FORMULATIONS FOR DYNAMIC LOT SIZING WITH SERVICE LEVELS

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Abstract. In this paper, we study deterministic dynamic lot-sizing problems with service level constraints on the total number of periods in which backorders can occur over the finite planning horizon. We give a natural mixed integer programming formulation for the single item problem (LS-SL-I) and study the structure of its solution. We show that an optimal solution to this problem can be found in $\mathcal{O}(n^2\kappa)$ time, where n is the planning horizon and $\kappa = \mathcal{O}(n)$ is the maximum number of periods in which demand can be backordered. Using the proposed shortest path algorithms, we develop alternative tight extended formulations for LS-SL-I and one of its relaxations, which we refer to as uncapacitated lot sizing with setups for stocks and backlogs. We show that this relaxation also appears as a substructure in a lot-sizing problem which limits the total amount of a period's demand met from a later period, across all periods. We report computational results that compare the natural and extended formulations on multi-item service-level constrained instances.

Keywords: Fixed-charge networks, lot sizing, service levels, extended formulation, shortest paths

1. Introduction

Suppliers are often expected to sign service level contracts with their customers to limit orders that are not shipped on time. Late shipments can be extremely disruptive to downstream businesses and frequent stock-outs can result in significant loss of customer goodwill. Suppliers often find it difficult to estimate such intangible components of their backorder costs. Thus, constraints such as the Type-I service level, which limit the proportion of time demand is backlogged, are increasingly used in practice [10]. In this paper, we study a deterministic uncapacitated lot-sizing problem with such a service level constraint, which we refer to as lot sizing with Type-I service levels (LS-SL-I). The duration of the service contract (i.e., the planning horizon) is n. The problem is to determine the lowest cost replenishment plan that meets the demand over the horizon at a predetermined service level, τ, where τ is defined as the proportion of time the demand is met over the horizon *(ready rate)*. In other words, letting $\kappa = \lfloor (1-\tau)n \rfloor$, the service level constraint stipulates that backorders occur in no more than κ out of the n periods. In addition, we consider a commonly used alternative service measure referred to as the Type-II service, which ensures that the percentage of demand met on time *(fill rate)* is above a predetermined threshold, γ. In contrast to Type-I service, Type-II service levels limit the quantity, not the frequency, of stock-outs.

In a dynamic environment, especially with seasonality and fluctuating energy prices, demands and production costs are time-varying. In such environments, even if the dynamic demands and costs are known with certainty, it might be economical to backorder part of the demand in periods when the costs are deemed too high. Similarly, in multi-item *capacitated* problems, the demand of a less profitable item (or a lower priority customer) may be backordered so that the demand of another item or customer is fulfilled on time. Traditionally, in deterministic inventory management models, such situations are dealt exclusively by the

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introduction of shortage penalties [17, 18], which suffer from difficulties in their accurate estimation. While service level constraints have been studied in stochastic inventory control theory (c.f. [10, 1]), most of these models consider service levels within a replenishment cycle. In contrast, our models account for service levels for entire planning horizon. To the best of our knowledge, there is no earlier work on incorporating service level considerations to deterministic dynamic lot-sizing problems.

1.1. **Related Literature.**

Wagner and Whitin [15], in their seminal paper, identify the structure of solutions and propose a dynamic programming algorithm for the uncapacitated lot-sizing problem without backlogging (ULS). Their work sparked interest in the study of a variety of lot-sizing problems. We refer the reader to [12] for an overview of mixed-integer programming approaches to lot-sizing problems.

Zangwill [17, 18] provides algorithms for the uncapacitated lot-sizing problem *with* backlogging (ULSB). In particular, [17] considers the case where the number of consecutive backlogs is limited. This problem can be modeled and solved easily using the $\mathcal{O}(n^2)$ dynamic program for ULSB. In contrast, in LS-SL-I, the total number of backorders in the horizon (not necessarily consecutive) is limited. As a result, LS-SL-I ensures a higher service level than the lot-sizing problem described in [17].

Eppen and Martin [4] derive strong extended formulations based on the shortest path formulation of various lot-sizing problems. Krarup and Bilde [7] propose the so-called facility location reformulation for ULS that solves the problem as a linear program. Pochet and Wolsey [11] propose linear programming reformulations and strong valid inequalities for ULSB. However, these inequalities are insufficient to describe the convex hull of solutions to the ULSB. More recently, Küçükyavuz and Pochet [8] propose a class of valid inequalities that subsume the inequalities of [11] and give the convex hull of solutions to the ULSB. Finally, van Vyve and Ortega [14] study an uncapacitated lot-sizing problem with no backlogs and fixed charges for carrying stocks (ULSW). They propose alternative extended formulations and valid inequalities that give the convex hull in the natural space. Traditionally, extended formulations have not been used directly in the solution of mixed-integer programs due to their large size, however, Wolsey [16] and Miller and Wolsey [9] make the case for using extended formulations directly in solving some classes of multi-item lot-sizing problems.

1.2. **Outline.**

First, we consider models for Type-I service in Section 2. In Section 2.1, we give a polynomial-time algorithm for LS-SL-I, based on a shortest path representation of the problem, which yields a tight and compact extended formulation for LS-SL-I. In Section 2.2, we develop alternative extended formulations for a relaxation of LS-SL-I, which we refer to as uncapacitated lot sizing with stock and backorder fixed costs (ULSBW). In Section 2.3 we present the results of our computational experiments on multi-item, capacitated, Type-I service-level constrained instances that demonstrate the effectiveness of the proposed reformulation. In Section 3 we consider formulations for lot sizing with Type-II constraints. Finally, in Section 4 we discuss open problems and an application of this model in a stochastic demand setting.

2. Type-I Service Levels

Consider a multi-item lot-sizing problem with Type-I service level constraints (MI-LS-SL-I), where m represents the number of items. Let $d_{it} \geq 0$ denote the known demand for item $i = 1, \dots, m$ in period $t = 1, \dots, n$ and $d_{ikt} := \sum_{j=k}^{t} d_{ij}$ for $k \leq t$ ($d_{ikt} = 0$ otherwise). Let f_{it} and a_{it} denote the order and stock fixed charge of item i in period t, respectively. Let c_{it} and h_{it} denote the unit cost of ordering and holding item i in period t, respectively. Throughout the paper, we make the realistic assumption that all costs are non-negative. Let x_{it}, w_{it}, u_{it} be 1 if production, stock or backlog for item i takes place in period t, respectively and 0 otherwise; and y_{it}, s_{it}, r_{it} be the amount of item i produced, stocked and backlogged in period t, respectively. Throughout, we let $[k, j]$ denote the interval $\{k, k+1, \dots, j\}$ for $k \leq j$ ($[k, j] = \emptyset$ if $k > j$). Without loss of generality, we assume that the starting and ending stocks and backlogs are zero.

Then MI-LS-SL-I is given by:

$$\min \sum_{i=1}^{m}\sum_{t=1}^{n}\left(f_{it}x_{it} + a_{it}w_{it} + c_{it}y_{it} + h_{it}s_{it}\right) \quad (1)$$

$$s_{it-1} + y_{it} - r_{i,t-1} = d_{it} + s_{it} - r_{it}, \qquad i \in [1,m], t \in [1,n] \quad (2)$$

$$y_{it} \leq d_{i1n}x_{it}, \qquad i \in [1,m], t \in [1,n] \quad (3)$$

$$s_{it} \leq d_{i(t+1)n}w_t, \qquad i \in [1,m], t \in [1,n] \quad (4)$$

$$r_{it} \leq d_{i1t}u_t, \qquad i \in [1,m], t \in [1,n] \quad (5)$$

$$\sum_{t=1}^{n} u_{it} \leq \kappa_i \qquad i \in [1,m] \quad (6)$$

$$s_{i0} = r_{i0} = s_{in} = r_{in} = 0 \qquad i \in [1,m] \quad (7)$$

$$x_{it}, u_{it}, w_{it} \in \{0,1\}, \qquad i \in [1,m], t \in [1,n] \quad (8)$$

$$y_{it}, s_{it}, r_{it} \geq 0, \qquad i \in [1,m], t \in [1,n] \quad (9)$$

$$(x,y,w,s,u,r) \in \mathcal{X}, \quad (10)$$

where $\mathcal{X}$ represents additional restrictions, such as capacity constraints, that capture the interaction between the multiple items. The objective (1) is to minimize the total cost of production and stocks. Equations (2) are flow balance constraints. Constraints (3)–(5) are setup constraints for production, stocks and backlogs, respectively. Inequality (6) is the service level constraint for each item, which limits the number of stockouts over the horizon. Note that if there exist measurable components for shortage fixed costs b_t or unit backorder costs g_t, they can easily be incorporated into the objective. However, we ignore such costs in this paper.

Observe that without additional restrictions given by $\mathcal{X}$, the problem decomposes into single item lot-sizing problem with a service level constraint, denoted by LS-SL-I. In this paper, we propose alternative formulations for the LS-SL-I substructure given by (2)–(9) for $m = 1$. Therefore, in the rest of the discussion, we drop the item index i from all costs and variables. In Section 4 we also show that LS-SL-I has a natural extension to a stochastic lot-sizing problem with Type-I service level constraint.

To motivate our model with service levels and compare it with a model that assumes shortage penalties, we give a small single-item example.

Example 1. To keep the example simple, we let $n = 5$ and $\tau = 0.5$ (i.e., $\kappa = 2$). An example with a more realistic service level τ can be constructed for a higher value of n. The problem data is given in Table 1.

Include Table 1 about here

The company estimates their shortage costs as $g = (15, 15, 15, 14, 18)$. Note that at each time period the shortage cost is greater than the holding cost. Let the optimal solution of a model using the shortage penalties instead of service levels be denoted by ULSB (uncapacitated lot sizing with backlogging). Also let the optimal solution without the artificial shortage cost, but with service level constraints be denoted by LS-SL-I. Then the optimal solutions are given in Table 2.

Include Table 2 about here

Note that with ULSB, there are more time periods with a stock-out. In fact, determining the right penalties to achieve a service level, τ, is a difficult inverse mixed-integer program.

2.1. **Solution Algorithm.** Throughout, we let X^F denote the set of solutions to a formulation F. For example, $X^{LS-SL-I}$ denotes the set of solutions to the single-item problem defined by (2)–(9) for $m = 1$. Let $\mathcal{Q} = \text{clconv}(X^{LS-SL-I})$, where $\text{clconv}(X)$ denotes the closure of the convex hull of set X. Next we describe the structure of the extreme point solutions of $\mathcal{Q}$ and show that they are a concatenation of *regeneration intervals* (c.f. [18]) of the form $\{i, j, k\}$ with $i \leq j \leq k$, where the incoming stock and outgoing backorder in period i and the outgoing stock and incoming backorder in period k are zero. The production in period j

satisfies its own demand and the demand backlogged in periods i through $j-1$, and the demand in periods $j+1$ through k from stocks. Let $N = [1, n]$.

Proposition 1. *The extreme points of $\mathcal{Q}$ are of the following form: For*

$$I = \{1 =: i_1, j_1, k_1, i_2, j_2, k_2, \ldots, i_l, j_l, k_l := n\} \subseteq N$$
$$U = [i_1, j_1 - 1] \cup [i_2, j_2 - 1] \cup \cdots \cup [i_l, j_l - 1]$$
$$W = [j_1, k_1 - 1] \cup [j_2, k_2 - 1] \cup \cdots \cup [j_l, k_l - 1]$$

where $k_0 := 0 < i_1 \leq j_1 \leq k_1 < i_2 \leq j_2 \leq k_2 < \cdots < i_l \leq j_l \leq k_l$ *and* $i_p = k_{p-1}+1$, $y_{j_p} = d_{i_p k_p}, \forall p = 1, \ldots, l$, $y_j = 0, \forall j \in N \backslash \{j_1, j_2, \ldots, j_l\}$; $x_j = 1, \forall j \in \{j_1, j_2, \ldots, j_l\} \cup I_1$, *where* $I_1 \subseteq N \backslash \{j_1, j_2, \ldots, j_l\}$ *and* $x_j = 0$ *for all other* j, $s_j r_j = 0$ *for* $j \in N$; $u_j = 1, \forall j \in U \cup I_2$, *where* $I_2 \subseteq N \backslash U$ *and* $u_j = 0$ *for all other* j; $w_j = 1, \forall j \in W \cup I_3$, *where* $I_3 \subseteq N \backslash W, w_j = 0$ *for all other* j; *and* $|I_2 \cup U| \leq \kappa$.

Proof. Note that Proposition 1 is equivalent to stating that in an extreme point solution $s_{t-1}y_t = s_{t-1}r_t = s_t r_t = y_t r_t = 0$ for all $t \in [1, n]$. The proof follows similarly to that of [18]. We will only show that $y_t r_t = 0$ for all $t \in [1, n]$. For contradiction, suppose that $\mathbf{x} = (x, y, w, s, u, r)$ is an extreme point of $\mathcal{Q}$ for which $y_t r_t > 0$ for some $t \in [1, n]$. Let $j > t$ be the first period after t with $y_j > 0$ (j exists as $r_t > 0$). Consider the feasible solution $\mathbf{x}^1 = (x, y^1, w, s, u, r^1)$ obtained by letting $y_t^1 = y_t + \varepsilon, y_j^1 = y_j - \varepsilon$ and $r_i^1 = r_i - \varepsilon$ for $i \in [t, j-1]$; and $\mathbf{x}^2 = (x, y^2, w, s, u, r^2)$ obtained by letting $y_t^2 = y_t - \varepsilon, y_j^2 = y_j + \varepsilon$ and $r_i^2 = r_i + \varepsilon$ for $i \in [t, j-1]$, for an infinitesimally small $\varepsilon > 0$, where all other components of $\mathbf{x}^1$ and $\mathbf{x}^2$ are the same as those of $\mathbf{x}$. Note that $\mathbf{x}^1$ and $\mathbf{x}^2$ are both feasible, and $\mathbf{x} = \frac{1}{2}\mathbf{x}^1 + \frac{1}{2}\mathbf{x}^2$, which contradicts the assumption that $\mathbf{x}$ is an extreme point. The proof of the statement that in all extreme points of $\mathcal{Q}$, we have $s_{t-1}y_t = s_{t-1}r_t = 0$ for all $t \in [1, n]$ is similar. □

Using the structure of optimal solutions, we develop a polynomial-time algorithm to solve LS-SL-I. To ease the exposition, we assume that $d_t > 0$ for all $t \in [1, n]$ throughout. Our results hold without this assumption, with minor modifications.

Proposition 2. *LS-SL-I can be solved in $\mathcal{O}(n^2\kappa)$ time.*

Proof. Note that, by assumption, $g_t = 0, h_t \geq 0$ for all $t \in [1, n]$. Therefore, there exists a bounded optimal solution. We give a shortest path formulation of the problem as depicted in Figure 1 for $n = 3, \kappa = 2$. Let $(\alpha)^+ := \max\{0, \alpha\}$. In this shortest path network, there are three types of nodes for each time period i and for each value of remaining backorders allowed j: (i, j); (i', j); and (i'', j), for $i \in [1, n], j \in [(\kappa - i + 1)^+, \kappa]$. The source node is $(1, \kappa)$, representing that κ backorder periods are allowed starting from time period 1. In addition, there is a dummy (sink) node $n + 1$, which is a conglomeration of nodes $(n + 1, j)$ for all $j \in [0, \kappa]$. There exists a *backlogging* arc from (i, j) to $(k', j - k + i)$ for all $k \in [i, n]$, and $i \in [1, n], j \in [\max\{k - i, (\kappa - i + 1)^+\}, \kappa]$, which represents producing in period k to satisfy demands in periods $[i, k - 1]$. The cost on this arc is $p(i, j, k', j-k+i) = c_k d_{i(k-1)}$. There exists a *production* arc from (i', j) to (i'', j) for all $i \in [1, n], j \in [(\kappa - i+1)^+, \kappa]$ with cost $p(i', j, i'', j) = f_i + c_i d_i$. A path visiting this arc represents production in period i. Finally, there exists an *inventory* arc from (i'', j) to (k, j) for $i \in [1, n-1], j \in [(\kappa - i + 1)^+, \kappa]$ and $k \in [i + 1, n + 1]$. Such an arc represents producing in period i to satisfy demands in periods $[i, k - 1]$. The cost on this arc is $p(i'', j, k, j) = c_i d_{(i+1)(k-1)} + \sum_{t=i}^{k-2} a_t + \sum_{t=i}^{k-2} h_t d_{(t+1)(k-1)}$ for $k > i + 1$. Finally, for $i \in [1, n], j \in [(\kappa - i + 1)^+, \kappa]$, the cost is $p(i'', j, i + 1, j) = 0$. For example, in Figure 1, the path $(1, 2) \to (2', 1) \to (2'', 1) \to (4, 1) =: 4$ represents a regeneration interval $\{1, 2, 3\}$, where production occurs in period 2 to satisfy demands in periods $[1, 3]$.

Include Figure 1 about here

There are $\mathcal{O}(n\kappa)$ nodes in this network, where κ is $\mathcal{O}(n)$. There are $\mathcal{O}(n)$ outgoing arcs per node. As the resulting shortest path network is a directed acyclic graph, an optimal solution can be found in $\mathcal{O}(n^2\kappa)$ time.

Finally, note that with appropriate preprocessing, all arc costs can be calculated in $\mathcal{O}(n^2\kappa)$ time in total. In the preprocessing stage, with one forward pass, we calculate d_{1j}, a_{1j}, h_{1j} for all $j \in [1,n]$, where $\alpha_{ij} := \sum_{t=i}^{j} \alpha_t$ for $i \leq j$ ($\alpha_{ij} = 0$ if $i > j$). As a result, any $\alpha_{ij} = \alpha_{1j} - \alpha_{1(i-1)}$ can be retrieved in unit time. Then, the terms $h'(i,k) := \sum_{t=i}^{k-1} h_t d_{(t+1)k}$ can be calculated for all $k > i$ in $\mathcal{O}(n^2)$ time. To see this, note that for a given i, starting from $k = i+1$, we have $h'(i,k) = h_i d_k$ and $h'(i,k+1) = h'(i,k) + h_{ik} d_{k+1}$ for $k > i+1$. □

Note that the assumption that $d_t > 0$ for all $t \in [1,n]$ can be relaxed with appropriate modifications. Suppose that there exists $t^* \in [i,k]$, where $k \in [i,n]$, and $i \in [1,n]$, such that $d_t = 0$ for all $t \in [i,t^*]$. Then the cost of the backlogging arc from (i,j) to $(k', j-k+i)$, for $j \in [\max\{k-i, (\kappa-i+1)^+\}, \kappa]$, is updated as $p(i,j,k',j-k+i) = c_k d_{i(k-1)} + \sum_{t=i}^{k-1} (a_t)^-$. A similar update should be made for the costs of the inventory arcs if there exists zero demand in some periods. Next, we give an $\mathcal{O}(n^2\kappa)$ extended formulation for the case with positive demands.

Let $\psi_{ijk} = 1$, for $i \in [1,n], j \in [0,\kappa]$ and $k \in [i,n]$ such that $j-k+i \geq 0$, if the shortest path visits the arc from (i,j) to $(k', j-k+i)$, and 0 otherwise. Let $z_{ij} = 1$ if the shortest path visits the arc from (i',j) to (i'',j) for all $i \in [1,n], j \in [0,\kappa]$, and 0 otherwise. Finally, let $\rho_{ijk} = 1$ if the shortest path visits the arc from (i'',j) to (k,j) for $i \in [1,n], j \in [0,\kappa], k \in [i+1,n+1]$. Consider the feasible set, X^{SP-SL}, given by the following extended formulation (SP-SL):

$$\min \sum_{t=1}^{n} (f_t x_t + a_t w_t + c_t y_t + h_t s_t)$$

$$\sum_{j=1}^{\kappa+1} \psi_{1\kappa j} = 1 \tag{11}$$

$$\sum_{k=1}^{i-1} \rho_{kji} - \sum_{k=i}^{\min(j+i,n)} \psi_{ijk} = 0, \qquad i \in [2,n], j \in [0,\kappa] \tag{12}$$

$$- \sum_{k \in [1,i]: j-k+i \leq \kappa} \psi_{k(j-k+i)i} + z_{ij} = 0, \qquad i \in [1,n], j \in [0,\kappa] \tag{13}$$

$$- z_{ij} + \sum_{k=i+1}^{n+1} \rho_{ijk} = 0, \qquad i \in [1,n], j \in [0,\kappa] \tag{14}$$

$$\sum_{i=1}^{n} \sum_{j=0}^{\kappa} \rho_{ij(n+1)} = 1 \tag{15}$$

$$y_t = \sum_{j=0}^{\kappa} d_t z_{tj} + \sum_{j=0}^{\kappa} \sum_{k=t+2}^{n+1} d_{t+1,k-1} \rho_{tjk} + \sum_{i=1}^{t-1} \sum_{j=t-i}^{\kappa} d_{i,t-1} \psi_{ijt}, \qquad t \in [1,n] \tag{16}$$

$$s_t = \sum_{i=1}^{t} \sum_{j=0}^{\kappa} \sum_{k=t+2}^{n+1} d_{t+1,k-1} \rho_{ijk}, \qquad t \in [1,n-1] \tag{17}$$

$$r_t = \sum_{k=t+1}^{n} \sum_{i=1}^{t} \sum_{j=k-i}^{\kappa} d_{i,t} \psi_{ijk}, \qquad t \in [1,n-1] \tag{18}$$

$$x_t \geq \sum_{j=0}^{\kappa} z_{tj}, \qquad t \in [1,n] \tag{19}$$

$$w_t \geq \sum_{i=1}^{t} \sum_{j=0}^{\kappa} \sum_{k=t+2}^{n+1} \rho_{ijk}, \qquad t \in [1, n-1] \tag{20}$$

$$u_t \geq \sum_{i=1}^{t} \sum_{k=t+1}^{n} \sum_{j=k-i}^{\kappa} \psi_{ijk}, \qquad t \in [1, n-1] \tag{21}$$

$$\rho_{ijk} = 0, \qquad i \in [1, \kappa], j \in [0, \kappa - i], k \in [i+1, n+1] \tag{22}$$

$$z_{ij} = 0, \qquad i \in [1, \kappa], j \in [0, \kappa - i] \tag{23}$$

$$\psi_{1jk} = 0, \qquad k \in [2, n], j \in [0, \kappa - 1] : k - j \leq 1 \tag{24}$$

$$s_n = r_n = 0$$

$$y, r, s \geq \mathbf{0},$$

$$\mathbf{0} \leq x, w, u, \rho, \psi, z \leq \mathbf{1}.$$

Theorem 3. *The extended formulation SP-SL provides a solution to LS-SL-I, when $d_t > 0$ for all $t \in [1, n]$.*

Proof. The proof follows from [4]. The extended formulation includes the flow balance equations (11)–(15) for the shortest path problem in Proposition 2 for nodes $(1, \kappa), (i, j), (i', j), (i'', j)$ and $(n+1, j)$, respectively. So they give integral (ψ, ρ, z). Constraints (16)–(21) define the relationship between the original variables and the new variables. For ease of exposition, we introduce variables for some arcs that do not exist, and set their values to zero in equations (22)–(24). The inventory setup variable w_t, when zero, blocks any flow on arcs corresponding to the variables $\rho_{ijk}, i \in [1, t], j \in [0, \kappa], k \in [t+2, n+1]$ and therefore for $a_t > 0$, (20) holds at equality. In addition, because we assume that $b_t = 0$, we may have $u_t = 1$ even if the right-hand-side of (21) is zero, but this does not affect the objective function. If this is the case, we postprocess the solution of SP-SL and let such $u_t = 0$, so that constraint (6) is not violated. As a result, (x, u, w) is also integral. □

2.2. **A Relaxation: Uncapacitated Lot Sizing with Setups for Stocks and Backorders.** We now study the problem obtained by relaxing the service level constraint (6), which we refer to as uncapacitated lot sizing with setups for stocks and backorders (ULSBW). Although SP-SL is a tight formulation for LS-SL, in our computational study (see Section 2.3), the SP-SL formulation turned out to be too large for some of the larger capacitated multi-item instances that we generated. One of our motivations to study ULSBW is to develop an extended formulation that has a smaller representation than SP-SL, and use this strong extended formulation as a relaxation to solve difficult capacitated multi-item lot sizing problem with service levels. ULSBW is an interesting problem in its own right, as it generalizes the uncapacitated lot sizing problem with backlogging studied in [11] and the uncapacitated lot sizing problem with stock fixed costs studied in [14]. Moreover, ULSBW also appears as a substructure in the Type-II service level model discussed in Section 3.

Let X^{ULSBW} denote the set of solutions obtained by relaxing (6) from the constraints (2)–(9). Also let $\mathcal{S} = \text{clconv}(X^{ULSBW})$. The extreme point structure of $\mathcal{S}$ is given in Proposition 1 with $\kappa = n$. We observe that ULSBW has a familiar single source fixed charge network flow structure. Figure 2 illustrates the network for a 5-period instance. In the figure, node 0 represents the source node and nodes 1 through 5 represent the demand nodes. There are three categories of arcs: production, stock and backlog, with the corresponding variables for period 1 indicated in Figure 2.

Include Figure 2 about here

The shortest path network for ULSBW, depicted in Figure 3, is a simplification of the LS-SL-I shortest path network (Figure 1) in which $\kappa = n$, so we do not keep track of the number of periods in which backorders take place. In addition, this shortest path network is similar to that of [11] for the uncapacitated lot sizing problem with backlogging (ULSB) *without* setups for stocks or backorders. However, the arc costs are different due to presence of setups on stocks and backorders.

We define the following variables for the shortest path formulation: ψ_{tl} is 1 if production in period t satisfies demand up to period l, $l \leq t$ and zero otherwise. ρ_{tl} is 1 if production in period t satisfies demand up to period $l \geq t$ and zero otherwise and z_{tt} is 1 if production in period t satisfies demand in t and zero otherwise.

The shortest path extended formulation (SP) for ULSBW can be written as follows:

$$\sum_{l=1}^{n} \psi_{l1} = 1 \tag{25}$$

$$\sum_{l=1}^{t-1} \rho_{l,t-1} - \sum_{l=t}^{n} \psi_{lt} = 0, \qquad t \in [2, n] \tag{26}$$

$$-\sum_{l=1}^{t} \psi_{tl} + z_{tt} = 0, \qquad t \in [1, n] \tag{27}$$

$$-z_{tt} + \sum_{l=t}^{n} \rho_{tl} = 0, \qquad t \in [1, n] \tag{28}$$

$$y_t = \sum_{l=t+1}^{n} d_{t+1,l}\rho_{tl} + \sum_{l=1}^{t-1} d_{l,t-1}\psi_{tl} + d_t z_{tt}, \qquad t \in [1, n]$$

$$s_t = \sum_{j=1}^{t} \sum_{l=t+1}^{n} d_{t+1,l}\rho_{jl}, \qquad t \in [1, n-1]$$

$$r_t = \sum_{j=t+1}^{n} \sum_{l=1}^{t} d_{lt}\psi_{jl}, \qquad t \in [1, n-1]$$

$$x_t \geq z_{tt}, \qquad t \in [1, n]$$

$$w_t \geq \sum_{j=1}^{t} \sum_{l=t+1}^{n} \rho_{jl}, \qquad t \in [1, n-1] \tag{29}$$

$$u_t \geq \sum_{j=1}^{t} \sum_{l=t+1}^{n} \psi_{lj}, \qquad t \in [1, n-1] \tag{30}$$

$$s_n = r_n = 0$$

$$y, r, s \geq \mathbf{0},$$

$$\mathbf{0} \leq z, x, w, u, \rho, \psi \leq \mathbf{1},$$

Include Figure 3 about here

Note that this formulation follows from the shortest path formulation of ULSB [11] with the additional inequalities (29)–(30). Also note that since we are interested in the feasible set X^{ULSBW}, we drop the objective function from the formulations for ULSBW. Constraints (25)–(28) are flow balance equations for the shortest path network for ULSBW. Let $\mathbf{x} = (x, y, w, s, u, r)$ and $\text{proj}_v(Y)$ denote the projection of the set Y on to the space of the v-variables. Following similar arguments to Theorem 3, we have the following result.

Proposition 4. $\text{proj}_{\mathbf{x}}(X^{SP}) = \mathcal{S}$.

Next, we let $z_{kt} = \sum_{l=t}^{n} \rho_{kl}, k \leq t$ and $z_{kt} = \sum_{l=1}^{t} \psi_{kl}, k \geq t$ as in [11]. Note that this definition implies that $\rho_{kt} = z_{kt} - z_{k,t+1}$ for $k \leq t$ and $\psi_{kt} = z_{kt} - z_{k,t-1}$ for $k \geq t$. Substituting for ρ_{kt} and ψ_{kt} in equation

(26) in SP, we obtain

$$\sum_{k=1}^{n}(z_{kt} - z_{k,t-1}) = 0, t \in [2, n]. \tag{31}$$

We also note that $\psi_{l1} = z_{l1}, l \in [1, n]$ and equation (25) in SP can be written as

$$\sum_{l=1}^{n} z_{l1} = 1 \tag{32}$$

Combining equations (31) and (32) we can eliminate ψ_{kt} and ρ_{kt} from SP to obtain the following formulation, which we denote by SPR:

$$\sum_{k=1}^{n} z_{kj} = 1, \qquad j \in [1, n] \tag{33}$$

$$y_i = \sum_{k=1}^{n} d_k z_{ik}, \qquad i \in [1, n] \tag{34}$$

$$s_t = \sum_{k=1}^{t} \sum_{j=t+1}^{n} d_j z_{kj}, \qquad t \in [1, n-1] \tag{35}$$

$$r_t = \sum_{k=t+1}^{n} \sum_{j=1}^{t} d_j z_{kj}, \qquad t \in [1, n-1] \tag{36}$$

$$x_t \geq z_{tt}, \qquad t \in [1, n] \tag{37}$$

$$w_t \geq \sum_{j=1}^{t} z_{j,t+1}, \qquad t \in [1, n-1] \tag{38}$$

$$u_t \geq \sum_{l=t+1}^{n} z_{lt}, \qquad t \in [1, n-1] \tag{39}$$

$$z_{t1} \leq z_{t2} \leq \cdots \leq z_{t,t-1} \leq z_{tt}, \qquad t \in [1, n] \tag{40}$$

$$z_{tt} \geq z_{t,t+1} \geq \cdots \geq z_{tn}, \qquad t \in [1, n] \tag{41}$$

$$s_n = r_n = 0, \tag{42}$$

$$\mathbf{0} \leq z, x, w, u \leq \mathbf{1}, \tag{43}$$

$$y, r, s \geq \mathbf{0}, \tag{44}$$

Here, the variables z_{kt} determine the fraction of demand in period t satisfied from period k. It immediately follows that, SPR, obtained through a change of variables, is also a tight extended formulation for ULSBW.

Proposition 5. $\text{proj}_{\mathbf{x}}(X^{SPR}) = \mathcal{S}$.

Now that we established that an optimal solution satisfies inequalities (40)–(41), the fact that SPR is a tight extended formulation for ULSBW also follows from [2].

Consider the following constraints,

$$x_i \geq z_{ij}, \qquad i \in [1, n], j \in [1, n] \tag{45}$$

$$w_t \geq \sum_{k=1}^{t} z_{kj}, \qquad j \in [1, n], t \in [1, j-1] \tag{46}$$

$$u_t \geq \sum_{k=t+1}^{n} z_{kj} \qquad j \in [1, n], t \in [j, n]. \tag{47}$$

It is easy to verify that if the constraints (37)–(41) are dropped from the SPR formulation and the constraints (45)–(47) are introduced, we obtain a relaxation of SPR. We call this relaxation the facility location (FL) formulation, because when $w = \mathbf{1}$ and $u = \mathbf{0}$, constraint (46) is redundant and backlogging does not occur. Therefore, we obtain the facility location formulation of Krarup and Bilde [7] proposed for the basic uncapacitated lot-sizing problem (ULS) without backlogging and without fixed charges on stocks. Krarup and Bilde [7] give sufficient conditions for their facility location formulation to be tight for ULS, however, their proof does not extend in the presence of $(w, u) \in \{\mathbf{0}, \mathbf{1}\}^{2n}$ and constraints (46)–(47).

Let $X^{FL} = \{(\mathbf{x}, z) : (33) - (36), (42) - (47)\}$. Also consider the relaxations $X^{FL^{\leq}} = \{(\mathbf{x}, z) : (33)\text{-}- (36), (40), (42) - (47)\}$ and $X^{FL^{\geq}} = \{(\mathbf{x}, z) : (33) - (36), (41) - (47)\}$. Clearly, $X^{SPR} \subseteq X^{FL^{\geq}} \subseteq X^{FL}$, and $X^{SPR} \subseteq X^{FL^{\leq}} \subseteq X^{FL}$. Our next result establishes a relationship between the relaxations $X^{FL}, X^{FL^{\geq}}$ and $X^{FL^{\leq}}$.

Proposition 6. $\text{proj}_{\mathbf{x}}(X^{FL}) = \text{proj}_{\mathbf{x}}(X^{FL^{\geq}}) = \text{proj}_{\mathbf{x}}(X^{FL^{\leq}})$.

Proof. The proof is provided in Appendix A. □

With a change of variables, the FL formulation is equivalent to the so-called multi-commodity reformulation. Whether the facility location formulation is tight for ULSBW is an open problem. If FL and thus the corresponding multi-commodity formulations are tight for ULSBW, then from the result of Rardin and Wolsey [13] it follows that multi-dicut inequalities obtained from the projection of the multi-commodity formulation are enough to solve the problem as a linear program, when all costs are nonnegative.

We end this section by noting that adding the cardinality constraint (6) to SP or SPR may give fractional optimal solutions.

2.3. **Computations.** In this section, we test the effectiveness of using the SPR extended formulation of ULSBW in solving multi-item lot-sizing instances with service levels (MI-LS-SL-I) given by (2)–(10). In particular, we let $\mathcal{X}$ be defined by the constraints

$$\sum_{i=1}^{m} x_{it} \leq 1 \qquad\qquad t \in [1, n] \tag{48}$$

where setup for only one item is allowed in a period.

In our experimental setup, we consider alternative time horizons $n \in \{60, 120\}$ and number of items $m \in \{3, 5\}$. The demands are generated from a discrete uniform distribution on the interval [10, 300]. We consider three service level parameters, $\pi \in \{10, 25, 40\}$ percent of the planning horizon, so $\pi = (1 - \tau)100$. Recall that $\kappa = \lfloor (1 - \tau)n \rfloor$. Unit inventory and production costs are generated using a discrete uniform distribution on the intervals [1, 5] and [1, 10], respectively. We let β denote the ratio of fixed and variable production costs and consider four different values, i.e., $\beta \in \{500, 1000, 2500, 5000\}$ and $f_t = \beta c_t$ for $t \in [1, n]$. Also, we let $a_t = 0$ for $t \in [1, n]$. Note that in models with service level restrictions, backlogging costs are often assumed to be immeasurable and are set to zero [10]. In contrast, backlogging models without service levels assume $g_t > h_t$, so that g_t is a "large enough" penalty including the intangible loss-of-goodwill cost to ensure an acceptable level of service.

For each combination of n, m, β, π we generate five instances and report averages. The problem instances are available at `http://ise.osu.edu/ISEFaculty/kucukyavuz/data/dls-sl_instances.zip`. One of our instances, called `lotsize`, also appears in the challenge category of MIPLIB 2010 [6]. We conduct all the experiments on a 2.66 GHz Intel Q9450 Core 2 Quad CPU with 4GB RAM. We use IBM ILOG CPLEX 12.2 as the MIP solver and impose a one hour time limit. We turn off CPLEX's parallel mode and use a single thread for the MIP solves. The remaining CPLEX parameters are set to their default values. We note that the default MIP GAP tolerance used by CPLEX is 10^{-4}, so the instances that reach an integral solution with an end gap of 0.01% are considered optimal.

In Tables 3 and 4 (for $n = 60$ and 120, respectively), we compare the strength of the natural formulation MI-LS-SL-I, and that of the SPR extended formulation for each item. In columns **S Gap**, we report the average integrality gap, which is $100 \times (\texttt{zub} - \texttt{zinit})/\texttt{zub}$, where zinit is the objective value of the initial LP relaxation and zub is the objective value of the best integer solution. In columns **E Gap**, we report the end gap at termination, which is $100 \times (\texttt{zub} - \texttt{zbest})/\texttt{zub}$, where zbest is the best lower bound available. Columns **T (sec)** and **Nodes** compare the solution time (in seconds), and the average number of branch-and-cut tree nodes explored, respectively.

Tables 3–4 clearly illustrate that the natural formulation of the test problems is very weak with over 70% initial gap for most instances. As a result, a majority of the instances cannot be solved within an hour and thousands of branch-and-cut nodes are explored. The end gaps at termination are quite high (1-18%). In contrast, using the extended formulations for each item, the initial integrality gap is less than 1% for all instances. As a result, all instances are solved to optimality within a few seconds and fewer than 250 branch-and-cut nodes. Comparing Tables 3 and 4 we observe that the extended formulation scales well when the problem size is increased (both with respect to the planning horizon n and the number of items m). The problem difficulty increases for the natural formulation with higher cost ratios, β, and higher service levels, τ (lower π).

We also tested the SP and FL formulations on our test instances. In all instances tested, the initial LP gaps were identical to those with the SPR formulation, and the solutions times and the branch-and-cut nodes explored are comparable. Because SP and SPR are equivalent formulations their similar performance is expected. That SPR and FL gave the same initial LP bounds provides empirical evidence to our conjecture that their projections onto the original space of variables are equivalent. We also tested the larger dynamic programming-based formulation (SP-SL) given in Section 2.1. This formulation gives tighter initial LP bounds, however it takes significantly longer to solve the instances with $n = 60$. For instances with $n = 120$, CPLEX encountered memory problems due to the size of the formulation. Therefore, we do not report our tests with the SP, FL and SP-SL formulations.

Include Table 3 about here

Include Table 4 about here

3. Type-II Service Levels

Next, we consider an alternative service measure, Type-II service or fill rate, that ensures that the percentage of the demand quantity met on time is above a given threshold γ. Throughout this section, we present the models for a single product. Let q_t denote the amount of period t's demand met from later periods. In our setting, we model Type-II service as $\sum_{t=1}^{n} q_t \leq (1-\gamma)d_{1n}$. (Note that the constraint $\sum_{t=1}^{n} r_t \leq (1-\gamma)d_{1n}$ limiting the total backorder quantity is not correct, because a demand backlogged for more than one period is multiply counted.) In order to correctly account for the portion of demand that is not met on time, we use the variables z_{kt} in the FL formulation in Section 2.2, which denote the fraction of demand in period t

satisfied from period k. Then, the single-item lot sizing with Type-II service (LS-SL-II) model is

$$\min \sum_{t=1}^{n} (f_t x_t + a_t w_t + c_t y_t + h_t s_t)$$

$$\sum_{k=1}^{n} z_{kj} = 1, \qquad j \in [1, n] \tag{49}$$

$$y_i = \sum_{k=1}^{n} d_k z_{ik}, \qquad i \in [1, n] \tag{50}$$

$$s_t = \sum_{k=1}^{t} \sum_{j=t+1}^{n} d_j z_{kj}, \qquad t \in [1, n-1] \tag{51}$$

$$r_t = \sum_{k=t+1}^{n} \sum_{j=1}^{t} d_j z_{kj}, \qquad t \in [1, n-1] \tag{52}$$

$$y_t \leq d_{1n} x_t, \qquad t \in [1, n] \tag{53}$$

$$s_t \leq d_{t+1,n} w_t \qquad t \in [1, n-1] \tag{54}$$

$$r_t \leq d_{1,t} u_t, \qquad t \in [1, n-1] \tag{55}$$

$$s_n = r_n = 0, \tag{56}$$

$$\sum_{t=1}^{n} \sum_{j=t+1}^{n} d_t z_{jt} \leq (1-\gamma) d_{1n} \tag{57}$$

$$w_t + u_t \leq 1 \qquad t \in [1, n-1] \tag{58}$$

$$\mathbf{0} \leq z, x, w, u \leq \mathbf{1}, \tag{59}$$

$$y, r, s \geq \mathbf{0}. \tag{60}$$

The service constraint is given by (57). It might appear that the variables u and the constraints (55) and (58) are not necessary in this formulation. We illustrate that both are necessary on Example 1.

Example 1 (cont). Rows in LS-SL-II-R in Table 5 gives the optimal solution (y, s, r) to LS-SL-II without the constraints (55) and (58). In this solution, $y_2, s_2, r_2 > 0$, which is not practically justifiable. Therefore, constraints (55) and (58) are needed to ensure that both stock and backlog does not occur in a given time period. The solution to LS-SL-II is also given in Table 5. From this solution, we observe that the extreme point optimal solution does not satisfy the properties of the regeneration intervals in Proposition 1, because $s_2 r_3 > 0$.

Include Table 5 about here

Note that when constraints (57) and (58) are relaxed, we obtain the ULSBW model and thus the constraints (45)-(47) from the facility location formulation of ULSBW are valid for LS-SL-II. Furthermore, it is easily seen that with (45)-(47) added, constraints (53)-(55) can be dropped from the LS-SL-II formulation. We refer to the resulting formulation as FL-II. Table 6 summarizes the results of our computational experiments with the comparison of LS-SL-II and FL-II on multi item instances drawn from instances generated for MI-LS-I for $n = 120$. We report our results for one instance for each setting of β, m, γ. From these results, we conclude that the instances of the natural formulation for Type-II service levels for multi-item problems are relatively easier than the instances of natural formulation for Type-I service level for CPLEX. We also observe that, similar to Type-I, the facility location formulation provides significantly tighter bounds and faster solution times for Type-II problems. We note that the instance 120.2500.5.25 could not be solved within an hour. For this instance, we indicate the end gap within the parenthesis next to the solution time.

Include Table 6 about here

4. Conclusions

In this paper, we introduce service levels into classical deterministic dynamic lot-sizing problems. In contrast to stochastic inventory theory, we consider a service level across the the planning or contract horizon. We establish polynomial solvability of the Type-I model and develop a tight extended formulation based on its shortest path representation. We study a related model, the uncapacitated lot-sizing problem with setups for stocks and backorders (ULSBW), which is a relaxation of the Type-I service level model. We develop alternate extended formulations for ULSBW and conclude from our computational study that these formulations are highly effective in solving the multi-item lot-sizing problems with service levels. Moreover, we show that the formulations for ULSBW are also useful for solving dynamic lot sizing problems with Type-II service level constraints.

Our study has also revealed a few interesting open problems. The convex hull description of ULSBW and LS-SL-I in the original space of variables is an open research question. The explicit description of inequalities defined by the projection of our extended formulations is one way to address this question. A related problem is the stochastic lot-sizing problem with service levels (SLS-SL-I), where demand is uncertain, and can be represented by a finite number of scenarios $\omega \in \Omega$, with probabilities p_ω. We consider a type-1 service level constraint in which the goal is to ensure that probability that the entire period demand is met from stock is at least $1-\alpha$, where $\alpha \in [0,1]$. In the deterministic equivalent of SLS-SL-I, we have the same set of variables for each scenario. For example, u_t^ω represents the binary setup variable for backlog in period t and scenario $\omega \in \Omega$. Then in addition to the flow balance inequalities and the variable upper bound constraints, the Type-I constraint is given by

$$\frac{\sum_{\omega \in \Omega} p_\omega \sum_{t=1}^n u_t^\omega}{n} \leq \alpha.$$

Furthermore, the deterministic equivalent of SLS-LS contains the so-called non-anticipativity constraints [3]. Therefore, the stochastic lot-sizing problem contains the LS-SL-I substructure and the results presented in this paper are potentially useful in the stochastic setting as well.

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Appendix A. Proof of Proposition 6

We show that the projection of a strengthened facility location reformulation, given by $X^{FL^{\geq}}$, onto the natural space, $\mathbf{x}$, is equivalent to that of the FL formulation. By a symmetric argument, we can also show that the projection of a related FL formulation, given by $X^{FL^{\leq}}$, onto $\mathbf{x}$ is equivalent to that of the FL formulation.

Proposition 6. $\text{proj}_{\mathbf{x}}(X^{FL}) = \text{proj}_{\mathbf{x}}(X^{FL^{\geq}})$.

Proof. Since X^{FL} is a relaxation of $X^{FL^{\geq}}$ we have $\text{proj}_{\mathbf{x}}(X^{FL^{\geq}}) \subseteq \text{proj}_{\mathbf{x}}(X^{FL})$. Let any $\mathbf{x} \in \text{proj}_{\mathbf{x}}(X^{FL})$ be given. Let $\mathbf{z} = \{z_{ij}, i \in [1,n], j \in [1,n]\}$. We show that $\text{proj}_{\mathbf{x}}(X^{FL^{\geq}}) \supseteq \text{proj}_{\mathbf{x}}(X^{FL})$ by showing that there exists a $\mathbf{z}$ such that $(\mathbf{x}, \mathbf{z}) \in X^{FL}$ and $\mathbf{z}$ satisfies (41).

Suppose that this claim is not true, i.e., suppose that for every $(\mathbf{x}, \mathbf{z}) \in X^{FL}$, there exist $l, k, l \leq k$ such that $z_{lk} < z_{l,k+1}$. Let l_z, k_z be the first index for which, $z_{l_z k_z} < z_{l_z, k_z+1}$. Also let v_z be the minimum $v \in [l_z, k_z]$ that satisfies $z_{l_z v} = z_{l_z, v+1} = \cdots = z_{l_z, k_z}$. Let $\gamma_z = z_{l_z, k_z+1} - z_{l_z k_z} > 0$ and if $v_z > l_z$, then let $\xi_z = z_{l_z, v_z-1} - z_{l_z, v_z} > 0$. For every $j \in [v_z, k_z]$, since $\sum_{i=1}^n z_{ij} = 1 = \sum_{i=1}^n z_{i,k_z+1}$, there exists τ_j such that $z_{\tau_j, j} > z_{\tau_j, k_z+1}$. Let τ_{j_z} be the minimum τ_j that satisfies $z_{\tau_j, j} > z_{\tau_j, k_z+1}$. We partition all $\mathbf{z}$ for which $(\mathbf{x}, \mathbf{z}) \in X^{FL}$ according to the values (l_z, k_z) and order the partitions by increasing values of l_z and k_z in that order of preference. Let $\hat{\mathbf{z}} = \{\hat{z}_{ij}, i \in [1,n], j \in [1,n]\}$ be a member of the last partition in the order such that z_{lk} is the largest among all members in the last partition, (l,k). Note that $\hat{\mathbf{z}}$ is well-defined for a given $\mathbf{x} \in \text{proj}_{\mathbf{x}}(X^{FL})$ and partition (l,k), and can be found by the (bounded) linear program

$$\max\{z_{lk} : (33) - (36), (42) - (47),$$
$$z_{ij} \geq z_{i,j+1}, \text{ for } i < l \text{ and } j \in [i, n-1]; \text{ or } i = l \text{ and } j \in [l, k-1]\}.$$

Henceforth, we drop the subscripts on $l_{\hat{z}}, k_{\hat{z}}, v_{\hat{z}}, \tau_{j_{\hat{z}}}$ and the indices l, k, v, τ_j will refer to $\hat{\mathbf{z}}$ unless otherwise stated. We observe that $\hat{\mathbf{z}}$ can fall under three cases. In each of the cases, we construct a feasible vector $\tilde{\mathbf{z}}$ such that $l_{\tilde{z}} \geq l_{\hat{z}}$, $k_{\tilde{z}} \geq l_{\hat{z}}$ and $\tilde{z}_{lk} > \hat{z}_{lk}$ which contradicts the assumption that $\hat{\mathbf{z}}$ falls in the last partition or that it has the largest z_{lk} value among all members in the last partition, (l,k). This proof technique is similar to that of van Vyve and Ortega [14] and Gade and Küçükyavuz [5] for the uncapacitated lot-sizing problem without backorders and with fixed costs on stocks. However, our construction of the feasible vector $\tilde{\mathbf{z}}$ is non-trivial and significantly different than the case without backorders.

Case 1. $\tau_k < l$. Let $\delta_k = \hat{z}_{\tau_k,k} - \hat{z}_{\tau_k,k+1} > 0$ and consider the following vector $\tilde{\mathbf{z}}$,

$$\tilde{z}_{ij} = \begin{cases} \hat{z}_{ij}, & i \neq l, \tau_k, \text{ or } j \notin [v, k+1], \\ \hat{z}_{ij} + \frac{\varepsilon}{d_{vk}}, & i = l, j \in [v,k], \\ \hat{z}_{ij} + \frac{\varepsilon}{d_{k+1}}, & i = \tau_k, j = k+1, \\ \hat{z}_{ij} - \frac{\varepsilon}{d_{vk}}, & i = \tau_k, j \in [v,k], \\ \hat{z}_{ij} - \frac{\varepsilon}{d_{k+1}}, & i = l, j = k+1. \end{cases}$$

The value of ε is chosen such that it is the largest number that satisfies (A.1)–(A.3):

$$\gamma \geq \frac{\varepsilon}{d_{vk}} + \frac{\varepsilon}{d_{k+1}} \tag{A.1}$$

$$\xi \geq \frac{\varepsilon}{d_{vk}}, \text{ if } v > l \tag{A.2}$$

$$\delta_k \geq \frac{\varepsilon}{d_{vk}} + \frac{\varepsilon}{d_{k+1}}. \tag{A.3}$$

This guarantees the following:

(1.a) $\tilde{z}_{ij} \geq \tilde{z}_{i,j+1}, i < l, j \in [i, n-1]$. The statement is trivially true for $i \neq \tau_k$ or for $j \notin [v, k]$ because $\tilde{z}_{ij} = \hat{z}_{ij}$ for such i, j and due to the choice of l. For $i = \tau_k, j \in [v, k-1]$ we have $\tilde{z}_{ij} \geq \tilde{z}_{i,j+1}$. For $i = \tau_k, j = k$, we have $\tilde{z}_{ij} = \hat{z}_{ij} - \frac{\varepsilon}{d_{vk}} = \hat{z}_{i,k+1} + \delta_k - \frac{\varepsilon}{d_{vk}} = \hat{z}_{i,k+1} + \frac{\varepsilon}{d_{k+1}} + \delta_k - \frac{\varepsilon}{d_{vk}} - \frac{\varepsilon}{d_{k+1}} \geq \tilde{z}_{i,k+1}$ using (A.3).

(1.b) $\tilde{z}_{lj} \geq \tilde{z}_{l,j+1}, l \leq j \leq v-1$, if $v > l$. This is true for $j \neq v-1$ by the choice of l and v. For $j = v-1$, we have $\tilde{z}_{l,v-1} = \hat{z}_{l,v-1} = \hat{z}_{lv} + \xi = \hat{z}_{lv} + \frac{\varepsilon}{d_{vk}} + \xi - \frac{\varepsilon}{d_{vk}} \geq \tilde{z}_{lv}$ using (A.2).

(1.c) $\tilde{z}_{lv} = \tilde{z}_{l,v+1} = \cdots = \tilde{z}_{lk}$. This holds since $\hat{z}_{lv} = \hat{z}_{l,v+1} = \cdots = \hat{z}_{lk}$ and $\tilde{z}_{lj} = \hat{z}_{lj} + \frac{\varepsilon}{d_{vk}}, j \in [v, k]$.

(1.d) $\tilde{z}_{lk} \leq \tilde{z}_{l,k+1}$. This is because $\tilde{z}_{lk} = \hat{z}_{lk} + \frac{\varepsilon}{d_{vk}} = \hat{z}_{l,k+1} - \gamma + \frac{\varepsilon}{d_{vk}} = \hat{z}_{l,k+1} - \frac{\varepsilon}{d_{k+1}} - \gamma + \frac{\varepsilon}{d_{vk}} + \frac{\varepsilon}{d_{k+1}} \leq \tilde{z}_{l,k+1}$, from (A.1).

(1.e) Since $\gamma, \xi, \delta_k > 0$, we have $\varepsilon > 0$ and so $\tilde{z}_{lk} > \hat{z}_{lk}$.

From (1.a)–(1.c), we ensure that $l_{\tilde{z}} \geq l_{\hat{z}}$ and $k_{\tilde{z}} \geq k_{\hat{z}}$. If (A.1) is satisfied at equality, then it follows that $\tilde{z}_{lk} = \tilde{z}_{l,k+1}$, so either $l_{\tilde{z}} = l_{\hat{z}}$ and $k_{\tilde{z}} > k_{\hat{z}}$, or $l_{\tilde{z}} > l_{\hat{z}}$, hence $\tilde{\mathbf{z}}$ falls in a partition after that of $\hat{\mathbf{z}}$, a contradiction. If (A.1) is not satisfied at equality, then either (A.2) or (A.3) is satisfied at equality. As a result, $l_{\tilde{z}} = l_{\hat{z}}$ and $k_{\tilde{z}} = k_{\hat{z}}$, but $\tilde{z}_{lk} > \hat{z}_{lk}$, from (1.e), contradicting the assumption that $\hat{\mathbf{z}}$ has the largest z_{lk} value among all members in the last partition, (l, k). Next, we show that $\tilde{\mathbf{z}}$ is feasible in X^{FL}.

Constraint (33) is trivially satisfied for $j \notin [v, k+1]$ since $\tilde{z}_{ij} = \hat{z}_{ij}$. For $j \in [v, k]$ we have,

$$\sum_{i=1}^{n} \tilde{z}_{ij} = \sum_{i \in [1,n] \setminus \{l, \tau_k\}} \tilde{z}_{ij} + \tilde{z}_{lj} + \tilde{z}_{\tau_k, j} = \sum_{i \in [1,n] \setminus \{l, \tau_k\}} \hat{z}_{ij} + \hat{z}_{lj} + \frac{\varepsilon}{d_{vk}} + \hat{z}_{\tau_k, j} - \frac{\varepsilon}{d_{vk}} = \sum_{i=1}^{n} \hat{z}_{ij} = 1.$$

For $j = k+1$,

$$\sum_{i=1}^{n} \tilde{z}_{ij} = \sum_{i \in [1,n] \setminus \{l, \tau_k\}} \tilde{z}_{ij} + \tilde{z}_{lj} + \tilde{z}_{\tau_k, j} = \sum_{i \in [1,n] \setminus \{l, \tau_k\}} \hat{z}_{ij} + \hat{z}_{lj} - \frac{\varepsilon}{d_{k+1}} + \hat{z}_{\tau_k, j} + \frac{\varepsilon}{d_{k+1}} = \sum_{i=1}^{n} \hat{z}_{ij} = 1.$$

Constraint (34) is trivially satisfied for $i \neq l, \tau_k$ since $\tilde{z}_{ij} = \hat{z}_{ij}$. For $i = l$,

$$\begin{aligned}\sum_{j=1}^{n} d_j \tilde{z}_{ij} &= \sum_{j \in [1,n] \setminus [v, k+1]} d_j \tilde{z}_{ij} + \sum_{j=v}^{k} d_j \tilde{z}_{ij} + d_{k+1} \tilde{z}_{i,k+1} \\ &= \sum_{j \in [1,n] \setminus [v, k+1]} d_j \hat{z}_{ij} + \sum_{j=v}^{k} d_j \left(\hat{z}_{ij} + \frac{\varepsilon}{d_{vk}} \right) + d_{k+1} \left(\hat{z}_{i,k+1} - \frac{\varepsilon}{d_{k+1}} \right) = \sum_{j=1}^{n} d_j \hat{z}_{ij} = y_i.\end{aligned}$$

For $i = \tau_k$,

$$\begin{aligned}\sum_{j=1}^{n} d_j \tilde{z}_{ij} &= \sum_{j \in [1,n] \setminus [v, k+1]} \tilde{z}_{ij} + \sum_{j=v}^{k} d_j \tilde{z}_{ij} + d_{k+1} \tilde{z}_{i,k+1} \\ &= \sum_{j \in [1,n] \setminus [v, k+1]} d_j \hat{z}_{ij} + \sum_{j=v}^{k} d_j \left(\hat{z}_{ij} - \frac{\varepsilon}{d_{vk}} \right) + d_{k+1} \left(\hat{z}_{ij} + \frac{\varepsilon}{d_{k+1}} \right) = \sum_{j=1}^{n} d_j \hat{z}_{ij} = y_i.\end{aligned}$$

Constraint (45) is trivially satisfied for $i \neq l, \tau_k$ or $j \notin [v, k+1]$. For $i = \tau_k, j \in [v, k], \tilde{z}_{ij} = \hat{z}_{ij} - \frac{\varepsilon}{d_{vk}} < \hat{z}_{ij} \leq x_i$. For $i = \tau_k, j = k+1, \tilde{z}_{ij} = \hat{z}_{ij} + \frac{\varepsilon}{d_{k+1}} = \hat{z}_{ik} - \delta_k + \frac{\varepsilon}{d_{k+1}} \leq \hat{z}_{ik} \leq x_i$, from (A.3). For $i = l, j \in [v, k], \tilde{z}_{ij} = \hat{z}_{ij} + \frac{\varepsilon}{d_{vk}} = \hat{z}_{i,k+1} - \gamma + \frac{\varepsilon}{d_{vk}} \leq x_i$, from (A.1). For $i = l, j = k+1$, $\tilde{z}_{ij} = \hat{z}_{ij} - \frac{\varepsilon}{d_{k+1}} < \hat{z}_{ij} \leq x_i$.

Constraint (46) is trivially satisfied for $j \notin [v, k+1]$ since $\tilde{z}_{ij} = \hat{z}_{ij}$. Similarly, the constraint holds for for $j \in [v, k], t < \tau_k$. For $j \in [v, k]$ and $\tau_k \leq t < l, \sum_{i=1}^{t} \tilde{z}_{ij} = \sum_{i \in [1,t] \setminus \{\tau_k\}} \tilde{z}_{ij} + \tilde{z}_{\tau_k, j} = \sum_{i \in [1,t] \setminus \{\tau_k\}} \hat{z}_{ij} + \hat{z}_{\tau_k, j} - \frac{\varepsilon}{d_{vk}} = \sum_{i=1}^{t} \hat{z}_{ij} - \frac{\varepsilon}{d_{vk}} < \sum_{i=1}^{t} \hat{z}_{ij} \leq w_t$. For $j \in [v, k]$ and $l \leq t \leq j, \sum_{i=1}^{t} \tilde{z}_{ij} = \sum_{i \in [1,t] \setminus \{l, \tau_k\}} \tilde{z}_{ij} + \tilde{z}_{\tau_k, j} + \tilde{z}_{lj} = \sum_{i \in [1,t] \setminus \{l, \tau_k\}} \hat{z}_{ij} + \hat{z}_{\tau_k, j} - \frac{\varepsilon}{d_{vk}} + \hat{z}_{lj} + \frac{\varepsilon}{d_{vk}} = \sum_{i=1}^{t} \hat{z}_{ij} \leq w_t$. For $j = k+1, \tau_k \leq t < l, \sum_{i=1}^{t} \tilde{z}_{ij} = \sum_{i \in [1,t] \setminus \{\tau_k\}} \hat{z}_{ij} + \hat{z}_{\tau_k, j} + \frac{\varepsilon}{d_{vk}} = \sum_{i \in [1,t] \setminus \{\tau_k\}} \hat{z}_{ij} + \hat{z}_{\tau_k, k} - \delta_k + \frac{\varepsilon}{d_{k+1}} < \sum_{i \in [1,t] \setminus \{\tau_k\}} \hat{z}_{ij} + \hat{z}_{\tau_k, k} \leq \sum_{i \in [1,t] \setminus \{\tau_k\}} \hat{z}_{ik} + \hat{z}_{\tau_k, k} = \sum_{i=1}^{t} \hat{z}_{ik} \leq w_t$, from (A.3). For $j = k+1, l \leq t \leq j, \sum_{i=1}^{t} \tilde{z}_{ij} = \sum_{i \in [1,t] \setminus \{l, \tau_k\}} \tilde{z}_{ij} + \tilde{z}_{\tau_k, j} + \tilde{z}_{lj} = \sum_{i \in [1,t] \setminus \{l, \tau_k\}} \hat{z}_{ij} + \hat{z}_{\tau_k, j} + \frac{\varepsilon}{d_{k+1}} + \hat{z}_{lj} - \frac{\varepsilon}{d_{k+1}} = \sum_{i=1}^{t} \hat{z}_{ij} \leq w_t$.

Constraint (47) is trivially satisfied since $\tilde{z}_{ij} = \hat{z}_{ij}, i > j$.

Case 2. $\tau_k > l$ and there exists $j \in [v, k-1]$ such that $\tau_j < l$. We define $p := \max\{j \in [v, k] : \tau_j < l\}$. Using the definition of p, we have

$$\hat{z}_{\tau_p, p+1} = \cdots = \hat{z}_{\tau_p, k} = \hat{z}_{\tau_p, k+1}, \tag{A.4}$$

since otherwise the definitions of τ_j, k and l will be violated. We define $\delta_j := \hat{z}_{\tau_j, j} - \hat{z}_{\tau_j, k+1} > 0$ for $j \in [p, k]$. Also, for $i \in [l+1, n]$ we let $T_i = \{j \in [p+1, k] : i = \tau_j\}$ and $\mathcal{D}_i = \sum_{j \in T_i} d_j$. Consider the following vector $\tilde{\mathbf{z}}$,

$$\tilde{z}_{ij} = \begin{cases} \hat{z}_{ij} + \frac{\varepsilon}{d_{vk}}, & \text{if } i = l, j \in [v, k], \\ \hat{z}_{ij} - \frac{\varepsilon}{d_{k+1}}, & \text{if } i = l, j = k+1, \\ \hat{z}_{ij} - \frac{\varepsilon}{d_{vk}}, & \text{if } i = \tau_p, j \in [v, p], \\ \hat{z}_{ij} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}}, & \text{if } i = \tau_p, j \in [p+1, k], \\ \hat{z}_{ij} - \frac{\varepsilon}{d_{vk}} - \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}}, & \text{if } j \in [p+1, k], i = \tau_j, \\ \hat{z}_{ij} + \frac{\mathcal{D}_i}{d_{k+1}} \left(\frac{\varepsilon}{d_{vk}} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} \right), & \text{if } j = k+1, i \in [l+1, k] : T_i \neq \emptyset \\ \hat{z}_{ij}, & \text{otherwise.} \end{cases}$$

The value of ε is chosen such that it is the largest number that satisfies,

$$\xi \geq \frac{\varepsilon}{d_{vk}} \text{ if } v > l \tag{A.5}$$

$$\delta_p \geq \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} + \frac{\varepsilon}{d_{vk}} \tag{A.6}$$

$$\delta_j \geq \frac{\mathcal{D}_{\tau_j}}{d_{k+1}} \left(\frac{\varepsilon}{d_{vk}} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} \right), j \in [p+1, k] \tag{A.7}$$

$$\gamma \geq \frac{\varepsilon}{d_{vk}} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} \tag{A.8}$$

$$\gamma \geq \frac{\varepsilon}{d_{vk}} + \frac{\varepsilon}{d_{k+1}}. \tag{A.9}$$

This guarantees the following:

(2.a) $\tilde{z}_{ij} \geq \tilde{z}_{i,j+1}$ for $i < l, j \in [i, n-1]$. The result trivially holds for $i \neq \tau_p, j \in [i, n-1]$. The result also holds for $i = \tau_p, j \in [v, p-1]$ since $\tilde{z}_{ij} = \hat{z}_{ij} - \frac{\varepsilon}{d_{vk}} \geq \hat{z}_{i,j+1} - \frac{\varepsilon}{d_{vk}} = \tilde{z}_{i,j+1}$. Similarly, the result holds for $i = \tau_p, j \in [p+1, k-1]$. For $i = \tau_p, j = p$ we have $\tilde{z}_{ij} = \hat{z}_{ij} - \frac{\varepsilon}{d_{vk}} = \hat{z}_{i,j+1} + \delta_p - \frac{\varepsilon}{d_{vk}} = \hat{z}_{i,j+1} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} + \delta_p - \frac{\varepsilon}{d_{vk}} - \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} = \tilde{z}_{i,j+1} + \delta_p - \frac{\varepsilon}{d_{vk}} - \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} \geq \tilde{z}_{i,j+1}$ using (A.4) and (A.6). For $i = \tau_p, j = k, \tilde{z}_{ij} = \hat{z}_{ij} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} = \hat{z}_{i,j+1} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} > \hat{z}_{i,j+1} = \tilde{z}_{i,j+1}$.

(2.b) $\tilde{z}_{lj} \geq \tilde{z}_{l,j+1}, l \leq j \leq v-1$, if $v > l$. This is true for $j \neq v-1$. For $j = v-1$, we have $\tilde{z}_{l,v-1} = \hat{z}_{l,v-1} = \hat{z}_{lv} + \xi = \hat{z}_{lv} + \frac{\varepsilon}{d_{vk}} + \xi - \frac{\varepsilon}{d_{vk}} \geq \tilde{z}_{lv}$ using (A.5).

(2.c) $\tilde{z}_{lv} = \tilde{z}_{l,v+1} = \cdots = \tilde{z}_{lk}$. This holds since $\hat{z}_{lv} = \hat{z}_{l,v+1} = \cdots = \hat{z}_{lk}$ and $\tilde{z}_{lj} = \hat{z}_{lj} + \frac{\varepsilon}{d_{vk}}, j \in [v, k]$.

(2.d) $\tilde{z}_{lk} \leq \tilde{z}_{l,k+1}$. This is because $\tilde{z}_{lk} = \hat{z}_{lk} + \frac{\varepsilon}{d_{vk}} = \hat{z}_{l,k+1} - \gamma + \frac{\varepsilon}{d_{vk}} = \tilde{z}_{l,k+1} - \frac{\varepsilon}{d_{k+1}} - \gamma + \frac{\varepsilon}{d_{vk}} + \frac{\varepsilon}{d_{k+1}} \leq \tilde{z}_{l,k+1}$, from (A.9).

(2.e) Since $\gamma, \xi, \delta_j, j \in [v, k] > 0$, we have $\varepsilon > 0$ and so $\tilde{z}_{lk} > \hat{z}_{lk}$.

From (2.a)–(2.c), we ensure that $l_{\tilde{z}} \geq l_{\hat{z}}$ and $k_{\tilde{z}} \geq k_{\hat{z}}$. If (A.9) is satisfied at equality, then it follows that $\tilde{z}_{lk} = \tilde{z}_{l,k+1}$, so either $l_{\tilde{z}} = l_{\hat{z}}$ and $k_{\tilde{z}} > k_{\hat{z}}$, or $l_{\tilde{z}} > l_{\hat{z}}$, hence $\tilde{\mathbf{z}}$ falls in a partition after that of $\hat{\mathbf{z}}$, a contradiction. If (A.9) is not satisfied at equality, then at least one of (A.5)–(A.8) is satisfied at equality. As a result, $l_{\tilde{z}} = l_{\hat{z}}$ and $k_{\tilde{z}} = k_{\hat{z}}$, but $\tilde{z}_{lk} > \hat{z}_{lk}$, from (2.e), contradicting the assumption that $\hat{\mathbf{z}}$ has the largest z_{lk} value among all members in the last partition, (l, k). Next, we show that $\tilde{\mathbf{z}}$ is feasible in X^{FL}.

Constraint (33) is trivially satisfied for $j \notin [v, k+1]$. For $j \in [v, p]$ we have $\sum_{i=1}^{n} \tilde{z}_{ij} = \sum_{i\in[1,n]\setminus\{l,\tau_p\}} \tilde{z}_{ij} + \tilde{z}_{lj} + \tilde{z}_{\tau_p,j} = \sum_{i\in[1,n]\setminus\{l,\tau_p\}} \hat{z}_{ij} + \hat{z}_{lj} + \frac{\varepsilon}{d_{vk}} + \hat{z}_{\tau_p,j} - \frac{\varepsilon}{d_{vk}} = \sum_{i=1}^{n} \hat{z}_{ij} = 1$. For $j \in [p+1, k]$,

$$\sum_{i=1}^{n} \tilde{z}_{ij} = \sum_{i\in[1,n]\setminus\{l,\tau_p,\tau_j\}} \hat{z}_{ij} + \hat{z}_{lj} + \frac{\varepsilon}{d_{vk}} + \hat{z}_{\tau_p,j} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} + \hat{z}_{\tau_j,j} - \frac{\varepsilon}{d_{vk}} - \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}}$$
$$= \sum_{i=1}^{n} \hat{z}_{ij} = 1.$$

For $j = k+1$, we have

$$\sum_{i=1}^{n} \tilde{z}_{ij} = \sum_{i\in[1,l-1] \text{ or } i\in[l+1,n]:T_i=\emptyset} \hat{z}_{ij} + \hat{z}_{lj} - \frac{\varepsilon}{d_{k+1}} + \sum_{i\in[l+1,n]:T_i\neq\emptyset} \left(\hat{z}_{ij} + \frac{\mathcal{D}_i}{d_{k+1}} \left(\frac{\varepsilon}{d_{vk}} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} \right) \right)$$
$$= \sum_{i=1}^{n} \hat{z}_{ij} - \frac{\varepsilon}{d_{k+1}} + \frac{\varepsilon d_{p+1,k}}{d_{k+1}} \left(\frac{1}{d_{vk}} + \frac{d_{vp}}{d_{p+1,k} d_{vk}} \right) = 1.$$

Constraint (34) is trivially satisfied for $i < l$ and $i \neq \tau_p$; or $i \in [l+1, n] : T_i = \emptyset$. For $i = l$, we have $\sum_{j=1}^{n} d_j \tilde{z}_{ij} = \sum_{j=1}^{n} d_j \hat{z}_{ij} + \sum_{j=v}^{k} d_j \frac{\varepsilon}{d_{vk}} - d_{k+1} \frac{\varepsilon}{d_{k+1}} = \sum_{j=1}^{n} d_j \hat{z}_{ij} = y_i$. For $i = \tau_p$, we have

$$\sum_{j=1}^{n} d_j \tilde{z}_{ij} = \sum_{j\in[1,n]\setminus[v,k]} d_j \hat{z}_{ij} + \sum_{j=v}^{k} d_j \hat{z}_{ij} - \sum_{j=v}^{p} d_j \frac{\varepsilon}{d_{vk}} + \sum_{j=p+1}^{k} d_j \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} = \sum_{j=1}^{n} d_j \hat{z}_{ij} = y_i.$$

For $i \in [l+1, n] : T_i \neq \emptyset$, we have

$$\sum_{j=1}^{n} d_j \tilde{z}_{ij} = \sum_{j\in[1,n]\setminus(T_i\cup\{k+1\})} d_j \hat{z}_{ij} + \sum_{j\in T_i} d_j \hat{z}_{ij} - \sum_{j\in T_i} d_j \left(\frac{\varepsilon}{d_{vk}} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} \right) + d_{k+1} \hat{z}_{i,k+1}$$
$$+ d_{k+1} \frac{\mathcal{D}_i}{d_{k+1}} \left(\frac{\varepsilon}{d_{vk}} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} \right) = \sum_{j=1}^{n} d_j \hat{z}_{ij} = y_i.$$

Constraint (45) is trivially satisfied for $i < l$ and $i \neq \tau_p$; or $i \in [l+1, n] : T_i \neq \emptyset$ and $j \notin T_i \cup \{k+1\}$; or $i \in [l+1, n] : T_i = \emptyset$; or $j \notin [v, k+1]$; or $i = \tau_p$ and $j = k+1$. For $i = l, j = k+1$, we have $\tilde{z}_{ij} = \hat{z}_{ij} - \frac{\varepsilon}{d_{vk}} < \hat{z}_{ij} \leq x_i$. For $i = l, j \in [v, k], \tilde{z}_{ij} = \hat{z}_{ij} + \frac{\varepsilon}{d_{vk}} = \hat{z}_{i,k+1} - \gamma + \frac{\varepsilon}{d_{vk}} \leq x_i$, from (A.9). For $i = \tau_p, j \in [v, p], \tilde{z}_{ij} = \hat{z}_{ij} - \frac{\varepsilon}{d_{vk}} < \hat{z}_{ij} \leq x_i$. For $i = \tau_p, j \in [p+1, k], \tilde{z}_{ij} = \hat{z}_{ij} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} = \hat{z}_{ip} - \delta_p + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} \leq \hat{z}_{ip} \leq x_i$ from (A.8). For $j \in [p+1, k], i = \tau_j, \tilde{z}_{ij} = \hat{z}_{ij} - \frac{\varepsilon}{d_{vk}} - \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} < \hat{z}_{ij} \leq x_i$. For $i \in [l+1, n] : T_i \neq \emptyset, j = k+1, \tilde{z}_{ij} = \hat{z}_{ij} + \frac{\mathcal{D}_i}{d_{k+1}} \left(\frac{\varepsilon}{d_{vk}} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} \right) = \hat{z}_{ij'} - \delta_{j'} + \frac{\mathcal{D}_i}{d_{k+1}} \left(\frac{\varepsilon}{d_{vk}} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} \right) \leq \hat{z}_{ij'} \leq x_i$, where $j' \in T_i$, from (A.7).

Constraint (46) is trivially satisfied for $t < \tau_p$; or $j \notin [v, k+1]$; or $j = k+1$ and $t < l$. For $\tau_p \leq t < l, j \in [v, p]$ we have, $\sum_{i=1}^{t} \tilde{z}_{ij} = \sum_{i\in[1,t]\setminus\{\tau_p\}} \hat{z}_{ij} + \hat{z}_{\tau_p,j} - \frac{\varepsilon}{d_{vk}} < \sum_{i=1}^{t} \hat{z}_{ij} \leq w_t$. For $l \leq t, j \in [v, p], \sum_{i=1}^{t} \tilde{z}_{ij} = \sum_{i\in[1,t]\setminus\{\tau_p,l\}} \hat{z}_{ij} + \hat{z}_{\tau_p,j} - \frac{\varepsilon}{d_{vk}} + \hat{z}_{lj} + \frac{\varepsilon}{d_{vk}} = \sum_{i=1}^{t} \hat{z}_{ij} \leq w_t$. For $\tau_p \leq t < l, j \in [p+1, k]$ we have $\sum_{i=1}^{t} \tilde{z}_{ij} = \sum_{i\in[1,t]\setminus\{\tau_p\}} \hat{z}_{ij} + \hat{z}_{\tau_p j} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} = \sum_{i\in[1,t]\setminus\{\tau_p\}} \hat{z}_{ij} + \hat{z}_{\tau_p,p} - \delta_p + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} \leq \sum_{i\in[1,t]\setminus\{\tau_p\}} \hat{z}_{ip} + \hat{z}_{\tau_p,p} = \sum_{i=1}^{t} \hat{z}_{ip} \leq w_t$, from the definition of l and (A.6). For $j \in [p+1, k], l \leq t < \tau_j, \sum_{i=1}^{t} \tilde{z}_{ij} = \sum_{i\in[1,t]\setminus\{\tau_p,l\}} \hat{z}_{ij} + \hat{z}_{\tau_p,j} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} + \hat{z}_{lj} + \frac{\varepsilon}{d_{vk}} \leq \sum_{i\in[1,t]\setminus\{\tau_p,l\}} \hat{z}_{i,k+1} + \hat{z}_{\tau_p,k+1} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} + \hat{z}_{l,k+1} - \gamma + \frac{\varepsilon}{d_{vk}} = \sum_{i=1}^{t} \hat{z}_{i,k+1} - \gamma + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} + \frac{\varepsilon}{d_{vk}} \leq \sum_{i=1}^{t} \hat{z}_{i,k+1} \leq w_t$, from the definition of τ_j, (A.4) and (A.8). For $j \in [p+1, k], \tau_j \leq t, \sum_{i=1}^{t} \tilde{z}_{ij} = \sum_{i\in[1,t]\setminus\{\tau_p,l,\tau_j\}} \hat{z}_{ij} + \hat{z}_{\tau_p,j} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} + \hat{z}_{lj} + \frac{\varepsilon}{d_{vk}} + \hat{z}_{\tau_j,j} - \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} - \frac{\varepsilon}{d_{vk}} =$

$\sum_{i=1}^{t} \hat{z}_{ij} \leq w_t$. For $j = k+1, l \leq t$, we have

$$\sum_{i=1}^{t} \tilde{z}_{ij} = \sum_{i\in[1,l-1] \text{ or } i\in[l+1,t]:T_i=\emptyset} \hat{z}_{ij} + \hat{z}_{l,j} - \frac{\varepsilon}{d_{k+1}} + \sum_{i\in[l+1,t]:T_i\neq\emptyset} \hat{z}_{ij}$$
$$+ \sum_{i\in[l+1,t]:T_i\neq\emptyset} \frac{\mathcal{D}_i}{d_{k+1}} \left(\frac{\varepsilon}{d_{vk}} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} \right).$$

Note that the last term in the expression above is no larger than $\frac{\varepsilon}{d_{k+1}}$ and thus $\sum_{i=1}^{t} \tilde{z}_{ij} \leq w_j$ holds for this case.

Constraint (47) is trivially satisfied for any $j \notin [v, k+1]$; and for t such that $t \geq \tau_q$ for all $q \in [v, k]$. Similarly, it is also satisfied for $j \in [v, p]$ by the choice of p; and if $j \in [p+1, k] : \tau_j \leq t$. For $j \in [p+1, k] : \tau_j > t$ we have $\sum_{i=t+1}^{n} \tilde{z}_{ij} = \sum_{i\in[t+1,n]\setminus\{\tau_j\}} \hat{z}_{ij} + \hat{z}_{\tau_j,j} - \frac{\varepsilon}{d_{vk}} - \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} < \sum_{i=t+1}^{n} \hat{z}_{ij} \leq u_t$. For $j = k+1$ and t such that there exists $q \in [p+1, k]$ with $\tau_q > t$, we have $\sum_{i=t+1}^{n} \tilde{z}_{ij} = \sum_{i=t+1}^{n} \hat{z}_{ij} + \sum_{i\in[t+1,n]:T_i\neq\emptyset} \frac{\mathcal{D}_i}{d_{k+1}} \left(\frac{\varepsilon}{d_{vk}} + \frac{\varepsilon d_{vp}}{d_{p+1,k} d_{vk}} \right) \leq \sum_{i\in[t+1,n]} \hat{z}_{iq} - \gamma + \frac{\varepsilon}{d_{k+1}} \leq \sum_{i\in[t+1,n]} \hat{z}_{iq} \leq u_t$. The first inequality follows because $\sum_{i=1}^{t} \hat{z}_{ij} \geq \sum_{i=1}^{t} \hat{z}_{iq} + \gamma$ and from equality (33), and the definitions of τ_q and γ.

Case 3. $\tau_j > l, \forall j \in [v, k]$. Recall that $\delta_j = \hat{z}_{\tau_j,j} - \hat{z}_{\tau_j,k+1}, j \in [v, k] > 0$ and for $i \in [l+1, n], T_i = \{j \in [v, k] : i = \tau_j\}$ and $\mathcal{D}_i = \sum_{j\in T_i} d_j$. Consider the following vector $\tilde{\mathbf{z}}$,

$$\tilde{z}_{ij} = \begin{cases} \hat{z}_{ij}, & i \neq l, i \in [l+1, n] : T_i = \emptyset \text{ or } j \notin [v, k+1] \\ \hat{z}_{ij} + \frac{\varepsilon}{d_{vk}}, & i = l, j \in [v, k] \\ \hat{z}_{ij} - \frac{\varepsilon}{d_{k+1}}, & i = l, j = k+1 \\ \hat{z}_{ij} - \frac{\varepsilon}{d_{vk}}, & j \in [v, k], i = \tau_j \\ \hat{z}_{ij} + \frac{\varepsilon \mathcal{D}_i}{d_{k+1} d_{vk}}, & j = k+1, i \in [l+1, n] : T_i \neq \emptyset. \end{cases} \tag{A.10}$$

The value of ε is chosen such that it is the largest number that satisfies:

$$\xi \geq \frac{\varepsilon}{d_{vk}}, \text{ if } v > l \tag{A.11}$$
$$\gamma \geq \frac{\varepsilon}{d_{vk}} + \frac{\varepsilon}{d_{k+1}} \tag{A.12}$$
$$\delta_j \geq \frac{\varepsilon}{d_{vk}} + \frac{\varepsilon \mathcal{D}_{\tau_j}}{d_{vk} d_{k+1}}, j \in [v, k]. \tag{A.13}$$

This guarantees the following:

(3.a) $\tilde{z}_{ij} \geq \tilde{z}_{i,j+1}$ for $i < l, j \in [i, n-1]$ since $\tilde{z}_{ij} = \hat{z}_{ij}$.
(3.b) $\tilde{z}_{lj} \geq \tilde{z}_{l,j+1}, l \leq j \leq v-1$, if $v > l$. This is true for $j \neq v-1$. For $j = v-1$, we have $\tilde{z}_{l,v-1} = \hat{z}_{l,v-1} = \hat{z}_{lv} + \xi = \hat{z}_{lv} + \frac{\varepsilon}{d_{vk}} + \xi - \frac{\varepsilon}{d_{vk}} \geq \tilde{z}_{lv}$ using (A.11).
(3.c) $\tilde{z}_{lv} = \tilde{z}_{l,v+1} = \cdots = \tilde{z}_{lk}$. This holds since $\hat{z}_{lv} = \hat{z}_{l,v+1} = \cdots = \hat{z}_{lk}$ and $\tilde{z}_{lj} = \hat{z}_{lj} + \frac{\varepsilon}{d_{vk}}, j \in [v, k]$.
(3.d) $\tilde{z}_{lk} \leq \tilde{z}_{l,k+1}$. This is because $\tilde{z}_{lk} = \hat{z}_{lk} + \frac{\varepsilon}{d_{vk}} = \hat{z}_{l,k+1} - \gamma + \frac{\varepsilon}{d_{vk}} = \tilde{z}_{l,k+1} - \frac{\varepsilon}{d_{k+1}} - \gamma + \frac{\varepsilon}{d_{vk}} + \frac{\varepsilon}{d_{k+1}} \leq \tilde{z}_{l,k+1}$, from (A.12).
(3.e) Since $\gamma, \xi, \delta_j, j \in [v, k] > 0$, we have $\varepsilon > 0$ and so $\tilde{z}_{lk} > \hat{z}_{lk}$.

From (3.a)–(3.c), we ensure that $l_{\tilde{z}} \geq l_{\hat{z}}$ and $k_{\tilde{z}} \geq k_{\hat{z}}$. If (A.12) is satisfied at equality, then from (3.d), it follows that $\tilde{z}_{lk} = \tilde{z}_{l,k+1}$, so either $l_{\tilde{z}} = l_{\hat{z}}$ and $k_{\tilde{z}} > k_{\hat{z}}$, or $l_{\tilde{z}} > l_{\hat{z}}$, hence $\tilde{\mathbf{z}}$ falls in a partition after that of $\hat{\mathbf{z}}$, a contradiction. If (A.12) is not satisfied at equality, then either (A.11) or (A.13) is satisfied at equality. As a result, $l_{\tilde{z}} = l_{\hat{z}}$ and $k_{\tilde{z}} = k_{\hat{z}}$, but $\tilde{z}_{lk} > \hat{z}_{lk}$, from (3.e), contradicting the assumption that $\hat{\mathbf{z}}$ has the largest z_{lk} value among all members in the last partition, (l, k). Next, we show that $\tilde{\mathbf{z}}$ is feasible in X^{FL}.

Constraint (33) is trivially satisfied for $j \notin [v, k+1]$. For $j \in [v,k]$ we have $\sum_{i=1}^{n} \tilde{z}_{ij} = \sum_{i \in [1,n] \setminus \{l, \tau_j\}} \hat{z}_{ij} + \hat{z}_{lj} + \frac{\varepsilon}{d_{vk}} + \hat{z}_{\tau_j,j} - \frac{\varepsilon}{d_{vk}} = \sum_{i=1}^{n} \hat{z}_{ij} = 1$. For $j = k+1$, we have

$$\begin{aligned}\sum_{i=1}^{n} \tilde{z}_{ij} &= \sum_{i \in [1,l-1] \text{ or } i \in [l+1,n]: T_i = \emptyset} \hat{z}_{ij} + \hat{z}_{lj} - \frac{\varepsilon}{d_{k+1}} + \sum_{i \in [l+1,n]: T_i \neq \emptyset} \left(\hat{z}_{ij} + \frac{\varepsilon \mathcal{D}_i}{d_{k+1} d_{vk}} \right) \\ &= \sum_{i=1}^{n} \hat{z}_{ij} - \frac{\varepsilon}{d_{k+1}} + \sum_{i \in [l+1,n]: T_i \neq \emptyset} \frac{\varepsilon \mathcal{D}_i}{d_{k+1} d_{vk}} \\ &= \sum_{i=1}^{n} \hat{z}_{ij} - \frac{\varepsilon}{d_{k+1}} + \frac{\varepsilon}{d_{k+1}} = 1.\end{aligned}$$

Constraint (34) is trivially satisfied for $i < l$, and for $i \in [l+1,n] : T_i = \emptyset$. For $i = l$,

$$\sum_{j=1}^{n} d_j \tilde{z}_{ij} = \sum_{j \in [1,n] \setminus [v,k+1]} d_j \hat{z}_{ij} + \sum_{j=v}^{k} d_j \left(\hat{z}_{ij} + \frac{\varepsilon}{d_{vk}} \right) + d_{k+1} \left(\hat{z}_{i,k+1} - \frac{\varepsilon}{d_{k+1}} \right) = \sum_{j=1}^{n} d_j \hat{z}_{ij} = y_i.$$

For $i \in [l+1,n] : T_i \neq \emptyset$, we have,

$$\begin{aligned}\sum_{j=1}^{n} d_j \tilde{z}_{ij} &= \sum_{j \in [i,n] \setminus T_i} d_j \hat{z}_{ij} + \sum_{j \in T_i} d_j \left(\hat{z}_{ij} + \frac{\varepsilon}{d_{vk}} \right) + d_{k+1} \left(\hat{z}_{i,k+1} - \frac{\varepsilon \mathcal{D}_i}{d_{k+1} d_{vk}} \right) \\ &= \sum_{j=1}^{n} d_j \hat{z}_{ij} + \frac{\varepsilon \mathcal{D}_i}{d_{vk}} - \frac{\varepsilon \mathcal{D}_i}{d_{vk}} = y_i.\end{aligned}$$

Constraint (45) is trivially satisfied for $i < l$; or $i \in [l+1,n] : T_i = \emptyset$; or $j \notin [v,k+1]$. For $i = l, j = k+1, \tilde{z}_{ij} = \hat{z}_{ij} - \frac{\varepsilon}{d_{k+1}} < \hat{z}_{ij} \leq x_i$. For, $i = l, j \in [v,k], \tilde{z}_{ij} = \hat{z}_{ij} + \frac{\varepsilon}{d_{vk}} = \hat{z}_{l,k+1} - \gamma + \frac{\varepsilon}{d_{vk}} < \hat{z}_{l,k+1} \leq x_i$, from (A.12). For $j \in [v,k], i = \tau_j, \tilde{z}_{ij} = \hat{z}_{ij} - \frac{\varepsilon}{d_{vk}} < \hat{z}_{ij} \leq x_i$. For $i \in [l+1,n] : T_i \neq \emptyset$ and $j = k+1$, we have, $\tilde{z}_{ij} = \hat{z}_{ij} + \frac{\varepsilon \mathcal{D}_i}{d_{k+1} d_{vk}} = \hat{z}_{ij'} - \delta_{j'} + \frac{\varepsilon \mathcal{D}_i}{d_{k+1} d_{vk}} \leq \hat{z}_{ij'} \leq x_i$, for some $j' \in T_i$, from (A.13).

Constraint (46) is trivially satisfied for $t < l$ or $j \notin [v,k+1]$. For $l \leq t, j = k+1$, we have, $\sum_{i=1}^{t} \tilde{z}_{ij} = \sum_{i=1}^{t} \hat{z}_{ij} - \frac{\varepsilon}{d_{k+1}} + \sum_{i \in [l+1,t]: T_i \neq \emptyset} \frac{\varepsilon \mathcal{D}_i}{d_{vk} d_{k+1}} \leq \sum_{i=1}^{t} \hat{z}_{ij} \leq w_t$. For $\tau_j > t \geq l, j \in [v,k]$, from the definition of τ_j and l, $\tilde{z}_{ij} = \hat{z}_{ij} = \hat{z}_{i,k+1}$ for $i < l$ and $\tilde{z}_{ij} \leq \tilde{z}_{i,k+1}$ for $i \in [l,t]$, hence, $\sum_{i=1}^{t} \tilde{z}_{ij} \leq \sum_{i=1}^{t} \tilde{z}_{i,k+1} \leq w_t$. Finally, for $j \in [v,k], l < \tau_j \leq t$, we have, $\sum_{i=1}^{t} \tilde{z}_{ij} = \sum_{i \in [1,t] \setminus \{l, \tau_j\}} \hat{z}_{ij} + \hat{z}_{\tau_j,j} + \hat{z}_{lj} + \frac{\varepsilon}{d_{vk}} - \frac{\varepsilon}{d_{vk}} = \sum_{i=1}^{t} \hat{z}_{ij} \leq w_t$.

Constraint (47) is trivially satisfied for any $j \notin [v,k+1]$; and for $j \in [v,k]$ such that $\tau_j \leq t$. For $j \in [v,k]$ such that $\tau_j > t$ we have $\sum_{i=t+1}^{n} \tilde{z}_{ij} = \sum_{i \in [t+1,n] \setminus \{\tau_j\}} \hat{z}_{ij} + \hat{z}_{\tau_j,j} - \frac{\varepsilon}{d_{vk}} < \sum_{i=t+1}^{n} \hat{z}_{ij} \leq u_t$. For $j = k+1$ and t such that there exists $q \in [v,k]$ with $\tau_q > t$, we have $\sum_{i=t+1}^{n} \tilde{z}_{ij} = \sum_{i=t+1}^{n} \hat{z}_{ij} + \sum_{i \in [t+1,n]: T_i \neq \emptyset} \frac{\varepsilon \mathcal{D}_i}{d_{k+1} d_{vk}} \leq \sum_{i \in [t+1,n]} \hat{z}_{iq} - \gamma + \frac{\varepsilon}{d_{k+1}} \leq \sum_{i \in [t+1,n]} \hat{z}_{iq} \leq u_t$. The first inequality follows because $\sum_{i=1}^{t} \hat{z}_{ij} \geq \sum_{i=1}^{t} \hat{z}_{iq} + \gamma$, and from equality (33) and the definitions of τ_q and γ.

In all cases, we showed that $\varepsilon > 0$ exists, and that constraints (33)–(34) and (45)–(47) are satisfied by $\tilde{\mathbf{z}}$. Finally, we need to show that (35)–(36) are also satisfied for $\tilde{\mathbf{z}}$ in all cases. First, note that for $t \in [1, n-1]$,

$$\sum_{k=t+1}^{n} \sum_{j=1}^{t} d_j \tilde{z}_{kj} \leq \sum_{k=t+1}^{n} \sum_{j=1}^{t} d_j \hat{z}_{kj} \leq r_t \tag{A.14}$$

in all cases. Thus, (36) is satisfied. For $t \in [1, n-1]$, inequality (35) can be rewritten as

$$\begin{aligned} s_t &\geq \sum_{k=1}^{t} \sum_{j=t+1}^{n} d_j z_{kj} \\ &= \sum_{k=1}^{t} (y_k - \sum_{j=1}^{t} d_j z_{kj}) \\ &= \sum_{k=1}^{t} y_k - \sum_{j=1}^{t} d_j \sum_{k=1}^{t} z_{kj} \\ &= \sum_{k=1}^{t} y_k - \sum_{j=1}^{t} d_j (1 - \sum_{k=t+1}^{n} z_{kj}) \\ &= \sum_{k=1}^{t} (y_k - d_k) + \sum_{k=t+1}^{n} \sum_{j=1}^{t} d_j z_{kj}. \end{aligned}$$

Using (A.14), we have $s_t \geq \sum_{k=1}^{t}(y_k - d_k) + \sum_{k=t+1}^{n} \sum_{j=1}^{t} d_j \hat{z}_{kj} \geq \sum_{k=1}^{t}(y_k - d_k) + \sum_{k=t+1}^{n} \sum_{j=1}^{t} d_j \tilde{z}_{kj}$. In addition, in all cases $\tilde{\mathbf{z}}$ falls into a partition in or after that of $\hat{\mathbf{z}}$ and $\tilde{z}_{lk} > \hat{z}_{lk}$. Hence, the proof of the proposition is complete.

□

The current update scheme cannot be used to prove the claim that $\text{proj}_{\mathbf{x}}(X^{SPR}) = \text{proj}_{\mathbf{x}}(X^{FL^{\leq}})$, which if true proves the conjecture that $\text{proj}_{\mathbf{x}}(X^{SPR}) = \text{proj}_{\mathbf{x}}(X^{FL})$. The update scheme for this claim, if one exists, appear to be significantly more complex.

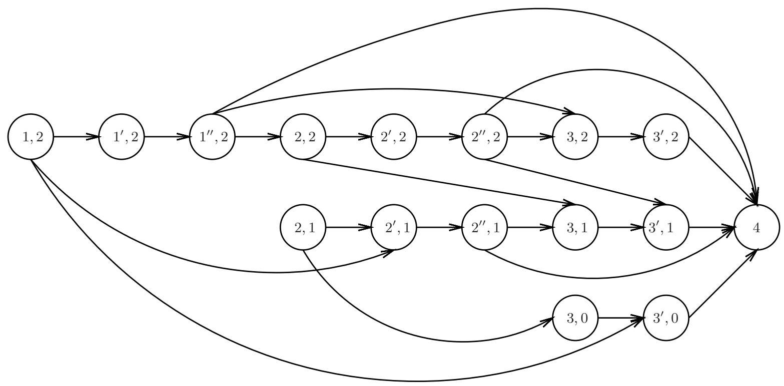

FIGURE 1. Shortest Path Representation of LS-SL for $n = 3, \kappa = 2$.

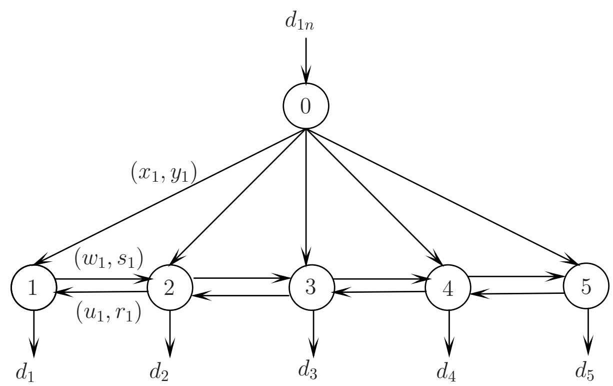

FIGURE 2. Fixed Charge Network Flow Representation of ULSBW for $n = 5$.

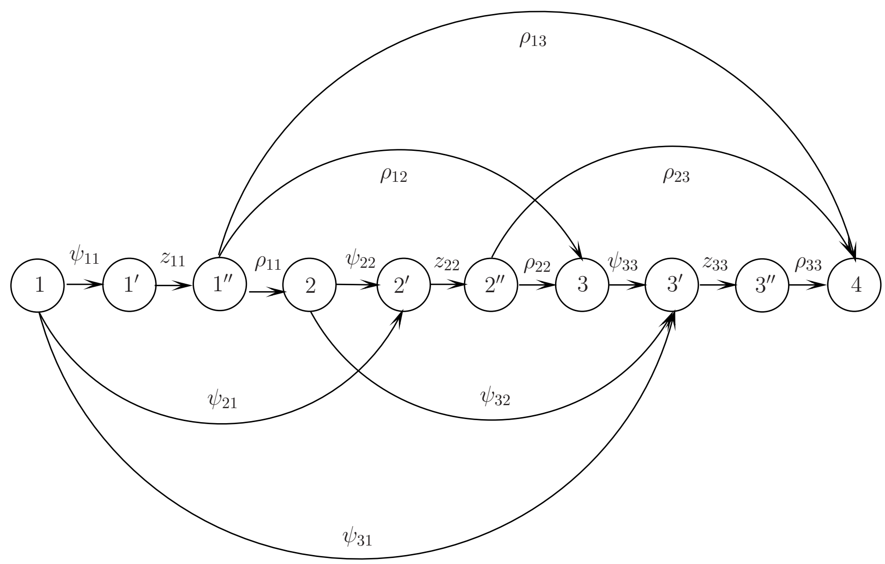

FIGURE 3. Shortest Path Network for ULSBW

Table 1. Problem data for Example 1

t	1	2	3	4	5
d	69	73	68	30	80
f	9000	4000	3000	2000	1000
c	100	78	85	63	95
h	10	8	10	4	8

Table 2. Optimal solutions for Example 1

Model	t	1	2	3	4	5
LS-SL	y	**0**	142	**0**	178	0
ULSB	y	**0**	**0**	**0**	320	0

Table 3. Comparison of MI-LS-SL and SPR Formulations for $n = 60$

	MI-LS-SL				SPR			
$n.m.\beta.\pi$	**S GAP**	**E GAP**	**T (sec)**	**Nodes**	**S GAP**	**E GAP**	**T (sec)**	**Nodes**
60.500.3.10	81.57%	3.11%	3600.0	1329242.6	0.21%	0.00%	0.6	17.6
60.500.3.25	76.02%	4.34%	3600.0	1818983.8	0.32%	0.00%	0.7	13.2
60.500.3.40	69.33%	1.98%	3123.5	1753791.2	0.20%	0.00%	0.7	12
60.500.5.10	83.47%	11.38%	3600.0	309144.8	0.31%	0.00%	1.6	14
60.500.5.25	79.02%	11.32%	3600.0	460147.4	0.35%	0.00%	2.6	50
60.500.5.40	70.35%	8.23%	3600.0	660910.6	0.46%	0.00%	3.5	72.2
60.1000.3.10	82.75%	4.47%	3600.0	1287002.4	0.17%	0.00%	0.5	7.6
60.1000.3.25	79.86%	5.19%	3600.0	1676558.8	0.53%	0.00%	0.8	16.8
60.1000.3.40	72.27%	2.93%	3157.3	1319453.0	0.31%	0.00%	0.7	12.6
60.1000.5.10	84.33%	10.74%	3600.0	402493.6	0.33%	0.00%	1.8	29.4
60.1000.5.25	81.29%	11.22%	3600.0	705396.2	0.32%	0.00%	1.9	37
60.1000.5.40	74.98%	10.06%	3600.0	682265.0	0.36%	0.00%	2.5	57.2
60.2500.3.10	86.07%	5.46%	3600.0	1033585.8	0.26%	0.00%	0.4	7.2
60.2500.3.25	83.77%	7.16%	3600.0	1313147.8	0.28%	0.00%	0.6	13.4
60.2500.3.40	78.14%	4.82%	3600.0	1575318.4	0.36%	0.00%	0.7	15.4
60.2500.5.10	86.88%	12.14%	3600.0	441599.4	0.52%	0.00%	2.0	56
60.2500.5.25	83.75%	12.34%	3600.0	566595.8	0.46%	0.00%	2.6	63.8
60.2500.5.40	77.59%	11.64%	3600.0	591281.8	0.32%	0.00%	2.2	50.4
60.5000.3.10	88.74%	6.01%	3600.0	875479.8	0.85%	0.00%	0.8	32.8
60.5000.3.25	83.61%	7.55%	3600.0	965516.2	0.47%	0.00%	0.7	18.6
60.5000.3.40	79.19%	5.53%	3600.0	1165189.4	0.41%	0.00%	0.7	22.8
60.5000.5.10	87.48%	11.05%	3600.0	450531.8	0.95%	0.00%	2.0	71.2
60.5000.5.25	84.71%	12.94%	3600.0	507513.2	0.72%	0.00%	2.8	117.2
60.5000.5.40	81.58%	13.22%	3600.0	546015.2	0.84%	0.00%	2.8	125.4

Table 4. Comparison of MI-LS-SL and SPR Formulations for $n = 120$

	MI-LS-SL				SPR			
$n.m.\beta.\pi$	**S GAP**	**E GAP**	**T (sec)**	**Nodes**	**S GAP**	**E GAP**	**T (sec)**	**Nodes**
120.500.3.10	84.90%	9.82%	3600.0	910724.0	0.07%	0.00%	2.2	5.8
120.500.3.25	78.65%	9.28%	3600.0	1196455.8	0.15%	0.00%	3.7	18
120.500.3.40	72.32%	7.46%	3600.0	1345806.6	0.12%	0.00%	3.0	21.6
120.500.5.10	85.57%	15.89%	3600.0	238873.2	0.08%	0.01%	8.2	22.4
120.500.5.25	81.75%	14.53%	3600.0	341460.0	0.17%	0.01%	16.2	104.2
120.500.5.40	73.62%	11.06%	3600.0	489091.8	0.09%	0.01%	12.6	58
120.1000.3.10	86.89%	12.30%	3600.0	984793.4	0.06%	0.00%	2.2	15.4
120.1000.3.25	83.06%	12.04%	3600.0	1249938.0	0.07%	0.00%	2.4	10.2
120.1000.3.40	77.60%	10.08%	3600.0	1124620.0	0.10%	0.00%	3.9	32.6
120.1000.5.10	87.68%	16.21%	3600.0	295196.2	0.12%	0.01%	8.3	30.6
120.1000.5.25	83.30%	15.35%	3600.0	408588.8	0.06%	0.00%	9.7	65.2
120.1000.5.40	77.17%	13.04%	3600.0	635322.4	0.16%	0.01%	16.1	206
120.2500.3.10	89.35%	13.79%	3600.0	769405.2	0.01%	0.00%	1.4	0.6
120.2500.3.25	85.63%	14.16%	3600.0	883662.2	0.13%	0.00%	3.3	34.8
120.2500.3.40	81.91%	11.77%	3600.0	1012314.2	0.14%	0.00%	3.3	27.2
120.2500.5.10	89.88%	17.77%	3600.0	324094.6	0.12%	0.00%	6.2	19.6
120.2500.5.25	87.29%	17.36%	3600.0	357905.0	0.13%	0.01%	12.0	95.2
120.2500.5.40	81.89%	14.68%	3600.0	482396.8	0.16%	0.01%	11.2	103.6
120.5000.3.10	90.75%	15.93%	3600.0	566162.6	0.09%	0.00%	2.2	4.4
120.5000.3.25	88.15%	16.71%	3600.0	548372.8	0.23%	0.00%	3.9	29.8
120.5000.3.40	83.67%	13.56%	3600.0	670782.0	0.26%	0.00%	4.5	54.8
120.5000.5.10	91.02%	18.34%	3600.0	326768.8	0.12%	0.00%	6.2	22.6
120.5000.5.25	88.02%	18.90%	3600.0	386560.4	0.14%	0.00%	10.4	127.8
120.5000.5.40	84.34%	16.96%	3600.0	416421.8	0.11%	0.01%	9.1	78.6

Table 5. Optimal solutions for Example 1 with type-II constraint

Model	t	1	2	3	4	5
LS-SL-II-R	y	0	114	0	206	0
	r	69	69	96	0	0
	s	0	41	0	80	0
LS-SL-II	y	0	183	0	137	0
	r	69	0	27	0	0
	s	0	41	0	80	0

Table 6. Results for $n = 120$

	MI-LS-SL-II			**FL-II**		
$n.\beta.m.(1-\gamma)$	Time (s)	Nodes	SGAP	Time (s)	Nodes	SGAP
120.500.3.10	117.9	272	54.59%	22.5	15	0.07%
120.500.3.25	114.4	156	56.98%	21.2	2	0.02%
120.500.3.40	89.4	141	55.87%	21.0	6	0.05%
120.500.5.10	761.3	533	58.92%	58.4	34	0.10%
120.500.5.25	1323.1	482	59.32%	59.2	12	0.02%
120.500.5.40	1202.7	899	58.04%	67.2	17	0.06%
120.1000.3.10	200.2	501	65.24%	29.1	6	0.02%
120.1000.3.25	175.4	378	64.99%	25.5	9	0.07%
120.1000.3.40	111.8	178	62.11%	27.4	12	0.17%
120.1000.5.10	1691.1	588	68.04%	49.4	15	0.05%
120.1000.5.25	940.9	709	66.94%	71.2	46	0.08%
120.1000.5.40	1223.2	513	66.34%	46.9	8	0.08%
120.2500.3.10	350.1	534	50.37%	32.0	12	0.05%
120.2500.3.25	382.7	609	47.59%	26.5	30	0.10%
120.2500.3.40	480.7	867	47.74%	23.7	17	0.14%
120.2500.5.10	1021.7	540	53.78%	63.6	9	0.05%
120.2500.5.25	3600.2(0.53%)	910	53.61%	67.0	25	0.07%
120.2500.5.40	1166.9	913	50.17%	69.1	32	0.17%
120.5000.3.10	124.0	234	70.83%	22.4	2	0.01%
120.5000.3.25	231.9	490	70.87%	18.8	0	0.01%
120.5000.3.40	263.8	751	69.20%	23.0	13	0.22%
120.5000.5.10	410.1	183	71.55%	50.6	15	0.06%
120.5000.5.25	669.9	613	70.48%	58.1	19	0.08%
120.5000.5.40	537.6	483	69.45%	53.0	13	0.08%