THE CENTRAL CURVE IN LINEAR PROGRAMMING

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Abstract. The central curve of a linear program is an algebraic curve specified by linear and quadratic constraints arising from complementary slackness. It is the union of the various central paths for minimizing or maximizing the cost function over any region in the associated hyperplane arrangement. We determine the degree, arithmetic genus and defining prime ideal of the central curve, thereby answering a question of Bayer and Lagarias. These invariants, along with the degree of the Gauss image of the curve, are expressed in terms of the matroid of the input matrix. Extending work of Dedieu, Malajovich and Shub, this yields an instance-specific bound on the total curvature of the central path, a quantity relevant for interior point methods. The global geometry of central curves is studied in detail.

1. Introduction

We consider the standard linear programming problem in its *primal* and *dual* formulation:

$$\text{(1)} \qquad \text{Maximize } \; \mathbf{c}^T\mathbf{x} \; \text{ subject to } \; A\mathbf{x} = \mathbf{b} \; \text{ and } \; \mathbf{x} \geq 0;$$

$$\text{(2)} \qquad \text{Minimize } \; \mathbf{b}^T\mathbf{y} \; \text{ subject to } \; A^T\mathbf{y} - \mathbf{s} = \mathbf{c} \; \text{ and } \; \mathbf{s} \geq 0.$$

Here A is a fixed matrix of rank d having n columns, while the vectors $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{b} \in \text{image}(A)$ may vary. Before describing our contributions, we review some basics from the theory of linear programming [22, 29]. The *logarithmic barrier function* for (1) is defined as

$$f_\lambda(\mathbf{x}) \;\; := \;\; \mathbf{c}^T\mathbf{x} + \lambda \sum_{i=1}^{n} \log x_i,$$

where $\lambda > 0$ is a real parameter. This specifies a family of optimization problems:

$$\text{(3)} \qquad \text{Maximize } \; f_\lambda(\mathbf{x}) \; \text{ subject to } \; A\mathbf{x} = \mathbf{b} \; \text{ and } \; \mathbf{x} \geq 0.$$

Since the function f_λ is strictly concave, it attains a unique maximum $\mathbf{x}^*(\lambda)$ in the interior of the feasible polytope $P = \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} : A\mathbf{x} = \mathbf{b}\}$. Note that $f_\lambda(\mathbf{x})$ tends to $-\infty$ when $\mathbf{x}$ approaches the boundary of P. The *primal central path* is the curve $\{\mathbf{x}^*(\lambda) \,|\, \lambda > 0\}$ inside the polytope P. There is an analogous logarithmic barrier function for the dual problem (2) and a corresponding *dual central path*. The central path connects the optimal solution of the linear program in question with its *analytic center*. It is homeomorphic to a line segment.

The *complementary slackness* condition says that the pair of optimal solutions, to the primal linear program (1) and to the dual linear program (2), are characterized by

$$\text{(4)} \qquad A\mathbf{x} = \mathbf{b}\,, \; A^T\mathbf{y} - \mathbf{s} = \mathbf{c}\,, \; \mathbf{x} \geq 0\,, \; \mathbf{s} \geq 0, \; \text{ and } \; x_i s_i = 0 \; \text{ for } \; i = 1, 2, \ldots, n.$$

The central path converges to the solution of this system of equations and inequalities:

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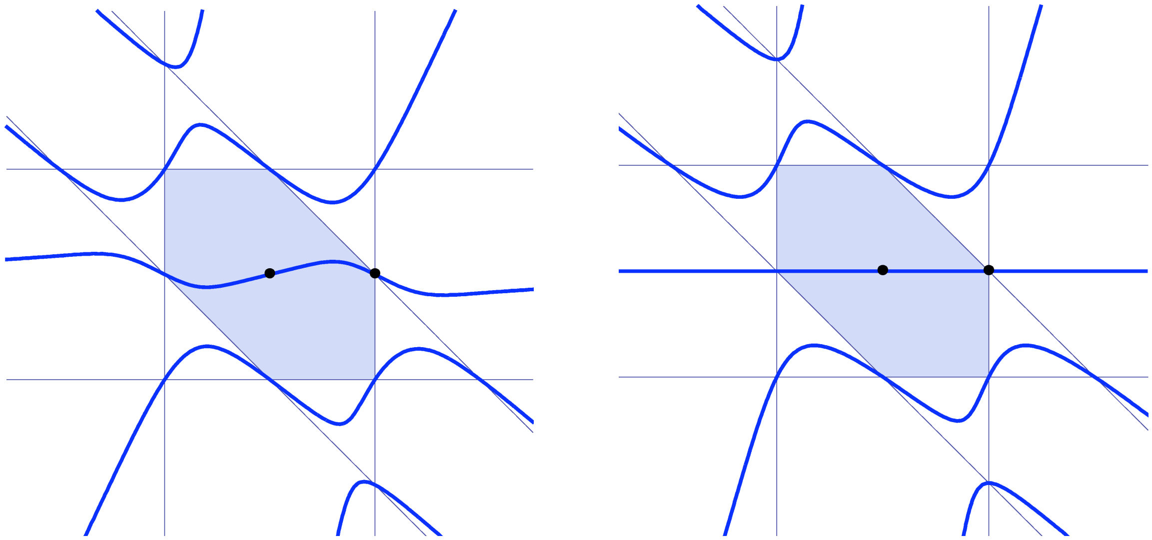


FIGURE 1. The central curve of a hexagon for two choices of the cost function

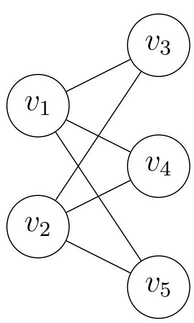
Theorem 1 (cf. [29]). *For all $\lambda > 0$, the system of polynomial equations*

$$A\mathbf{x} = \mathbf{b}\,,\;\; A^T\mathbf{y} - \mathbf{s} = \mathbf{c}, \;\; \textit{and } x_i s_i = \lambda \textit{ for } i = 1, 2, \ldots, n, \tag{5}$$

has a unique real solution $(\mathbf{x}^*(\lambda), \mathbf{y}^*(\lambda), \mathbf{s}^*(\lambda))$ with the properties $\mathbf{x}^*(\lambda) > 0$ and $\mathbf{s}^*(\lambda) > 0$. The point $\mathbf{x}^*(\lambda)$ is the optimal solution of (3)*. The limit point $(\mathbf{x}^*(0), \mathbf{y}^*(0), \mathbf{s}^*(0))$ of these solutions for $\lambda \to 0$ is the unique solution of the complementary slackness constraints* (4)*.*

Our object of study in this paper is the set of *all* solutions of the equations (5), not just those whose coordinates are real and positive. For general $\mathbf{b}$ and $\mathbf{c}$, this set is the following irreducible algebraic curve. The *central curve* is the Zariski closure of the central path, that is, the smallest algebraic variety in $(\mathbf{x}, \mathbf{y}, \mathbf{s})$-space, $\mathbb{R}^{2n+d}$, that contains the central path. The *primal central curve* in $\mathbb{R}^n$ is obtained by projecting the central curve into $\mathbf{x}$-space. We can similarly define the *dual central curve* by projecting into $\mathbf{y}$-space or into $\mathbf{s}$-space.

Example 2. Figure 1 depicts the primal central curve for a small *transportation problem*. Here A is the 5×6 node-edge matrix of the complete bipartite graph $K_{2,3}$, as shown below:



$$\longleftrightarrow \qquad \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Here $n = 6$ and $d = 4$ because A has rank 4. We return to this example in Section 4. □

As seen in Figure 1, the primal central curve contains the central paths of every polytope in the arrangement in $\{A\mathbf{x} = \mathbf{b}\}$ defined by the coordinate hyperplanes $\{x_i = 0\}$. The union over all central curves, as the right hand side $\mathbf{b}$ varies, is an algebraic variety of dimension $d + 1$, called the *central sheet*, which will play an important role. Our analysis will rely on recent advances on the understanding of algebras generated by reciprocals of linear forms as presented in [5, 20, 27]. Matroid theory will be our language for working with these objects.

The algebro-geometric study of central paths was pioneered by Bayer and Lagarias [2, 3]. Their 1989 articles are part of the early history of interior point methods. They observed (on pages 569-571 of [3]) that the central path defines an irreducible algebraic curve in **x**-space or **y**-space, and they identified a complete intersection that has the central curve as an irreducible component. The last sentence of [3, §11] states the open problem of identifying polynomials that cut out the central curve, without any extraneous components. It is worth stressing that while one can easily find equations for the central curve from the first derivative optimality conditions on the barrier function, those yield, in general, extra spurious solutions.

In Section 4 of this article we present a complete solution to the Bayer-Lagarias problem. Under the assumption that $\mathbf{b}$ and $\mathbf{c}$ are general, while A is fixed and possibly special, we determine the prime ideal of all polynomials that vanish on the primal central curve. We express the degree of this curve as a matroid invariant read from the linear program. This yields a tight upper bound $\binom{n-1}{d}$ for the degree. For instance, the curves in Figure 1 have degree 5. We also determine the Hilbert series and arithmetic genus of its closure in $\mathbb{P}^{n-1}$.

In Section 6, we give an entirely symmetric description of the primal-dual central curve inside a product of two projective spaces. This leads to a range of results on the global geometry of our curves. In particular, we explain precisely how the central curve passes through all vertices of the hyperplane arrangement and through all the analytic centers.

In practical computations, the optimal solution to (1) is found by following a piecewise-linear approximation to the central path. Different strategies for generating the step-by-step moves correspond to different interior point methods. One way to estimate the number of Newton steps needed to reach the optimal solution is to bound the *total curvature* of the central path. This has been investigated by many authors (see e.g. [8, 18, 24, 30, 32]), the idea being that curves with small curvature are easier to approximate with line segments.

Section 5 develops our approach to estimating the total curvature of the central curve. Dedieu, Malajovich and Shub [8] noted that the total curvature of any curve $\mathcal{C}$ coincides with the arc length of the image of $\mathcal{C}$ under the *Gauss map*. Hence any bound on the degree of the *Gauss curve* translates into a bound on the total curvature. Our main result in Section 5 is a very precise bound for the degree of the Gauss curve arising from any linear program.

Our formulas and bounds in Sections 4, 5, and 6 are expressed in the language of matroid theory. A particularly important role is played by matroid invariants, such as the *Tutte polynomial*, that are associated with the matrix A. Section 3 is devoted to an introductory exposition of the required background from matroid theory and geometric combinatorics.

Section 2 offers an analysis of central curves in the plane, with emphasis on the dual formulation ($d = 2$). We shall see that planar central curves are *Vinnikov curves* [31] of degree $n-1$ that are derived from an arrangement of n lines by taking a *Renegar derivative* [21]. The total curvature of a plane curve can be bounded in terms of its number of real inflection points, and we shall derive a new bound from a classical formula due to Felix Klein [15].

What got us started on this project was our desire to understand the "snakes" of Deza, Terlaky and Zinchenko [10]. We close the introduction by presenting their curve for $n = 6$.

Example 3. Let $n = 6$, $d = 2$ and fix the following matrix, right hand side and cost vector:

$$A = \begin{pmatrix} 0 & -1 & 1 & -1 & 1 & -1 \\ -1 & \frac{1}{10} & \frac{1}{3} & \frac{100}{11} & \frac{1000}{11} & \frac{10000}{11} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

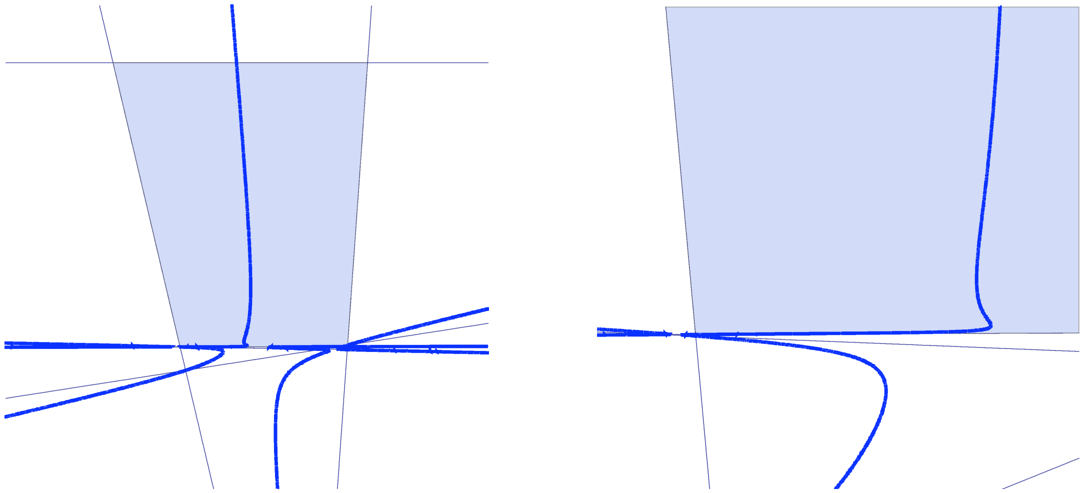


FIGURE 2. The DTZ snake with 6 constraints. On the left, a global view of the polygon and its central curve with the line $y_2 = 1$ appearing as part of the curve. On the right a close-up of the central path and its inflection points.

$$\mathbf{c}^T = \begin{pmatrix} -1 & -\frac{1}{2} & -\frac{1}{3} & -\frac{449989}{990000} & -\frac{359989}{792000} & -\frac{299989}{660000} \end{pmatrix}.$$

The resulting linear program, in its dual formulation (2), is precisely the instance in [10, Figure 2, page 218]. We redrew the central curve in Figure 2. The hexagon $P^*_{6,2}$ shown there equals $\{\mathbf{y} \in \mathbb{R}^2 : \mathbf{y}A \geq \mathbf{c}^T\}$. The analytic center of $P^*_{6,2}$ is a point with approximate coordinates $\mathbf{y} = (-0.027978..., 0.778637...)$. It has algebraic degree 10 over $\mathbb{Q}$, which indicates the level of difficulty to write exact coordinates. The optimal solution is the vertex with rational coordinates $\mathbf{y} = (y_1, y_2) = (-\frac{599700011}{1800660000}, -\frac{519989}{600220000}) = (-0.033304..., -0.00086...)$.

Following [10], we sampled many points along the central path, and we found that the total curvature of the central path equals 13.375481417.... This measurement concerns only the part of the central curve that goes from the analytic center to the optimum. Our algebraic recipe (7) for computing the central curve leads to the following polynomial:

$$(y_2 - 1)\,(276051888000000000000000\, y_2^4 + 227839918953600000000000\, y_2^3 y_1 - 1559398946696532000000000\, y_2^3 + 1688399343321073200000000 y_1 y_2^2 + 877170099134709108180000 y_2^2 - 3511691013758400000000000 y_1^2 y_2^2 - 3246213267594419313170 y_2 + 1118321629244980654800\, y_1 y_2 + 255847482441540000000\, y_1^2 y_2 - 513584318016000000000\, y_1^3 y_2 + 63370354950967001400\, y_1 + 776239200000000000000\, y_1^4 - 138563517603436200000\, y_1^2 + 2915896048475466550 - 385758735120000000000\, y_1^3).$$

This polynomial of degree 5 has a linear factor $y_2 - 1$ because the vector $\mathbf{b}$ that specifies the objective function in this dual formulation is parallel to the first column of A. Thus the curve in Figure 2 has degree 4, and its defining irreducible polynomial is the second factor. When the cost vector $\mathbf{b}$ is replaced by a vector that is not parallel to a column of A then the output of the same calculation (to be explained in Section 4) is an irreducible polynomial of degree 5. In other words, for almost all choices of $\mathbf{b}$, the central curve is a quintic curve.

While most studies in optimization focus only on just the small portion of the curve that runs from the analytic center to the optimum, we argue here that the algebraic geometry of the entire curve reveals a more complete and interesting picture. The entire central curve is a quintic that passes through all vertices of the line arrangement defined by the six edges of the polygon. As we shall see, the central curve passes through the analytic centers of all bounded cells (Theorem 28) and it is topologically a nested set of ovals (Proposition 4). $\square$

2. Plane Curves

When the central curve lives in a plane, the curve is cut out by a single polynomial equation. This occurs for the dual curve when $d = 2$ and the primal curve when $n = d - 2$. We now focus on the dual curve ($d = 2$). This serves as a warm-up to the full derivation of all equations in Section 4. In this section we derive the equations of the central curve from first principles, we show that these curves are hyperbolic and Renegar derivatives of a product of lines, and we use this structure to bound the average total curvature of the curve.

Let $A = (a_{ij})$ be a fixed $2 \times n$ matrix of rank 2, and consider arbitrary vectors $\mathbf{b} = (b_1, b_2)^T \in \mathbb{R}^2$ and $\mathbf{c} = (c_1, \ldots, c_n)^T \in \mathbb{R}^n$. Here the $\mathbf{y}$-space is the plane $\mathbb{R}^2$ with coordinates (y_1, y_2). The central curve is the Zariski closure in this plane of the parametrized path

$$\mathbf{y}^*(\lambda) \;=\; \underset{\mathbf{y}\in\mathbb{R}^2}{\operatorname{argmin}} \;\; b_1 y_1 + b_2 y_2 - \lambda \sum_{i=1}^{n} \log(a_{1i} y_1 + a_{2i} y_2 + c_i).$$

The conditions for optimality are obtained by setting the first partial derivatives to zero:

$$0 \;=\; b_1 - \lambda \sum_{i=1}^{n} \frac{a_{1i}}{a_{1i}y_1 + a_{2i}y_2 + c_i} \qquad \text{and} \qquad 0 \;=\; b_2 - \lambda \sum_{i=1}^{n} \frac{a_{2i}}{a_{1i}y_1 + a_{2i}y_2 + c_i}.$$

Multiplying these equations by b_2/λ or b_1/λ gives

$$\frac{b_1 b_2}{\lambda} \;=\; \sum_{i=1}^{n} \frac{b_2 a_{1i}}{a_{1i}y_1 + a_{2i}y_2 + c_i} \;=\; \sum_{i=1}^{n} \frac{b_1 a_{2i}}{a_{1i}y_1 + a_{2i}y_2 + c_i}. \tag{6}$$

This eliminates the parameter λ and we are left with the equation on the right. By clearing denominators, we get a single polynomial C that vanishes on the central curve in $\mathbf{y}$-space:

$$C(y_1, y_2) \;=\; \sum_{i\in\mathcal{I}} (b_1 a_{2i} - b_2 a_{1i}) \prod_{j\in\mathcal{I}\backslash\{i\}} (a_{1j} y_1 + a_{2j} y_2 + c_j), \tag{7}$$

where $\mathcal{I} = \{i \,:\, b_1 a_{2i} - b_2 a_{1i} \neq 0\}$. We see that the degree of $C(\mathbf{y})$ is $|\mathcal{I}| - 1$. This equals $n-1$ for generic $\mathbf{b}$. In our derivation we assumed that λ is non-zero but the resulting equation is valid on the Zariski closure, which includes the important points with parameter $\lambda = 0$.

We consider the closure $\mathcal{C}$ of the central curve in the complex projective plane $\mathbb{P}^2$ with coordinates $[y_0 : y_1 : y_2]$. Thus $\mathcal{C}$ is the complex projective curve defined by $y_0^{|\mathcal{I}|-1} C(\frac{y_1}{y_0}, \frac{y_2}{y_0})$.

Proposition 4. *The curve $\mathcal{C}$ is* hyperbolic *with respect to the point $[0 : -b_2 : b_1]$. This means that every line in $\mathbb{P}^2(\mathbb{R})$ passing through this special point meets $\mathcal{C}$ only in real points.*

Proof. Any line passing through the point $[0 : -b_2 : b_1]$ (except $\{y_0 = 0\}$) has the form $\{b_1 y_1 + b_2 y_2 = b_0 y_0\}$ for some $b_0 \in \mathbb{R}$. On such a line the objective function value of our linear program (2) is constant. See the left picture in Figure 3. We shall see in Remark 29 that, for any $b_0 \in \mathbb{R}$, the line meets $\mathcal{C}$ in $d_{\mathbf{s}} = \deg(\mathcal{C})$ real points. This proves the claim. □

Hyperbolic curves are also known as *Vinnikov curves*, in light of Vinnikov's seminal work [17, 31] relating them to semidefinite programming [23]. Semidefinite programming has been generalized to hyperbolic programming, in the work of Renegar [21] and others. A key construction in hyperbolic programming is the Renegar derivative which creates a (hyperbolic) polynomial of degree $D-1$ from any (hyperbolic) polynomial of degree D. To be

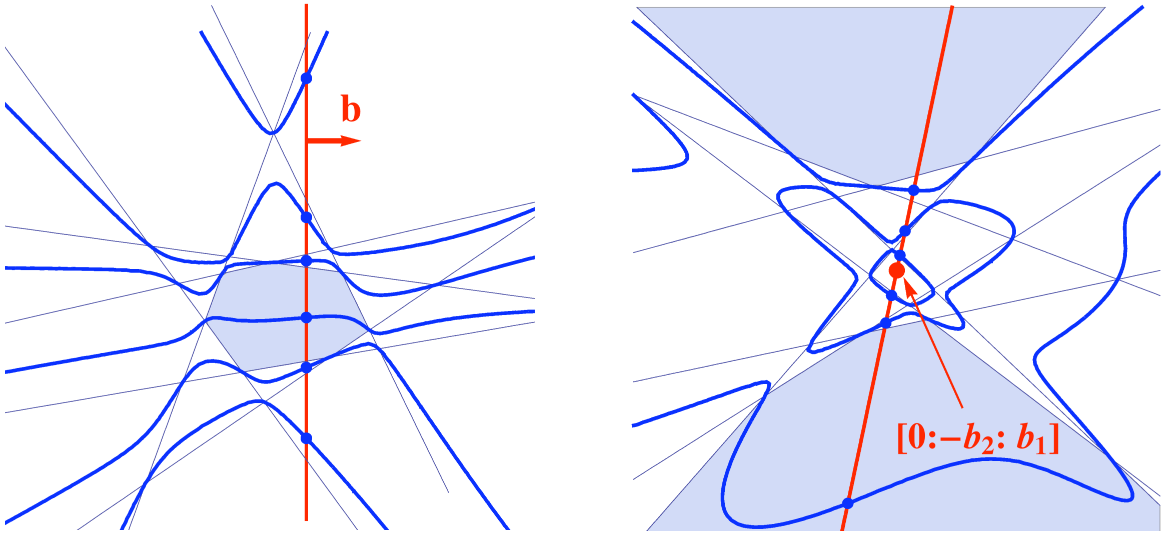


FIGURE 3. The degree-6 central path of a planar 7-gon in the affine charts $\{y_0 = 1\}$ and $\{y_2 = 1\}$. Every line passing through $[0 : -b_2 : b_1]$ intersects the curve in 6 real points, showing the real curve to be 3 completely-nested ovals.

precise, the *Renegar derivative* of a homogeneous polynomial f with respect to a point $\mathbf{e}$ is

$$R_{\mathbf{e}}f(\mathbf{y}) \;=\; \left(\frac{\partial}{\partial t} f(\mathbf{y}+t\mathbf{e})\right)|_{t=0}.$$

Renegar derivatives correspond to the *polar curves* of classical algebraic geometry [11, §1.1].

The Renegar derivative of $f = \prod_{i\in\mathcal{I}}(a_{1i}y_1 + a_{2i}y_2 + c_iy_0)$ with $\mathbf{e} = (0, -b_2, b_1)$ is seen to be

$$(8) \qquad R_{\mathbf{e}}f(\mathbf{y}) \;=\; \sum_{i\in\mathcal{I}}(b_1a_{1i} - b_2a_{2i}) \prod_{j\in\mathcal{I}\backslash\{i\}}(a_{1j}y_1 + a_{2j}y_2 + c_jy_0) \;=\; C(\mathbf{y}).$$

In words: the central curve $\mathcal{C}$ is the Renegar derivative, taken with respect to the cost function, of the product of the linear forms that define the convex polygon of feasible points.

The product of linear forms $f = \prod_i(a_{1i}y_1 + a_{2i}y_2 + c_iy_0)$ is a hyperbolic polynomial. Renegar [21] shows that if f is hyperbolic with respect to $\mathbf{e}$ then so is $R_{\mathbf{e}}f$. This yields a second proof for Proposition 4, stating that $C(\mathbf{y})$ is hyperbolic with respect to $[0 : -b_2 : b_1]$.

Proposition 4 is visualized in Figure 3. The picture on the right is obtained from the picture on the left by a projective transformation. The point at infinity which represents the cost function is now in the center of the diagram. In this rendition, the central curve consists of three nested ovals around that point, highlighting the salient features of a Vinnikov curve. This beautiful geometry is found not just in the dual picture but also in the primal picture:

Remark 5. If $d = n - 2$ then the primal central curve lies in the plane $\{A\mathbf{x} = \mathbf{b}\}$. The conditions for optimality of (1) state that the vector $\nabla(\sum_i \log(x_i)) = (x_1^{-1}, \ldots, x_n^{-1})$ is in the span of $\mathbf{c}$ and the rows of A. The Zariski closure of such $\mathbf{x}$ is the *central sheet*, to be studied in Section 4. Here, the central sheet is the hypersurface in $\mathbb{R}^n$ with defining polynomial

$$(9) \qquad \det\begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ c_1 & c_2 & \cdots & c_n \\ x_1^{-1} & x_2^{-1} & \cdots & x_n^{-1}\end{pmatrix} \cdot \prod_{i\in\mathcal{I}} x_i,$$

where A_i is the ith column of A and $\mathcal{I} = \{ i : \{\binom{A_j}{\mathbf{c}_j}\}_{j\in[n]\setminus i}$ are linearly independent$\}$. We see that the degree of this hypersurface is $|\mathcal{I}| - 1$, so it is $n-1$ for generic A. Intersecting this surface with the plane $\{A\cdot\mathbf{x} = \mathbf{b}\}$ gives the primal central curve, which is hence a curve of degree $|\mathcal{I}| - 1$. The corresponding complex projective curve in $\mathbb{P}^2 = \{A\mathbf{x} = x_0\mathbf{b}\} \subset \mathbb{P}^{n-1}$ is hyperbolic with respect to the point $[0 : \mathbf{v}]$ in $\mathbb{P}^{n-1}$, where $\mathbf{v}$ spans the kernel of $\binom{A}{\mathbf{c}}$. □

It is of importance for interior point algorithms to know the exact total curvature, formally introduced in equation (27), of the central path of a linear program (see [8, 18, 24, 30, 32]). Deza *et al.* [10] proved that even for $d = 2$ the total curvature grows linearly in n, and they conjectured that the total curvature is no more than $2\pi n$. They named this conjecture the *continuous Hirsch conjecture* because of its similarity with the discrete simplex method analogue (see [9]). In Section 5 we derive general bounds for total curvature, but for plane curves, we can exploit an additional geometric feature, namely, *inflection points*.

Benedetti and Dedò [4] derived a general bound for the total curvature of a real plane curve in terms of its number of inflection points and its degree. We can make this very explicit for our central path $\{\mathbf{y}^*(\lambda) : \lambda \in \mathbb{R}_{\geq 0}\}$. Its total curvature is bounded above by

$$\text{total curvature of the central path} \quad \leq \quad \pi \cdot (\text{the number of inflection points} + 1). \tag{10}$$

To see this, note that the Gauss map γ takes the curve into S^1. As λ decreases from ∞ to 0, the cost function $\mathbf{b}^T\mathbf{y}^*(\lambda)$ strictly increases. This implies that, for any point $\mathbf{y}^*(\lambda)$ on the curve, its image under the Gauss map has positive inner product with $\mathbf{b}$, that is, $\mathbf{b}^T\gamma(\mathbf{y}^*(\lambda)) \geq 0$. Thus the image of the Gauss map is restricted to a half circle of S^1, and it cannot wrap around S^1. This shows that the Gauss map can achieve a length of at most π before it must "change direction", which happens only at inflection points of the curve.

It is known that the total number of (complex) inflection points of a plane curve of degree D is at most $3D(D-2)$. For real inflection points, there is an even better bound:

Proposition 6 (A classical result of Felix Klein [15])**.**
The number of real inflection points of a plane curve of degree D is at most $D(D-2)$.

This provides only a quadratic bound for the total curvature of the central path in terms of its degree, but it does allow us to improve known bounds for the average total curvature. The *average total curvature* of the central curve of a hyperplane arrangement is the average, over all bounded regions of the arrangement, of the total curvature of the central curve in that region. Dedieu *et al.* [8] proved that the average total curvature in a simple arrangement (*i.e.* for a generic matrix A) defined by n hyperplanes in dimension d is not greater than $2\pi d$. When $d = 2$, we can use Proposition 6 to improve this bound by a factor of two.

Theorem 7. *The average total curvature of a central curve in the plane is at most 2π.*

Proof. The central curve for n general lines in $\mathbb{R}^2$ has degree $n-1$, and it consists of $n-1$ connected components. From the argument above and Klein's theorem, this shows that

$$\begin{aligned}\sum_{i=1}^{n-1}(\text{curvature of the } i\text{th component}) \;&\leq\; \sum_{i=1}^{n-1}\pi(\#\text{inflection points on the } i\text{th component} + 1)\\ &\leq\; \pi(n-1)(n-2).\end{aligned}$$

Our arrangement of n general lines has $\binom{n-1}{2}$ bounded regions. The average total curvature over each of these regions is therefore at most $\pi(n-1)(n-2)/\binom{n-1}{2} = 2\pi$. □

To bound the curvature of just the central path, we need to bound the number of inflection points appearing on that piece of the curve. This leads to the following open problem:

Question 8. What is the largest number of inflection points on a single oval of a (hyperbolic) curve of degree D in the real plane? In particular, is this number *linear* in the degree D?

It has been noted in the literature on interior point algorithms (e.g. [18,30]) that the final stretch of the central path, shortly before reaching a non-degenerate optimal solution, is close to linear. In other words, locally, at a simple vertex of our arrangement, the central curve is well approximated by its tangent line. In closing, we wish to point out a way to see this tangent line in our derivation of the equation (7). Namely, let i and j be the indices of the two lines passing through that simple vertex. Then the equation (6) takes the following form:

$$\frac{b_1 a_{2i} - b_2 a_{1i}}{a_{1i}y_1 + a_{2i}y_2 + c_i} \quad = \quad \frac{b_1 a_{2j} - b_2 a_{1j}}{a_{1j}y_1 + a_{2j}y_2 + c_j} \quad + \quad \text{much smaller terms.} \tag{11}$$

Dropping the small terms and clearing denominators reveals the equation of the tangent line.

3. Concepts from Matroid Theory

We have seen in the previous section that the geometry of a central curve in the plane is intimately connected to that of the underlying arrangement of constraint lines. For instance, the degree of the central curve, $|\mathcal{I}|-1$, is one less than the number of constraints not parallel to the cost function. This systematic study of this kind of combinatorial information, encoded in a geometric configuration of vectors or hyperplanes, is the subject of *matroid theory*.

Matroid theory will be crucial for stating and proving our results in the rest of this paper. This section offers an exposition of the relevant concepts. Of course, there is already a well-established connection between matroid theory and linear optimization (e.g., as outlined in [16] or in oriented matroid programming [1]). Our paper sets up yet another connection.

The material that follows is well-known in algebraic combinatorics, but less so in optimization, so we aim to be reasonably self-contained. Most missing details can be found in [6, 7] and the other surveys in the same series. We consider an r-dimensional linear subspace $\mathcal{L}$ of the vector space K^n with its fixed standard basis. Here K is any field. Typically, $\mathcal{L}$ will be given to us as the row space of an $r \times n$-matrix. The kernel of that matrix is denoted by $\mathcal{L}^\perp$. This is a subspace of dimension $n-r$ in K^n. We write $x_1, \ldots, x_n$ for the restriction of the standard coordinates on K^n to the subspace $\mathcal{L}$, and $s_1, \ldots, s_n$ for their restriction to $\mathcal{L}^\perp$.

The two subspaces $\mathcal{L}$ and $\mathcal{L}^\perp$ specify a dual pair of matroids, denoted $M(\mathcal{L})$ and $M(\mathcal{L}^\perp)$, on the set $[n] = \{1, \ldots, n\}$. The matroid $M(\mathcal{L})$ has rank r and its dual $M(\mathcal{L}^\perp) = M(\mathcal{L})^*$ has rank $n-r$. We now define the first matroid $M = M(\mathcal{L})$ by way of its *independent sets*. A subset I of $[n]$ is *independent* in M if the linear forms in $\{x_i : i \in I\}$ are linearly independent on $\mathcal{L}$. Maximal independent sets are called *bases*. These all have cardinality r. A subset I is *dependent* if it is not independent. It is a *circuit* if it is minimally dependent. For example, if K is an infinite field and $\mathcal{L}$ is a general subspace then we get the *uniform matroid* $M = U_{r,n}$ whose bases are all r-subsets in $[n]$ and whose circuits are all $(r+1)$-subsets of $[n]$. The bases of the dual matroid M^* are the complementary sets $[n]\backslash B$ where B is any basis of M.

The most prominent invariant in the theory of matroids is the Tutte polynomial (see [7]). To define this, we assume the usual order $1 < 2 < \cdots < n$ on the ground set $[n]$, but it turns out that the definition is independent of which ordering is chosen. Let B be a basis of M and consider any element $p \in [n]\backslash B$. Then the set $B \cup \{p\}$ is dependent, and it contains a unique circuit C. The circuit C contains p. We say that p is *externally active* for B if p is the least element in C, in symbols, $p = \min(C)$. Similarly, an element $p \in B$ is called *internally active* if p is the least element that completes the independent set $B\backslash\{p\}$ to a basis of the matroid. Let $ia(B,w)$ and $ea(B,w)$ denote the number of internally and externally active elements associated to the basis B. Then the *Tutte polynomial* of M is the bivariate polynomial

$$T_M(x,y) \quad = \sum_{B\in\mathcal{B}(M)} x^{ia(B,w)}y^{ea(B,w)}.$$

It satisfies the duality relation $T_{M^*}(x,y) = T_M(y,x)$; see [7, Proposition 6.2.4].

An important specialization of the Tutte polynomial is the *characteristic polynomial*

$$\chi_M(t) \;=\; (-1)^r \cdot T_M(1-t,0).$$

This univariate polynomial plays a key role in the enumerative theory of hyperplane arrangements [33]. The characteristic polynomial can be rewritten in terms of the lattice of flats of the matroid. Its coefficients are values of the *Möbius function* λ_M. An important number for us is the *Möbius invariant* [7, Eq. (6.21)]. It is defined in terms of the following evaluations:

$$\mu(M) \;=\; (-1)^r \cdot T_M(1,0) \;=\; \chi_M(0) \;=\; \lambda_M(\emptyset,[n]). \tag{12}$$

Throughout this paper we use the absolute value $|\mu(M)|$ and call it the *Möbius number* of M.

In algebraic combinatorics, one regards the independent sets of the matroid M as the faces in a simplicial complex of dimension $r-1$. We write $f_i(M)$ for the number of i-dimensional faces of this *independence complex* of M. Equivalently, $f_i(M)$ is the number of independent sets of cardinality $i+1$. By [6, Proposition 7.4.7 (i)], the (reduced) *Euler characteristic*, $-1+f_0(M)-f_1(M)+\cdots+f_{r-1}(M)$, of the independence complex of a matroid M coincides up to sign with the Möbius invariant of the dual matroid M^*:

$$\mu(M^*) \quad = \quad (-1)^{r-1}\left(-1+\sum_{i=0}^{r-1}(-1)^i f_i(M)\right). \tag{13}$$

We apply this to compute the Möbius number of the uniform matroid $M = U_{r,n}$ as follows:

$$|\mu(U_{r,n})| \quad = \quad |\mu(U^*_{n-r,n})| \quad = \quad \sum_{i=-1}^{n-r-1}(-1)^{n-r+i+1}\binom{n}{i+1} \quad = \quad \binom{n-1}{r-1}. \tag{14}$$

This binomial coefficient is an upper bound on $|\mu(M)|$ for any rank r matroid M on $[n]$.

A useful characterization of the Möbius number involves another simplicial complex on $[n]$ associated with the matroid M. As before, we fix the standard ordering $1 < 2 < \cdots < n$ of $[n]$, but any other ordering will do as well. A *broken circuit* of M is any subset of $[n]$ of the form $C\backslash\{\min(C)\}$ where C is a circuit. The *broken circuit complex* of M is the simplicial complex $\mathrm{Br}(M)$ whose minimal non-faces are the broken circuits. Hence, a subset of $[n]$ is a face of $\mathrm{Br}(M)$ if it does not contain any broken circuit. It is known that $\mathrm{Br}(M)$ is a shellable simplicial complex of dimension $r-1$ (see Theorem 7.4.3 in [6]). We can recover the Möbius number of M (not that of M^*) as follows. Let $f_i = f_i(\mathrm{Br}(M))$ denote the number of i-dimensional faces of the broken circuit complex $\mathrm{Br}(M)$. The corresponding h-vector $(h_0, h_1, \ldots, h_{r-1})$ can be read off from any shelling (cf. [6, §7.2] and [25, §2]). It satisfies

$$\sum_{i=0}^{r-1}\frac{f_{i-1}z^i}{(1-z)^i} \quad = \quad \frac{h_0 + h_1 z + h_2 z^2 + \cdots + h_{r-1}z^{r-1}}{(1-z)^r}. \tag{15}$$

The relation between f-vector and h-vector holds for any simplicial complex [25]. The next identity follows from [6, Eq. (7.15)] and the discussion on shelling polynomials in [6, §7.2]:

$$h_0 + h_1 z + h_2 z^2 + \cdots + h_{r-1} z^{r-1} \quad = \quad z^r \cdot T_M(1/z, 0). \tag{16}$$

The rational function (15) is the *Hilbert series* (see [25]) of the *Stanley-Reisner ring* of the broken circuit complex $\mathrm{Br}(M)$. The defining ideal of the Stanley-Reisner ring is generated by the monomials $\prod_{i \in C \backslash \{\min(C)\}} x_i$ representing broken circuits. Proudfoot and Speyer [20] constructed a *broken circuit ring*, which is the quotient of $K[x_1, \ldots, x_n]$ modulo a prime ideal whose initial ideal is precisely this monomial ideal. Hence (15) is also the Hilbert series of the ring in [20]. In particular, the Möbius number is the common degree of both rings:

$$|\mu(M)| \quad = \quad h_0 + h_1 + h_2 + \cdots + h_{r-1}. \tag{17}$$

This result is obtained from setting $z = 1$ in (16) and applying the identity (12).

The Möbius number is important to us because it computes the degree of the central curve of the primal linear program (1). See Theorem 12 below. The matroid we need there has rank $r = d+1$ and it is denoted $M_{A,\mathbf{c}}$. The corresponding r-dimensional vector subspace of K^n is denoted $\mathcal{L}_{A,\mathbf{c}}$. It is spanned by the rows of A and the vector $\mathbf{c}$. We use the notation

$$|\mu(A, \mathbf{c})| \;:=\; |\mu(M_{A,\mathbf{c}})| \;=\; |\mu(M(\mathcal{L}_{A,\mathbf{c}}))|. \tag{18}$$

The constraint matrix A has real entries and it has n columns and rank d. We write $\mathbb{Q}(A)$ for the subfield of $\mathbb{R}$ generated by the entries of A. In Section 4 we shall assume that the coordinates b_i and c_j of the right hand side $\mathbf{b}$ and the cost vector $\mathbf{c}$ are algebraically independent over $\mathbb{Q}(A)$, and we work over the rational function field $K = \mathbb{Q}(A)(\mathbf{b}, \mathbf{c})$ generated by these coordinates. This will ensure that all our algebraic results derived remain valid under almost all other specializations $K \to \mathbb{R}$ of these coordinates to the field of real numbers.

We now present a geometric interpretation of the Möbius number $|\mu(M)|$ in terms of hyperplane arrangements. The arrangements we discuss often appear in linear programming, in the context of pivoting algorithms, such as the criss-cross method [12]. Fix any r-dimensional linear subspace $\mathcal{L} \subset \mathbb{R}^n$ and the associated rank r matroid $M = M(\mathcal{L})$. For our particular application in Section 4, we would take $r = d+1$, $\mathcal{L} = \mathcal{L}_{A,\mathbf{c}}$ and $M = M_{A,\mathbf{c}}$.

Let $\mathbf{u}$ be a generic vector in $\mathbb{R}^n$ and consider the $(n-r)$-dimensional affine space $\mathcal{L}^\perp + \mathbf{u}$ of $\mathbb{R}^n$. The equations $x_i = 0$ define n hyperplanes in this affine space. These hyperplanes do not meet in a common point. In fact, the resulting arrangement $\{x_i = 0\}_{i \in [n]}$ in $\mathcal{L}^\perp + \mathbf{u}$ is *simple*, which means that no point lies on more than $n - r$ of the n hyperplanes. The vertices of this hyperplane arrangement are in bijection with the bases of the matroid M. The complements of the hyperplanes are convex polyhedra; they are the *regions* of the arrangement. Each region is either bounded or unbounded, and we are interested in the bounded regions. These bounded regions are the feasibility regions for the linear programs with various sign restrictions on the variables x_i (one of the regions is $x_i \geq 0$ for all i). Proposition 6.6.2 in [7], which is based on results of Zaslavsky [33], establishes the following:

$$|\mu(M)| \;=\; \#\,\text{bounded regions of the hyperplane arrangement } \{x_i = 0\}_{i \in [n]} \text{ in } \mathcal{L}^\perp + \mathbf{u}. \tag{19}$$

For $M = M_{A,\mathbf{c}}$, the affine linear space $\mathcal{L}^\perp + \mathbf{u}$ is cut out by the equations $\binom{A}{\mathbf{c}}\mathbf{x} = \binom{A}{\mathbf{c}}\mathbf{u}$. It is instructive to examine equation (19) for the case of the uniform matroid $M = U_{r,n}$. Here we are given n general hyperplanes through the origin in $\mathbb{R}^{n-r}$, and we replace each of them by a random parallel translate. The resulting arrangement of n affine hyperplanes in $\mathbb{R}^{n-r}$ creates precisely $\binom{n-1}{r-1}$ bounded regions, as promised by the conjunction of (14) and (19).

For example, if $r = 1$ then $|\mu(U_{1,n})| = 1$, since n hyperplanes in $\mathbb{R}^{n-1}$ can create only one bounded region. At the other hand, if $r = n-1$ then our n affine hyperplanes are just n points on a line and these will create $|\mu(U_{n-1,n})| = n-1$ bounded line segments.

For a relevant instance of the latter case consider Example 2, with A the displayed $5{\times}6$-matrix of rank $d = 4$, or the instance in Figure 3. Here, $n = 6$, $r = d{+}1 = 5$, and $M_{A,\mathbf{c}} = U_{5,6}$ is the uniform matroid. Its Möbius number equals $|\mu(A, \mathbf{c})| = |\mu(U_{5,6})| = 5$. This number 5 counts the bounded segments on the vertical red line on the left in Figure 3. Note that the relevant matroid for Example 2 is not, as one might expect, the graphic matroid of $K_{2,3}$. For higher-dimensional problems the matroids $M_{A,\mathbf{c}}$ we encounter are typically non-uniform.

4. Equations defining the central curve

In this section we determine the prime ideal of the central curve of the primal linear program (1). As a consequence we obtain explicit formulas for the degree, genus and Hilbert function of the projective closure of the primal central curve. These results resolve the problem stated by Bayer and Lagarias at the end of [3, §11]. Let $\mathcal{L}_{A,\mathbf{c}}$ be the subspace of K^n spanned by the rows of A and the vector $\mathbf{c}$. Our ground field is $K = \mathbb{Q}(A)(\mathbf{b}, \mathbf{c})$ as above. We define the *central sheet* to be the coordinate-wise reciprocal $\mathcal{L}^{-1}_{A,\mathbf{c}}$ of that linear subspace. In precise terms, we define $\mathcal{L}^{-1}_{A,\mathbf{c}}$ to be the Zariski closure in the affine space $\mathbb{C}^n$ of the set

$$\left\{ \left(\frac{1}{u_1}, \frac{1}{u_2}, \ldots, \frac{1}{u_n} \right) \in \mathbb{C}^n \;:\; (u_1, u_2, \ldots, u_n) \in \mathcal{L}_{A,\mathbf{c}} \quad \text{and} \quad u_i \neq 0 \;\text{ for }\; i = 1, \ldots, n \right\}. \tag{20}$$

Lemma 9. *The Zariski closure of the primal central path* $\{\mathbf{x}^*(\lambda) : \lambda \in \mathbb{R}_{\geq 0}\}$ *is equal to the intersection of the central sheet* $\mathcal{L}^{-1}_{A,\mathbf{c}}$ *with the affine-linear subspace defined by* $A \cdot \mathbf{x} = \mathbf{b}$*.*

Proof. We eliminate $\mathbf{s}, \mathbf{y}$ and λ from the equations $A^T\mathbf{y} - \mathbf{s} = \mathbf{c}$ and $x_i s_i = \lambda$ as follows. We first replace the coordinates of $\mathbf{s}$ by $s_i = \lambda / x_i$. The linear system becomes $A^T\mathbf{y} - \lambda\mathbf{x}^{-1} = \mathbf{c}$. This condition means that $\mathbf{x}^{-1} = (\frac{1}{x_1}, \ldots, \frac{1}{x_n})$ lies in the linear space $\mathcal{L}_{A,\mathbf{c}}$ spanned by $\mathbf{c}$ and the rows of A. The result of the elimination says that $\mathbf{x}$ lies in the central sheet $\mathcal{L}^{-1}_{A,\mathbf{c}}$. □

The linear space $\{A\mathbf{x} = \mathbf{b}\}$ has dimension $n - d$, and we write $I_{A,\mathbf{b}}$ for its linear ideal. The central sheet $\mathcal{L}^{-1}_{A,\mathbf{c}}$ is an irreducible variety of dimension $d+1$, and we write $J_{A,\mathbf{c}}$ for its prime ideal. Both $I_{A,\mathbf{b}}$ and $J_{A,\mathbf{c}}$ are ideals in $K[x_1, \ldots, x_n]$. We argue the following is true:

Lemma 10. *The prime ideal of polynomials that vanish on the central curve* $\mathcal{C}$ *is* $I_{A,\mathbf{b}} + J_{A,\mathbf{c}}$*. The degree of both* $\mathcal{C}$ *and the central sheet* $\mathcal{L}^{-1}_{A,\mathbf{c}}$ *coincides with the Möbius number* $|\mu(A, \mathbf{c})|$*.*

Proof. The intersection of the linear space $\{A\mathbf{x} = \mathbf{b}\}$ with the central sheet is the variety of the ideal $I_{A,\mathbf{b}} + J_{A,\mathbf{c}}$. This ideal is prime because $\mathbf{b}$ and $\mathbf{c}$ are generic over $\mathbb{Q}(A)$. The intersection is the central curve. In Proposition 11 we show that the degree of the central sheet is $|\mu(A, \mathbf{c})|$, so here it only remains to show that this is the degree of the central curve as well. For a generic vector $(\mathbf{b}, c_0) \in \mathbb{R}^{d+1}$, we consider the hyperplane arrangement induced by $\{x_i = 0\}$ in the affine space $\{\binom{A}{\mathbf{c}}\mathbf{x} = \binom{\mathbf{b}}{c_0}\}$. The number of bounded regions of this hyperplane arrangement equals the Möbius number $|\mu(A, \mathbf{c})|$, as seen in (19).

Note that $|\mu(A, \mathbf{c})|$ does not depend on $\mathbf{c}$, since this vector is generic over $\mathbb{Q}(A)$. Each of the $|\mu(A, \mathbf{c})|$ bounded regions in the $(n-d-1)$-dimensional affine space $\{\binom{A}{\mathbf{c}}\mathbf{x} = \binom{\mathbf{b}}{c_0}\}$ contains a unique point maximizing $\sum_i \log(|x_i|)$. This point is the *analytic center* of that region. Each such analytic center lies in $\mathcal{L}_{A,\mathbf{c}}^{-1}$, and thus on the central curve. This shows that the intersection of the central curve with the plane $\{\mathbf{c}^T\mathbf{x} = c_0\}$ contains $|\mu(A,\mathbf{c})|$ points.

Bézout's Theorem implies that the degree of a variety $V \subset \mathbb{C}^n$ is an upper bound for the degree of its intersection $V \cap H$ with an affine-linear space H, provided that $n + \dim(V \cap H) = \dim(V) + \dim(H)$. We use this theorem for two inequalities; first, that the degree of $\mathcal{L}_{A,\mathbf{c}}^{-1}$ bounds the degree of the central curve $\mathcal{C}$, and second that the degree of $\mathcal{C}$ bounds the number of its intersection points with $\{\mathbf{c}\cdot\mathbf{x} = c_0\}$. To summarize, we have shown:

$$|\mu(A,\mathbf{c})| \;\leq\; \#(\mathcal{C} \cap \{\mathbf{c}\cdot x = c_0\}) \;\leq\; \deg(\mathcal{C}) \;\leq\; \deg(\mathcal{L}_{A,\mathbf{c}}^{-1}) \;=\; |\mu(A,\mathbf{c})|.$$

From this we conclude that $|\mu(A,\mathbf{c})|$ is the degree of the primal central curve $\mathcal{C}$. □

At this point we are left with the problem of computing the minimal generators and the degree of the homogeneous ideal $J_{A,c}$. Luckily, this has already been done for us in the literature through matroid tools. The following proposition was proved by Proudfoot and Speyer [20] and it refines an earlier result of Terao [27]. See also [5]. The paper [26] indicates how our results can be extended from linear programming to semidefinite programming.

Proposition 11 (Proudfoot-Speyer [20])**.** *The degree of the central sheet $\mathcal{L}_{A,\mathbf{c}}^{-1}$, regarded as a variety in complex projective space, coincides with the Möbius number $|\mu(A,\mathbf{c})|$. Its prime ideal $J_{A,\mathbf{c}}$ is generated by a universal Gröbner basis consisting of all homogeneous polynomials*

$$\sum_{i\in \mathrm{supp}(v)} v_i \cdot \prod_{j \in \mathrm{supp}(v)\backslash\{i\}} x_j, \tag{21}$$

where $\sum v_i x_i$ runs over non-zero linear forms of minimal support that vanish on $\mathcal{L}_{A,\mathbf{c}}$.

Proof. The construction in [20] associates the ring $K[x_1,\ldots,x_n]/J_{A,\mathbf{c}}$ to the linear subspace $\mathcal{L}_{A,\mathbf{c}}$ of K^n. Theorem 4 of [20] says that the homogeneous polynomials (21) form a universal Gröbner bases for $J_{A,\mathbf{c}}$. As argued in [20, Lemma 2], this means that the ring degenerates to the Stanley-Reisner ring of the broken circuit complex $\mathrm{Br}(M_{A,\mathbf{c}})$. Hence, by our discussion in Section 3, or by [20, Prop. 7], the Hilbert series of $K[x_1,\ldots,x_n]/J_{A,\mathbf{c}}$ is the rational function (15), and the degree of $J_{A,\mathbf{c}}$ equals $|\mu(A,\mathbf{c})|$ as seen in (17). The ideal $J_{A,\mathbf{c}}$ is radical, since its initial ideal is square-free, and hence it is prime because its variety $\mathcal{L}_{A,\mathbf{c}}^{-1}$ is irreducible. □

The polynomials in (21) correspond to the circuits of the matroid $M_{A,\mathbf{c}}$. There is at most one circuit contained in each $(d+2)$-subset of $\{x_1,\ldots,x_n\}$, so their number is at most $\binom{n}{d+2}$. If the matrix A is generic then $M_{A,\mathbf{c}}$ is uniform and, by (14), its Möbius number equals

$$|\mu(A,\mathbf{c})| \;=\; \binom{n-1}{d}.$$

For arbitrary matrices A, this binomial coefficient furnishes an upper bound on the Möbius number $|\mu(A,\mathbf{c})|$. We are now prepared to conclude with the main theorem of this section. The analogous equations for the dual central curve are given in Proposition 22 in Section 6.

Theorem 12. *The degree of the primal central path of (1) is the Möbius number $|\mu(A,\mathbf{c})|$ and is hence at most $\binom{n-1}{d}$. The prime ideal of polynomials that vanish on the primal central path is generated by the circuit polynomials (21) and the d linear polynomials in $A\mathbf{x}-\mathbf{b}$.*

Proof. This is an immediate consequence of Lemmas 9 and 10 and Proposition 11. □

It is convenient to write the circuit equations (21) in the following determinantal representation. Suppose that A has format $d \times n$ and its rows are linearly independent. Then the linear forms of minimal support that vanish on $\mathcal{L}_{A,\mathbf{c}}$ are the $(d+2) \times (d+2)$-minors of the $(d+2) \times n$ matrix $\begin{pmatrix} A \\ \mathbf{c} \\ \mathbf{x} \end{pmatrix}$. This gives the following concise description of our prime ideal $J_{A,\mathbf{c}}$:

$$J_{A,\mathbf{c}} \quad = \quad I_{\mathrm{num},d+2} \begin{pmatrix} A \\ \mathbf{c} \\ \mathbf{x}^{-1} \end{pmatrix} \tag{22}$$

where $\mathbf{x}^{-1} = (x_1^{-1}, \ldots, x_n^{-1})$ and the operator $I_{\mathrm{num},d+2}$ extracts the numerators of the $(d{+}2) \times (d{+}2)$-minors of the matrix. Note that there are $\binom{n}{d+2}$ such minors but they need not be distinct. For example, the generator of $J_{A,\mathbf{c}}$ coming from the leftmost minor equals

$$\det \begin{pmatrix} A_1 & A_2 & \ldots & A_{d+2} \\ c_1 & c_2 & \ldots & c_{d+2} \\ x_1^{-1} & x_2^{-1} & \ldots & x_{d+2}^{-1} \end{pmatrix} \cdot \prod_{i \in \mathcal{I}} x_i,$$

where $\mathcal{I}$ is the lexicographically earliest circuit of the matroid $M_{A,\mathbf{c}}$.

Example 13. Let $d = 4, n = 6$ and A the matrix in Example 2. The linear ideal is

$$I_{A,\mathbf{b}} \quad = \quad \langle\, x_1 + x_2 + x_3 - b_1\,,\; x_4 + x_5 + x_6 - b_2\,,\; x_1 + x_4 - b_3\,,\; x_2 + x_5 - b_4\,\rangle.$$

The central sheet $\mathcal{L}_{A,\mathbf{c}}^{-1}$ is the quintic hypersurface whose defining polynomial is

$$f_{A,\mathbf{c}}(\mathbf{x}) \quad = \quad \det \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ x_1^{-1} & x_2^{-1} & x_3^{-1} & x_4^{-1} & x_5^{-1} & x_6^{-1} \end{pmatrix} \cdot x_1 x_2 x_3 x_4 x_5 x_6. \tag{23}$$

The primal central curve is the plane quintic defined by the ideal $I_{A,\mathbf{b}} + \langle f_{A,\mathbf{c}} \rangle$. This ideal is prime for general choices of $\mathbf{b}$ and $\mathbf{c}$. However, this may fail for special values: the quintic on the left in Figure 1 is irreducible but that on the right decomposes into a quartic and a line. For a concrete numerical example we set $b_1 = b_2 = 3$ and $b_3 = b_4 = b_5 = 2$. Then the transportation polygon P is the regular hexagon depicted in Figure 1. Its vertices are

$$\begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}. \tag{24}$$

Consider the two transportation problems (1) given by $\mathbf{c} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ and $\mathbf{c}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$. In both cases, the last matrix in (24) is the unique optimal solution. Modulo the linear ideal $I_{A,\mathbf{b}}$ we can write the quintics $f_{A,\mathbf{c}}$ and $f_{A,\mathbf{c}'}$ as polynomials in only two variables x_1 and x_2:

$$\begin{aligned} f_{A,\mathbf{c}} \quad &= \quad 3x_1^4 x_2 + 5x_1^3 x_2^2 - 2x_1 x_2^4 - 3x_1^4 - 22x_1^3 x_2 - 15x_1^2 x_2^2 + 8x_1 x_2^3 + 2x_2^4 \\ &\qquad +18x_1^3 + 45x_1^2 x_2 - 12x_2^3 - 33x_1^2 - 22x_1 x_2 + 22x_2^2 + 18x_1 - 12x_2, \\ f_{A,\mathbf{c}'} \quad &= \quad (x_2 - 1) \cdot (2x_1^4 + 4x_1^3 x_2 + x_1^2 x_2^2 - x_1 x_2^3 - 12x_1^3 - 14x_1^2 x_2 + x_1 x_2^2 \\ &\qquad + x_2^3 + 22x_1^2 + 10x_1 x_2 - 5x_2^2 - 12x_1 + 6x_2). \end{aligned}$$

Both quintics pass through all intersection points of the arrangement of six lines. The cost matrix $\mathbf{c}$ exemplifies the generic behavior, when the quintic curve is irreducible. On the other hand, the central path for $\mathbf{c}$ is a segment on the horizontal line $x_2 = 1$ in Figure 1. □

In the remainder of this section we consider the question of what happens to the central sheet, and hence to the central path, when the cost function $\mathbf{c}$ degenerates to one of the unit vectors e_i. Geometrically this means that the cost vector becomes normal to one of the constraint hyperplanes, and the curve reflects this by breaking into irreducible components. What follows is independent of the rest of the paper and can be skipped upon first reading.

To set up our degeneration in proper algebraic terms, we work over the field $K\{\{t\}\}$ of Puiseux series over the field $K = \mathbb{Q}(A)(\mathbf{b}, \mathbf{c})$ that was used above. The field $K\{\{t\}\}$ comes with a natural t-adic valuation. Passing to the special fiber represents the process of letting the parameter t tend to 0. Our cost vector $\mathbf{c}$ has its coordinates in the Puiseux series field:

$$\mathbf{c} \;=\; \big(t^{w_1}, t^{w_2}, \ldots, t^{w_{n-1}}, 1\big) \tag{25}$$

Here $w_1 > w_2 > \cdots > w_{n-1} > 0$ are any rational numbers. We are interested in the special fiber of the central sheet $\mathcal{L}_{A,\mathbf{c}}^{-1}$. This represents the limit of the central sheet as t approaches 0. This induces a degeneration of the central curve $\mathcal{L}_{A,\mathbf{c}}^{-1} \cap \{A\mathbf{x} = \mathbf{b}\}$. We wish to see how, in that limit, the central curve breaks into irreducible curves in the affine space $\{A\mathbf{x} = \mathbf{b}\}$.

The ideal defining the special fiber of $J_{A,\mathbf{c}}$ is denoted $\mathrm{in}(J_{A,\mathbf{c}}) = J_{A,\mathbf{c}}|_{t=0}$. By a combinatorial argument as in [20], the maximal minors in (22) have the Gröbner basis property for this degeneration. Hence we obtain the prime ideal of the flat family by simply dividing each such minor by a non-negative power of t. This observation implies the following result:

Theorem 14. *The central sheet $\mathcal{L}_{A,\mathbf{c}}^{-1}$ degenerates into a reduced union of central sheets of smaller linear programming instances. More precisely, the ideal $\mathrm{in}(J_{A,\mathbf{c}})$ is radical, and it has the following representation as an intersection of ideals that are prime when A is generic:*

$$\mathrm{in}(J_{A,\mathbf{c}}) \quad = \quad \bigcap_{i=d}^{n-1} \left(I_{\mathrm{num},d+1} \begin{pmatrix} A_1 & A_2 & \cdots & A_i \\ x_1^{-1} & x_2^{-1} & \cdots & x_i^{-1} \end{pmatrix} + \langle x_{i+2}, x_{i+3}, \ldots, x_n \rangle \right) \tag{26}$$

Proof sketch. The Gröbner basis property says that $\mathrm{in}(J_{A,\mathbf{c}})$ is generated by the polynomials obtained from the maximal minors of (22) by dividing by powers of t and then setting t to zero. The resulting polynomials factor, and this factorization shows that they lie in each of the ideals on the right hand side of (26). Conversely, each element in the product of the ideals on the right hand side is seen to lie in $\mathrm{in}(J_{A,\mathbf{c}})$. To complete the proof, it then suffices to note that $\mathrm{in}(J_{A,\mathbf{c}})$ is radical because its generators form a square-free Gröbner basis. □

Example 15. Let $n = 6$ and $d = 3$. The matrix A might represent the three-dimensional Klee-Minty cube. The decomposition of the initial ideal in (26) has three components:

$$\mathrm{in}(J_{A,\mathbf{c}}) \;=\; \langle x_5,\, x_6 \rangle \cap \langle \det\begin{pmatrix} x_1A_1 & x_2A_2 & x_3A_3 & x_4A_4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, x_6 \,\rangle \cap I_{\mathrm{num},4}\begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ x_1^{-1} & x_2^{-1} & x_3^{-1} & x_4^{-1} & x_5^{-1} \end{pmatrix}.$$

For general A, the ideal $J_{A,\mathbf{c}}$ defines an irreducible curve of degree 10, namely the central path, in each of the 3-planes $\{A\mathbf{x} = \mathbf{b}\}$. The three curves in its degeneration above are irreducible of degrees 1, 3 and 6 respectively. The first is one of the lines in the arrangement of six facet planes, the second curve is the central path inside the facet defined by $x_6 = 0$, and the last curve is the central path of the polytope obtained by removing that facet. □

In general, we can visualize the degenerated central path in the following geometric fashion. We first flow from the analytic center of the polytope to the analytic center of its last facet. Then we iterate and flow from the analytic center of the facet to the analytic center of its last facet, which is a ridge of the original polytope. Then we continue inside that ridge, etc.

5. The Gauss Curve of the Central Path

The total curvature of the central path is an important quantity for the estimation of the running time of interior point methods in linear programming. In this section we relate the algebraic framework developed so far to the problem of bounding the total curvature. The relevant geometry was pioneered by Dedieu, Malajovich and Shub [8]. Following their approach, we consider the *Gauss curve* associated with the primal central path. The Gauss curve is the image of the central curve under the Gauss map, and its arc length is precisely the total curvature of the central path. Moreover, the arc length of the Gauss curve can be bounded in terms of its degree. An estimate of that degree, via the multihomogeneous Bézout Theorem, was the workhorse in [8]. Our main result here is a much more precise bound, in terms of matroid invariants, for the degree of the Gauss curve of the primal central curve. As a corollary we obtain a new upper bound on the total curvature of that central curve.

We begin our investigation by reviewing definitions from elementary differential geometry. Consider an arbitrary curve $[a,b] \to \mathbb{R}^n$, $t \mapsto f(t)$, whose parameterization is twice differentiable and whose derivative $f'(t)$ is a non-zero vector for all parameter values $t \in [a,b]$. This curve has an associated *Gauss map* into the unit sphere S^{m-1}, which is defined as

$$\gamma \,:\, [a,b] \to S^{m-1}\,,\; t \mapsto \frac{f'(t)}{||f'(t)||}.$$

The image $\gamma = \gamma([a,b])$ of the Gauss map in S^{m-1} is called the *Gauss curve* of the given curve f. In our situation, the curve f is algebraic, with known defining polynomial equations, and it makes sense to consider the *projective Gauss curve* in complex projective space $\mathbb{P}^{m-1}$. By this we mean the Zariski closure of the image of the Gauss curve under the double-cover map $S^{m-1} \to \mathbb{P}^{m-1}$. If $m = 2$, so that $\mathcal{C}$ is a non-linear plane curve, then the Gauss curve traces out several arcs on the unit curve S^1, and the projective Gauss curve is the entire projective line $\mathbb{P}^1$. Here, the line $\mathbb{P}^1$ comes with a natural multiplicity, to be derived in Example 19.

If $m = 3$ then the Gauss curve lies on the unit sphere S^2 and the projective Gauss curve lives in the projective plane $\mathbb{P}^2$. Since a curve in 3-space typically has parallel tangent lines, the Gauss curve is here expected to have singularities, even if f is a smooth curve.

The *total curvature* K of our curve f is defined to be the arc length of its associated Gauss curve γ; see [8, §3]. This quantity admits the following expression as an integral:

$$K \;:=\; \int_a^b ||\frac{d\gamma(t)}{dt}||dt. \tag{27}$$

The total curvature of the central path is a very interesting number for linear programming.

The *degree* of the Gauss curve $\gamma(t)$ is defined as the maximum number of intersection points, counting multiplicities, with any hyperplane in $\mathbb{R}^m$, or equivalently, with any equator in S^{m-1}. This (geometric) degree is bounded above by the (algebraic) degree of the projective Gauss curve in $\mathbb{P}^{m-1}$. The latter can be computed exactly, from any polynomial representation of $\mathcal{C}$, using standard methods of computer algebra. Throughout this section, by *degree* we mean the degree of the image of curve in $\mathbb{P}^{m-1}$ multiplied by the degree of the map that takes $\mathcal{C}$ onto $\gamma(\mathcal{C})$. From now on we use the notation $\deg(\gamma(\mathcal{C}))$ for that number.

Proposition 16. [8, Corollary 4.3] *The total curvature of any real algebraic curve $\mathcal{C}$ in $\mathbb{R}^m$ is bounded above by π times the degree of its projective Gauss curve in $\mathbb{P}^{m-1}$. In symbols,*

$$K \;\leq\; \pi \cdot \deg(\gamma(\mathcal{C})).$$

We now present our main result in this section, which concerns the degree of the projective Gauss curve $\gamma(\mathcal{C})$, when $\mathcal{C}$ is the central curve of a linear program in primal formulation. As before, A is an arbitrary real matrix of rank d having n columns, but the cost vector $\mathbf{c}$ and the right hand side $\mathbf{b}$ are generic over $\mathbb{Q}(A)$. The curve $\mathcal{C}$ lives in an $(n-d)$-dimensional affine subspace of $\mathbb{R}^n$, which we identify with $\mathbb{R}^{n-d}$, so that $\gamma(\mathcal{C})$ is a curve in $\mathbb{P}^{n-d-1}$.

Let $M_{A,\mathbf{c}}$ denote the matroid of rank $d+1$ on the ground set $[n]=\{1,\ldots,n\}$ associated with the matrix $\binom{A}{\mathbf{c}}$. We write $(h_0, h_1, ..., h_d)$ for the h-vector of the broken circuit complex of $M_{A,\mathbf{c}}$, as defined in (15). In the generic case, when $M_{A,\mathbf{c}} = U_{d+1,n}$ is the uniform matroid, the maximal simplices in $\mathrm{Br}(M_{A,\mathbf{c}})$ are $\{1, j_1, \ldots, j_d\}$ where $2 \leq j_1 < \cdots < j_d \leq n$. In that case, the coordinates of the h-vector are found to be $h_i = \binom{n-d+i-2}{i}$. For special matrices A, this simplicial complex gets replaced by a pure shellable subcomplex of the same dimension, so the h-vector (weakly) decreases in each entry. Hence, the following always holds:

$$h_i \;\leq\; \binom{n-d+i-2}{i} \qquad \text{for } i = 0, 1, \ldots, d. \tag{28}$$

As indicated, this inequality holds with equality when $M_{A,\mathbf{c}}$ is the uniform matroid.

Theorem 17. *The degree of the projective Gauss curve of the primal central curve* $\mathcal{C}$ *satisfies*

$$\deg(\gamma(\mathcal{C})) \;\leq\; 2 \cdot \sum_{i=1}^{d} i \cdot h_i. \tag{29}$$

In particular, we have the following upper bound which is tight for generic matrices A*:*

$$\deg(\gamma(\mathcal{C})) \quad \leq \quad 2 \cdot (n-d-1) \cdot \binom{n-1}{d-1}. \tag{30}$$

The difference between the bound in (29) and the degree of $\gamma(\mathcal{C})$ can be explained in terms of singularities the curve $\mathcal{C}$ may have on the hyperplane at infinity. The relevant algebraic geometry will be seen in the proof of Theorem 17, which we shall present after an example.

Example 18. Let $d = 3$ and $n = 6$. We first assume that A is a generic 3×6-matrix. The arrangement of six facet planes creates 10 bounded regions. The primal central curve $\mathcal{C}$ has degree $\binom{6-1}{3} = 10$. It passes through the $\binom{6}{3} = 20$ vertices of the arrangements. In-between it visits the 10 analytic centers of the bounded regions. Here the curve $\mathcal{C}$ is smooth and its genus is 11. This number is seen from the formula (33) below. The corresponding Gauss curve in the projective plane has degree $2 \cdot 10 + 2 \cdot \mathrm{genus}(\mathcal{C}) - 2 = 40$, as given by the right hand side of (30). Hence the total curvature of the central curve $\mathcal{C}$ is bounded above by 40π.

Next we consider the classical Klee-Minty cube in 3-space. It is given by the constraints

$$0 \leq x \leq 1\,, \;\; \epsilon x \leq y \leq 1 - \epsilon x, \quad \text{and} \quad \epsilon y \leq z \leq 1 - \epsilon y. \tag{31}$$

In its primal formulation, the general linear program over this polytope is given by the matrix

$$\begin{pmatrix} A \\ \mathbf{c} \end{pmatrix} \quad = \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 2\epsilon & 0 & 1 & 1 & 0 & 0 \\ 2\epsilon^2 & 0 & 2\epsilon & 0 & 1 & 1 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{pmatrix}.$$

Here ϵ is a small positive real constant. The above 4×6-matrix is not generic, and its associated matroid $M_{A,\mathbf{c}}$ is not uniform. It has exactly one non-basis, and so the h-vector equals $(h_0, h_1, h_2, h_3) = (1, 2, 3, 3)$. The central curve $\mathcal{C}$ has degree $\sum_{i=0}^{3} h_i = 9$. In the coordinates used in (31), the curve is defined by the 5×5-minors of the 5×6-matrix which is obtained from the 4×6-matrix $\binom{A}{\mathbf{c}}$ by adding one row consisting of reciprocal facet equations:

$$\big(\, x^{-1},\, (1-x)^{-1},\, (y-\epsilon x)^{-1},\, (1-y-\epsilon x)^{-1},\, (z-\epsilon y)^{-1},\, (1-z-\epsilon y)^{-1}\,\big).$$

According to Theorem 17, the degree of the Gauss curve $\gamma(\mathcal{C})$ in $\mathbb{P}^2$ is bounded above by $2\sum_{i=1}^{3} i \cdot h_i = 34$. A computation using `Macaulay2` [13] reveals that $\text{degree}(\gamma(\mathcal{C})) = 32$. ☐

Proof of Theorem 17. For the proof we shall use the *generalized Plücker formula for curves*:

$$\deg(\gamma(\mathcal{C})) \;=\; 2\cdot\deg(\mathcal{C}) + 2\cdot\text{genus}(\mathcal{C}) - 2 - \kappa. \tag{32}$$

The formula in (32) is obtained from [19, Thm. (3.2)] by setting $m = 1$ or from [14, Eq. (4.26)] by setting $k = 0$. The quantity κ is a non-negative integer and it measures the singularities of the curve $\mathcal{C}$. We have $\kappa = 0$ whenever $\mathcal{C}$ is a smooth curve, and this happens in our application when $M_{A,\mathbf{c}}$ is the uniform matroid. In general, we may have singularities at infinity because here the real affine curve $\mathcal{C}$ has to be replaced by its closure in complex projective space $\mathbb{P}^{n-d}$, which is the projectivization of the affine space defined by $A\mathbf{x} = \mathbf{b}$. The degree and genus on the right hand side of (32) refer to that projective curve in $\mathbb{P}^{n-d}$.

The references above actually give the degree of the *tangent developable* of the projective curve $\mathcal{C}$, but we see that this equals the degree of the Gauss curve. The tangent developable is the surface obtained by taking the union of all tangent lines at points in $\mathcal{C}$. The projective Gauss curve $\gamma(\mathcal{C})$ is obtained from the tangent developable by intersecting it with a hyperplane, namely, the hyperplane at infinity, representing the directions of lines.

In the formula (32), the symbol $\text{genus}(\mathcal{C})$ refers to the arithmetic genus of the curve. We shall now compute this arithmetic genus for primal central curve $\mathcal{C}$. For this we use the formula for the Hilbert series of the central sheet due to Terao, in Theorem 1.2 on page 551 of [27]. See the recent work of Berget [5] for a nice proof of a more general statement.

As seen in the proof of Proposition 11, the Hilbert series of its coordinate ring equals

$$\frac{h_0 + h_1 z + h_2 z^2 + \cdots + h_d z^d}{(1-z)^{d+2}}.$$

The central curve $\mathcal{C}$ is obtained from the central sheet by intersection with a general linear subspace of dimension $n-d$. The (projective closure of the) central sheet is arithmetically Cohen-Macaulay since it has a flat degeneration to a shellable simplicial complex, as shown by Proudfoot and Speyer [20]. We conclude that the Hilbert series of the central curve $\mathcal{C}$ is

$$\frac{h_0 + h_1 z + h_2 z^2 + \cdots + h_d z^d}{(1-z)^2} \;=\; \sum_{m\geq d}\Big[(\sum_{i=0}^{d} h_i)\cdot m \,+\, \sum_{j=0}^{d}(1-j)h_j\Big] z^m \;+\; O(z^{d-1}).$$

The parenthesized expression is the Hilbert polynomial of the projective curve $\mathcal{C}$. The degree of $\mathcal{C}$ is the coefficient of m, and we recover our result relating the degree and Möbius number:

$$\text{degree}(\mathcal{C}) \;=\; |\mu(A,\mathbf{c})| \;=\; \sum_{i=0}^{d} h_i.$$

The arithmetic genus of the curve $\mathcal{C}$ is one minus the constant term of its Hilbert polynomial:

$$\text{genus}(\mathcal{C}) \;=\; 1 - \sum_{j=0}^{d}(1-j)h_j. \tag{33}$$

We now see that our assertion (29) follows directly from the generalized Plücker formula (32).

When d and n are fixed, then the degree and genus of $\mathcal{C}$ are maximal when the matrix A is generic. In this case, h_i equals the right hand side of (28), and we need to sum these binomial coefficients times two. Hence, our second assertion (30) follows from the identify

$$\sum_{i=0}^{d} i \cdot \binom{n-d+i-2}{i} \quad = \quad (n-d-1) \cdot \binom{n-1}{d-1}.$$

This complete the proof of Theorem 17. □

Example 19. Let $d = n-2$ and suppose A is generic. Here, the primal central curve $\mathcal{C}$ is a plane curve. Our h-vector equals $(1, 1, ..., 1)$. Theorem 17 reveals that the degree of $\mathcal{C}$ is $d+1$ and the genus of $\mathcal{C}$ is $\binom{d}{2}$. In this case, the Gauss curve $\gamma(\mathcal{C})$ is the projective line $\mathbb{P}^1$, but regarded with multiplicity $\deg(\gamma(\mathcal{C})) = (d+1)d$. This number is the degree of the projectively dual curve $\mathcal{C}^\vee$. The identity (32) specializes to the Plücker formula for plane curves, which expresses the degree of $\mathcal{C}^\vee$ in terms of the degree and the singularities of $\mathcal{C}$. □

In the next section we shall establish a dictionary that translates between the primal and the dual central curve. As we shall see, all our results hold essentially verbatim for the dual central curves. In particular, the discussion in Example 19 above applies also to the situation of Section 2, where we discussed dual central curves that live in the plane $(d = 2)$, such as:

Example 20. Consider the DTZ snake in Figure 2. The curve shown there has degree 4 and its projective closure $\mathcal{C}$ is smooth in $\mathbb{P}^2$. So, we have $\deg(\gamma(\mathcal{C})) = 12$, and Proposition 16 would give the upper bound 12π on the total curvature of the full central curve in $\mathbb{R}^2$. □

We close this section by showing how to compute the Gauss curve for a non-planar instance.

Example 21. Let $n = 5, d = 2$ and $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$. The primal central curve has degree 6 and its equations are obtained by clearing denominators in the $4{\times}4$-minors of

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ (x-g_1)^{-1} & (-2x+y-g_2)^{-1} & (x-2y+z-g_3)^{-1} & (y-2z-g_4)^{-1} & (z-g_5)^{-1} \end{pmatrix}.$$

The c_i and g_j are random constants representing the cost function and right hand side of (1). To be precise, the vector $\mathbf{g} = (g_1, g_2, g_3, g_4, g_5)^T$ satisfies $A\mathbf{g} = \mathbf{b}$ as in Section 6. Writing `I` for the ideal of these polynomials, the following one-line command in the computer algebra system `Macaulay2` [13] computes the defining polynomial of the Gauss curve in $\mathbb{P}^2$:

```
eliminate({x,y,z},I+minors(1,matrix{{u,v,w}}*diff(matrix{{x},{y},{z}},gens I)))
```

The output is a homogeneous polynomial of degree 16 in the coordinates u, v, w on $\mathbb{P}^2$. Note that $\deg(\gamma(\mathcal{C})) = 16$ is consistent with Theorem 17 because $h = (h_0, h_1, h_2) = (1, 2, 3)$. □

6. Global Geometry of the Central Curve

In this section we return to the central path in its original primal-dual formulation, and we study its geometric properties. We shall study how the central curve connects the vertices of the hyperplane arrangement with the analytic centers of its bounded regions. This picture behaves well under duality, as the vertices of the two arrangements are in natural bijection.

We begin by offering an algebraic representation of the primal-dual central curve that is more symmetric than that given in the Introduction. Let $\mathcal{L}_A$ denote the row space of the matrix A and $\mathcal{L}_A^\perp$ its orthogonal complement in $\mathbb{R}^n$. We also fix a vector $\mathbf{g} \in \mathbb{R}^n$ such that $A\mathbf{g} = \mathbf{b}$. By eliminating $\mathbf{y}$ from the system (5) in Theorem 1, we see that the primal-dual central path $(\mathbf{x}^*(\lambda), \mathbf{s}^*(\lambda))$ has the following description that is symmetric under duality:

$$\mathbf{x} \in \mathcal{L}_A^\perp + \mathbf{g}\,, \quad \mathbf{s} \in \mathcal{L}_A + \mathbf{c} \quad \text{and} \quad x_1 s_1 = x_2 s_2 = \cdots = x_n s_n = \lambda. \tag{34}$$

The implicit (*i.e.* λ-free) representation of the primal-dual central curve is simply obtained by erasing the very last equality "$= \lambda$" in (34). Its prime ideal is generated by the quadrics $x_i s_i - x_j s_j$ and the affine-linear equations defining $\mathcal{L}_A^\perp + \mathbf{g}$ in $\mathbf{x}$-space and $\mathcal{L}_A + \mathbf{c}$ in $\mathbf{s}$-space.

The symmetric description of the central path in (34) lets us write down the statements from Section 4 for the dual version. For example, we derive equations for the dual central curve in $\mathbf{s}$-space or $\mathbf{y}$-space as follows. Let B be any $(n-d) \times n$ matrix whose rows span the kernel of A. In symbols, $\mathcal{L}_B = \mathcal{L}_A^\perp$. The (dual) central curve in $\mathbf{s}$-space is obtained by intersecting the d-dimensional affine space $\mathcal{L}_A + \mathbf{c} = \{\, \mathbf{s} \in \mathbb{R}^n \,:\, B\mathbf{s} = B\mathbf{c} \,\}$ with the central sheet $\mathcal{L}_{B,\mathbf{g}}^{-1}$ (20). To obtain the central curve in $\mathbf{y}$-space, we substitute $s_i = \sum_{j=1}^d a_{ji} y_j + c_i$ in the equations defining $\mathcal{L}_{B,\mathbf{g}}^{-1}$. This gives dual formulations of Theorems 12 and 17:

Corollary 22. *The degree of the dual central curve of (2) equals the Möbius number* $|\mu(B, \mathbf{g})|$ *and is hence at most* $\binom{n-1}{d-1}$. *The prime ideal of polynomials that vanish on the central path is generated by the circuit polynomials (21), but now associated with the space generated by the rows of* B *and the vector* $\mathbf{g}$, *and the* $n-d$ *linear equations in* $\mathbf{s}$ *given by* $B\mathbf{s} = B\mathbf{c}$.

Corollary 23. *The degree* $\deg(\gamma(\mathcal{C}))$ *of the Gauss image of the dual central curve* $\mathcal{C}$ *is at most* $\sum_i i \cdot h_i$, *where* $h = h(\mathrm{Br}(M_{B,\mathbf{g}}))$. *This implies the bound* $\deg(\gamma(\mathcal{C})) \leq 2 \cdot (d-1) \cdot \binom{n-1}{d}$.

In algebraic geometry, it is more natural to replace each of the affine spaces in (34) by a complex projective space $\mathbb{P}^n$, and to study the closure $\mathcal{C}$ of the central curve in $\mathbb{P}^n \times \mathbb{P}^n$. Algebraically, we use homogeneous coordinates $[x_0 : x_1 : \cdots : x_n]$ and $[s_0 : s_1 : \cdots : s_n]$. Writing $\mathbf{x}$ and $\mathbf{s}$ for the corresponding column vectors of length $n+1$, we represent

$$\mathcal{L}_A^\perp + \mathbf{g} \quad \text{by} \quad \{\mathbf{x} \in \mathbb{P}^n : (-\mathbf{b}, A) \cdot \mathbf{x} = 0\} \quad \text{and} \quad \mathcal{L}_A + \mathbf{c} \quad \text{by} \quad \{\mathbf{s} \in \mathbb{P}^n : (-B\mathbf{c}, B) \cdot \mathbf{s} = 0\}.$$

The projective primal-dual central curve $\mathcal{C}$ is an irreducible curve in $\mathbb{P}^n \times \mathbb{P}^n$. Its bi-homogeneous prime ideal in $K[x_0, x_1, \ldots, x_n, s_0, s_1, \ldots, s_n]$ can be computed by the process of saturation. Namely, we compute it as the saturation with respect to $\langle x_0 s_0 \rangle$ of the ideal generated by the above linear forms together with the bi-homogeneous forms $x_i s_i - x_j s_j$.

Example 24. Let $d = 2, n = 4$. Fix 2×4-matrices $A = (a_{ij})$ and $B = (b_{ij})$ such that $\mathcal{L}_B = \mathcal{L}_A^\perp$. We start with the ideal J in $K[x_0, \ldots, x_4, s_0, \ldots, s_4]$ generated by

$$\begin{gathered}
a_{11}(x_1 - g_1 x_0) + a_{12}(x_2 - g_2 x_0) + a_{13}(x_3 - g_3 x_0) + a_{14}(x_4 - g_4 x_0),\\
a_{21}(x_1 - g_1 x_0) + a_{22}(x_2 - g_2 x_0) + a_{23}(x_3 - g_3 x_0) + a_{24}(x_4 - g_4 x_0),\\
b_{11}(s_1 - c_1 s_0) + b_{12}(s_2 - c_2 s_0) + b_{13}(s_3 - c_3 s_0) + b_{14}(s_4 - c_4 s_0),\\
b_{21}(s_1 - g_1 s_0) + b_{22}(s_2 - c_2 s_0) + b_{23}(s_3 - c_3 s_0) + b_{24}(s_4 - c_4 s_0),\\
s_1 x_1 - s_2 x_2, \quad s_2 x_2 - s_3 x_3, \quad s_3 x_3 - s_4 x_4.
\end{gathered}$$

The central curve $\mathcal{C}$ is irreducible in $\mathbb{P}^4 \times \mathbb{P}^4$. It has degree $(3,3)$ unless A is very special. The prime ideal of $\mathcal{C}$ is computed as the saturation $(J : \langle x_0 s_0 \rangle^\infty)$. We find that this ideal has two minimal generators in addition to the seven above. These are cubic polynomials in $\mathbf{x}$ and in $\mathbf{s}$, which define the primal and dual central curves. They are shown in Figure 4. $\square$

Returning to the general case, the primal-dual curve $\mathcal{C}$ is always irreducible, by definition. Since it lives in $\mathbb{P}^n \times \mathbb{P}^n$, its *degree* is now a pair of integers $(d_{\mathbf{x}}, d_{\mathbf{s}})$. These two integers can be defined geometrically: $d_{\mathbf{x}}$ is the number of solutions of a general equation $\sum_{i=0}^n \alpha_i x_i = 0$ on the curve $\mathcal{C}$, and $d_{\mathbf{s}}$ is the number of solutions of a general equation $\sum_{i=0}^n \beta_i s_i = 0$ on $\mathcal{C}$.

Corollary 25. *Let* $\mathbf{c}$ *and* $\mathbf{g}$ *be generic vectors in* $\mathbb{R}^n$ *and let* $(d_{\mathbf{x}}, d_{\mathbf{s}})$ *be the degree of the projective primal-dual central curve* $\mathcal{C} \subset \mathbb{P}^n \times \mathbb{P}^n$. *This degree is given by our two Möbius numbers, namely* $d_{\mathbf{x}} = |\mu(A, \mathbf{c})|$ *and* $d_{\mathbf{s}} = |\mu(B, \mathbf{g})|$. *These numbers are defined in* (18).

Proof. The projection from the primal-dual central curve onto its image in either $\mathbf{x}$-space or $\mathbf{s}$-space is birational. For instance, if $\mathbf{x}$ is a general point on the primal central curve then the corresponding point $\mathbf{s}$ is uniquely obtained by solving the linear equations $x_i s_i = x_j s_j$ on $\mathcal{L}_A + \mathbf{c}$. Likewise, given a general point $\mathbf{s}$ on the dual central curve we can recover the unique $\mathbf{x}$ such that $(\mathbf{x}, \mathbf{s}) \in \mathcal{C}$. This implies that the intersections in $\mathbb{P}^n \times \mathbb{P}^n$ that define $d_{\mathbf{x}}$ and $d_{\mathbf{s}}$ are equivalent to intersecting the primal or dual central curve with a general hyperplane in $\mathbb{P}^n$, and the number of points on that intersection is the respective Möbius number. □

Next we discuss the geometry of this correspondence between the primal and dual curves at their special points, namely vertices and analytic centers of the relevant hyperplane arrangements. These special points are given by intersecting $\mathcal{C}$ with certain bilinear equations. The sum of our two Möbius numbers, $d_{\mathbf{x}} + d_{\mathbf{s}}$, is the number of solutions of a general bilinear equation $\sum_{i,j} \gamma_{ij} x_i s_j = 0$ on the primal-dual central curve $\mathcal{C}$. Two special choices of such bilinear equations are of particular interest to us, namely, the bilinear equation $x_0 s_0 = 0$ and the bilinear equation $x_i s_i = 0$ for any $i \geq 1$. Note that the choice of the index i does not matter for the second equation because $x_i s_i = x_j s_j$ holds on the curve.

Let us first observe what happens in $\mathbb{P}^n \times \mathbb{P}^n$ when the parameter λ becomes 0. The corresponding points on the primal-dual curve $\mathcal{C}$ are found by solving the equations $x_1 s_1 = 0$ on $\mathcal{C}$. Its points are the solutions of the n equations $x_1 s_1 = x_2 s_2 = \cdots = x_n s_n = 0$ on the n-dimensional subvariety $(\mathcal{L}_A^\perp + \mathbf{g}) \times (\mathcal{L}_A + \mathbf{c})$ of $\mathbb{P}^n \times \mathbb{P}^n$. This intersection now contains many points in the product of affine spaces, away from the hyperplanes $\{x_0 = 0\}$ and $\{s_0 = 0\}$. We find the points by solving the linear equations $x_{i_1} = \cdots = x_{i_d} = 0$ on $\mathcal{L}_A^\perp + \mathbf{g}$ and $s_{j_1} = \cdots = s_{j_{n-d}} = 0$ on $\mathcal{L}_A + \mathbf{c}$, where $\{i_1, \ldots, i_d\}$ runs over all bases of the matroid $M(\mathcal{L}_A)$ and $\{j_1, \ldots, j_{n-d}\}$ is the complementary basis of the dual matroid $M(\mathcal{L}_A)^* = M(\mathcal{L}_B)$. These points represent vertices in the hyperplane arrangements $\mathcal{H}$ and $\mathcal{H}^*$, where

$$\mathcal{H} \text{ denotes } \{x_i = 0\}_{i \in [n]} \text{ in } \mathcal{L}_A^\perp + \mathbf{g} \quad \text{and} \quad \mathcal{H}^* \text{ denotes } \{s_i = 0\}_{i \in [n]} \text{ in } \mathcal{L}_A + \mathbf{c}.$$

The vertices come in pairs corresponding to complementary bases, so the points with parameter $\lambda = 0$ on the primal-dual central curve $\mathcal{C}$ are the pairs $(\mathbf{x}, \mathbf{s})$ where $\mathbf{x}$ is a vertex in the hyperplane arrangement $\mathcal{H}$ and $\mathbf{s}$ is the complementary vertex in the dual arrangement $\mathcal{H}^*$.

Imposing the equation $x_0 s_0 = 0$ means setting $\lambda = \infty$ in the parametric representation of the central curve, and the points thus obtained have the following geometric interpretation in terms of bounded regions of the hyperplane arrangements $\mathcal{H}$ and $\mathcal{H}^*$. We recall that the *analytic center* of the polytope $P = \{A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is the unique point in the interior of P that maximizes the concave function $\sum_{i=1}^n \log(x_i)$. The algebraic condition that characterizes the analytic center is that the gradient of $\sum_{i=1}^n \log(x_i)$, which is $\mathbf{x}^{-1}$, is orthogonal to the affine-linear space $\mathcal{L}_A^\perp + \mathbf{g} = \{A\mathbf{x} = \mathbf{b}\}$. This means that the vector $\mathbf{x}^{-1}$ lies in the row span $\mathcal{L}_A$ of A. Let $\mathcal{L}_A^{-1}$ denote the coordinatewise reciprocal of $\mathcal{L}_A$. By passing to the Zariski closure, we regard $\mathcal{L}_A^{-1}$ as a subvariety in the projective space $\mathbb{P}^n$.

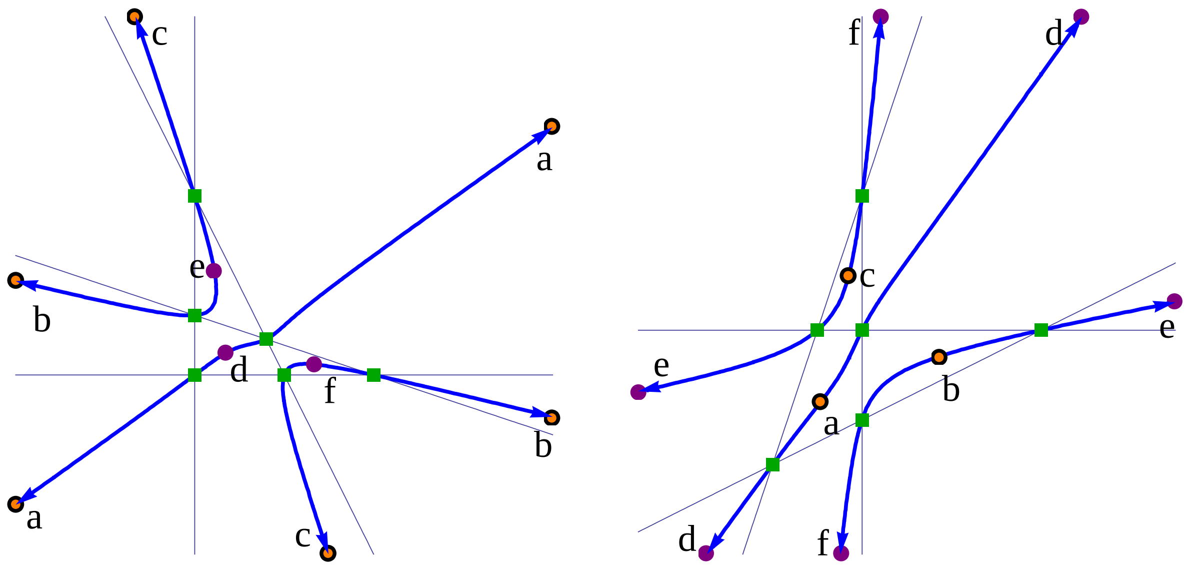


FIGURE 4. Correspondence of vertices and analytic centers in the two projections of a primal-dual central curve. Here both curves are Vinnikov cubics.

Proposition 26. *The intersection* $\mathcal{L}_A^{-1} \cap (\mathcal{L}_A^{\perp} + \mathbf{g})$ *defines a zero-dimensional reduced subscheme of the affine space* $\mathbb{P}^n \backslash \{x_0 = 0\}$*. All its points are defined over* $\mathbb{R}$*. They are the analytic centers of the polytopes that form the bounded regions of the arrangement* $\mathcal{H}$*.*

Proof. The analytic center of each bounded region is a point in the variety $\mathcal{L}_A^{-1} \cap (\mathcal{L}_A^{\perp}+\mathbf{g})$, by the gradient argument in the paragraph above. This gives us $|\mu(A)|$ real points of intersection on $\mathcal{L}_A^{-1} \cap (\mathcal{L}_A^{\perp} + \mathbf{g})$. By Proposition 11, we know that the degree of $\mathcal{L}_A^{-1}$ is $|\mu(A)|$. This shows that these real points are all the intersection points (over $\mathbb{C}$) and they occur with multiplicity one. This argument closely follows the proof of Lemma 10. □

Naturally, the dual statement holds verbatim, and we shall now state it explicitly.

Proposition 27. *The intersection* $(\mathcal{L}_A^{\perp})^{-1} \cap (\mathcal{L}_A + \mathbf{c})$ *defines a zero-dimensional reduced subscheme of the affine space* $\mathbb{P}^n \backslash \{s_0 = 0\}$*. All its points are defined over* $\mathbb{R}$*. They are the analytic centers of the polytopes that form the bounded regions of the dual arrangement* $\mathcal{H}^*$*.*

The above picture of the curve $\mathcal{C}$ in $\mathbb{P}^n \times \mathbb{P}^n$ reveals the geometric correspondence between special points on the primal and dual curves, coming from $\lambda = 0$ and $\lambda = \infty$. We summarize our discussion with the following theorem on the global geometry of the primal-dual central curve. Figure 4 serves as an illustration of this global geometry for the case $n = 4$ and $d = 2$.

Theorem 28. *The primal central curve in* $\mathbf{x}$*-space* $\mathbb{R}^n$ *passes through all vertices of the arrangement* $\mathcal{H}$*. In between these vertices, it passes through the analytic centers of the bounded regions in* $\mathcal{H}$*. Similarly, the dual central curve in* $\mathbf{s}$*-space passes through all vertices and analytic centers of* $\mathcal{H}^*$*. Vertices of* $\mathcal{H}$ *in the primal curve correspond to vertices of* $\mathcal{H}^*$ *in the dual curve. The analytic centers of bounded regions of* $\mathcal{H}$ *correspond to points on the dual curve in* $\mathbf{s}$*-space at the hyperplane* $\{s_0 = 0\}$*, and the analytic centers of bounded regions of* $\mathcal{H}^*$ *correspond to points on the primal curve in* $\mathbf{x}$*-space at the hyperplane* $\{x_0 = 0\}$*.*

The primal central curve misses precisely one of the antipodal pairs of unbounded regions of $\mathcal{H}$. It corresponds to the region in the induced arrangement at infinity that contains the point representing the cost function $\mathbf{c}$. For a visualization see the picture of the central curve in Figure 3. Here a projective transformation of $\mathbb{P}^2$ moves the line from infinity into $\mathbb{R}^2$.

The points described in Propositions 26 and 27 are precisely those points on the primal-dual central curve $\mathcal{C}$ for which the parameter λ becomes ∞. Equivalently, in its embedding in $\mathbb{P}^n \times \mathbb{P}^n$, these are solutions of the equation $x_0 s_0 = 0$ on the curve $\mathcal{C}$. Note, however, that for special choices of A, the projective curve $\mathcal{C}$ will pass though points with $x_0 = s_0 = 0$. Such points, which lie on the hyperplanes at infinity in both projective spaces, are entirely independent of the choice of $\mathbf{c}$ and $\mathbf{g}$. Indeed, they are the solutions of the equations

$$\mathbf{s} \in \mathcal{L}_A = \ker B\,, \quad \mathbf{x} \in \mathcal{L}_A^{\perp} = \ker A, \quad \text{and} \quad x_1 s_1 = x_2 s_2 = \cdots = x_n s_n = 0. \tag{35}$$

The solutions to these equations form the *disjoint support variety* in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, which contains pairs of vectors in the two spaces $\mathcal{L}_A$ and $\mathcal{L}_A^{\perp}$ whose respective supports are disjoint.

When studying the global geometry of the primal-dual central curve, it is useful to start with the case when the constraint matrix A is generic. In that case, our matroids are uniform, namely $M(\mathcal{L}_A) = U_{d,n}$ and $M(\mathcal{L}_B) = U_{n-d,n}$, and the disjoint support variety (35) is empty. This condition ensures that the intersections of the curve $\mathcal{C}$ with both the hypersurfaces $\{x_0 s_0 = 0\}$ and $\{x_1 s_1 = 0\}$ in $\mathbb{P}^n \times \mathbb{P}^n$ is reduced, zero-dimensional and fully real. The number of points in these intersections is the common number of bases in the two matroids:

$$d_{\mathbf{x}} + d_{\mathbf{s}} \;=\; \binom{n-1}{d} + \binom{n-1}{d-1} \;=\; \binom{n}{d} \;=\; \binom{n}{n-d}.$$

The intersection points of $\mathcal{C}$ with $\{x_0 s_0 = 0\}$ are the pairs $(\mathbf{x}, \mathbf{s})$ where either $\mathbf{x}$ is an analytic center in $\mathcal{H}$ and $\mathbf{s}$ lies at infinity in the dual central curve, or $\mathbf{x}$ lies at infinity in the primal central curve and $\mathbf{s}$ is an analytic center in $\mathcal{H}^*$. The intersection points of $\mathcal{C}$ with $\{x_1 s_1 = 0\}$ are the pairs $(\mathbf{x}, \mathbf{s})$ where $\mathbf{x}$ is a vertex in $\mathcal{H}$ and $\mathbf{s}$ is a vertex in $\mathcal{H}^*$. If we now degenerate the generic matrix A into a more special matrix, then some of the above points representing vertices and analytic centers degenerate to points on the disjoint support variety (35). Figure 4 visualizes the above correspondences for the case $n = 4$ and $d = 2$.

In Theorem 28 we did not mention the degree of the primal or dual central curve. For the sake of completeness, here is a brief discussion of the geometric meaning of the degree $d_{\mathbf{x}}$:

Remark 29. Consider the intersection of the primal central path with a general level set $\{\mathbf{c}^T\mathbf{x} = c_0\}$ of the linear objective function $\mathbf{c}$. As c_0 varies, we obtain a family of parallel hyperplanes. Each hyperplane meets the curve in precisely $d_{\mathbf{x}}$ points, all of which have real coordinates. These points are the analytic centers of the $(n-d-1)$-dimensional polytopes obtained as the bounded regions of the induced arrangement of n hyperplanes $\{x_i = 0\}$ in the affine space $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b},\, \mathbf{c}^T\mathbf{x} = c_0\}$. We can see $d_{\mathbf{x}}$ as the number of $(n-d-1)$-dimensional bounded regions in the restriction of the arrangement $\mathcal{H}$ to a general level hyperplane $\{\mathbf{c}^T\mathbf{x} = c_0\}$. In particular, this gives a one-dimensional family of hyperplanes all of whose intersection points with the central curve are real. □

We now offer a few remarks relating our algebro-geometric results to the classical theory of linear programming, which was the language used at the opening of this article. Strong linear programming duality [1, 22, 29] says that the optimal points of the pair of linear programs (1) and (2) are precisely the feasible points satisfying $\mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} = 0$. We prefer to think of the optimal solutions as points of intersection of the central curve with a particular bilinear hypersurface $x_i s_i = 0$. Indeed, any point $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ of the primal-dual central path satisfies $\mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} = n \cdot \lambda$. It follows that, as $\lambda \to \infty$, at least one of $\mathbf{y}$ or $\mathbf{x}$ must approach its respective hyperplane at infinity, as seen in Theorem 28. One usually thinks of the analytic center of the polytope $P = \{A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ as the unique point of P maximizing the concave function $\sum_{i=1}^{n} \log(x_i)$. In this article we considered the analytic centers of all polytopes in $\{A\mathbf{x} = \mathbf{b}\}$ obtained by putting different sign conditions on $\mathbf{x}$, that is, analytic centers of the bounded regions in $\mathcal{H}$. Proposition 27 says that these are found by intersecting the primal-dual central curve with a hyperplane (at infinity) in $\mathbf{y}$-space. There is of course a parallel picture, seen in Figure 4, for the corresponding arrangement $\mathcal{H}^*$ of the dual problem.

A main theme in this paper is that projective algebraic geometry provides an alternative view on optimality and duality in optimization, as well as powerful tools for analyzing interior point methods. This parallels the discussion of semidefinite programming in [23]. See also [26] for a statistical perspective on analytic centers and central curves in the semidefinite context.

Our algebraic methods in Section 5 resulted in the first instance-specific bound for the total curvature of the central curve. This raises the question whether one can derive a similar bound for the central path within a single feasibility region, which is tied to the investigations in [9, 10]. That our bounds on curvature are expressed in the language of matroid theory was surely no surprise to those familiar with oriented matroid programming and its beautiful duality [1], hinted at in Figure 4. We conclude with another concrete and pertinent question:

Question 30. How tight is the bound (29) for the total curvature of the central curve?

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