^{l} \lambda_j \delta_j v_j^i \ ,$$

for some $u \in U$, and $\lambda_j \in \mathbb{R}_+$ and $\delta_j \in \{-1,1\}$, $j = 1,\ldots,l$; and in this representation it must also hold that $\sum_{j=1}^{l} \lambda_j > 0$. Then with (28)

$$h(\delta)\,x = h(\delta)\,x_0 + \sum_{j=1}^{l} \lambda_j \delta_j h(\delta)\,v_j > h(\delta)\,x_0 \ge \lfloor h(\delta)\,x_0 \rfloor \ .$$

This implies that also every point in $(a^i x = a^i x_0) \setminus V_R$ is separated by some Gomory-Chvátal cut $h(\delta)\,x \le \lfloor h(\delta)\,x_0 \rfloor$ and, thus,

$$\Big((a^i x = a^i x_0) \cap H\Big) \subseteq V_R \ . \tag{32}$$

As every hyperplane $\big(h(\delta)\,x = \lfloor h(\delta)\,x_0 \rfloor\,\big)$ is parallel to V_R, either every point in V_R satisfies the corresponding inequality or every point in V_R violates it. Therefore,

$$\Big((ax = a_P) \cap H\Big) = \Big((a^i x = a^i x_0) \cap H\Big) \in \{\emptyset, V_R\} \ \ .$$

Observe furthermore that every minimal face of $\big((a^i x \leq a^i x_0) \cap H\big)$ is also a minimal face of $\big((ax \leq a_P) \cap H\big)$ and vice versa. Consequently,

$$\Big((a^i x \leq \lfloor a^i x_0 \rfloor\,) \cap H\Big) \subseteq \Big((a^i x \leq a^i x_0) \cap H\Big) = \Big((ax \leq a_P) \cap H\Big) \subseteq (ax \leq a_P) \ \ .$$

It follows that a^i and the vectors $h(\delta)$, for $\delta \in \{-1,1\}^l$, form the desired set S of the lemma.

Case (c): In the remainder of the proof, we consider the case $(ax = a_P) \cap \mathbb{Q}^n = \emptyset$. Let $u_1, \ldots, u_k \in \mathbb{Z}^n$ be a maximal set of linearly independent integral vectors such that $au_i = 0$ for $i = 1, \ldots, k$. Let $U := \operatorname{span}(u_1, \ldots, u_k)$ and note that $U = \{0\}$ is possible. Furthermore, take an arbitrary point $x_0 \in F$. Since P is bounded, there exists a constant $R_1 > 0$ such that for every $x \in F$ there is an $u \in U$ such that

$$x \in x_0 + u + B(0, R_1) \ \ . \tag{33}$$

Let us fix an $R \geq R_1 c_1 / c$, where c and c_1 are the constants from property (ix) in Lemma 5.2 and Lemma 4.3, respectively. Now consider the sequences that exist according to Lemma 5.2 for $V = (ax = 0)$ and R .

Property (ii) from Lemma 5.2 implies that if there exists an index i and an integer a_0^i such that

$$a_0^i + 1 > \max\left\{a^i x \mid x \in P\right\} = \max\left\{a^i x \mid x \in F\right\} \geq \min\left\{a^i x \mid x \in F\right\} > a_0^i \ \ ,$$

then $a^i x \leq a_0^i$ is a Gomory-Chvátal cut for P with the property that every point in F violates the cut and such that

$$(a^i x \leq a_0^i) \cap F = \emptyset \ \ .$$

In particular, one can then find an $\varepsilon_1 > 0$ such that $\big(P \cap (a^i x \leq a_0^i)\big) \subseteq (ax \leq a_P - \varepsilon_1)$. This implies that there exists a rational polyhedron $Q \supseteq P$ such that $(a^i x \leq a_0^i)$ is also a Gomory-Chvátal cut for Q and such that $Q \cap (a^i x \leq a_0^i) \subseteq (ax \leq a_P)$. The facet normals of Q together with a^i imply the desired set S of the lemma.

Let us assume in the remainder of the proof of part (c) that for every i, there exists an integer a_0^i such that

$$F \cap (a^i x = a_0^i) \neq \emptyset \ \ .$$

Let $y^i \in F \cap (a^i x = a_0^i)$. Since $\gcd(a^i) = 1$ according to property (i) of Lemma 5.2, there exists an $z_0^i \in (a^i x = a_0^i) \cap \mathbb{Z}^n$. We have

$$(a^i x = a_0^i) = z_0^i + \operatorname{span}(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i) \ \ .$$

Let $\tilde{x}_0^i$ denote the projection of x_0 onto the hyperplane $(a^i x = a_0^i)$, that is,

$$\tilde{x}_0^i = x_0 + \frac{a_0^i - a^i x_0}{\|a^i\|^2}\, a^i \ . \tag{34}$$

Note that because of property (iii) in Lemma 5.2 and the boundedness of P,

$$\begin{aligned} \left|a_0^i - a^i x_0\right| &= \left|a^i y^i - a^i x_0\right| = \left\|a^i\right\| \left|\bar{a}^i y^i - \bar{a}^i x_0 + (\bar{a} x_0 - \bar{a} y^i)\right| \\ &\leq \left\|a^i\right\| \left\|\bar{a}^i - \bar{a}\right\| \left\|x_0 - y^i\right\| \longrightarrow 0 \ . \end{aligned} \tag{35}$$

We can assume w.l.o.g. that the point $z_0^i \in (a^i x = a_0^i) \cap \mathbb{Z}^n$ is chosen such that there exist numbers $\gamma_1^i, \ldots, \gamma_k^i, \mu_1^i, \ldots, \mu_l^i \in [0,1]$ such that

$$\tilde{x}_0^i = z_0^i + \gamma_1^i u_1 + \ldots + \gamma_k^i u_k + \mu_1^i v_1^i + \ldots + \mu_l^i v_l^i \ . \tag{36}$$

This is because $\tilde{x}_0^i \in (a^i x = a_0^i)$ can be written as

$$\begin{aligned} \tilde{x}_0^i &= z_0^i + \sum_{p=1}^{k} \gamma_p^i u_p + \sum_{j=1}^{l} \mu_j^i v_j^i \\ &= \left(z_0^i + \sum_{p=1}^{k} \left\lfloor \gamma_p^i \right\rfloor u_p + \sum_{j=1}^{l} \left\lfloor \mu_j^i \right\rfloor v_j^i\right) + \sum_{p=1}^{k} (\gamma_p^i - \left\lfloor \gamma_p^i \right\rfloor) u_p + \sum_{j=1}^{l} (\mu_j^i - \left\lfloor \mu_j^i \right\rfloor) v_j^i \ , \end{aligned}$$

and

$$\left(z_0^i + \sum_{p=1}^{k} \left\lfloor \gamma_p^i \right\rfloor u_p + \sum_{j=1}^{l} \left\lfloor \mu_j^i \right\rfloor v_j^i\right) \in \left(a^i x = a_0^i\right) \cap \mathbb{Z}^n \ .$$

Figure 2 illustrated the described situation.

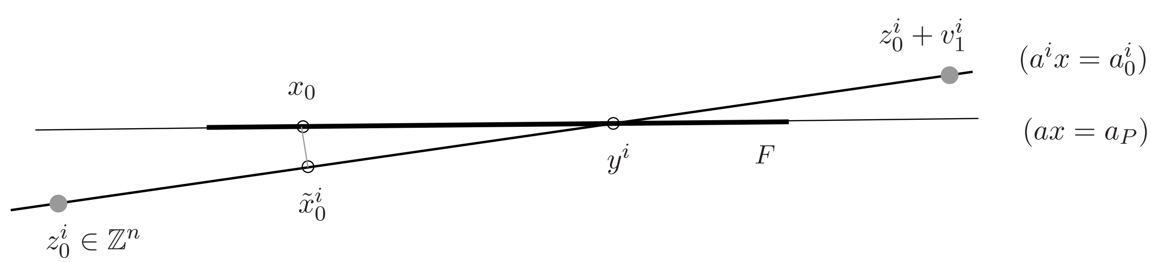

Figure 2: Illustration of the situation in part (c) of the proof of Lemma 5.7 in the special case that for every i there exists an integer a_0^i such that $F \cap (a^i x = a_0^i) \neq \emptyset$.

Next, we show that $\tilde{x}_0^i$, and therefore also x_0, is far away from any integer point in the hyperplane $\left(a^i x = a_0^i\right)$.

Claim 5.8 *Any vertex f^i of the parallelepiped*

$$z_0^i + \bar{\Pi}(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i)$$

satisfies $\left\|x_0 - f^i\right\| \longrightarrow \infty$.

Proof of claim. As F is bounded and as x_0 and y^i are points in F, there exists a constant K_1 such that for all i, $\left\|x_0 - y^i\right\| \leq K_1$. Then

$$\left\|f^i - y^i\right\| = \left\|f^i - x_0 + x_0 - y^i\right\| \leq \left\|f^i - x_0\right\| + \left\|x_0 - y^i\right\| \leq \left\|f^i - x_0\right\| + K_1$$

implies $\left\|f^i - x_0\right\| \geq \left\|f^i - y^i\right\| - K_1$. Hence, in order to show the claim it suffices to prove $\left\|f^i - y^i\right\| \longrightarrow \infty$. Suppose that there exists some positive constant $K_2 > 0$ such that for all i we have $\left\|f^i - y^i\right\| \leq K_2$. Note that then $f^i \in B(x_0, K_1 + K_2) \cap \mathbb{Z}^n$. Let $\tilde{f}^i$ denote the projection of f^i onto the hyperplane $(ax = a_P)$, that is, $\tilde{f}^i + \lambda a = f^i$, where $\lambda = (af^i - a_P)/\left\|a\right\|^2 = (af^i - ay^i)/\left\|a\right\|^2$. Since $f^i \in \mathbb{Z}^n$ and $f^i \notin (ax = a_P)$ (remember that $(ax = a_P) \cap \mathbb{Q}^n = \emptyset$) and since the number of integer points in $B(x_0, K_1 + K_2)$ is finite, there must exist some positive number D such that $\left\|f^i - \tilde{f}^i\right\| \geq D$, for every i. However, with property (iii) from Lemma 5.2 and using $\bar{a}^i(f^i - y^i) = 0$, we get

$$\begin{aligned}
\left\|f^i - \tilde{f}^i\right\| &= |\lambda| \left\|a\right\| = \frac{\left|af^i - ay^i\right|}{\left\|a\right\|} = \left|\bar{a}(f^i - y^i) - \bar{a}^i(f^i - y^i)\right| \\
&\leq \left\|\bar{a} - \bar{a}^i\right\| \left\|f^i - y^i\right\| \leq K_2 \left\|\bar{a} - \bar{a}^i\right\| \longrightarrow 0 \ ,
\end{aligned}$$

which is a contradiction. □

As the above claim implies that $\tilde{x}_0^i$ is far away from any integer point in the hyperplane $\left(a^i x = a_0^i\right)$, it is intuitive that not all the coefficients μ_j^i in the representation (36) can be close to 0 or 1. We formally prove this observation in the next claim.

Claim 5.9 *Let $K > 1$ be a constant. There exists an integer $N_1 = N_1(K)$ such that for every $i \geq N_1$, there exists an index $j \in \{1, \ldots, l\}$ such that the coefficient μ_j^i in (36) satisfies*

$$\frac{K}{q_j^i} \leq \mu_j^i \leq 1 - \frac{K}{q_j^i} \ .$$

Proof of claim. By Claim 5.8, any vertex f^i of the parallelepiped

$$z_0^i + \bar{\Pi}(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i)$$

satisfies $\left\|x_0 - f^i\right\| \longrightarrow \infty$. Therefore,

$$\left\|x_0 - f^i\right\| \leq \left\|x_0 - \tilde{x}_0^i\right\| + \left\|\tilde{x}_0^i - f^i\right\| \longrightarrow \infty \ .$$

Because (35) implies $\left\|x_0 - \tilde{x}_0^i\right\| \longrightarrow 0$, we must have

$$\left\|\tilde{x}_0^i - f^i\right\| \longrightarrow \infty \ .$$

In particular, there exists a number N_1 such that for all $i \geq N_1$

$$\left\|\tilde{x}_0^i - f^i\right\| > \sum_{p=1}^{k} \|u_p\| + 2KRl \ .$$

Now let $i \geq N_1$ and assume that there are index sets J_1^i and J_2^i such that $J_1^i \cup J_2^i = \{1, \ldots, l\}$ and such that for every index $j \in J_1^i$, we have $0 \leq \mu_j^i < K/q_j^i$, and for every index $j \in J_2^i$, it holds that $0 \leq 1 - \mu_j^i < K/q_j^i$. For the vertex

$$f^i = z_0^i + \sum_{j \in J_2^i} v_j^i \ ,$$

of the parallelepiped it follows with property (vi) from Lemma 5.2 that

$$\begin{aligned}
\left\|\tilde{x}_0^i - f^i\right\| &= \left\|\sum_{p=1}^{k} \gamma_p^i u_p + \mu_1^i \delta_1^i v_1^i + \ldots + \mu_l^i \delta_l^i v_l^i - \sum_{j \in J_2^i} \delta_j^i v_j^i\right\| \\
&\leq \sum_{p=1}^{k} \|u_p\| + \left\|\sum_{j \in J_1^i} \mu_j^i \delta_j^i v_j^i - \sum_{j \in J_2^i} (1 - \mu_j^i) \delta_j^i v_j^i\right\| \\
&\leq \sum_{p=1}^{k} \|u_p\| + \sum_{j \in J_1^i} \mu_j^i \left\|v_j^i\right\| + \sum_{j \in J_2^i} (1 - \mu_j^i) \left\|v_j^i\right\| \\
&\leq \sum_{p=1}^{k} \|u_p\| + K \sum_{j=1}^{l} \frac{\left\|v_j^i\right\|}{q_j^i} \longrightarrow \sum_{p=1}^{k} \|u_p\| + KRl \ ,
\end{aligned}$$

which is a contradiction. $\square$

The next technical claim is needed to choose a proper parameter α for the vectors $h_{\alpha^i}^i(\delta)$ in Lemma 5.2 that give rise to appropriate Gomory-Chvátal cuts.

Claim 5.10 *Let $K > 1$, $\mu \in [0, 1]$ and $q \in \mathbb{R}$ such that $q \geq 2K$ and $K/q \leq \mu \leq 1 - K/q$. Then there exist integers p_1 and p_2 such that $1 \leq p_1 \leq q/(2K)$ and*

$$p_2 + 1/4 \leq \mu p_1 \leq (p_2 + 1) - 1/4 \ .$$

Proof of claim. We consider three cases. If $1/4 \leq \mu \leq 1 - 1/4$, then $p_1 = 1$ and $p_2 = 0$ satisfy the conditions of the claim. If $\mu < 1/4$, there must exist an integer p such that $1/4 \leq \mu p \leq 1/2 \leq 1 - 1/4$. Then

$$1 \leq \frac{1}{4\mu} \leq p \leq \frac{1}{2\mu} \leq \frac{q}{2K} \ ,$$

and we can set $p_1 = p$ and $p_2 = 0$. Finally, if $\mu > 1 - 1/4$, then $1 - \mu < 1/4$ and there must exist an integer p such that $1/4 \leq (1 - \mu)p \leq 1/2$. Then

$$1 \leq \frac{1}{4(1-\mu)} \leq p \leq \frac{1}{2(1-\mu)} \leq \frac{q}{2K} \ .$$

For $p_1 = p$ and $p_2 = p - 1$, we get

$$p_2 + 1/4 < p - 1/2 \leq \mu p \leq p - 1/4 = (p_2 + 1) - 1/4 \ .$$

□

For the remainder, let us fix a constant K such that $K > 8(2+l)$. For large enough i, the assumptions of Claim 5.10 are satisfied, that is, $q^i_j \geq 2K$ for every $j = 1, \ldots, l$. Then Claims 5.9 and 5.10 imply that there exists an integer $N(K)$ such that for every $i \geq N(K)$, there exists an index $s \in \{1, \ldots, l\}$ and integer numbers p^i_1 and p^i_2 such that

$$1 \leq p^i_1 \leq \frac{q^i_s}{2K}$$

and

$$p^i_2 + \frac{1}{4} \leq \mu^i_s p^i_1 \leq (p^i_2 + 1) - \frac{1}{4} \ .$$

Note that we can write the positive integer p^i_1 as $\lfloor \bar{\alpha}^i_s q^i_s \rfloor$ for some scalar $\bar{\alpha}^i_s$. That is, there exist a number $0 < \bar{\alpha}^i_s < 1/K$ and an integer p^i such that

$$p^i + 1/4 \leq \mu^i_s \lfloor \bar{\alpha}^i_s q^i_s \rfloor \leq (p^i + 1) - 1/4 \ . \tag{37}$$

Define $\alpha^i \in \mathbb{R}^l_+$ by

$$\alpha^i_j = \begin{cases} \bar{\alpha}^i_s, & \text{if } j = s \\ 0, & \text{otherwise} \ . \end{cases}$$

Note that $\|\alpha^i\|_\infty \leq 1$. Now take $h^i := h^i_{\alpha^i}(\bar{\delta})$ according to Lemma 5.2 from property (x) for $\bar{\delta} =$

$(1,\ldots,1)$. For some sufficiently large number N_2, we can assume that for every $i \geq N_2$,

$$h^i \perp u_p \qquad \text{for } p = 1,\ldots,k \tag{38}$$

$$\alpha_j^i - 1/K \leq w_j^i\, h^i \leq \alpha_j^i + 1/K \qquad \text{for } j = 1,\ldots,l \tag{39}$$

$$h^i\, v_j^i = \lfloor \alpha_j^i q_j^i \rfloor \qquad \text{for } j = 1,\ldots,l \tag{40}$$

$$0 \geq r_s\, h^i \qquad \text{for } s = 1,\ldots,m \tag{41}$$

$$\left| h^i\, a^i \right| \leq C \left\| a^i \right\|^2 \ , \tag{42}$$

where $C > 0$ is a constant. By (33) and arguing as in part (b), R has been chosen large enough so that every point $x \in F$ can be written as

$$x = x_0 + u + \sum_{j=1}^{l} \lambda_j \delta_j w_j^i \ ,$$

for some $u \in U$ and $\lambda_j \in [0,1]$ and $\delta_j \in \{-1,1\}$ for $j = 1,\ldots,l$. With (34), (36), (38) and (40), we get for every $x \in F$,

$$\begin{aligned} h^i x &= h^i x_0 + h^i u + \sum_{j=1}^{l} \lambda_j h^i \delta_j w_j^i = h^i x_0 + \sum_{j=1}^{l} \lambda_j h^i \delta_j w_j^i \\ &= h^i z_0^i + \mu_1^i\, h^i\, v_1^i + \ldots + \mu_l^i\, h^i\, v_l^i - \frac{a_0^i - a^i x_0}{\|a^i\|^2}\, h^i a^i + \sum_{j=1}^{l} \lambda_j h^i \delta_j w_j^i \\ &= h^i z_0^i + \mu_1^i \lfloor \alpha_1^i q_1^i \rfloor + \ldots + \mu_l^i \lfloor \alpha_l^i q_l^i \rfloor - \frac{a_0^i - a^i x_0}{\|a^i\|^2}\, h^i a^i + \sum_{j=1}^{l} \lambda_j h^i \delta_j w_j^i \\ &= h^i z_0^i + \mu_s^i \lfloor \bar{\alpha}_s^i q_s^i \rfloor - \frac{a_0^i - a^i x_0}{\|a^i\|^2}\, h^i a^i + \sum_{j=1}^{l} \lambda_j h^i \delta_j w_j^i \ . \end{aligned}$$

For large enough i we get with (35) and (42),

$$\left| \frac{a_0^i - a^i x_0}{\|a^i\|^2}\, h^i a^i \right| \leq 1/K \ .$$

Consequently, with (39) and $0 \leq \bar{\alpha}_s^i < 1/K$, we obtain

$$\begin{aligned} \left| -\frac{a_0^i - a^i x_0}{\|a^i\|^2}\, h^i a^i + \sum_{j=1}^{l} \lambda_j h^i \delta_j w_j \right| &\leq \left| \frac{a_0^i - a^i x_0}{\|a^i\|^2}\, h^i a^i \right| + \sum_{j=1}^{l} \left| h^i w_j \right| \\ &\leq \frac{1}{K} + \bar{\alpha}_s^i + \frac{l}{K} \leq \frac{2+l}{K} < \frac{1}{8} \ . \end{aligned}$$

This implies that for every $x \in F$,

$$h^i z_0^i + \mu_s^i \lfloor \bar{\alpha}_s^i q_s^i \rfloor - \frac{1}{8} \leq h^i x \leq h^i z_0^i + \mu_s^i \lfloor \bar{\alpha}_s^i q_s^i \rfloor + \frac{1}{8} \ ,$$

and with (37), it follows that for every $x \in F$,

$$(h^i z_0^i + p^i) + 1/8 \leq h^i x \leq (h^i z_0^i + p^i + 1) - 1/8 \ . \tag{43}$$

Now observe that (41) implies that

$$\max\left\{h^i x \mid x \in P\right\} = \max\left\{h^i x \mid x \in F\right\} \ .$$

Therefore, using (43) and the fact that $z_0^i \in \mathbb{Z}^n$, we have that $h^i x \leq h^i z_0^i + p^i$ is a Gomory-Chvátal cut for P. Moreover, (43) implies that this cut is violated by every point in F, that is,

$$\left(h^i x \leq h^i z_0^i + p^i\right) \cap F = \emptyset \ .$$

Arguing as at the beginning of part (c), we can find a rational polyhedron $Q \supseteq P$ such that $(h^i x \leq h^i z_0^i + p^i)$ is also a Gomory-Chvátal cut for Q and such that

$$Q \cap \left(h^i x \leq h^i z_0^i + p^i\right) \subseteq (ax \leq a_P) \ .$$

The facet normals of Q together with h^i imply the desired set S of the lemma. ☐

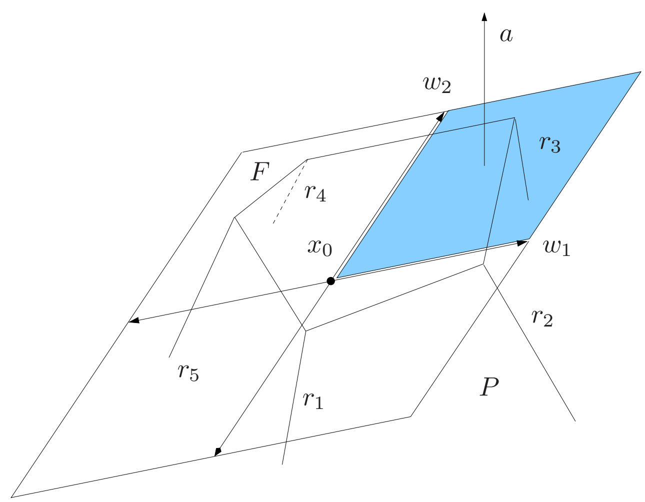

Figure 3: The hyperplane defining the facet F of P contains a single rational point x_0 and is, therefore, spanned by two non-rational vectors. F can be covered by 2^2 parallelepipeds, and for each parallelepiped there exists a single Gomory-Chvátal cut that separates all points of F that do not belong to $V_R = \{x_0\}$.

As the proof of the above lemma shows, for every non-rational face-defining inequality $ax \leq a_P$ of P, the Gomory-Chvátal procedure will separate every point in $P \cap (ax = a_P)$ that is not contained in the maximal rational affine subspace of $(ax = a_P)$.

Corollary 5.11 *Let P be a polytope and let $F = P \cap (ax = a_P)$ be a face of P. If V_R denotes the maximal rational affine subspace of $(ax = a_P)$, then $P' \cap F \subseteq V_R$.*

Lemma 5.7 gives us the tools to complete the first step of the main proof.

Corollary 5.12 *Let P be a polytope in $\mathbb{R}^n$. Then there exists a finite set $S \subseteq \mathbb{Z}^n$ such that $C_S(P) \subseteq P$.*

Proof. Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for some matrix A and some vector b. Let A^1 denote the set of vectors corresponding to rows of A that define rational facet-defining inequalities of P and let A^2 denote the set of vectors associated with the non-rational facet-defining inequalities of P. By means of Lemma 5.7, for every non-rational facet-defining inequality $ax \leq a_P$ of P, there exists a finite set $S_a \subseteq \mathbb{Z}^n$ such that $C_{S_a}(P) \subseteq (ax \leq a_P)$. Therefore, the finite set

$$S := \left(\bigcup_{a \in A^2} S_a \right) \cup A^1$$

satisfies $C_S(P) \subseteq P$. □

5.2 Step 2

In this section, we show a property of the Gomory-Chvátal closure that is sometimes referred to as *homogeneity*: The Gomory-Chvátal closure of a face of a polytope is equal to the intersection of the Gomory-Chvátal closure of the polytope with the face. This property is well-known for rational polytopes (see, e.g., Schrijver 1986), but to our knowledge, has not yet been shown for non-rational polytopes.

Lemma 5.13 *Let P be a polytope and let $F = P \cap (ax = a_P)$ be a face of P. Let V_R denote the maximal rational affine subspace of $(ax = a_P)$ and assume that $V_R \neq \emptyset$. If $cx \leq \lfloor c_F \rfloor$ is a Gomory-Chvátal cut for F and facet-defining for F', then there exists a Gomory-Chvátal cut $\bar{c}x \leq \lfloor \bar{c}_P \rfloor$ for P such that*

$$(ax = a_P) \cap V_R \cap (\bar{c}x \leq \lfloor \bar{c}_P \rfloor) = (ax = a_P) \cap V_R \cap (cx \leq \lfloor c_F \rfloor) \ .$$

Proof. Let $V_R = x_0 + \mathrm{span}(u_1, \ldots, u_k)$, where $x_0 \in (ax = a_P) \cap \mathbb{Q}^n$ and $u_1, \ldots, u_k \in \mathbb{Z}^n$, $k \leq n - 1$. Note that $V_R = \{x_0\}$ is possible. Furthermore, assume that $P \subseteq (ax \leq a_P)$. Now consider a Gomory-Chvátal cut $cx \leq \lfloor c_F \rfloor$ for F that is facet-defining for F'. Moreover, assume that $\hat{x}$ is a vertex of F that maximizes c over F. Let $r_1, \ldots, r_m$ denote all edge directions of P that emanate from vertices in F to vertices of P that are not in F. Note that for $s = 1, \dots, m$,

$$r_s\, a < 0 \ . \tag{44}$$

According to Corollary 4.6, there exists a sequence $\{a^i\} \subseteq \mathbb{Z}^n$ such that $a^i \perp u_j$, for $j = 1, \dots, k$, and such that

$$\left\|a^i\right\| \, \left\|\bar{a}^i - \bar{a}\right\| \quad \longrightarrow \quad 0 \ , \tag{45}$$

where $\bar{a}^i = a^i / \left\|a^i\right\|$ and $\bar{a} = a / \|a\|$. As $r_s \bar{a} < 0$ by (44), it follows with (45) that there exists a constant $\beta > 0$ such that $r_s \bar{a}^i \leq -\beta$ for large enough i. Hence, noting that $\left\|a^i\right\| \longrightarrow \infty$ because of $a \in \mathbb{R}^n \setminus \mathbb{Q}^n$, there exists a constant $\beta > 0$ and an $N_1 \in \mathbb{N}$ such that $r_s a^i \leq -\beta$ for all $s = 1, \dots, m$ and $i \geq N_1$. Let $M := \max_{s=1,\dots,m}\{c\, r_s\}$. If $M \leq 0$, then $\hat{x}$ also maximizes c over P and, hence, $cx \leq \lfloor c_F \rfloor$ is a Gomory-Chvátal cut for P. Therefore, assume that $M > 0$. Let $p \in \mathbb{Z}^n$ and $q \in \mathbb{N}$ with $q \geq 1$ such that $x_0 = p/q$. We define the constant $K := q\lceil \frac{1}{\beta} M \rceil$ and vectors $\vec{c}^{\,i} := c + K\, a^i$ for every $i \geq N_1$. Note that $K \in \mathbb{Z}$ and therefore $\vec{c}^{\,i} \in \mathbb{Z}^n$. We have for $s = 1, \dots, m$,

$$r_s \vec{c}^{\,i} = r_s(c + K\, a^i) \leq r_s c - K\beta \leq 0 \ ,$$

which implies that for $i \geq N_1$, the vector $\vec{c}^{\,i}$ is maximized over P by a point in F. Now let $\hat{x}^i \in \arg\max\left\{a^i x \mid x \in F\right\}$. We obtain for every $i \geq N_1$,

$$\begin{aligned} \max\left\{\vec{c}^{\,i} x \mid x \in P\right\} &= \max\left\{\vec{c}^{\,i} x \mid x \in F\right\} \\ &\leq \max\left\{cx \mid x \in F\right\} + K \max\left\{a^i x \mid x \in F\right\} \\ &= c\hat{x} + K a^i \hat{x}^i \\ &= c_F + K a^i x_0 + K a^i(\hat{x}^i - x_0) \ . \end{aligned}$$

With (45), the boundedness of F, and $a\hat{x}^i = ax_0 = a_P$, we get

$$\left|a^i(\hat{x}^i - x_0)\right| = \left\|a^i\right\| \left|(\bar{a}^i - \bar{a})(\hat{x}^i - x_0)\right| \leq \left\|a^i\right\| \left\|\bar{a}^i - \bar{a}\right\| \left\|\hat{x}^i - x_0\right\| \longrightarrow 0 \ .$$

Therefore, for any $\varepsilon > 0$, there exists an $N_\varepsilon \in \mathbb{N}$ such that $\left|K a^i(\hat{x}^i - x_0)\right| \leq \varepsilon$ for all $i \geq N_\varepsilon$. In particular, we can choose i large enough so that

$$\vec{c}^{\,i}_P = \max\left\{\vec{c}^{\,i} x \mid x \in P\right\} < \lfloor c_F \rfloor + K a^i x_0 + 1 \ .$$

Observe that $K a^i x_0 \in \mathbb{Z}$. Consequently,

$$\vec{c}^{\,i}\, x \leq \left\lfloor \vec{c}^{\,i}_P \right\rfloor \leq \lfloor c_F \rfloor + K a^i x_0 \tag{46}$$

is a Gomory-Chvátal cut for P. Furthermore, it has to hold that $\left\lfloor \vec{c}^{\,i}_P \right\rfloor = \lfloor c_F \rfloor + K a^i x_0$: First, observe that Corollary 5.12 implies that $F' \subseteq F$ and, therefore, $F' \subseteq P' \cap F$. As $cx \leq \lfloor c_F \rfloor$ is by assumption facet-defining for F', there must exist a point $\tilde{x} \in F'$ such that $c\tilde{x} = \lfloor c_F \rfloor$. Note that $F' \subseteq V_R$, according to Corollary 5.11, implies that $\tilde{x} \in V_R$ and, thus, $a^i\tilde{x} = a^i x_0$. Furthermore, we have $\tilde{x} \in P' \cap F$ because of $F' \subseteq P' \cap F$. In particular, $\tilde{x}$ satisfies the Gomory-Chvátal cut $\bar{c}^i x \leq \lfloor \bar{c}^i_P \rfloor$. Consequently,

$$\bar{c}^i \tilde{x} = c\tilde{x} + Ka^i\tilde{x} = \lfloor c_F \rfloor + Ka^i x_0 \leq \lfloor \bar{c}^i_P \rfloor \quad .$$

Together with (46), we obtain $\lfloor \bar{c}^i_P \rfloor = \lfloor c_F \rfloor + Ka^i x_0$. It follows that

$$\begin{aligned} \left(\bar{c}^i\, x \leq \lfloor \bar{c}^i_P \rfloor\,\right) \cap \left(a^i x = a^i x_0\right) &= \left(cx + Ka^i x \leq \lfloor c_F \rfloor + Ka^i x_0\right) \cap \left(a^i x = a^i x_0\right) \\ &= \left(cx \leq \lfloor c_F \rfloor\,\right) \cap \left(a^i x = a^i x_0\right) \quad . \end{aligned}$$

As $V_R \subseteq (a^i x = a^i x_0)$, this implies for $\bar{c} := \bar{c}^i$ for some large enough i,

$$\left(\bar{c}\, x \leq \lfloor \bar{c}_P \rfloor\,\right) \cap V_R = \left(cx \leq \lfloor c_F \rfloor\,\right) \cap V_R \quad .$$

The lemma follows. □

With this observation, we can prove the homogeneity property for non-rational polytopes.

Corollary 5.14 *Let P be a polytope and let F be a face of P. Then $F' = P' \cap F$.*

Proof. For the first direction $F' \subseteq P' \cap F$, observe that $F \subseteq P$ implies $F' \subseteq P'$. Furthermore, $F' \subseteq F$ because of Corollary 5.12. Hence, $F' \subseteq P' \cap F$.

For the second direction, let $F = P \cap (ax = a_P)$ be a face of P and let $cx \leq \lfloor c_F \rfloor$ be a Gomory-Chvátal cut for F that is facet-defining for F'. If $(ax = a_P) \cap \mathbb{Q}^n = \emptyset$, Corollary 5.11 implies $P' \cap F = \emptyset \subseteq F'$. Therefore, assume that $(ax = a_P) \cap \mathbb{Q}^n \neq \emptyset$, that is, the maximal rational affine subspace V_R of $(ax = a_P)$ is non-empty. By Lemma 5.13, there exists a Gomory-Chvátal cut for P that satisfies

$$(ax = a_P) \cap V_R \cap \left(\bar{c}x \leq \lfloor \bar{c}_P \rfloor\,\right) = (ax = a_P) \cap V_R \cap \left(cx \leq \lfloor c_F \rfloor\,\right) \quad .$$

Together with Corollary 5.11, that is, $P' \cap F \subseteq V_R$, we obtain

$$P' \cap F \subseteq \left(cx \leq \lfloor c_F \rfloor\,\right) \quad .$$

□

5.3 Step 3

A lemma similar to the following has been shown in Dadush, Dey, and Vielma (2010b). Here, we state an extension to non-full-dimensional convex sets.

Lemma 5.15 *Let K be a convex and compact set in $\mathbb{R}^n$. If there exists a finite set $S \subseteq \mathbb{Z}^n$ such that*

(i) $C_S(K) \subseteq K$,

(ii) $C_S(K) \cap \operatorname{rbd}(K) \subseteq K'$,

then K' is a rational polytope.

Proof. As $C_S(K)$ is a rational polytope, we can assume that $\operatorname{aff}(C_S(K)) = w_0 + W$, where $w_0 \in \mathbb{Q}^n$ and where W is a rational linear vector space. Let $\mathcal{V}$ denote the finite set of vertices of $C_S(K)$. Because of assumption (ii), any Gomory-Chvátal cut for K that separates a point in $C_S(K) \setminus K'$ must also separate a vertex in $\mathcal{V} \setminus \operatorname{rbd}(K)$. We will show that for each of the finitely many vertices of $C_S(K)$ in the relative interior of K one only has to consider a finite set of Gomory-Chvátal cuts.

First, observe that because of $\mathcal{V} \setminus \operatorname{rbd}(K) \subseteq \operatorname{ri}(K)$ and since the number of vertices of $C_S(K)$ is finite, there exists an $\varepsilon > 0$ such that for every $v \in \mathcal{V} \setminus \operatorname{rbd}(K)$,

$$\big(v + B(0,\varepsilon)\big) \cap \operatorname{aff}(K) \subseteq K \ . \tag{47}$$

Consequently,

$$\big(v + B(0,\varepsilon)\big) \cap \operatorname{aff}\big(C_S(K)\big) \subseteq K \ . \tag{48}$$

Now let us fix a vertex v of $C_S(K)$ in the relative interior of K, that is, $v \in \mathcal{V} \setminus \operatorname{rbd}(K)$. Furthermore, let $c \in \mathbb{Z}^n$. We will consider two cases, depending on whether K is full-dimensional or not. If $\dim(K) = n$, then $\operatorname{aff}(K) = \mathbb{R}^n$ and with (47),

$$\big(v + B(0,\varepsilon)\big) \subseteq K \ .$$

We get

$$\begin{aligned} \lfloor c_K \rfloor &= \lfloor \max\{cx \mid x \in K\} \rfloor \geq \max\{cx \mid x \in K\} - 1 \\ &\geq c\,v + \max\big\{cx \mid x \in B(0,\varepsilon)\big\} - 1 = c\,v + c\left(\varepsilon\,\frac{c}{\|c\|}\right) - 1 = c\,v + \varepsilon\,\|c\| - 1 \ . \end{aligned}$$

If $\|c\| \geq 1/\varepsilon$, the Gomory-Chvátal cut associated with the normal vector c does not separate the vertex v. Hence, we only need to consider Gomory-Chvátal cuts that correspond to vectors c such that $\|c\| < 1/\varepsilon$ and their number is finite. This completes the proof for the case that K is full-dimensional.

In the remainder of the proof, let us assume that $\dim(K) < n$ and, therefore, $\dim(\operatorname{aff}(C_S(K)) =: k < n$. Since $\dim(W) = k$, we can rename the indices such that there exist integers p_{ij} and $q_{ij} \geq 1$, for $i = 1, \ldots, n-k$ and $j = 1, \ldots, k$, such that for every $x \in W$,

$$x_{k+i} = \sum_{j=1}^{k} \frac{p_{ij}}{q_{ij}}\, x_j \ .$$

In words, any point in W is uniquely determined by its first k components. Moreover, we can find an upper bound for the norm of each point $x \in W$ that is a function of the norm of the vector $(x_1, \ldots, x_k)$, that is, the restriction of x to its first k components: Since

$$\|x\|^2 = x_1^2 + \ldots + x_k^2 + \left(\sum_{j=1}^{k} \frac{p_{1j}}{q_{1j}} x_j \right)^2 + \ldots + \left(\sum_{j=1}^{k} \frac{p_{n-k,j}}{q_{n-k,j}} x_j \right)^2 ,$$

there exist rational constants $\alpha_i > 0$, for $i = 1, \ldots, k$, and α_{ij}, for $1 \leq i < j \leq k$, such that

$$\|x\|^2 = \alpha_1 x_1^2 + \ldots + \alpha_k x_k^2 + \sum_{1 \leq i < j \leq k} \alpha_{ij} x_i x_j .$$

Using the relation

$$\alpha_{ij}\, x_i\, x_j \leq \frac{1}{2} |\alpha_{ij}|\, x_i^2 + \frac{1}{2} |\alpha_{ij}|\, x_j^2 ,$$

and defining $\alpha_{ji} := \alpha_{ij}$ for every $1 \leq i < j \leq k$, we obtain

$$\|x\|^2 \leq \left(\alpha_1 + \frac{1}{2} \sum_{j=2}^{k} |\alpha_{1j}| \right) x_1^2 + \ldots + \left(\alpha_k + \frac{1}{2} \sum_{j=1}^{k-1} |\alpha_{kj}| \right) x_k^2 .$$

Let $\alpha := \max_{i=1,\ldots,k} \left\{ \alpha_i + \frac{1}{2} \sum_{j=1, j \neq i}^{k} |\alpha_{ij}| \right\}$. Then α is a constant that only depends on W. For any $x \in W$, we have

$$\|x\| \leq \sqrt{\alpha}\; \|(x_1, \ldots, x_k)\| .$$

Moreover,

$$c\,x = c_1 x_1 + \ldots + c_k x_k + \sum_{i=1}^{n-k} c_{k+i} \left(\sum_{j=1}^{k} \frac{p_{ij}}{q_{ij}} x_j \right) = \sum_{j=1}^{k} \left(c_j + \sum_{i=1}^{n-k} \frac{p_{ij}}{q_{ij}} c_{k+i} \right) x_j .$$

Let $L : \mathbb{R}^n \to \mathbb{R}^k$ denote the affine map that is defined for $j = 1, \ldots, k$ by

$$L_j(x) := x_j + \sum_{i=1}^{n-k} \frac{p_{ij}}{q_{ij}} x_{k+i} . \tag{49}$$

Then for every $x \in W$,

$$cx = \sum_{j=1}^{k} L_j(c)\, x_j = L(c)\, (x_1, \ldots, x_k) . \tag{50}$$

Let $x^c = (x_1^c, \ldots, x_n^c) \in W$ such that $(x_1^c, \ldots, x_k^c) = L(c)$. Observe that

$$\|x^c\| \leq \sqrt{\alpha}\; \|(x_1^c, \ldots, x_k^c)\| = \sqrt{\alpha}\; \|L(c)\|$$

implies

$$\frac{1}{\sqrt{\alpha}\,\|L(c)\|}\,x^c \in B(0,1)\cap W \ .$$

Using $\operatorname{aff}(C_S(K)) = v + W$, we get

$$v + \frac{\varepsilon}{\sqrt{\alpha}\,\|L(c)\|}x^c \in \Big((v + B(0,\varepsilon)) \cap \operatorname{aff}\big(C_S(K)\big)\Big) \ ,$$

and by (48),

$$v + \frac{\varepsilon}{\sqrt{\alpha}\,\|L(c)\|}x^c \in K \ .$$

Therefore, and with (50), we get

$$\begin{aligned} \lfloor c_K \rfloor &= \lfloor \max\{cx \mid x \in K\} \rfloor \geq \max\{cx \mid x \in K\} - 1 \\ &\geq c\,v + \frac{\varepsilon}{\sqrt{\alpha}\,\|L(c)\|}\,c\,x^c - 1 = c\,v + \frac{\varepsilon}{\sqrt{\alpha}}\,\|L(c)\| - 1 \ . \end{aligned}$$

For $\|L(c)\| \geq \sqrt{\alpha}/\varepsilon$, the Gomory-Chvátal cut associated with c does not separate v. Because of (49), for each $j = 1,\ldots,k$, there exists an integer $q_j \geq 1$ such that $L_j(c)$ is a multiple of $1/q_j$. Therefore, the number of vectors $L(c)$ in $\mathbb{R}^k$ with $\|L(c)\| < \sqrt{\alpha}/\varepsilon$ is finite. However, there is an infinite number of integral vectors c in $\mathbb{R}^n$ that are mapped to the same rational vector $L(c)$ in $\mathbb{R}^k$. Let $\mathcal{A}$ denote the set of rational vectors $a \in \mathbb{R}^k$ such that a_j is a multiple of q_j, for $j = 1,\ldots,k$, and such that $\|a\| < \sqrt{\alpha}/\varepsilon$. For every $a \in \mathcal{A}$, we define

$$N(a) := \{c \in \mathbb{Z}^n \mid L(c) = a\} \ .$$

Let

$$c^a \in \arg\min_{c\in N(a)} \big\{ \lfloor c_K \rfloor - cv \big\} \ .$$

Observe, that c^a is well-defined: Since $v \in K$, we have for any $c \in N(a)$,

$$\lfloor c_K \rfloor - cv \geq \max\{cx \mid x \in K\} - 1 - cv \geq -1 \ .$$

Furthermore, as v is a vertex of the rational polytope $C_S(K)$, it holds that $v \in \mathbb{Q}^n$. Hence, there exist an integer vector $\bar{v} \in \mathbb{Z}^n$ and an integer $q_v \geq 1$ such that $v = \bar{v}/q_v$. Consequently, the set $\big\{ \lfloor c_K \rfloor - cv \mid c \in N(a)\big\}$ contains only multiples of q_v and is bounded from below.

Finally, observe that the Gomory-Chvátal cut $c^a\,x \leq \lfloor c^a_K \rfloor$ dominates every other Gomory-Chvátal cut associated with a vector in $N(a)$ in $\operatorname{aff}(C_S(K))$: For this, consider an arbitrary point $x \in \operatorname{aff}(C_S(K))$. We can write $x = v + w$, for some $w = (w_1,\ldots,w_n) \in W$. If, using (50),

$$c^a x = c^a v + c^a w = c^a v + L(c^a)(w_1,\ldots,w_k) = c^a v + a(w_1,\ldots,w_k) \leq \lfloor c^a_K \rfloor \ ,$$

then

$$a(w_1, \ldots, w_k) \leq \lfloor c^a_K \rfloor - c^a v \ .$$

By the definition of c^a, it follows that for every $c \in N(a)$,

$$cx = cv + cw = cv + a(w_1, \ldots, w_k) \leq cv + \lfloor c_K \rfloor - cv = \lfloor c_K \rfloor \ .$$

That is, if x satisfies the Gomory-Chvátal cut $c^a\, x \leq \lfloor c^a_K \rfloor$, it also satisfies every Gomory-Chvátal cut $cx \leq \lfloor c_K \rfloor$ with $c \in N(a)$. Consequently, for each vector $a \in \mathcal{A}$, we only need to consider a single Gomory-Chvátal cut. Since $|\mathcal{A}|$ is finite, this completes the proof. □

5.4 Step 4

In this section, we show that the Gomory-Chvátal closure of any polytope is a rational polytope.

Theorem 5.16 *The Gomory-Chvátal closure P' of a non-rational polytope P is a rational polytope.*

Proof. The proof is by induction on the dimension $d \leq n$ of $P \subseteq \mathbb{R}^n$. Let $n \geq 1$ be arbitrary. The base case, $d = 0$, is trivially true. Therefore, assume that $d \geq 1$. By Corollary 5.12, we know that there exists a finite set $S_1 \subseteq \mathbb{Z}^n$ such that

$$C_{S_1}(P) \subseteq P \ .$$

Let $\{F_i\}_{i \in I}$ denote the set of facets of P and assume that $F^i = P \cap (a^i x = a^i_P)$. By the induction assumption for $d-1$, we know that F'_i is a rational polytope for every $i \in I$. That is, there exists a finite set $S_i \subseteq \mathbb{Z}^n$ such that

$$C_{S_i}(F_i) = F'_i \ .$$

According to Lemma 5.13, we can find for every Gomory-Chvátal cut for F_i that is facet-defining for F'_i a Gomory-Chvátal cut for P that has the same impact on the maximal rational affine subspace of $(a^i x = a^i_P)$. Furthermore, by Corollary 5.11, F'_i is contained in this rational affine subspace. Hence, for every $i \in I$, there exists a finite set $\bar{S}_i \subseteq \mathbb{Z}^n$ such that

$$C_{\bar{S}_i}(P) \cap F_i = F'_i \ .$$

Because $\mathrm{rbd}(P) = \cup_{i \in I} F_i$, the set $S = S_1 \cup \big(\cup_{i \in I} \bar{S}_i \big)$ satisfies

$$\begin{aligned} C_S(P) &\subseteq P \ , \\ C_S(P) \cap \mathrm{rbd}(P) &\subseteq P' \ . \end{aligned}$$

By Lemma 5.15, P' is a rational polytope. □

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