Bad semidefinite programs: they all look the same

Gábor Pataki*

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Abstract

We call a conic linear system

$$Ax \leq_K b \tag{P}$$

well behaved, if for all c objective functions the value of $\sup \{ \langle c, x \rangle \mid Ax \leq_K b \}$ and of its dual agree, and the latter is attained, when finite. We call (P) *badly behaved,* when not well behaved.

We give simple conditions for a conic system to be well- and badly behaved, and exactly characterize such systems over a broad and important class of cones, called *nice* cones. We characterize badly behaved semidefinite systems via certain excluded matrices, which are easy to spot in all such systems that appear in the literature.

We show how to reformulate semidefinite systems in a certain standard form. The reformulation allows us

- to prove that the question

  "Is a semidefinite system well behaved?"

  is in $\mathcal{NP} \cap$ co-$\mathcal{NP}$ in the real number model of computing, and to verify the status of a system in polynomial time, by elementary arguments;
- to deduce that in well-behaved semidefinite systems we can choose optimal dual matrices with a predefined block-diagonal structure for *all* objective functions;
- to systematically generate all well behaved systems by a simple algorithm.

Our main tool is one of our recent results on the closedness of the linear image of a closed, convex cone.

Key words: semidefinite programming; duality; closedness of the linear image; badly behaved instances

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1 Introduction

Conic linear programs provide a natural framework to study duality in convex programs, and model a wide variety of practical optimization problems. We consider a primal-dual pair of conic LPs

$$(P_c) \quad \begin{array}{ll} \sup & \langle c, x \rangle \\ s.t. & Ax \leq_K b \end{array} \qquad\qquad \begin{array}{ll} \inf & \langle b, y \rangle \\ s.t. & y \geq_{K^*} 0 \\ & A^* y = c, \end{array} \quad (D_c)$$

*Department of Statistics and Operations Research, University of North Carolina at Chapel Hill

where X and Y are finite dimensional euclidean spaces, $A : X \to Y$ is a linear map, A^* is its adjoint, $K \subseteq Y$ is a closed, convex cone, and K^* is its dual cone. We write $s \leq_K t$ to denote $t - s \in K$, and note that we parametrize the conic LPs by the primal objective function.

Semidefinite programs (SDPs) and second order conic programs (SOCPs) are particularly useful, due to their modeling power, and efficient solvability. We refer to [5, 30, 44, 1, 41, 14] for general references on SDP and SOCP, to [21] for a useful website, and to [8, 43, 38, 4, 27] for treatments of conic duality theory. The recent book [28] is a very nice treatment of the connection of semidefinite programming and algebraic geometry.

When the optimal values of (P_c) and (D_c) agree, an optimal solution of (D_c) can serve as a proof of (near) optimality of a solution of (P_c), since the the weak duality inequality

$$\langle c, x\rangle \leq \langle b, y\rangle$$

holds for a pair of feasible solutions. However, when K is not polyhedral, the optimal values of (P_c) and (D_c) may differ, and/or may not be attained. All the above pathologies – nonattainment, and positive gaps – appear in semidefinite and second order conic programming.

While the pathologies in the SDPs in the literature vary, the underlying *systems* look curiously similar, and this fact motivated our research. We define the primal conic system as

$$Ax \leq_K b, \tag{P}$$

and assume it to be feasible. We say that (P) is *well-behaved*, if for all c objective functions the values of (P_c) and (D_c) agree, and the dual value is attained, when it is finite. We say that (P) is *badly behaved*, when not well-behaved. The focus of this paper is succinct and efficiently verifiable characterizations of when (P) is well- and badly behaved.

To sketch our main results, we let $\mathcal{R}(A, b)$ stand for the the rangespace of the operator (A, b), and choose $z \in K$ in the relative interior of the set of all slack vectors in (P). We call z a maximum slack. We let

$$\operatorname{dir}(z, K) \;=\; \{\, y \mid x + \epsilon y \in K \text{ for some } \epsilon > 0 \,\}$$

be the set of feasible directions at z in K and $\operatorname{cl}\operatorname{dir}(z, K)$ its closure. Theorem 3, with our main characterizations of well- and badly behaved conic systems proves that the basic condition:

$$\mathcal{R}(A, b) \cap (\operatorname{cl}\operatorname{dir}(z, K) \setminus \operatorname{dir}(z, K)) = \emptyset \tag{*}$$

is always necessary for (P) to be well-behaved, and necessary and sufficient when K belongs to the class of *nice* cones. Essentially all cones are nice, for which we can efficiently solve (P_c), for instance, polyhedral, semidefinite, and second order cones.

We say that (P) is strictly feasible, if some feasible slack is in the relative interior of K. We know that (P) is well behaved, when strictly feasible, or if K is polyhedral. Both of these conditions imply (*), so Theorem 3 provides a natural generalization.

For a semidefinite system the difference set in (*) is particularly simple, so in Theorem 4 we characterize badly behaved semidefinite systems in terms of the excluded matrices

$$Z \;=\; \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \; V \;=\; \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \tag{1.1}$$

where Z is assumed to be a maximum rank slack matrix, V is a linear combination of the constraint matrices, V_{22} is positive semidefinite, and the rangespace of V_{12}^T is not contained in the rangespace of V_{22}. These matrices are easy to spot in all badly behaved instances in the literature; for instance, in the system

$$x_1 \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.2}$$

where α is some real number, the right hand side matrix serves as Z, and the constraint matrix on the left serves as V.

Badly behaved second order conic systems have a similarly simple characterization. For the sake of brevity, we do not include this result in the current version of the paper, but it can be found in the online version at `http://arxiv.org/abs/1112.1436`.

Our main complexity result (Theorems 6, 7, and 8) shows that the recognition problem for well- (and badly) behaved semidefinite systems is in $\mathcal{NP} \cap \text{co-}\mathcal{NP}$ in the real number model of computing. The main certificate is a certain equivalent reformulation, whose status we can check in polynomial time, using only elementary linear algebra, and knowing that a strictly feasible system is well-behaved. The reformulation has other uses: we deduce that in a well-behaved system we can choose optimal dual matrices for *all* objective functions with a predefined block-diagonal structure, and present an algorithm to systematically generate *all* well behaved semidefinite systems.

We also find, in Corollary 4, that all badly behaved semidefinite systems can be reduced to the system (1.2), via natural elementary operations.

We now review relevant literature in detail. The algorithm of Borwein and Wolkowicz in [12, 11] converts (P) into a strictly feasible system by a sequence of reduction steps. Ramana's dual, proposed in [36] for semidefinite programs, has the same value as (P_c), and attains it, even when (P_c) is not strictly feasible. His result implies that semidefinite feasibility is in $\mathcal{NP} \cap \text{co-}\mathcal{NP}$ in the real number model of computing.

Ramana, Tunçel, and Wolkowicz in [37] proved the correctness of Ramana's dual from the facial reduction algorithm of [12, 11], hence showed that the two – seemingly unrelated – approaches are equivalent. A nice complementary approach for a gapfree dual is the conic expansion method of Luo, Sturm, and Zhang([27]). Waki and Muramatsu [48] recently proved the equivalence of conic expansion to their simplified facial reduction approach. Klep and Schweighofer in [25] constructed a Ramana type dual, which, interestingly, is based on ideas from algebraic geometry, not convex analysis. The paper [35] ([32] was its preliminary version) also describes a facial reduction algorithm, and generalizations of Ramana's dual over nice cones, with a semidefinite constraint replaced by a general tangent space type constraint.

Closed cone constraint qualifications (CQs) – see Duffin et al [17], Jeyakumar et al [24] and Bot and Wanka [13] – connect the duality properties of a constraint set to the closedness of a certain related set. These CQs are powerful, and very general, as they guarantee that the problem

$$\inf \{ f(x) \mid x \in C, \, g(x) \in -K \},$$

and its Lagrange dual have the same value, and the latter attains it. Here K is a closed convex cone, C a closed convex set, and g is a K-convex function. In particular, Condition (2) in our Theorem 3 specializes to Corollary 1 in [23] when K is the semidefinite cone, and Theorem 3.2 in [13] gives an equivalent condition, stated given in a different form.

A pathology of infeasible systems is the lack of a Farkas' lemma certificate. Waki [47] describes a method for generating such instances, and Lourenco et al [26] an analysis via an error bound based reduction approach.

Our main results hinge on Theorem 1 from [33], on whether the linear image of a closed convex cone is closed. While this question is fundamental, there is surprisingly little literature on the subject. For related results, we refer to Waksman and Epelman [49]; Auslender [2]; and Bauschke and Borwein [3] for the closedness of the continuous image of a closed, convex cone. For perturbation results we refer to Borwein and Moors [10]. For a more general problem, whether the intersection of an infinite sequence of nested sets is nonempty, Bertsekas and Tseng [7] gave a sufficient condition.

Schurr et al in [42] found characterizations of *universal duality*, when strong duality holds for all right hand sides, and objective functions. Tunçel and Wolkowicz in [46] related the lack of strict complementarity in a homogeneous conic linear system to the existence of an objective function with a positive gap. For a study on which cones allow a positive gap, see Nemirovskii and Shapiro [29], and Zalinescu [51].

The rest of the paper is organized as follows. In Section 2 we present, and prove Theorem 3, on when

(P) is well- or badly behaved. In Section 3 we characterize well- and badly behaved semidefinite systems in Theorem 4, and Theorem 5, respectively. In Section 4 we present our reformulations of semidefinite systems, and our complexity results. In Section 5 we conclude: we show how to reduce all badly behaved semidefinite systems to the system (1.2), via a sequence of natural operations, and how to characterize well and badly behaved conic linear systems in a different form.

1.1 Notation, and preliminaries

We used [39, 22, 9, 4] as references in convex analysis.

If x and y are elements of the same Euclidean space, we sometimes write x^*y for $\langle x, y\rangle$. For a set C we denote its linear span, the orthogonal complement of its linear span, its closure, and interior by $\operatorname{lin} C$, $C^\perp$, $\operatorname{cl} C$, and $\operatorname{int} C$, respectively. For a convex set C we denote its relative interior by $\operatorname{ri} C$. For a convex set C, and $x \in C$ we define

$$\begin{aligned} \operatorname{dir}(x, C) &= \{\, y \,|\, x + \epsilon y \in C \text{ for some } \epsilon > 0 \,\}, && (1.3)\\ \operatorname{ldir}(x, C) &= \operatorname{dir}(x, C) \cap - \operatorname{dir}(x, C), && (1.4)\\ \tan(x, C) &= \operatorname{cl} \operatorname{dir}(x, C) \cap - \operatorname{cl} \operatorname{dir}(x, C). && (1.5) \end{aligned}$$

Here $\operatorname{dir}(x, C)$ is the set of feasible directions at x in C, $\operatorname{ldir}(x, C)$ is the lineality space of $\operatorname{dir}(x, C)$, and $\tan(x, C)$ is the tangent space at x in C.

A set C is called a *cone,* if $\lambda x \in C$ holds for all $x \in C$, and $\lambda \geq 0$. Let us fix a closed convex cone C. We define its dual cone as

$$C^* = \{\, y \,|\, \langle y, x\rangle \geq 0 \;\forall x \in C \}.$$

For E, a convex subset of C we say that E is a *face* of C, if $x_1, x_2 \in C$, and $1/2(x_1 + x_2) \in E$ implies that x_1 and x_2 are in C.

For an E face of C we define its *conjugate face* as

$$E^\triangle = C^* \cap E^\perp.$$

Similarly, the conjugate face of a G face of C^* is

$$G^\triangle = C \cap G^\perp.$$

For $x \in C$, and $u \in C^*$ we say that u is *strictly complementary to* x, if $x \in \operatorname{ri} E$ for some E face of C (i.e., E is the smallest face of C that contains x), and $u \in \operatorname{ri} E^\triangle$. It is possible that u is strictly complementary to x, but x is not strictly complementary to u. The reason is that $(E^\triangle)^\triangle$ is the smallest exposed face of C that contains E, i.e., the smallest face of C that is the intersection of C with a supporting hyperplane, and it only equals E, when E itself is exposed. In the semidefinite and second order cones all faces are exposed, so in these cones strict complementarity is a symmetric concept.

We say that a closed convex cone C is *nice*, if

$$C^* + E^\perp \text{ is closed for all } E \text{ faces of } C.$$

Nice cones first appear in the papers of Borwein and Wolkowicz [12, 11], where the authors prove that polyhedral, semidefinite, and p-order cones are nice. The geometric and dual geometric cones, introduced by Glineur in [20] are also nice. Chua and Tunçel in Proposition 4 in [16] proved that the intersection of a nice cone with a linear subspace, and the linear preimage of a nice cone are nice. By their intersection result homogeneous cones are nice, as they are the intersection of a semidefinite cone with a linear subspace (see Chua [15] and Faybusovich [19]). Their preimage result implies that the intersection of nice cones is nice ([34], Proposition 5). In [34] we characterized nice cones, proved that they must be facially exposed, and conjectured that all facially exposed cones are nice. However, Roshchina [40] disproved this conjecture.

We state the following lemma for convenience:

Lemma 1. *Let C be a closed convex cone, $x \in C,$ and E the smallest face of C that contains x. Then the following hold:*

$$\begin{aligned} \operatorname{dir}(x, C) &= C + \operatorname{lin} E, & (1.6) \\ \operatorname{ldir}(x, C) &= \operatorname{lin} E, & (1.7) \\ \operatorname{cl}\operatorname{dir}(x, C) &= (E^{\triangle})^*, & (1.8) \\ \tan(x, C) &= (E^{\triangle})^{\perp}. & (1.9) \end{aligned}$$

Proof Statements (1.6) and (1.8) are in Lemma 3.2.1 in [31] (in Lemma 2.7 in the online version). Statement (1.9) was also proved there, under the assumption that C is nice. In fact, it easily follows from (1.8) and (1.5) in general.

In (1.7) the containment $\supseteq$ is trivial. To see $\subseteq$ let $y \in \operatorname{ldir}(x, C),$ then $x \pm \epsilon y \in C$ for some $\epsilon > 0$. Hence $x \pm \epsilon y \in E,$ so $\epsilon y \in \operatorname{lin} E,$ and this completes the proof. □

The following question is fundamental in convex analysis: when is the linear image of a closed convex cone closed? We state a short version of Theorem 1.1 from [33], which gives easily checkable conditions which are "almost" necessary and sufficient.

Theorem 1. *Let M be a linear map, C a closed convex cone, and $x \in \operatorname{ri}(C \cap \mathcal{R}(M))$. Then conditions (1) and (2) below are equivalent to each other, and* necessary *for the closedness of M^*C^*.*

(1) $\mathcal{R}(M) \cap (\operatorname{cl}\operatorname{dir}(x, C) \setminus \operatorname{dir}(x, C)) = \emptyset$.

(2) There is $u \in \mathcal{N}(M^*) \cap C^*$ strictly complementary to $x,$ and

$$\mathcal{R}(M) \cap (\tan(x, C) \setminus \operatorname{ldir}(x, C)) = \emptyset.$$

Also, let E be the smallest face of C that contains x. If $C^* + E^{\perp}$ is closed, then conditions (1) and (2) are each sufficient *for the closedness of M^*C^*.*

□

The second part of condition (2) in Theorem 1 appears slightly differently in in Theorem 1.1 in [33] as

$$\mathcal{R}(M) \cap ((E^{\triangle})^{\perp} \setminus \operatorname{lin} E) = \emptyset.$$

However, Lemma 1 implies $(E^{\triangle})^{\perp} = \tan(x, C),$ and $\operatorname{lin} E = \operatorname{ldir}(x, C)$.

We note that Theorem 1 implies the sufficiency of two classical conditions for the closedness of M^*C^*, and gives necessary and sufficient conditions for nice cones:

Corollary 1. *Let M and C be as in Theorem 1. Then the following hold:*

(1) If C is polyhedral, then M^*C^* is closed.

(2) If $\mathcal{R}(M) \cap \operatorname{ri} C \neq \emptyset,$ then M^*C^* is closed.

(3) If C is nice, then conditions (1) and (2) in Theorem 1 are each necessary and sufficient for the closedness of M^*C^*.

Proof Let x and E be as in Theorem 1. If C is polyhedral, then so are the sets $C^* + E^{\perp},$ and $\operatorname{dir}(x, C)$, which are hence closed. So the sufficiency of condition (1) in Theorem 1 proves the closedness of M^*C^*.

If $\mathcal{R}(M) \cap \operatorname{ri} C \neq \emptyset,$ then Theorem 6.5 in [39] implies $x \in \operatorname{ri} C,$ hence $E = C.$ Therefore $C^* + E^\perp = C^*,$ and $\operatorname{dir}(x, C) = \operatorname{lin} C,$ and both of these sets are closed. Again, the sufficiency of condition (1) in Theorem 1 implies the closedness of $M^* C^*.$

If C is nice, then $C^* + E^\perp$ is closed for *all* E faces of $C,$ and this shows that conditions (1) and (2) in Theorem 1 are each necessary and sufficient for the closedness of $M^* C^*.$ $\square$

For the program (D_c) we define solutions that are only "nearly" feasible: we say that $\{y_i\} \subseteq K^*$ is an *asymptotically feasible (AF)* solution to (D_c) if $A^* y_i \to c$, and the *asymptotic value of* (D_c) is

$$\operatorname{aval}(D_c) \quad = \quad \inf\{\, \liminf_i b^* y_i \,|\, \{y_i\} \text{ is asymptotically feasible to } (D_c)\,\}.$$

Clearly, if (D_c) is asymptotically feasible, then there is an AF solution $\{y_i\} \subseteq K^*$ with $\lim b^* y_i = \operatorname{aval}(D_c).$

Theorem 2. *(Duffin [18]) Problem* (P_c) *is feasible with* $\operatorname{val}(P_c) < +\infty,$ *iff* (D_c) *is asymptotically feasible with* $\operatorname{aval}(D_c) > -\infty,$ *and if these equivalent statements hold, then*

$$\operatorname{val}(P_c) = \operatorname{aval}(D_c).$$

$\square$

We denote by $\mathcal{S}^n$ the set of n by n symmetric matrices, and by $\mathcal{S}^n_+$ the set of n by n symmetric positive semidefinite matrices. For symmetric matrices X and Y we write $X \preceq Y$ to denote that $Y - X$ is positive semidefinite, and define their inner product as

$$X \bullet Y = \sum_{i,j=1}^n x_{ij} y_{ij},$$

where x_{ij} and y_{ij} denote the components of X and $Y,$ resp. We will use the identity

$$T^T Y T \bullet T^{-1} X T^{-T} \; = \; Y \bullet X, \tag{1.10}$$

where T is any invertible matrix.

As in [1], juxtaposing, or commas denote concatenation of matrices along rows: if $A_1, \dots, A_k$ are matrices each with m rows, and $n_1, \dots, n_k$ columns, respectively, then $(A_1 \; ... \; A_k) = (A_1, \dots, A_k)$ has m rows, and $n_1 + \cdots + n_k$ columns. We use semicolons to denote concatenation of matrices along columns. That is, if $A_1, \dots, A_k$ are matrices each with n columns, and $m_1, \dots, m_k$ rows, respectively, then $(A_1; \dots; A_k) = (A_1^T, \dots, A_k^T)^T$ is a matrix with n columns, and $m_1 + \cdots + m_k$ rows.

We similarly denote concatenation of elements of sets, and of operators. If $x_i \in X_i$ for sets X_i $(i = 1, \dots, k),$ then $(x_1; \dots; x_k)$ is the corresponding element of $X_1 \times \dots \times X_k$. If X, Y, A and b are as in the definition of the conic system (P), then the operator

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

maps from $X \times \mathbb{R}$ to $Y \times \mathbb{R}$, by assigning $(Ax + bx_0; x_0)$ to $(x; x_0)$, and

$$\begin{pmatrix} A^* & 0 \\ b^* & 1 \end{pmatrix}$$

denotes its adjoint operator. For matrices A_1 and $A_2,$ we define

$$A_1 \oplus A_2 \; = \; \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

For sets of matrices X_1 and X_2 we define

$$X_1 \oplus X_2 \; = \; \{\, A_1 \oplus A_2 \,|\, A_1 \in X_1, \, A_2 \in X_2 \,\}.$$

For instance, $\mathcal{S}^r_+ \oplus \{0\}$ (where the order of the 0 matrix will be clear from the context) stands for the set of matrices whose upper left r by r block is positive semidefinite, and the rest of the components are zero.

2 Characterizing when (P) is well and badly behaved

We assume throughout that (P) is feasible, and we need the following

Definition 1. *A* maximum slack *in (P) is a vector in*

$$\mathrm{ri}\,\{\, z \,|\, z = b - Ax,\ z \in K \,\} \ = \ \mathrm{ri}\,(\,(\mathcal{R}(A) + b) \cap K\,).$$

We will use the fact that for $z \in K$ the sets $\mathrm{dir}(z,K)$, $\mathrm{ldir}(z,K)$, and $\tan(z,K)$ (as defined in (1.3)) depend only the smallest face of K that contains z, cf. Lemma 1.

Theorem 3. *Let z be a maximum slack in (P). Consider the statements*

(1) The system (P) is well-behaved.

(2) The set
$$\begin{pmatrix} A^* & 0 \\ b^* & 1 \end{pmatrix} \begin{pmatrix} K^* \\ \mathbb{R}_+ \end{pmatrix}$$
is closed.

(3) The set
$$\begin{pmatrix} A^* \\ b^* \end{pmatrix} K^*$$
is closed.

(4) $\mathcal{R}(A,b) \cap (\mathrm{cl}\,\mathrm{dir}(z,K) \setminus \mathrm{dir}(z,K)) = \emptyset$.

(5) There is $u \in \mathcal{N}((A,b)^*) \cap K^*$ strictly complementary to z, and
$$\mathcal{R}(A,b) \cap (\tan(z,K) \setminus \mathrm{ldir}(z,K)) = \emptyset.$$

Among them the following relations hold:

$$\begin{array}{ccccc} (1) & \Leftrightarrow & (2) & \Leftarrow & (3) \\ & & \Downarrow & & \\ & & (4) & \Leftrightarrow & (5) \end{array}$$

Also, let F be the smallest face of K that contains z. If the set $K^* + F^\perp$ is closed, then (1) through (5) are all equivalent.

$\square$

Below we show that Theorem 3 unifies two classical, seemingly unrelated *sufficient* conditions for (P) to be well behaved, and gives *necessary and sufficient* conditions when K is nice. We omit the proof, as it is analogous to the proof of Corollary 1.

Corollary 2. *The following hold:*

(1) If K is polyhedral, then (P) is well behaved.

(2) If $z \in \mathrm{ri}\,K$, i.e., Slater's condition holds, then (P) is well behaved.

(3) If K is nice, then conditions (2) through (5) are all equivalent to each other, and with the well behaved nature of P.

$\square$

First define the sets

$$\begin{array}{rcl} S_1 & = & (\mathcal{R}(A)+b)\cap K, \\ S_2 & = & \mathcal{R}(A,b)\cap K, \\ S_3 & = & \mathcal{R}\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \cap \begin{pmatrix} K \\ \mathbb{R}_+ \end{pmatrix}. \end{array} \tag{2.11}$$

Lemma 2. *Let z and F be as in the statement of Theorem 3, $s_2 \in \operatorname{ri} S_2,$ and $(s_3; s_0) \in \operatorname{ri} S_3.$ Then*

(1) $s_2 \in \operatorname{ri} F.$

(2) $s_3 \in \operatorname{ri} F$ *and* $s_0 > 0.$

(3) $\operatorname{dir}(z, K) \; = \; \operatorname{dir}(s_2, K) \; = \; \operatorname{dir}(s_3, K).$

Proof We first prove

$$\operatorname{ri} S_1 \cap \operatorname{ri} S_2 \neq \emptyset. \tag{2.12}$$

Let $s_1 \in \operatorname{ri} S_1,$ $s_2 \in \operatorname{ri} S_2,$ and let us write

$$s_1 \;=\; Ax_1 + b, \tag{2.13}$$

$$s_2 \;=\; Ax_2 + by_2 \tag{2.14}$$

for some x_1, x_2 and y_2. Since $s_1 \in S_2,$ by Theorem 6.1 in [39] the line-segment $(s_1, s_2]$ is contained in $\operatorname{ri} S_2,$ so for some small $\epsilon > 0$ we have

$$\begin{array}{rcl} s_1(\epsilon) & := & (1-\epsilon)s_1 + \epsilon s_2 \\ & = & A[(1-\epsilon)x_1 + \epsilon x_2] + b[(1-\epsilon)+\epsilon y_2] \;\in\; \operatorname{ri} S_2. \end{array}$$

Define

$$s_1'(\epsilon) \;=\; \frac{1}{1-\epsilon+\epsilon y_2} s_1(\epsilon). \tag{2.15}$$

If ϵ is small enough, then the denominator in (2.15) is positive, so $s_1'(\epsilon) \in S_1$. Also, S_2 is a cone, and $s_1(\epsilon)$ is in $\operatorname{ri} S_2,$ hence so is $s_1'(\epsilon)$. Hence we have

$$s_1'(\epsilon) \in S_1 \cap \operatorname{ri} S_2. \tag{2.16}$$

Also, since $s_1'(\epsilon) \to s_1,$ as $\epsilon \to 0,$ and $s_1 \in \operatorname{ri} S_1,$ by (2.16) we obtain

$$s_1'(\epsilon) \in \operatorname{ri} S_1 \cap \operatorname{ri} S_2,$$

for small enough $\epsilon > 0,$ and this completes the proof of (2.12).

Since $\operatorname{ri} S_1$ is relatively open, and convex, by Theorem 18.2 in [39] it is contained in the relative interior of a face of K. Since $\operatorname{ri} S_1$ and $\operatorname{ri} F$ intersect (in z), this face is $F,$ so $\operatorname{ri} S_1 \subseteq \operatorname{ri} F$. Similarly, $\operatorname{ri} S_2$ is contained in the relative interior of a face of K; by (2.12) and $\operatorname{ri} S_1 \subseteq \operatorname{ri} F,$ we have $\operatorname{ri} S_2 \subseteq \operatorname{ri} F$. This finishes the proof of (1) in Lemma 2.

In (2) we first prove $s_0 > 0$. Since $(s_3; s_0) \in S_3,$ there is x_3 that satisfies

$$s_3 \;=\; Ax_3 + bs_0. \tag{2.17}$$

Let $s_1 \in S_1$. Expressing s_1 as in (2.13) shows $(s_1; 1) \in S_3$. Hence by Theorem 6.4 in [39] the line-segment from $(s_1; 1)$ to $(s_3; s_0)$ can be extended past $(s_3; s_0)$ in S_3, i.e.,

$$(1+\epsilon)(s_3; s_0) - \epsilon(s_1; 1) \;\in\; S_3, \tag{2.18}$$

for some small $\epsilon > 0$, hence $s_0 > 0$.

Define $s_3' := s_3/s_0$. Since S_3 is a cone, and $(s_3; s_0)$ is in $\operatorname{ri} S_3$, so is $(s_3'; 1)$. So (2.18) holds with $(s_3'; 1)$ in place of $(s_3; s_0)$, i.e., there is a small $\epsilon > 0$ such that

$$(1+\epsilon)(s_3'; 1) - \epsilon(s_1; 1) \;=\; ((1+\epsilon)s_3' - \epsilon s_1; 1) \in S_3. \tag{2.19}$$

Dividing in (2.17) by s_0 we find $s_3' \in S_1$. Also, (2.19) implies

$$(1+\epsilon)s_3' - \epsilon s_1 \in S_1,$$

hence (again by Theorem 6.4 in [39]) $s_3' \in \operatorname{ri} S_1$. Since $\operatorname{ri} S_1 \subseteq \operatorname{ri} F$, we obtain $s_3' \in \operatorname{ri} F$, hence $s_3 \in \operatorname{ri} F$, and this finishes the proof of (2).

Statement (3) then follows from using Lemma 1 with the fact that the minimal face of K that contains z, s_2, and s_3 is the same, namely F. □

Proof of Theorem 3: In this proof, for a C closed convex cone, and $x \in C$ we use the notation

$$\operatorname{frdir}(x, C) \;=\; \operatorname{cl} \operatorname{dir}(x, C) \setminus \operatorname{dir}(x, C). \tag{2.20}$$

Proof of (3) ⇒ (1) and (2) ⇒ (1) Suppose that (3) or (2) holds. We prove that whenever (P_c) is bounded, the dual (D_c) has a solution that attains the same value.

Let c be an objective vector, such that $c_0 := \operatorname{val}(P_c)$ is finite. Then $\operatorname{aval}(D_c) = c_0$ holds by Theorem 2, so there is $\{y_i\} \subseteq K^*$ s.t. $A^* y_i \to c$, and $b^* y_i \to c_0$, in other words,

$$\begin{pmatrix} c \\ c_0 \end{pmatrix} \in \operatorname{cl} \begin{pmatrix} A^* \\ b^* \end{pmatrix} K^*. \tag{2.21}$$

By the closedness of the set in (2.21), there exists $y \in K^*$ such that $A^* y = c$, and $b^* y = c_0$. Thus the implication (3) ⇒ (1) follows.

To prove (2) ⇒ (1), note that

$$\operatorname{cl} \begin{pmatrix} A^* \\ b^* \end{pmatrix} K^* \subseteq \operatorname{cl} \begin{pmatrix} A^* & 0 \\ b^* & 1 \end{pmatrix} \begin{pmatrix} K^* \\ \mathbb{R}_+ \end{pmatrix}. \tag{2.22}$$

Since the second set in (2.22) is closed, there exists $y \in K^*$, and $s \in \mathbb{R}_+$ such that $A^* y = c$, and $b^* y + s = c_0$. We must have $s = 0$, since $s > 0$ would contradict weak duality. Therefore y is a feasible solution to (D_c) with $b^* y = c_0$, and this completes the proof.

Proof of ¬(2) ⇒ ¬(1) Now let c and c_0 be arbitrary, and suppose they satisfy

$$\begin{pmatrix} c \\ c_0 \end{pmatrix} \in \operatorname{cl} \begin{pmatrix} A^* & 0 \\ b^* & 1 \end{pmatrix} \begin{pmatrix} K^* \\ \mathbb{R}_+ \end{pmatrix}, \tag{2.23}$$

and

$$\begin{pmatrix} c \\ c_0 \end{pmatrix} \notin \begin{pmatrix} A^* & 0 \\ b^* & 1 \end{pmatrix} \begin{pmatrix} K^* \\ \mathbb{R}_+ \end{pmatrix}. \tag{2.24}$$

By (2.23) there is $\{(y_i, s_i)\} \subseteq K^* \times \mathbb{R}_+$ s.t. $A^* y_i \to c$, and $b^* y_i + s_i \to c_0$. Hence $\operatorname{aval}(D_c) \leq c_0$. Also, Theorem 2 implies $\operatorname{val}(P_c) = \operatorname{aval}(D_c)$, so

$$\operatorname{val}(P_c) \;\leq\; c_0.$$

However, (2.24) implies that no feasible solution of (D_c) can have value $\leq c_0$. Hence either $\mathrm{val}(D_c) > c_0$ (this includes the case $\mathrm{val}(D_c) = +\infty$, i.e., when (D_c) is infeasible), or $\mathrm{val}(D_c)$ is not attained. In other words, c is a "bad" objective function.

Proof of $(2) \Rightarrow (4)$ **and of** $(2) \Leftrightarrow (4)$ **when** $K^* + F^\perp$ **is closed:** Let $(s_3; s_0) \in \mathrm{ri}\, S_3$, and G the smallest face of $K \times \mathbb{R}_+$ that contains $(s_3; s_0)$. Consider the statement

$$\mathrm{frdir}((s_3; s_0), K \times \mathbb{R}_+) \cap \mathcal{R}\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} = \emptyset. \tag{2.25}$$

By Theorem 1 with

$$M = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix},\ C = K \times \mathbb{R}_+,\ E = G$$

we have that (2) implies (2.25), and it is equivalent to it, when $(K \times \mathbb{R}_+)^* + G^\perp$ is closed. Obviously $(K \times \mathbb{R}_+)^* = K^* \times \mathbb{R}_+$. We claim

$$K^* \times \mathbb{R}_+ + G^\perp \,\text{is closed} \;\Leftrightarrow\; K^* + F^\perp \,\text{is closed.} \tag{2.26}$$

Indeed, by (2) in Lemma 2 we have $(s_3; s_0) \in \mathrm{ri}(F \times \mathbb{R}_+)$, so $G = F \times \mathbb{R}_+$. Since F and $\mathbb{R}_+$ are cones, we find $G^\perp = F^\perp \times \{0\}$, and this proves (2.26). (In fact, even if s_0 were 0, then $G^\perp = F^\perp \times \mathbb{R}$ would hold, and (2.26) would still be true, since all cones contained in the real line are closed.)

Next we claim

$$\mathrm{frdir}((s_3; s_0), K \times \mathbb{R}_+) = \mathrm{frdir}(s_3, K) \times \mathbb{R}. \tag{2.27}$$

Indeed, by definition, the set on the left hand side of (2.27) is the union of the two sets

$$\mathrm{frdir}(s_3, K) \times \mathrm{cl}\,\mathrm{dir}(s_0, \mathbb{R}_+)$$

and

$$\mathrm{cl}\,\mathrm{dir}(s_3, K) \times \mathrm{frdir}(s_0, \mathbb{R}_+).$$

But $s_0 > 0$ by Lemma 2, so $\mathrm{dir}(s_0, \mathbb{R}_+) = \mathrm{cl}\,\mathrm{dir}(s_0, \mathbb{R}_+) = \mathbb{R}$. So the second of these sets is empty, and the first is equal to the set on the right hand side of (2.27).

Given (2.27), statement (2.25) is equivalent to

$$\mathrm{frdir}(s_3, K) \cap \mathcal{R}(A, b) = \emptyset. \tag{2.28}$$

By (3) in Lemma 2 we obtain $\mathrm{frdir}(z, K) = \mathrm{frdir}(s_3, K)$, so (2.28) is equivalent to (4), and this completes the proof.

Proof of $(3) \Rightarrow (4) \Leftrightarrow (5)$ **and of** $(3) \Leftrightarrow (4) \Leftrightarrow (5)$ **when** $K^* + F^\perp$ **is closed:**

Let $s_2 \in \mathrm{ri}\, S_2$, and consider the statements (4-s_2) and (5-s_2) obtained from (4) and (5) by replacing z with s_2. By part (1) in Lemma 2 we have $s_2 \in \mathrm{ri}\, F$. Therefore Theorem 1 with $M = (A, b)$, $C = K$, $E = F$ proves

$$(3) \Rightarrow \text{(4-}s_2\text{)} \Leftrightarrow \text{(5-}s_2\text{)},$$

and that these statements are equivalent, when $K^* + F^\perp$ is closed. By part (3) in Lemma 2, and the definition of the sets $\mathrm{ldir}(z, K)$ and $\tan(z, K)$ in (1.4)-(1.5) we have that (4-s_2) is equivalent to (4) and (5-s_2) to (5), and this completes the proof.

$\square$

3 Characterizing badly, and well-behaved semidefinite systems

In this section we specialize the results of Section 2 to exactly characterize when the semidefinite system

$$\sum_{i=1}^{m} x_i A_i \preceq B, \tag{P_{SD}}$$

is badly- or well-behaved. Here $A_1, \ldots, A_m$ and B are symmetric matrices, which we assume to be linearly independent, i.e., only their trivial linear combination yields the zero matrix.

Given the semidefinite system (P_{SD}) we consider an SDP of the form

$$\sup \{ \sum_{i=1}^{m} c_i x_i \mid x \text{ is feasible in } (P_{SD}) \}. \tag{3.29}$$

The dual of this problem is

$$\inf \{ B \bullet Y \mid Y \succeq 0, \, A_i \bullet Y = c_i \, (i = 1, \ldots, m) \}.$$

Also, a maximum slack as defined in Definition 1 is a slack matrix with maximum *rank* in the semidefinite system (P_{SD}), and the cone of positive semidefinite matrices is nice [12, 11].

We make the following

Assumption 1. *The maximum rank slack in (P_{SD}) is*

$$Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.30}$$

We can satisfy Assumption 1 by applying the rotation $Q^T()Q$ to all A_i and $B,$ where Q is a matrix of suitably scaled eigenvectors of the maximum rank slack.

In the interest of the reader we first state and illustrate the main results, then prove them.

Theorem 4. *The system (P_{SD}) is badly behaved, if and only if there is a matrix V which is a linear combination of the A_i and B of the form*

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \tag{3.31}$$

where V_{11} is r by $r,$ $V_{22} \succeq 0,$ and $\mathcal{R}(V_{12}^T) \not\subseteq \mathcal{R}(V_{22}).$ □

The Z and V matrices provide a *certificate* of the bad behavior of (P_{SD}).

Example 1. In the problem

$$\begin{array}{ll} \sup & x_1 \\ s.t. & x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \tag{3.32}$$

the only feasible solution is $x_1 = 0$. The dual program, where we denote the components of Y by $y_{ij},$ is equivalent to

$$\begin{array}{ll} \inf & y_{11} \\ s.t. & \begin{pmatrix} y_{11} & 1/2 \\ 1/2 & y_{22} \end{pmatrix} \succeq 0, \end{array}$$

which has a 0 infimum, but does not attain it.

Here the certificates of the bad behavior of the system in (3.32) are

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \, V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Example 2. The problem

$$\begin{array}{ll} \sup & x_2 \\ s.t. & x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \tag{3.33}$$

again has an attained 0 supremum. The reader can easily check that the value of the dual program is 1, (and it is attained), so there is a finite, positive duality gap.

In (3.33) the right hand side is the maximum slack, and we can choose the coefficient matrix of x_2 as the V matrix of Theorem 4.

Well behaved systems have a similarly simple characterization:

Theorem 5. *The system (P_{SD}) is well-behaved, if and only if conditions (1) and (2) below hold:*

(1) there is a matrix U of the form

$$U = \begin{pmatrix} 0 & 0 \\ 0 & U_{22} \end{pmatrix}, \tag{3.34}$$

with $U_{22} \succ 0$ and

$$A_1 \bullet U = \ldots = A_m \bullet U = B \bullet U = 0. \tag{3.35}$$

(2) For all V matrices, which are a linear combination of the A_i and B and are of the form

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & 0 \end{pmatrix},$$

with $V_{11} \in \mathcal{S}^r$, we must have $V_{12} = 0$.

The next example illustrates both well- and badly behaved semidefinite systems:

Example 3. Consider the system

$$x_1 \begin{pmatrix} 1 & 1 & \alpha \\ 1 & 1 & 1 \\ \alpha & 1 & -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.36}$$

where α is some real number. Let us denote by A_i the constraint matrices on the left, and by B the right hand side matrix. The matrix $U = 0 \oplus I_2$ is orthogonal to all constraint matrices. Hence condition (1) in Theorem 5 holds, and B is the maximum rank slack in (3.36).

If $\alpha = 1,$ then (3.36) is well-behaved, and condition (2) of Theorem 5 is easy to verify.

If $\alpha \neq 1,$ then

$$V := A_1 - A_2 - A_3 = \begin{pmatrix} 1 & 0 & \alpha - 1 \\ 0 & 0 & 0 \\ \alpha - 1 & 0 & 0 \end{pmatrix}$$

is a certificate matrix as required by Theorem 4, hence (3.36) is badly behaved.

We return to this example in Section 4. We use it to illustrate that we can verify the bad or good behavior of semidefinite systems *without referring to Theorems 4 or 5,* using only elementary arguments.

We remark that for semidefinite systems that are strictly feasible, a matrix similar to the V matrix in Theorem 4 can make sure that the optimal primal-dual solution pair fails strict complementarity; see [50].

Next we prove Theorems 4 and 5. We first collect some results on the geometry of the semidefinite cone:

Lemma 3. *Let Z be a positive semidefinite matrix of the form (3.30). Recall the definition of the set of feasible directions, and related sets from (1.3)-(1.5). Then*

$$\operatorname{ldir}(Z, \mathcal{S}^n_+) = \mathcal{S}^r \oplus \{0\}, \tag{3.37}$$

$$\operatorname{cl}\operatorname{dir}(Z, \mathcal{S}^n_+) = \left\{ \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{pmatrix} \,\middle|\, Y_{22} \in \mathcal{S}^{n-r}_+ \right\}, \tag{3.38}$$

$$\tan(Z, \mathcal{S}^n_+) = \left\{ \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & 0 \end{pmatrix} \,\middle|\, Y_{11} \in \mathcal{S}^r \right\}, \tag{3.39}$$

$$\operatorname{dir}(Z, \mathcal{S}^n_+) = \left\{ \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{pmatrix} \,\middle|\, Y_{22} \in \mathcal{S}^{n-r}_+, \mathcal{R}(Y_{12}^T) \subseteq \mathcal{R}(Y_{22}) \right\}. \tag{3.40}$$

Proof Let F be the smallest face of $\mathcal{S}^n_+$ that contains Z. Then clearly $F = \mathcal{S}^r_+ \oplus \{0\}$, and $F^\triangle = \{\, 0 \,\} \oplus \mathcal{S}^{n-r}_+$. Hence $(F^\triangle)^*$ is the set on the right hand side in equation (3.38), and Lemma 1 with $C = \mathcal{S}^n_+$, $x = Z$, $E = F$ proves statements (3.37)-(3.39).

Next, fix $Y \in \operatorname{cl}\operatorname{dir}(Z, \mathcal{S}^n_+)$, and partition it as in the right hand side set in (3.38). Then (3.40) is equivalent to

$$Y \in \operatorname{dir}(Z, \mathcal{S}^n_+) \Leftrightarrow \mathcal{R}(Y_{12}^T) \subseteq \mathcal{R}(Y_{22}). \tag{3.41}$$

Let P be an orthogonal matrix, such that $P^T Y_{22} P = I_s \oplus 0,$ where s is the number of positive eigenvalues of Y_{22} and $T = I_r \oplus P$.

Define

$$V := T^T Y T = \begin{pmatrix} Y_{11} & Y_{12}P \\ P^T Y_{12}^T & P^T Y_{22} P \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12}P \\ P^T Y_{12}^T & I_s \oplus 0 \end{pmatrix}.$$

Next we claim

$$Y \in \operatorname{dir}(Z, \mathcal{S}^n_+) \Leftrightarrow V \in \operatorname{dir}(Z, \mathcal{S}^n_+), \tag{3.42}$$

$$\mathcal{R}(Y_{12}^T) \subseteq \mathcal{R}(Y_{22}) \Leftrightarrow \mathcal{R}(P^T Y_{12}^T) \subseteq \mathcal{R}(P^T Y_{22} P). \tag{3.43}$$

Indeed, (3.42) follows from $T^T Z T = Z,$ and the definition of feasible directions. As to (3.43), the left hand side statement holds, iff there is a matrix D with

$$Y_{12}^T = Y_{22} D, \tag{3.44}$$

and the right hand side statement holds, iff a matrix D' such that

$$P^T Y_{12}^T = P^T Y_{22} P D'. \tag{3.45}$$

If D satisfies (3.44), then $D' := P^{-1} D$ satisfies (3.45). Conversely, if (3.45) holds for D', then $D := PD'$ verifies (3.44).

Partition $Y_{12}P$ as $(V_{12}, V_{13}),$ so that V_{12} has s columns; then (3.43) is equivalent to $V_{13} = 0$. So we only need to prove

$$V \in \operatorname{dir}(Z, \mathcal{S}^n_+) \Leftrightarrow V_{13} = 0. \tag{3.46}$$

Consider the matrix $Z + \epsilon V$ for some $\epsilon > 0$. If $V_{13} \neq 0$, then $Z + \epsilon V$ is not positive semidefinite for any $\epsilon > 0$, and this proves the direction $\Rightarrow$. As to $\Leftarrow$, if $V_{13} = 0$, then by the Schur-complement condition for positive semidefiniteness we have that $Z + \epsilon V \succeq 0$ iff

$$(I_r + \epsilon V_{11}) - (\epsilon V_{12})(\epsilon I_s)^{-1}(\epsilon V_{12}^T) \succeq 0,$$

and the latter is clearly true for some small $\epsilon > 0$. $\square$

Proof of Theorem 4 The equivalence (1) $\Leftrightarrow$ (4) in Theorem 3 shows that (P_{SD}) is badly behaved, iff there is a matrix V, which is a linear combination of the A_i and of B, and also satisfies

$$V \in \operatorname{cl} \operatorname{dir}(Z, \mathcal{S}_+^n) \setminus \operatorname{dir}(Z, \mathcal{S}_+^n).$$

Then our statement follows from parts (3.38) and (3.40) in Lemma 3. $\square$

Proof of Theorem 5 We use the equivalence (1) $\Leftrightarrow$ (5) in Theorem 3. Let F be the smallest face of $\mathcal{S}_+^n$ that contains Z. Then $F = \mathcal{S}_+^r \oplus \{0\}$, and $F^\triangle = \{\, 0\,\} \oplus \mathcal{S}_+^{n-r}$, hence a $U \succeq 0$ is strictly complementary to Z if and only if

$$U \;=\; \begin{pmatrix} 0 & 0 \\ 0 & U_{22} \end{pmatrix}, \text{ with } U_{22} \succ 0.$$

So the first part of (5) in Theorem 3 holds iff there is such a U that satisfies (3.35). By (3.37) and (3.39) in Lemma 3 the second part of condition (5) in Theorem 3 holds, if and only if all V matrices, which are a linear combination of the A_i and B, and are of the form

$$V \;=\; \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & 0 \end{pmatrix},$$

satisfy $V_{12} = 0$. $\square$

4 Reformulations of semidefinite systems. Proving membership of well-behaved systems in $\mathcal{NP} \cap \text{co-}\mathcal{NP}$

In this section we prove that the question

> "Is (P_{SD}) well-behaved?"

is in $\mathcal{NP} \cap \text{co-}\mathcal{NP}$ in the real number model of computing.

A key part of the certificates is a standard form reformulation of (P_{SD}) with a strictly feasible block, and some variables identically zero. We construct the reformulation, so it is well-behaved, if and only if (P_{SD}) is. The reformulation is presented in Theorem 6 for badly behaved systems, and in Theorem 7 for well behaved systems. If the reformulation is badly behaved, we can verify this by elementary linear algebra; if well behaved, we need the additional fact that a strictly feasible system is well-behaved.

Hence, while it is nontrivial to prove that the standard reformulation exists, its use to verify the behavior of (P_{SD}) is straightforward.

The reformulation also allow us to prove that if (P_{SD}) is well behaved, then for *all* objective functions there is an optimal dual matrix with a predefined block-diagonal structure; and to systematically generate *all* well behaved semidefinite systems.

Definition 2. A semidefinite system
$$\sum_{i=1}^{m} x_i A_i' \preceq B', \tag{P_{SD}'}$$
is a reformulation of (P_{SD}) if it is obtained from it by a sequence of the following operations:

(1) Apply a rotation $T^T()T$ to all A_i and B, where $T = I_r \oplus M$, and M is an orthogonal matrix.

(2) Replace B by $B + \sum_{j=1}^{m} \mu_j A_j$, where $\mu \in \mathbb{R}^m$.

(3) Replace A_i by $\sum_{j=1}^{m} \lambda_j A_j$, where $i \in \{1, \ldots, m\}$, $\lambda \in \mathbb{R}^m$, $\lambda_i \neq 0$.

(4) Exchange A_i and A_j, where $i, j \in \{1, \ldots, m\}$.

It is clear that in all reformulations of (P_{SD}) the maximum rank slack is the same. Also, note that (P_{SD}) is a reformulation of itself, and also a reformulation of (P_{SD}') in Definition 2.

For convenience, let us recall the SDP
$$\sup \{ \sum_{i=1}^{m} c_i x_i \,|\, x \,\text{is feasible in}\, (P_{SD}) \}. \tag{4.47}$$

Definition 3. The SDP
$$\sup \{ \sum_{i=1}^{m} c_i' x_i \,|\, x \,\text{is feasible in}\, (P_{SD}') \} \tag{4.48}$$
is a reformulation of (4.47) if

(1) (P_{SD}') is a reformulation of (P_{SD}); and

(2) c' is obtained from c by the same type (3) and (4) operations, in the same sequence, that produced (P_{SD}') from (P_{SD}).

That is, if we replaced A_i by $\sum_{j=1}^{n} \lambda_j A_j$ in a type (3) operation, then at the same time we replace c_i by $\sum_{j=1}^{n} \lambda_j c_j$. If we switched A_i and A_j in a type (4) operation, then at the same time we switch c_i and c_j.

Lemma 4. *The system (P_{SD}) is well-behaved if and only if its reformulations are.*

Proof Let (P_{SD}') be a reformulation of (P_{SD}). We can assume without loss of generality that to obtain (P_{SD}') we used operations (1) and (2) only once, in the beginning, with corresponding rotation matrix, say, T and vector $\mu \in \mathbb{R}^m$.

Let c be an objective function such that the value of (4.47) is finite, and let c' be such that (4.48) is a reformulation of (4.47). Denoting the values of these problems by v and v' we clearly have
$$v' = v + \sum_{j=1}^{m} \mu_j c_j.$$
Using identity (1.10) we find that $Y \succeq 0$ is feasible to the dual of (4.47) with value, say, α, if and only of if $T^{-1} Y T^{-T}$ is feasible to the dual of (4.48) with value $\alpha + \sum_{j=1}^{m} \mu_j c_j$.

Hence if (P_{SD}') is well-behaved, then so is (P_{SD}). Noting that (P_{SD}) is also a reformulation of (P_{SD}') proves the reverse implication, and completes the proof. $\square$

For clarity, we reformulate (P_{SD}) in two steps. The first step, presented in Lemma 5, provides easy to verify certificates to prove that Z is a maximum slack. These certificates are constructed by a facial reduction algorithm (see [12, 11, 48, 35]). In Lemma 5 we only use rotations, i.e., type (1) operations in Definition 2.

Lemma 5. *The system (P_{SD}) has a reformulation of the form (P'_{SD}), in which the maximality of the slack Z is proven by symmetric matrices of the form*

$$Y_i = \begin{array}{c} \begin{array}{ccc} & {\scriptstyle r_i} & {\scriptstyle r_{i-1}+\cdots+r_1} \end{array} \\ \left(\begin{array}{ccc} 0 & 0 & \times \\ 0 & I & \times \\ \times & \times & \times \end{array} \right) \end{array} (i = 1, \ldots, \ell)$$

where $r_1, \ldots, r_\ell$ are positive integers, $r_1 + \cdots + r_\ell = n - r,$ and the Y_i are orthogonal to all constraint matrices in (P'_{SD}). Here the $\times$ symbols denote blocks with arbitrary elements in the Y_i matrices.

Proof We first show that the Y_i matrices indeed prove that Z is maximal. Suppose that S is a feasible slack. By $Y_1 \bullet S = 0$ and $S \succeq 0$ we find that the last r_1 rows and columns of S are zero; then, by $Y_2 \bullet S = 0$ and $S \succeq 0$ we have that the next r_2 rows and columns of S are zero, and so on.

To find the reformulation, we start with no Y_i matrices, and the original system (P_{SD}). Suppose that at some point we have a reformulation of the form (P'_{SD}) and matrices $Y_1, \ldots, Y_i$ of the required form. For brevity, let $s_i = r_1 + \cdots + r_i$. We clearly must have $s_i \leq n - r$. If $s_i = n - r,$ we set $\ell = i,$ and stop.

Otherwise, define the cone $K = \mathcal{S}^n_+ \cap Y_1^\perp \cdots \cap Y_i^\perp$. Clearly, K and its dual cone K^* are of the form

$$K \;=\; \mathcal{S}^{n-s_i}_+ \oplus \{0\},\; K^* \;=\; \left\{ \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{pmatrix} \,\Big|\, Y_{11} \in \mathcal{S}^{n-s_i}_+ \right\}.$$

Define the affine subspace

$$H \;=\; \mathrm{lin}\,\{\, A'_1, \ldots, A'_m \,\} + B'.$$

Since Z is also a maximum rank slack in (P'_{SD}), and $s_i < n - r,$ we have $H \cap \mathrm{ri}\, K = \emptyset,$ hence $H^\perp \cap (K^* \setminus K^\perp) \neq \emptyset$ by a classic theorem of the alternative (see e.g. [6]).

Let

$$Y_{i+1} \in H^\perp \cap (K^* \setminus K^\perp).$$

Since $Y_{i+1} \bullet Z = 0,$ we have

$$Y_{i+1} = \begin{array}{c} \begin{array}{ccc} {\scriptstyle r} & & {\scriptstyle s_i} \end{array} \\ \begin{pmatrix} 0 & 0 & \times \\ 0 & Y' & \times \\ \times & \times & \times \end{pmatrix} \end{array}$$

for some $Y' \succeq 0$. As $Y_{i+1} \notin K^\perp,$ the number of positive eigenvalues of Y', which we denote by r_{i+1}, is positive.

Let Q be an orthogonal matrix such that $Q^T Y' Q = 0 \oplus I_{r_{i+1}}$, and $T = I_r \oplus Q \oplus I_{s_i}$.

We apply the rotation $T^T()T$ to Y_j for $j = 1, \ldots, i+1$, and the rotation $T^{-1}()T^{-T}$ to all constraint matrices in (P'_{SD}).

By (1.10) the new Y_i are orthogonal to all constraint matrices in the new system. By the form of T the new Y_{i+1} is in the form as the lemma requires, and the identity block in $Y_1, \ldots, Y_i$ remains the same. $\square$

To find the *final* reformulation of (P_{SD}) (for easily checking its well- or badly behaved nature) we start with the system given by Lemma 5 and further reformulate it using only operations (2), (3), and (4) in Definition 2. Thus the same Y_i matrices prove the maximality of the Z slack. The reformulations, which are in a different shape for well- and for badly behaved systems, consist of a strictly feasible part, and a part with identically zero variables.

Theorem 6 shows the reformulation for badly behaved systems, and a trivial objective function that proves the bad behavior. The last constraint matrix on the left hand side of the reformulation will be a V certificate matrix as given in Theorem 4.

Theorem 6. *The system (P_{SD}) is badly behaved if and only if it has a reformulation*

$$\sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z, \qquad (P_{\text{SD,bad}})$$

where

(1) the matrix Z is the maximum rank slack, and this is verified by matrices $Y_1, \dots, Y_\ell,$ as given by Lemma 5.

(2) The matrices

$$\begin{pmatrix} G_i \\ H_i \end{pmatrix} \ (i = k+1, \dots, m)$$

are linearly independent.

(3) $k < m$ and $H_m \succeq 0$.

Proof (If) It suffices to prove that $(P_{\text{SD,bad}})$ is badly behaved. We first prove that if x is a feasible solution in $(P_{\text{SD,bad}})$, then

$$x_{k+1} \;=\; \dots \;=\; x_m \;=\; 0. \tag{4.49}$$

Let x be such a solution, with a corresponding slack S. The last $n-r$ columns and rows of S must be zero; otherwise a convex combination of S and Z would give a slack with larger rank than Z. Hence our claim follows from condition (2).

Next, consider the SDP

$$\sup\{\, -x_m \mid x \text{ is feasible in } (P_{\text{SD,bad}})\,\}.$$

By (4.49) its optimal value is 0. We prove that its dual cannot have a feasible solution with value 0. Indeed, suppose that

$$Y \;=\; \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{pmatrix}$$

is such a solution. By $Y \bullet Z = 0$ we get $Y_{11} = 0,$ hence $Y_{12} = 0$. Thus the inner product of Y with the mth constraint matrix in $(P_{\text{SD,bad}})$ equals $Y_{22} \bullet H_m \geq 0,$ a contradiction.

(Only if) To construct the reformulation, we start with the system given by Lemma 5. By applying an operation of type (2) we first replace B' by Z. Since $(P_{\text{SD,bad}})$ is badly behaved, there is a certificate matrix

$$V \;=\; \lambda_0 Z + \sum_{i=1}^{m} \lambda_i A_i', = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix},$$

as Theorem 4 guarantees, where V_{11} is r by $r,$ $V_{22} \succeq 0,$ and $\mathcal{R}(V_{12}^T) \not\subseteq \mathcal{R}(V_{22})$. By the form of Z it holds that $V' = V - \lambda_0 Z$ is also a certificate matrix, which satisfies the same conditions. Note that the submatrix of V' made up of the last $n-r$ columns of V' must be nonzero.

Suppose $\lambda_i \neq 0$ for some $i \in \{1, \dots, m\}$. We reformulate our system by replacing A_i' by $V',$ then switch A_i' and A_m'. For brevity, we still call the constraint matrices A_i'.

Next we choose a maximal subset of the A_i' matrices so their blocks comprising the last $n-r$ columns are linearly independent. We choose A_m' as one of these matrices, and permute the matrices so this subset becomes $A_{k+1}', \dots, A_m'$ for some $k \geq 0$.

We finally add suitable multiples of $A_{k+1}', \dots, A_m'$ to the other A_i' to zero out the last $n-r$ columns and rows of the latter, and arrive at the required reformulation. $\square$

Theorem 7 shows a similar reformulation, and a fundamental block-diagonality property of well-behaved systems:

Theorem 7. *The system (P_{SD}) is well-behaved if and only if it has a reformulation*

$$\sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z, \qquad (P_{\mathrm{SD,good}})$$

where

(1) the matrix Z is the maximum rank slack.

(2) The matrices $H_i \, (i = k+1, \ldots, m)$ are linearly independent.

(3) $H_{k+1} \bullet I = \cdots = H_m \bullet I = 0.$

Also, if (P_{SD}) is well behaved, then for all $c \in \mathbb{R}^m$ for which the value of (4.47) is finite, there is an optimal dual matrix in $\mathcal{S}_+^r \oplus \mathcal{S}_+^{n-r}$.

Proof We first prove the "If" part, and the block-diagonality result. Suppose that (P_{SD}) has a reformulation of the form ($P_{\mathrm{SD,good}}$), and let $c \in \mathbb{R}^m$ be such that the value of (4.47) is finite, and let us denote this value by v. Let $c' \in \mathbb{R}^m$ be such that

$$\sup \{ \sum_{i=1}^{m} c_i' x_i \,|\, x \text{ is feasible in } (P_{\mathrm{SD,good}}) \} \qquad (4.50)$$

is a reformulation of (4.47), and denote the optimal value of (4.50) by v'. An argument like in the proof of Theorem 6 proves that $x_{k+1} = \cdots = x_m = 0$ holds for any x feasible in (4.50), hence the optimal value of

$$\sup \{ \sum_{i=1}^{k} c_i' x_i \,|\, \sum_{i=1}^{k} x_i F_i \preceq I \} \qquad (4.51)$$

is also v'. Since (4.51) satisfies Slater's condition, there is Y_{11} feasible to its dual with $Y_{11} \bullet I_r = v'$.

As the H_i are linearly independent, there is $P \in \mathcal{S}^{n-r}$ (possibly not psd) with

$$H_i \bullet P = c_i' \, (i = k+1, \ldots, m). \qquad (4.52)$$

Let us choose $\lambda \geq 0$ such that $Y_{22} := P + \lambda I$ is positive semidefinite. By Condition (3) the matrix Y_{22} also satisfies the constraints (4.52), hence $Y := Y_{11} \oplus Y_{22}$ is an optimal solution to the dual of (4.50), and it attains the optimal value v'.

Assume without loss of generality that ($P_{\mathrm{SD,good}}$) was obtained from (P_{SD}) by applying operations (1) and (2) only once, at the beginning, with a transformation matrix $T = I_r \oplus M$, and a vector $\mu \in \mathbb{R}^m$. The proof of Lemma 4 implies that $Y_{11} \oplus M Y_{22} M^T$ is optimal to the dual of (4.47) and attains the optimal value v. This completes the proof.

Proof of "Only if" We again start with the system (P_{SD}') that Lemma 5 provides. Notice that the U matrix of Theorem 5 became the $Y_1 = 0 \oplus I_{n-r}$ matrix of Lemma 5, after we rotated it. By using operation (2) we replace B' by Z.

We choose a maximal subset of the A_i' whose lower principal $(n-r) \times (n-r)$ blocks are linearly independent. We permute the A_i', if needed, to make this subset $A_{k+1}', \ldots, A_m'$ for some $k \geq 0$.

We finally add multiples of $A_{k+1}', \ldots, A_m'$ to $A_1', \ldots, A_k'$ to zero out their lower principal $(n-r) \times (n-r)$ block. By Theorem 5 the upper right $r \times (n-r)$ block of $A_1', \ldots, A_k'$ and the symmetric counterpart also becomes zero. Thus we reformulated (P_{SD}), as required. □

We note that in the case of a badly behaved semidefinite system we may need several of the Y_i matrices to prove the maximality of the slack Z; when (P_{SD}) is well behaved, we only need Y_1.

In Examples 1 and 2 the corresponding systems are already in the form of $(P_{\mathrm{SD,bad}})$.

Example 3 continued To bring the system (3.36) into the form of $(P_{\mathrm{SD,bad}})$ or $(P_{\mathrm{SD,good}})$ we first replace A_1 by $A_1 - A_2 - A_3$ to obtain

$$x_1 \begin{pmatrix} 1 & 0 & \alpha-1 \\ 0 & 0 & 0 \\ \alpha-1 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.53}$$

If $\alpha = 1,$ i.e., the system is well-behaved, then (4.53) is in the form of $(P_{\mathrm{SD,good}})$.

If $\alpha \neq 1,$ i.e., the system is badly behaved, then we also switch A_1 and A_3 to get the reformulation in the form $(P_{\mathrm{SD,bad}})$:

$$x_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 & 0 & \alpha-1 \\ 0 & 0 & 0 \\ \alpha-1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.54}$$

The proof of Theorem 6 implies that (4.54) with the objective function $c = (0, 0, -1)$ yields a 0 primal objective value, but no dual solution attains the same value.

We can recover the original system (3.36) by exchanging the first and third constraint matrices in (4.54), then adding the second and the third constraint matrices to the first. Hence the objective function $(-1, 0, 0)$ over the original system (3.36) also yields a 0 primal objective value, but no dual solution has the same value. □

Let us consider now the following algorithm to generate a well-behaved system:

(1) Choose integers n, r, m and k with

$$n \geq 1,\, 0 \leq r \leq n,\, m - k \leq (n-r)(n-r+1)/2.$$

(2) Choose $Z = I_r \oplus 0$ as the right hand side matrix in $(P_{\mathrm{SD,good}})$.

(3) Choose the last $m - k$ constraint matrices in $(P_{\mathrm{SD,good}})$ by choosing the H_i to satisfy conditions (2) and (3) in Theorem 7, and the F_i and G_i as arbitrary.

(4) Choose linearly independent matrices $F_1, \dots, F_k$ to form the first k matrices in $(P_{\mathrm{SD,good}})$.

Theorem 7 directly implies that this simple algorithm can generate all well behaved systems:

Corollary 3. *Suppose that a semidefinite system of the form (P_{SD}) is well behaved. Then (P_{SD}) is a reformulation of a possible output of the above algorithm.*

Our main complexity result is summarized in Theorem 8:

Theorem 8. *The question*

"Is (P_{SD}) well behaved?"

is in $\mathcal{NP} \cap \text{co-}\mathcal{NP}$ in the real number model of computing.

Proof The certificates to check the status of (P_{SD}) are:

- A reformulation of the form ($P_{\text{SD,bad}}$) or ($P_{\text{SD,good}}$);
- The Y_i matrices to verify that $Z,$ the right hand side matrix in ($P_{\text{SD,bad}}$) or ($P_{\text{SD,good}}$), is indeed a maximum rank slack;
- A matrix $T = I_r \oplus M,$ where M is an orthogonal matrix.

Let us denote the matrices on the left hand side of ($P_{\text{SD,bad}}$) or ($P_{\text{SD,good}}$) by A_i'. The verifier first checks that

$$Z = T^T(B + \sum_{i=1}^m \mu_i A_i)T$$

holds for some $\mu \in \mathbb{R}^m$, and that the linear span of $\{T^T A_1 T, \dots, T^T A_m T\}$ and of $\{A_1', \dots, A_m'\}$ agree. This proves that ($P_{\text{SD,bad}}$) or ($P_{\text{SD,good}}$) is indeed a reformulation.

He/she then checks the properties of ($P_{\text{SD,bad}}$) or ($P_{\text{SD,good}}$) as given in Theorems 6 or 7. Then the proof of the "If" part in these results shows that these systems are well- or badly behaved. $\square$

5 Concluding remarks

We can complement Theorem 4 by the following

Corollary 4. *Suppose that in addition to the operations of Definition 2 we allow a sequence of the following:*

(1) Delete row i and column i from all matrices, where $i \in \{1, \dots, n\}$.

(2) Delete a constraint matrix.

Then any badly behaved semidefinite system can be brought to the form of system (1.2).

Proof Suppose that (P_{SD}) is badly behaved. We first add multiples of the A_i to B to make sure that the right hand side is the maximum slack. Then we can assume that the V certificate matrix of Theorem 4 is the linear combination of the A_i only; we reformulate, so V becomes a constraint matrix.

As we proved in Lemma 3 by a rotation we can bring V to the form

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{12}^T & I_s & 0 \\ V_{13}^T & 0 & 0 \end{pmatrix},$$

where $V_{13} \neq 0$. Suppose that $v_{ij} \neq 0,$ where $1 \leq i \leq r$ and $r + s + 1 \leq j \leq n$. We rescale V to get $v_{ij} = 1$, then delete all rows and columns from the constraint matrices whose index is not i nor $j,$ to obtain system (1.2). $\square$

Excluded minor results in graph theory, such as Kuratowski's theorem, show that a graph lacks a certain fundamental property, if and only if it can be reduced to a minimal such graph by a sequence of elementary operations. Corollary 4 resembles such results, since system (1.2) is trivially badly behaved.

We can define the well- or badly behaved nature of conic linear systems in a different form, and characterize such systems. For instance, we call the dual system

$$A^* y = c,\ y \in K^*, \tag{5.55}$$

well-behaved, if for all b dual objective functions the values of (D_c) and (P_c) agree, and the latter is attained, when this value is finite. System (5.55) can be recast in the primal form

$$Bx \leq_K y_0, \tag{5.56}$$

where B and y_0 satisfy $\mathcal{R}(B) = \mathcal{N}(A^*)$ and $A^* y_0 = c$. It is straightforward to show that (5.55) is well-behaved, if and only if (5.56) is, and to translate the conditions in Theorem 3 to characterize when (5.55) is well- or badly behaved. We leave the details to the reader. Thus, for equality constrained semidefinite systems we can obtain the following result:

Theorem 9. *Suppose that in the system*

$$Y \succeq 0,\ A_i \bullet Y = c_i \ (i = 1, \dots, m) \tag{5.57}$$

the maximum rank feasible matrix is of the form

$$\bar{Y} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Then (5.57) is badly behaved if and only if there is a matrix V and a real number λ such that

$$A_i \bullet V = \lambda c_i \ (i = 1, \dots, m),$$

and

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \tag{5.58}$$

where V_{11} is r by r, $V_{22} \succeq 0$, and $\mathcal{R}(V_{12}^T) \not\subseteq \mathcal{R}(V_{22})$. □

We can apply similar arguments to conic linear systems in a subspace form

$$K \cap (L + x_0),$$

to characterize their well- or badly behaved status.

A Appendix

In the appendix we present two further examples of badly behaved semidefinite systems. In Example 4 it is difficult to verify the bad behavior of the original system, whereas we can easily check the bad behavior of the standard form reformulation in the form of $(P_{\mathrm{SD,bad}})$.

The system in Example 5 is obtained by slightly modifying Tunçel's example on page 43 in [45]. It is already in the standard form of $(P_{\mathrm{SD,bad}})$. It is interesting, since here *all* matrices on the left hand side can serve as a V matrix; also, a small modification yields a well behaved system.

Example 4. Consider the semidefinite system

$$\begin{gathered} x_1 \begin{pmatrix} -3 & 3 & 2 & 20 & 15 \\ 3 & -7 & -2 & -4 & 6 \\ 2 & -2 & 2 & 9 & 7 \\ 20 & -4 & 9 & -2 & 5 \\ 15 & 6 & 7 & 5 & 2 \end{pmatrix} + x_2 \begin{pmatrix} -5 & 5 & 5 & 24 & 25 \\ 5 & -6 & -5 & -4 & 14 \\ 5 & -5 & 4 & 16 & 11 \\ 24 & -4 & 16 & -5 & 6 \\ 25 & 14 & 11 & 6 & 5 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 0 & 0 & 4 & 2 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 4 & -1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \end{pmatrix} \\ + x_4 \begin{pmatrix} -1 & 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 0 & 2 \\ 1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 2 & -1 & 0 \\ 3 & 2 & 1 & 0 & 1 \end{pmatrix} + x_5 \begin{pmatrix} -2 & 2 & 1 & 8 & 6 \\ 2 & -2 & -1 & -1 & 4 \\ 1 & -1 & 2 & 4 & 3 \\ 8 & -1 & 4 & -1 & 2 \\ 6 & 4 & 3 & 2 & 1 \end{pmatrix} \preceq \begin{pmatrix} -18 & 19 & 15 & 92 & 83 \\ 19 & -24 & -15 & -15 & 46 \\ 15 & -15 & 16 & 53 & 38 \\ 92 & -15 & 53 & -15 & 23 \\ 83 & 46 & 38 & 23 & 15 \end{pmatrix} \end{gathered} \tag{A.59}$$

This system is badly behaved, although from this form it would be difficult to tell.

To bring it into the form of $(P_{\text{SD,bad}})$, let us denote the constraint matrices on the left by A_i $(i = 1, \ldots, 5)$, and the right hand side matrix by B, and perform the following operations:

$$\begin{array}{rcl} B & := & B - A_1 - 2A_2 - 3A_5 \\ A_5 & := & A_5 - 2A_3 - A_4 \\ A_1 & := & A_1 - 5A_3 - 2A_4 - A_5 \\ A_2 & := & A_2 - 6A_3 - 5A_4 - 2A_5 \end{array}$$

We obtain the system

$$\begin{array}{l} x_1 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 0 & 0 & 4 & 2 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 4 & -1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \end{pmatrix} \\ + x_4 \begin{pmatrix} -1 & 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 0 & 2 \\ 1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 2 & -1 & 0 \\ 3 & 2 & 1 & 0 & 1 \end{pmatrix} + x_5 \begin{pmatrix} -1 & 2 & 0 & 0 & -1 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{array} \tag{A.60}$$

In both the original, and the transformed systems the matrices

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; Y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

prove that $Z = I_2 \oplus 0,$ i.e., the right hand side in (A.60) is the maximum rank slack.

Following the proof of the "If" implication in Theorem 6, it is trivial to see that the objective function $\sup -x_5$ yields a value of 0 over (A.60), but no dual solution attains the same value.

We can recover the system (A.59) from (A.60) by performing the operations

$$\begin{array}{rcl} B & := & B + A_1 + 2A_2 + 23A_3 + 15A_4 + 8A_5 \\ A_5 & := & A_5 + 2A_3 + A_4 \\ A_1 & := & A_1 + 3A_3 + A_4 \\ A_2 & := & A_2 + 2A_3 + 3A_4 \end{array}$$

on (A.60). Hence the proof of Lemma 4 implies that the optimal value of (A.59) with the same objective function is $-8,$ but no dual solution attains the same value.

Example 5. For a positive integer n let us define the n by n symmetric unit matrices $E_i = e_i e_i^T,$ and $E_{ij} = e_i e_j^T + e_j e_i^T,$ if $i \neq j.$

Let $n \geq 4,$ and define the system

$$\sum_{i=2}^{n-1} x_i A_i \preceq B, \tag{A.61}$$

where $A_i = E_i + E_{1,i+1}\,(i = 2,\ldots,n-1)$, and $B = E_{11}$. (We number the matrices from 2 to $n-1$ for convenience.)

This system is badly behaved. First note that the matrices

$$Y_n = E_n,\, Y_i = 2E_{n-i+1} - E_{1,n-i+2}(i = 2,\ldots,n-1)$$

prove that B is a maximum rank slack, and each of the A_i matrices can serve as a V matrix of Theorem 4. This system is in the form of $(P_{\mathrm{SD,bad}})$.

If we change the A_i to $A_i = E_i - E_{i+1} + E_{1,i+1}$, then the system (A.61) becomes well behaved. It is in the form of $(P_{\mathrm{SD,good}})$, with $Y_1 = 0 \oplus I_{n-1}$ proving that B is a maximum rank slack.

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