Minimax and risk averse multistage stochastic programming

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Abstract. In this paper we study relations between the minimax, risk averse and nested formulations of multistage stochastic programming problems. In particular, we discuss conditions for time consistency of such formulations of stochastic problems. We also describe a connection between law invariant coherent risk measures and the corresponding sets of probability measures in their dual representation. Finally, we discuss a minimax approach with moment constraints to the classical inventory model.

Key Words: stochastic programming, dynamic equations, robust optimization, coherent risk measures, risk averse stochastic optimization, problem of moments, inventory model.

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1 Introduction

One of the criticisms of the stochastic programming approach to optimization under uncertainty is that the assumption of knowing the probability distribution of the uncertain parameters could be quite unrealistic. On the other hand, the worst case approach of robust optimization could be too conservative (for a thorough discussion of robust optimization we refer to Ben-Tal, El Ghaoui and Nemirovski [3]). A possible compromise between these two extremes could be a minimax approach to stochastic programming where the worst case expected value optimization is performed with respect to a specified family of probability distributions. This approach has a long history and was already discussed in Žáčková [22] more than 40 years ago.

Another criticism of stochastic programming is that the optimization on average does not take into account the involved risk of possible deviations from the expected value. A risk averse approach to stochastic optimization was initiated by Markowitz [9] in the context of portfolio selection. These two approaches - the minimax and risk averse - to stochastic optimization were separate entities for a long time. In a pioneering paper by Artzner et al [1] an axiomatic approach to risk averse optimization was suggested and among other things it was shown that the minimax and risk averse approaches in a sense are dual to each other.

As of today the minimax and risk averse approaches to stochastic optimization are reasonably well understood for static models. The situation is considerably more delicate in dynamic settings. Multistage robust optimization, under the name "adjustable robust optimization", was initiated in Ben-Tal, Goryashko, Guslitzer and Nemirovski [2], robust dynamic programming and robust control of Markov decision processes were discussed in Iyengar [7] and Nilim and El Ghaoui [8]. Dynamic programming equations for risk averse optimization were derived in Ruszczyński and Shapiro [16]. It turns out that some suggested approaches to dynamic risk averse optimization are not time consistent (cf., [19]). For a discussion of time consistency concepts we may refer to [5],[17] and references therein. As far as we know time consistency was not discussed in the context of minimax multistage stochastic programming.

This paper is organized as follows. In the next section we give a quick introduction to risk neutral multistage stochastic programming. For a detail discussion of this topic we may refer, e.g., to [20]. In section 3 we discuss static and dynamic coherent risk measures. In particular we describe a connection between law invariant coherent risk measures and the corresponding sets of probability measures in their dual representation (Theorem 3.2). The main development is presented in section 4. In that section we study connections between the minimax, risk averse and nested formulations of multistage stochastic programming problems. Finally, in section 5 we give examples and applications of the general theory. In particular, we discuss a minimax approach to the classical inventory model.

We use the following notation throughout the paper. For random variables X and Y we denote by $\mathbb{E}[X|Y]$ or $\mathbb{E}_{|Y}[X]$ the conditional expectation of X given Y. We use the same notation ξ for a random vector and its particular realization, which of these two meanings will be used in a specific situation will be clear from the context. For a process $\xi_1, \xi_2, ...,$ and positive integers $s \le t$ we denote by $\xi_{[s,t]} := (\xi_s, ..., \xi_t)$ history of the process from time s to time t. In particular, $\xi_{[t]} := \xi_{[1,t]} = (\xi_1, ..., \xi_t)$ denotes history of the process up to time t. By $\Delta(\xi)$ we denote measure of mass one concentrated at point ξ.

2 Risk Neutral Formulation

In a generic form a T-stage stochastic programming problem can be written as

$$\begin{array}{cl}\underset{x_1,x_2(\cdot),\dots,x_T(\cdot)}{\text{Min}} & \mathbb{E}\big[F_1(x_1)+F_2(x_2(\xi_{[2]}),\xi_2)+\ldots+F_T\left(x_T(\xi_{[T]}),\xi_T\right)\big] \\ \text{s.t.} & x_1\in\mathcal{X}_1,\ x_t(\xi_{[t]})\in\mathcal{X}_t(x_{t-1}(\xi_{[t-1]}),\xi_t),\ t=2,\dots,T.\end{array} \tag{2.1}$$

Here $\xi_1,\xi_2,\dots,\xi_T$ is a random data process, $x_t\in\mathbb{R}^{n_t}$, $t=1,\dots,T$, are decision variables, $F_t:\mathbb{R}^{n_t}\times\mathbb{R}^{d_t}\to\mathbb{R}$ are measurable functions and $\mathcal{X}_t:\mathbb{R}^{n_{t-1}}\times\mathbb{R}^{d_t}\rightrightarrows\mathbb{R}^{n_t}$, $t=2,\dots,T$, are measurable closed valued multifunctions (point-to-set mappings). The first stage data, i.e., the vector ξ_1, the function $F_1:\mathbb{R}^{n_1}\to\mathbb{R}$, and the set $\mathcal{X}_1\subset\mathbb{R}^{n_1}$ are deterministic. In particular, the multistage problem is *linear* if the objective functions and the constraint functions are linear, that is

$$\begin{array}{l}F_t(x_t,\xi_t):=c_t^{\mathsf{T}}x_t,\ \ \mathcal{X}_1:=\{x_1:A_1x_1=b_1,\ x_1\geq 0\}\,,\\ \mathcal{X}_t(x_{t-1},\xi_t):=\{x_t:B_tx_{t-1}+A_tx_t=b_t,\ x_t\geq 0\}\,,\ t=2,\dots,T,\end{array} \tag{2.2}$$

where $\xi_1:=(c_1,A_1,b_1)$ and $\xi_t:=(c_t,B_t,A_t,b_t)\in\mathbb{R}^{d_t}$, $t=2,\dots,T$, are data vectors some/all elements of which can be random.

Optimization in (2.1) is performed over feasible *policies*. A policy is a sequence of (measurable) functions $x_t=x_t(\xi_{[t]})$, $t=1,\dots,T$. Each $x_t(\xi_{[t]})$ is a function of the data process $\xi_{[t]}$ up to time t, this ensures the *nonanticipativity* property of a considered policy. A policy[1] $x_t(\cdot):\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_t}\to\mathbb{R}^{n_t}$, $t=1,\dots,T$, is said to be *feasible* if it satisfies the feasibility constraints for almost every realization of the random data process. It could be noted that since policies are elements of appropriate functional spaces, formulation (2.1) leads to an infinite dimensional optimization problem, unless the data process $\xi_1,\dots,\xi_T$ has a finite number of realizations (called scenarios).

Recall that if X and Y are two random variables, then $\mathbb{E}[X]=\mathbb{E}\{\mathbb{E}[X|Y]\}$, i.e., average of averages is the total average. Therefore we can write the expectation in (2.1) as

$$\begin{array}{l}\mathbb{E}\left[F_1(x_1)+F_2(x_2(\xi_{[2]}),\xi_2)+\ldots+F_{T-1}\left(x_{T-1}(\xi_{[T-1]}),\xi_{T-1}\right)+F_T\left(x_T(\xi_{[T]}),\xi_T\right)\right]\\ =\mathbb{E}_{|\xi_1}\Big[\cdots\,\mathbb{E}_{|\xi_{[T-2]}}\big[\mathbb{E}_{|\xi_{[T-1]}}[F_1(x_1)+F_2(x_2(\xi_{[2]}),\xi_2)+\ldots+F_{T-1}\left(x_{T-1}(\xi_{[T-1]}),\xi_{T-1}\right)\\ \quad+F_T\left(x_T(\xi_{[T]}),\xi_T\right)]\big]\Big]\\ =F_1(x_1)+\mathbb{E}_{|\xi_1}\Big[F_2(x_2(\xi_{[2]}),\xi_2)+\ldots+\mathbb{E}_{|\xi_{[T-2]}}\big[F_{T-1}\left(x_{T-1}(\xi_{[T-1]}),\xi_{T-1}\right)\big]\\ \quad+\mathbb{E}_{|\xi_{[T-1]}}\big[F_T\left(x_T(\xi_{[T]}),\xi_T\right)\big]\Big].\end{array} \tag{2.3}$$

This, together with an interchangeability property of the expectation and minimization operators (e.g., [13, Theorem 14.60]), leads to the following nested formulation of the multistage problem (2.1)

$$\underset{x_1\in\mathcal{X}_1}{\text{Min}}\ F_1(x_1)+\mathbb{E}_{|\xi_1}\left[\inf_{x_2\in\mathcal{X}_2(x_1,\xi_2)}F_2(x_2,\xi_2)+\mathbb{E}_{|\xi_{[2]}}\Big[\cdots+\mathbb{E}_{|\xi_{[T-1]}}\big[\inf_{x_T\in\mathcal{X}_T(x_{T-1},\xi_T)}F_T(x_T,\xi_T)\big]\Big]\right]. \tag{2.4}$$

Of course, since ξ_1 is deterministic, $\mathbb{E}_{|\xi_1}[\,\cdot\,]=\mathbb{E}[\,\cdot\,]$. We write it here in the conditional form for the uniformity of notation.

This decomposition property of the expectation operator is a basis for deriving the *dynamic programming* equations. That is, going backward in time the so-called *cost-to-go* (also called *value*) functions are defined recursively for $t=T,...,2$, as follows

$$V_t\left(x_{t-1},\xi_{[t]}\right)=\inf_{x_t\in\mathcal{X}_t(x_{t-1},\xi_t)}\left\{F_t(x_t,\xi_t)+\mathcal{V}_{t+1}\left(x_t,\xi_{[t]}\right)\right\}, \tag{2.5}$$

[1] In order to distinguish between a function $x_t(\xi_{[t]})$ and a vector $x_t\in\mathbb{R}^{n_t}$ we often write $x_t(\cdot)$ to emphasize that this denotes a function.

where

$$\mathcal{V}_{t+1}\left(x_t, \xi_{[t]}\right) = \mathbb{E}\left\{V_{t+1}\left(x_t, \xi_{[t+1]}\right) \middle| \xi_{[t]}\right\}, \tag{2.6}$$

with $\mathcal{V}_{T+1}(\cdot,\cdot) \equiv 0$ by definition. At the first stage the following problem should be solved

$$\operatorname*{Min}_{x_1 \in \mathcal{X}_1} F_1(x_1) + \mathbb{E}\left[V_2(x_1, \xi_2)\right]. \tag{2.7}$$

The optimal value of the first stage problem (2.7) gives the optimal value of the corresponding multistage problem formulated in the form (2.1), or equivalently in the form (2.4).

A policy $\bar{x}_t(\xi_{[t]})$, $t = 1,\dots,T$, is *optimal* if $\bar{x}_1$ is an optimal solution of the first stage problem (2.7) and for $t = 2,\dots,T$,

$$\bar{x}_t(\xi_{[t]}) \in \arg \min_{x_t \in \mathcal{X}_t\left(\bar{x}_{t-1}(\xi_{[t-1]}), \xi_t\right)} \left\{F_t(x_t, \xi_t) + \mathcal{V}_{t+1}\left(x_t, \xi_{[t]}\right)\right\}, \quad \text{w.p.1.} \tag{2.8}$$

In the dynamic programming formulation the problem is reduced to solving a family of finite dimensional problems (2.5)–(2.6).

It is said that the random process $\xi_1, ..., \xi_T$ is *stagewise independent* if random vector ξ_{t+1} is independent of $\xi_{[t]}$, $t = 1, ..., T-1$. In case of stagewise independence the (expected value) cost-to-go function

$$\mathcal{V}_T(x_{T-1}, \xi_{[T-1]}) = \mathbb{E}\left[V_T(x_{T-1}, \xi_T) \middle| \xi_{[T-1]}\right] \tag{2.9}$$

does not depend on $\xi_{[T-1]}$. By induction in t going backward in time, it can be shown that:

- If the data process is stagewise independent, then the (expected value) cost-to-go functions $\mathcal{V}_t(x_t)$, $t = 2, ..., T$, do not depend on the data process and equations (2.5) take the form

$$V_t(x_{t-1}, \xi_t) = \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \left\{F_t(x_t, \xi_t) + \mathcal{V}_{t+1}(x_t)\right\}. \tag{2.10}$$

In formulation (2.1) the expectations are taken with respect to a specified probability distribution of the random process $\xi_1, ..., \xi_T$. The optimization is performed on average and does not take into account risk of a possible deviation from the average for a particular realization of the data process. Therefore formulation (2.1) is referred to as *risk neutral*.

3 Risk Measures

In order to proceed to a risk averse formulation of multistage programs we need to discuss the following concept of so-called coherent risk measures. Consider a probability space $(\Omega, \mathcal{F}, P)$. To measurable functions $Z : \Omega \to \mathbb{R}$ we refer as *random variables*. With every random variable $Z = Z(\omega)$ we associate a number, denoted $\rho(Z)$, indicating our preference between possible realizations of random variables. That is, $\rho(\cdot)$ is a real valued function defined on a space of measurable functions $Z : \Omega \to \mathbb{R}$. We refer to $\rho(\cdot)$ as a *risk measure*. For example, we can employ the expected value $\rho(Z) := \mathbb{E}_P[Z]$ as a risk measure. The term "risk measure" is somewhat unfortunate since it could be confused with the concept of probability measures. However, it became quite standard, so we will use it here.

We have to specify a space of random variables on which a considered risk measure will be defined. In that respect it is natural to consider spaces $L_p(\Omega, \mathcal{F}, P)$ of random variables $Z(\omega)$ having finite p-th order moment, $p \in [1, \infty)$. Note that two random variables $Z(\omega)$ and $Z'(\omega)$ are undistinguishable if $Z(\omega) = Z'(\omega)$ for a.e. $\omega \in \Omega$ (i.e., for all $\omega \in \Omega$ except on a set of P-measure zero). Therefore $L_p(\Omega, \mathcal{F}, P)$ consists of classes of random variables $Z(\omega)$ such that $Z(\omega)$ and $Z'(\omega)$ belong to the same class if $Z(\omega) = Z'(\omega)$ for a.e. $\omega \in \Omega$, and $\mathbb{E}|Z|^p = \int_\Omega |Z(\omega)|^p dP(\omega)$ is finite. The space $L_p(\Omega,\mathcal{F},P)$ equipped with the norm $\|Z\|_p := \left(\int_\Omega |Z(\omega)|^p dP(\omega)\right)^{1/p}$ becomes a Banach space. We also consider space $L_\infty(\Omega,\mathcal{F},P)$ of essentially bounded functions. That is, $L_\infty(\Omega,\mathcal{F},P)$ consists of random variables with finite sup-norm $\|Z\|_\infty := \operatorname{ess\,sup}|Z|$, where the essential supremum of a random variable $Z(\omega)$ is defined as

$$\operatorname{ess\,sup}(Z) := \inf\left\{\sup_{\omega\in\Omega} Z'(\omega) : Z'(\omega) = Z(\omega) \text{ a.e. } \omega\in\Omega\right\}. \tag{3.1}$$

A set $\mathfrak{A} \subset L_p(\Omega,\mathcal{F},P)$ is said to be *bounded* if there exists constant $c > 0$ such that $\|Z\|_p \le c$ for all $Z \in \mathfrak{A}$. Unless stated otherwise all topological statements related to the space $L_p(\Omega,\mathcal{F},P)$ will be made with respect to its strong (norm) topology.

Formally, risk measure is a real valued function $\rho : \mathcal{Z} \to \mathbb{R}$, where $\mathcal{Z} := L_p(\Omega,\mathcal{F},P)$ for some $p \in [1,\infty]$. It is also possible to consider risk measures taking values $\rho(Z) = +\infty$ for some $Z \in \mathcal{Z}$. However, with virtually every interesting risk measure is associated in a natural way an $L_p(\Omega,\mathcal{F},P)$ space on which it is finite valued. It was suggested in Artzner et al [1] that a "good" risk measure should satisfy the following axioms, and such risk measures were called *coherent.*

(A1) *Monotonicity:* If $Z, Z' \in \mathcal{Z}$ and $Z \succeq Z'$, then $\rho(Z) \ge \rho(Z')$.

(A2) *Convexity:*

$$\rho(tZ + (1-t)Z') \le t\rho(Z) + (1-t)\rho(Z')$$

for all $Z, Z' \in \mathcal{Z}$ and all $t \in [0,1]$.

(A3) *Translation Equivariance:* If $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\rho(Z + a) = \rho(Z) + a$.

(A4) *Positive Homogeneity:* If $t \ge 0$ and $Z \in \mathcal{Z}$, then $\rho(tZ) = t\rho(Z)$.

Here the notation $Z \succeq Z'$ means that $Z(\omega) \ge Z'(\omega)$ for a.e. $\omega \in \Omega$. Monotonicity property (axiom (A1)) is a natural condition that a risk measure should satisfy (recall that we deal here with minimization rather than maximization formulations of optimization problems). Convexity property is also a natural one. Because of (A4) the convexity axiom (A2) holds iff the following subadditivity property holds

$$\rho(Z + Z') \le \rho(Z) + \rho(Z'), \ \ \forall\, Z, Z' \in \mathcal{Z}. \tag{3.2}$$

That is, risk of the sum of two random variables is not bigger than the sum of risks. Axioms (A3) and (A4) postulate position and scale properties, respectively, of risk measures. We refer to [6],[12],[20] for a thorough discussion of coherent risk measures.

We have the following basic duality result associated with coherent risk measures. With each space $\mathcal{Z} := L_p(\Omega,\mathcal{F},P)$, $p \in [1,\infty)$, is associated its dual space $\mathcal{Z}^* := L_q(\Omega,\mathcal{F},P)$, where $q \in (1,\infty]$ is such that $1/p + 1/q = 1$. For $Z \in \mathcal{Z}$ and $\zeta \in \mathcal{Z}^*$ their scalar product is defined as

$$\langle Z, \zeta\rangle := \int_\Omega Z(\omega)\zeta(\omega)dP(\omega). \tag{3.3}$$

We denote by

$$\mathfrak{P} := \left\{\zeta \in \mathcal{Z}^* : \int_\Omega \zeta(\omega)dP(\omega) = 1,\ \zeta \succeq 0\right\} \tag{3.4}$$

the set of probability density functions in the dual space $\mathcal{Z}^*$.

Theorem 3.1 *Let $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$, and $\rho : \mathcal{Z} \to \mathbb{R}$ be a coherent risk measure. Then ρ is continuous (in the norm topology of $\mathcal{Z}$) and there exists a bounded closed convex set $\mathfrak{A} \subset \mathfrak{P}$ such that*

$$\rho(Z) = \sup_{\zeta \in \mathfrak{A}} \langle Z, \zeta \rangle, \quad \forall Z \in \mathcal{Z}. \tag{3.5}$$

Moreover, the set $\mathfrak{A}$ can be written in the form

$$\mathfrak{A} = \left\{ \zeta \in \mathfrak{P} : \langle Z, \zeta \rangle \le \rho(Z), \ \ \forall Z \in \mathcal{Z} \right\}. \tag{3.6}$$

Conversely if the representation (3.5) *holds for some nonempty bounded set $\mathfrak{A} \subset \mathfrak{P}$, then ρ is a (real valued) coherent risk measure.*

The dual representation (3.5) follows from the classical Fenchel-Moreau theorem. Originally it was derived in [1], and the following up literature (cf., [6]), for space $\mathcal{Z} := L_\infty(\Omega, \mathcal{F}, P)$. For general spaces $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$ this representation was obtained in [15] and it was shown there that monotonicity (axiom (A1)) and convexity (axiom (A2)) imply continuity of the (real valued) risk measure ρ. Note that if the representation (3.5) holds for some bounded set $\mathfrak{A}$, then it also holds if the set $\mathfrak{A}$ is replaced by the topological closure of its convex hull. Therefore, without loss of generality, it suffices to consider only bounded closed convex sets $\mathfrak{A}$.

For $\zeta \in \mathfrak{P}$ the scalar product $\langle Z, \zeta \rangle$ can be understood as the expectation $\mathbb{E}_Q[Z]$ taken with respect to the probability measure $dQ = \zeta dP$. Therefore the representation (3.5) can be written as

$$\rho(Z) = \sup_{Q \in \mathfrak{Q}} \mathbb{E}_Q[Z], \ \ \forall Z \in \mathcal{Z}, \tag{3.7}$$

where $\mathfrak{Q} := \{Q : dQ = \zeta dP, \ \zeta \in \mathfrak{A}\}$. Recall that if P and Q are two measures on $(\Omega, \mathcal{F})$, then it is said that Q is *absolutely continuous* with respect to P if $A \in \mathcal{F}$ and $P(A) = 0$ implies that $Q(A) = 0$. The Radon-Nikodym theorem says that Q is absolutely continuous with respect to P iff there exists a function $\eta : \Omega \to \mathbb{R}_+$ (density function) such that $Q(A) = \int_A \eta dP$ for every $A \in \mathcal{F}$. Therefore the result of theorem 3.1 can be interpreted as follows.

- Let $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$. Then a risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is coherent iff there exists a set $\mathfrak{Q}$ of absolutely continuous with respect to P probability measures such that the set of densities $\left\{ \frac{dQ}{dP} : Q \in \mathfrak{Q} \right\}$ forms a bounded set in the dual space $\mathcal{Z}^*$ and the representation (3.7) holds.

Let us consider some examples. The following risk measure is called the *mean-upper-semideviation* risk measure of order $p \in [1, \infty)$:

$$\rho(Z) := \mathbb{E}[Z] + \lambda \left(\mathbb{E}\left[[Z - \mathbb{E}[Z]]_+^p \right] \right)^{1/p}. \tag{3.8}$$

In the second term of the right hand side of (3.8), the excess of Z over its expectation is penalized. In order for this risk measure to be real valued it is natural to take $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$. For any $\lambda \in [0, 1]$ this risk measure is coherent and has the dual representation (3.5) with the set

$$\mathfrak{A} = \left\{ \zeta' \in \mathcal{Z}^* : \zeta' = 1 + \zeta - \mathbb{E}[\zeta], \ \|\zeta\|_q \le \lambda, \ \zeta \succeq 0 \right\}. \tag{3.9}$$

Note that the above set $\mathfrak{A}$ is a bounded convex closed subset of the dual space $\mathcal{Z}^* = L_q(\Omega, \mathcal{F}, P)$.

An important example of risk measure is *Value-at-Risk* measure

$$\mathsf{V@R}_\alpha(Z) := \inf\left\{ z : \mathsf{Pr}(Z \le z) \ge 1 - \alpha \right\}, \ \ \alpha \in (0, 1). \tag{3.10}$$

That is, $\mathsf{V@R}_\alpha(Z) = H^{-1}(1-\alpha)$ is the left side $(1-\alpha)$-quantile of the distribution of Z. Here $H(z) := \Pr(Z \le z)$ if the *cumulative distribution function* (cdf) of Z and

$$H^{-1}(\gamma) := \inf\left\{z : H(z) \ge \gamma\right\}$$

for $\gamma \in (0,1)$. For $\gamma = 0$ the corresponding left side quantile $H^{-1}(0) = -\infty$, and by the definition $H^{-1}(1) = +\infty$ if $Z(\omega)$ is unbounded from above. The $\mathsf{V@R}_\alpha$ risk measure is not coherent, it satisfies axioms (A1),(A3) and (A4) but is not necessarily convex, i.e., it does not possess the subadditivity property (3.2).

An important example of coherent risk measure is the *Average Value-at-Risk* measure

$$\mathsf{AV@R}_\alpha(Z) := \inf_{z\in\mathbb{R}} \left\{z + \alpha^{-1}\mathbb{E}[Z-z]_+\right\}, \quad \alpha \in (0,1]. \tag{3.11}$$

It is natural to take here $\mathcal{Z} := L_1(\Omega, \mathcal{F}, P)$. This risk measure is also known under the names Expected Shortfall, Expected Tail Loss and Conditional Value-at-Risk. It is possible to show that the set of minimizers of the right hand side of (3.11) is formed by $(1-\alpha)$-quantiles of the distribution of Z. In particular $z^* = \mathsf{V@R}_\alpha(Z)$ is such a minimizer. It follows that $\mathsf{AV@R}_\alpha(Z) \ge \mathsf{V@R}_\alpha(Z)$. Also it follows from (3.11) that

$$\mathsf{AV@R}_{\alpha_1}(Z) \ge \mathsf{AV@R}_{\alpha_2}(Z), \;\; 0 < \alpha_1 \le \alpha_2 \le 1. \tag{3.12}$$

The dual representation (3.5) for $\rho(Z) := \mathsf{AV@R}_\alpha(Z)$ holds with the set

$$\mathfrak{A} = \left\{\zeta \in L_\infty(\Omega, \mathcal{F}, P) : \zeta(\omega) \in [0, \alpha^{-1}] \text{ a.e. } \omega \in \Omega, \; \mathbb{E}[\zeta] = 1\right\}. \tag{3.13}$$

Note that the above set $\mathfrak{A}$ is a bounded closed subset of the dual space $\mathcal{Z}^* = L_\infty(\Omega, \mathcal{F}, P)$. If $\alpha = 1$, then the set $\mathfrak{A}$ consists of unique point $\zeta(\omega) \equiv 1$. That is, $\mathsf{AV@R}_1(Z) = \mathbb{E}[Z]$, this can be verified directly from the definition (3.11). For α tending to zero we have the following limit

$$\lim_{\alpha\downarrow 0} \mathsf{AV@R}_\alpha(Z) = \operatorname{ess\,sup}(Z). \tag{3.14}$$

In order for the risk measure $\rho(Z) := \operatorname{ess\,sup}(Z)$ to be finite valued it should be considered on the space $\mathcal{Z} := L_\infty(\Omega, \mathcal{F}, P)$; defined on that space this risk measure is coherent.

In both examples considered above the risk measures are functions of the distribution of the random variable Z. Such risk measures are called law invariant. Recall that two random variables Z and Z' have the same distribution if their cumulative distribution functions are equal to each other, i.e., $\Pr(Z \le z) = \Pr(Z' \le z)$ for all $z \in \mathbb{R}$. We write this relation as $Z \overset{\mathcal{D}}{\sim} Z'$.

Definition 3.1 It is said that a risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is *law invariant* if for any $Z, Z' \in \mathcal{Z}$ such that $Z \overset{\mathcal{D}}{\sim} Z'$ it follows that $\rho(Z) = \rho(Z')$.

Suppose for the moment that the set $\Omega = \{\omega_1, ..., \omega_K\}$ is finite with respective probabilities $p_1, ..., p_K$ such that any partial sums of p_k are different, i.e., $\sum_{k\in\mathcal{I}} p_k = \sum_{k\in\mathcal{J}} p_k$ for $\mathcal{I}, \mathcal{J} \subset \{1, ..., K\}$ only if $\mathcal{I} = \mathcal{J}$. Then $Z, Z' : \Omega \to \mathbb{R}$ have the same distribution only if $Z(\omega) = Z'(\omega)$ for all $\omega \in \Omega$. In that case any risk measure, defined on the space of random variables $Z : \Omega \to \mathbb{R}$, is law invariant. Therefore, for a meaningful discussion of law invariant risk measures it is natural to consider nonatomic probability spaces. It is said that measure P, and hence the space $(\Omega, \mathcal{F}, P)$, is *nonatomic* if any set $A \in \mathcal{F}$ of positive measure $P(A)$ contains a subset $B \in \mathcal{F}$ such that $P(A) > P(B) > 0$.

A natural question is how law invariance can be described in terms of the set $\mathfrak{A}$ in the dual representation (3.5). Let $T : \Omega \to \Omega$ be one-to-one onto mapping, i.e., $T(\omega) = T(\omega')$ iff $\omega = \omega'$ and

$T(\Omega) = \Omega$. It is said that T is a *measure-preserving transformation* if image $T(A) = \{T(\omega) : \omega \in A\}$ of any measurable set $A \in \mathcal{F}$ is also measurable and $P(A) = P(T(A))$ (see, e.g., [4, p.311]). Let us denote by

$$\mathfrak{G} := \{\text{the set of one-to-one onto measure-preserving transformations } T : \Omega \to \Omega\}.$$

We have that if $T \in \mathfrak{G}$, then $T^{-1} \in \mathfrak{G}$; and if $T_1, T_2 \in \mathfrak{G}$, then their composition[2] $T_1 \circ T_2 \in \mathfrak{G}$. That is, $\mathfrak{G}$ forms a group of transformations.

Theorem 3.2 *Suppose that the probability space* $(\Omega, \mathcal{F}, P)$ *is nonatomic. Then a coherent risk measure* $\rho : \mathcal{Z} \to \mathbb{R}$ *is law invariant iff the set* $\mathfrak{A}$ *in the dual representation* (3.5) *is invariant with respect to measure-preserving transformations, i.e., iff for any* $\zeta \in \mathfrak{A}$ *and any* $T \in \mathfrak{G}$ *and* $\zeta' := \zeta \circ T$ *it follows that* $\zeta' \in \mathfrak{A}$.

Proof. Let $T \in \mathfrak{G}$ and $\zeta \in \mathfrak{A}$. Consider $\zeta' := \zeta \circ T$. For $Z \in \mathcal{Z}$ we have

$$\langle Z, \zeta' \rangle = \int_\Omega Z(\omega)\zeta(T(\omega))dP(\omega) = \int_\Omega Z(T^{-1}(\omega))\zeta(\omega)dQ(\omega) = \langle Z', \zeta \rangle \tag{3.15}$$

where $Q = PT^{-1} = P$ and $Z' := Z \circ T^{-1}$. Since T is measure-preserving we have that $Z \overset{\mathcal{D}}{\sim} Z'$ and since ρ is law invariant, it follows that $\rho(Z) = \rho(Z')$. Therefore by (3.6) we obtain that $\zeta' \in \mathfrak{A}$.

Conversely suppose that $\zeta \circ T \in \mathfrak{A}$ for any $\zeta \in \mathfrak{A}$ and any $T \in \mathfrak{G}$. Let Z, Z' be two random variables having the same distribution. Since the probability space $(\Omega, \mathcal{F}, P)$ is nonatomic, there is $T \in \mathfrak{G}$ such that $Z' = Z \circ T$. For $\varepsilon > 0$ let $\zeta \in \mathfrak{A}$ be such that $\rho(Z') \le \langle Z', \zeta \rangle + \varepsilon$. By (3.15) and since $\zeta' \in \mathfrak{A}$ it follows that

$$\rho(Z') \le \langle Z', \zeta \rangle + \varepsilon = \langle Z, \zeta' \rangle + \varepsilon \le \rho(Z) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain that $\rho(Z') \le \rho(Z)$. The other inequality $\rho(Z') \ge \rho(Z)$ can be obtained in the same way and hence $\rho(Z') = \rho(Z)$. This competes the proof. ∎

With every law invariant risk measure ρ is associated its conditional analogue. That is, let Z be a random variable and Y be a random vector. Since $\rho(Z)$ is a function of the distribution of Z we can consider value of ρ at the conditional distribution of Z given $Y = y$, which we write as $\rho(Z|Y = y)$. Note that $\rho(Z|Y = y) = \phi(y)$ is a function of y, and hence $\phi(Y)$ is a random variable. We denote this random variable $\phi(Y)$ as $\rho(Z|Y)$ or $\rho_{|Y}(Z)$ and refer to $\rho_{|Y}(\cdot)$ as *conditional risk measure*. Of course, if Z and Y are independent, then distribution of Z does not depend on Y and hence in that case $\rho_{|Y}(Z) = \rho(Z)$.

For example the conditional analogue of the mean-upper semideviation risk measure (3.8) is

$$\rho_{|Y}(Z) = \mathbb{E}_{|Y}[Z] + \lambda \left(\mathbb{E}_{|Y} \left[\left[Z - \mathbb{E}_{|Y}[Z] \right]_+^p \right] \right)^{1/p}. \tag{3.16}$$

The conditional analogue of the *Average Value-at-Risk* measure is

$$\mathsf{AV@R}_{\alpha|Y}(Z) = \inf_{z \in \mathbb{R}} \left\{ z + \alpha^{-1} \mathbb{E}_{|Y}[Z - z]_+ \right\}, \quad \alpha \in (0, 1]. \tag{3.17}$$

The set of minimizers of the right hand side of (3.17) is given by $(1-\alpha)$-quantiles of the conditional distribution of Z, given Y, and is a function of Y.

There is an alternative, and in a sense equivalent, approach to defining conditional risk measures which is based on an axiomatic method and using sequences of nested sigma algebras (cf.,

[2]Composition $T = T_1 \circ T_2$ of two mappings is the mapping $T(\omega) = T_1(T_2(\omega))$.

[11],[16]). By considering sigma subalgebra of $\mathcal{F}$ generated by Y, the above approach of conditional distributions can be equivalently described in terms of the axiomatic approach. Both approaches have advantages and disadvantages. The above approach is more intuitive, although is restricted to *law invariant* risk measures. Also some properties could be easier seen in one approach than the other.

Since $\rho(Z|Y)$ is a random variable, we can condition it on another random vector X. That is, we can consider the following conditional risk measure $\rho[\rho(Z|Y)|X]$. We refer to this (conditional) risk measure as the composite risk measure and sometimes write it as $\rho_{|X} \circ \rho_{|Y}(Z)$. In particular, we can consider the composition $\rho \circ \rho_{|Y}$. The composite risk measure $\rho \circ \rho_{|Y}$ inherits many properties of ρ. If ρ is a law invariant coherent risk measure, then so is the composite risk measure $\rho \circ \rho_{|Y}$.

The composite risk measures $\rho \circ \rho_{|Y}$ can be quite complicated and difficult to write explicitly (cf., [16, section 5]). In general it does *not* hold that

$$\rho \circ \rho_{|Y} = \rho. \tag{3.18}$$

For example, for nonconstant Y equation (3.18) does not hold for $\rho := \mathsf{AV@R}_\alpha$ with $\alpha \in (0,1)$. Of course, if Z and Y are independent, then $\rho(Z|Y) = \rho(Z)$ and hence $\rho \circ \rho_{|Y}(Z) = \rho(Z)$, provided ρ is coherent. In particular, (3.18) holds if Y is constant and hence Z is independent of Y for any $Z \in \mathcal{Z}$. This also shows that the composite risk measure $\rho \circ \rho_{|Y}(\cdot)$ depends on Y. Equation (3.18) holds for any Y in at least in two cases, namely for $\rho(\cdot) := \mathbb{E}(\cdot)$ and $\rho(\cdot) := \operatorname{ess\,sup}(\cdot)$.

4 Minimax and Risk Averse Multistage Programming

Consider the following minimax extension of the risk neutral formulation (2.1) of multistage stochastic programs:

$$\begin{array}{cl} \underset{x_1, x_2(\cdot), \dots, x_T(\cdot)}{\operatorname{Min}} & \sup\limits_{Q \in \mathfrak{M}} \Big\{ \mathbb{E}_Q \big[F_1(x_1) + F_2(x_2(\xi_{[2]}), \xi_2) + \ldots + F_T \left(x_T(\xi_{[T]}), \xi_T \right) \big] \Big\} \\ \text{s.t.} & x_1 \in \mathcal{X}_1, \ x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}), \xi_t), \ t = 2, \dots, T. \end{array} \tag{4.1}$$

Here $\mathfrak{M}$ is a set of probability measures associated with vector $(\xi_2, ..., \xi_T) \in \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_T}$. We assume that probability measures of the set $\mathfrak{M}$ are supported on a closed set $\Xi \subset \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_T}$, i.e., for every $Q \in \mathfrak{M}$ it holds that Q-almost surely $(\xi_2, ..., \xi_T) \in \Xi$. As in the risk neutral case the minimization in (4.1) is performed over policies satisfying the feasibility constraints Q-almost surely for every $Q \in \mathfrak{M}$. The set $\mathfrak{M}$ can be viewed as the uncertainty set of probability measures and formulation (4.1) as hedging against a worst possible distribution. Of course, if $\mathfrak{M} = \{P\}$ is a singleton, then (4.1) becomes the risk neutral formulation (2.1).

Let P be a (reference) probability measure[3] on the set $\Xi \subset \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_T}$ equipped with its Borel sigma algebra $\mathcal{B}$ and let $\mathcal{Z} := L_p(\Xi, \mathcal{B}, P)$. That is, for $p \in [1, \infty)$ the space $\mathcal{Z}$ consists of measurable functions $Z(\cdot) : \Xi \to \mathbb{R}$ viewed as random variables having finite p-th order moment (with respect to the reference probability measure P), and for $p = \infty$ this is the space of essentially bounded measurable functions. Consider a coherent risk measure $\rho : \mathcal{Z} \to \mathbb{R}$. The corresponding *risk averse* multistage problem can be written as

$$\begin{array}{cl} \underset{x_1, x_2(\cdot), \dots, x_T(\cdot)}{\operatorname{Min}} & \rho \Big[F_1(x_1) + F_2(x_2(\xi_{[2]}), \xi_2) + \ldots + F_T \left(x_T(\xi_{[T]}), \xi_T \right) \Big] \\ \text{s.t.} & x_1 \in \mathcal{X}_1, \ x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}), \xi_t), \ t = 2, \dots, T. \end{array} \tag{4.2}$$

[3]Unless stated otherwise expectations and probabilistic statements will be made here with respect to the reference measure P.

The optimization in (4.2) is performed over policies satisfying the feasibility constraints for P-almost every realization of the data process and such that the function (random variable)

$$Z(\xi_{[T]}) := F_1(x_1) + F_2(x_2(\xi_{[2]}), \xi_2) + \ldots + F_T\left(x_T(\xi_{[T]}), \xi_T\right) \tag{4.3}$$

belongs to the considered space $\mathcal{Z}$.

Using dual representation (3.7) we can write the risk measure ρ as

$$\rho(Z) = \sup_{Q \in \mathfrak{Q}} \mathbb{E}_Q[Z], \ \ \forall Z \in \mathcal{Z}, \tag{4.4}$$

and hence problem (4.2) can be represented in the minimax form (4.1) with $\mathfrak{M} = \mathfrak{Q}$. There is a slight difference between respective formulations (4.1) and (4.2) of robust multistage programs - the set $\mathfrak{Q}$ consists of probability measures on $(\Xi, \mathcal{B})$ which are *absolutely continuous* with respect to the reference measure P, while we didn't make such assumption for the set $\mathfrak{M}$. However, at this point this is not essential, we will discuss this later.

In order to write dynamic programming equations for problems (4.1) and (4.2) we need a decomposable structure similar to (2.3) for the expectation operator. At every stage $t = 2, ..., T$ of the process we know the past, i.e., we observe a realization $\xi_{[t]}$ of the data process. For observed at stage t realization $\xi_{[t]}$ we need to define what do we optimize in the future stages. From the point of view of the minimax formulation (4.1) we need to specify conditional distribution of $\xi_{[t+1,T]}$ given $\xi_{[t]}$ for every probability distribution $Q \in \mathfrak{M}$ of $\xi_{[T]} = (\xi_{[t]}, \xi_{[t+1,T]})$.

Consider a linear space $\mathcal{Z}$ of measurable functions $Z(\cdot) : \Xi \to \mathbb{R}$, for example take $\mathcal{Z} := L_p(\Xi, \mathcal{B}, P)$, and sequence of spaces $\mathcal{Z}_1 \subset \mathcal{Z}_2 \subset \cdots \subset \mathcal{Z}_T$ with $\mathcal{Z}_t$ being the space of functions $Z \in \mathcal{Z}$ such that $Z(\xi_{[T]})$ does not depend on $\xi_{t+1}, ..., \xi_T$; with some abuse of notation we write such functions as $Z_t(\xi_{[t]})$. In particular, $\mathcal{Z}_T = \mathcal{Z}$ and $\mathcal{Z}_1$ is the space of constants and can be identified with $\mathbb{R}$. It could be noted that functions $Z_t \in \mathcal{Z}_t$ are defined on the set

$$\Xi_t := \left\{\xi_{[t]} \in \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_t} : \exists\, \xi'_{[T]} \in \Xi \text{ such that } \xi_{[t]} = \xi'_{[t]}\right\},$$

which is the projection of Ξ onto $\mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_t}$.

Consider sequence of mappings $\varrho_{t,T}(\cdot) : \mathcal{Z} \to \mathcal{Z}_t$, $t = 1, ..., T-1$, defined as

$$[\varrho_{t,T}(Z)](\xi_{[t]}) := \sup_{Q \in \mathfrak{M}} \mathbb{E}_{Q|\xi_{[t]}}\left[Z(\xi_{[T]})\right], \ \ Z \in \mathcal{Z}, \tag{4.5}$$

where the notation $\mathbb{E}_{Q|\xi_{[t]}}$ means that the expectation is conditional on $\xi_{[t]}$ and with respect to probability distribution Q of $\xi_{[T]} = (\xi_{[t]}, \xi_{[t+1,T]})$. We assume that the maximum in the right hand side of (4.5) is finite valued. Restricted to the space $\mathcal{Z}_{t+1} \subset \mathcal{Z}$ the mapping $\varrho_{t,T}$ will be denoted ρ_t, i.e., $\rho_t : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$ is given by

$$[\rho_t(Z_{t+1})](\xi_{[t]}) = \sup_{Q \in \mathfrak{M}} \mathbb{E}_{Q|\xi_{[t]}}\left[Z_{t+1}(\xi_{[t+1]})\right], \ \ Z_{t+1} \in \mathcal{Z}_{t+1}. \tag{4.6}$$

We also use notation $\varrho_{t,T|\xi_{[t]}}(Z)$ and $\rho_{t|\xi_{[t]}}(Z_{t+1})$ for $[\varrho_{t,T}(Z)](\xi_{[t]})$ and $[\rho_t(Z_{t+1})](\xi_{[t]})$, respectively. In a sense mappings ρ_t can be viewed as conditional risk mappings discussed in [11],[16], where such mappings were introduced in an axiomatic way (see section 5 for a further discussion).

After observing value $\xi_{[t]}$ of the data process at stage t, it is natural to perform future optimization at later stages using the conditional distributions of $\xi_{[t+1,T]}$ given $\xi_{[t]}$, that is with respect to $\varrho_{t,T|\xi_{[t]}}(\cdot)$. This motivates to consider the composite function

$$\bar{\varrho}(Z) := \varrho_{1,T}(\varrho_{2,T} \ldots (\varrho_{T-1,T}(Z)) \ldots), \ \ Z \in \mathcal{Z}, \tag{4.7}$$

denoted $\bar{\varrho} = \varrho_{1,T} \circ \varrho_{2,T} \circ \cdots \circ \varrho_{T-1,T}$. Note that mappings $\varrho_{t,T}(\cdot)$ and $\rho_t(\cdot)$ do coincide on $\mathcal{Z}_{t+1}$, and hence $\bar{\varrho} = \rho_1 \circ \rho_2 \circ \cdots \circ \rho_{T-1}$ as well. Since $\mathcal{Z}_1$ can be identified with $\mathbb{R}$, we can view $\bar{\varrho} : \mathcal{Z} \to \mathbb{R}$ as a real valued function, i.e., as a risk measure.

Consider risk measure $\rho(\cdot)$ in the form (4.4), this risk measure is coherent by Theorem 3.1. The respective composite risk measure $\bar{\varrho} = \rho_1 \circ \rho_2 \circ \cdots \circ \rho_{T-1}$ is also coherent. For the composite risk measure $\bar{\varrho}$ the corresponding risk averse problem can be written in the following nested form similar to (2.4):

$$\operatorname*{Min}_{x_1 \in \mathcal{X}_1} F_1(x_1) + \rho_{1|\xi_{[1]}} \left[\inf_{x_2 \in \mathcal{X}_2(x_1,\xi_2)} F_2(x_2,\xi_2) + \rho_{2|\xi_{[2]}} \left[\cdots + \rho_{T-1|\xi_{[T-1]}} \left[\inf_{x_T \in \mathcal{X}_T(x_{T-1},\xi_T)} F_T(x_T,\xi_T) \right] \right] \right], \tag{4.8}$$

(cf., [16]). Note that each mapping ρ_t, $t = 1, ..., T-1$, in (4.8) can be equivalently replaced by the respective mapping $\varrho_{t,T}$.

The risk measure ρ is not necessarily the same as the associated composite risk measure $\bar{\varrho}$, and formulations (4.1) and (4.8) are not necessarily equivalent.

- From the point of view of information at stage t - observed realization $\xi_{[t]}$ of the data process and the corresponding conditional distributions at future stages - the nested formulation (4.8) is *time consistent*. Therefore from this point of view the minimax formulation (4.1) (the risk averse formulation (4.2)) is time consistent iff it is equivalent to the nested formulation (4.8), in particular if $\rho(\cdot) = \bar{\varrho}(\cdot)$.

Some risk averse formulations are time consistent and some are not (cf., [19]). For a discussion and survey of time consistency concepts we may refer to [5],[17]; we will discuss this further in the next section.

For the nested formulation (4.8) it is possible to write dynamic programming equations in a way similar to (2.5)–(2.6) (cf., [16]). That is, equation (2.6) should be replaced by the equation

$$\mathcal{V}_{t+1}\left(x_t, \xi_{[t]}\right) = \rho_{t|\xi_{[t]}} \left[V_{t+1}\left(x_t, \xi_{[t+1]}\right) \right], \tag{4.9}$$

while equation (2.5) remains the same. Similar to the risk neutral case, the cost-to-go (value) functions $\mathcal{V}_{t+1}\left(x_t, \xi_{[t]}\right)$ do not depend on $\xi_{[t]}$ if the data process is stagewise independent. Here the stagewise independence means that $\xi_{[t+1,T]}$ is independent of $\xi_{[t]}$ for every distribution $Q \in \mathfrak{M}$ of $\xi_{[T]}$ and $t = 1, ..., T-1$. In terms of the set $\mathfrak{M}$ the stagewise independence means that

$$\mathfrak{M} = \left\{ Q = Q_2 \times \cdots \times Q_T : Q_t \in \mathcal{M}_t,\ t = 2, ..., T \right\}, \tag{4.10}$$

where for $t = 2, ..., T$, the set $\mathcal{M}_t$ is a set of probability measures on a (closed) set $\Xi_t \subset \mathbb{R}^{d_t}$ equipped with its Borel sigma algebra $\mathcal{B}_t$. Note that here measures $Q \in \mathfrak{M}$ are defined on the set $\Xi = \Xi_1 \times \cdots \times \Xi_T$.

In order to see a relation between formulation (4.1) (formulation (4.2)) and the corresponding nested formulation (4.8) let us observe the following. For $Z \in \mathcal{Z}$, we can write

$$\mathbb{E}_Q[Z(\xi_{[T]})] = \mathbb{E}_{Q|\xi_1}\Big[\cdots \mathbb{E}_{Q|\xi_{[T-2]}}\big[\mathbb{E}_{Q|\xi_{[T-1]}}[Z(\xi_{[T]})] \big] \cdots \Big],$$

and hence for $\rho(\cdot) = \sup_{Q \in \mathfrak{M}} \mathbb{E}_Q[\,\cdot\,]$ we have

$$\begin{aligned} \rho(Z) &= \sup_{Q \in \mathfrak{M}} \mathbb{E}_{Q|\xi_1}\Big[\cdots \mathbb{E}_{Q|\xi_{[T-2]}}\big[\mathbb{E}_{Q|\xi_{[T-1]}}[Z(\xi_{[T]})] \big] \cdots \Big] \\ &\leq \sup_{Q \in \mathfrak{M}} \mathbb{E}_{Q|\xi_1}\Big[\cdots \sup_{Q \in \mathfrak{M}} \mathbb{E}_{Q|\xi_{[T-2]}}\big[\sup_{Q \in \mathfrak{M}} \mathbb{E}_{Q|\xi_{[T-1]}}[Z(\xi_{[T]})] \big] \cdots \Big] \\ &= \rho_1 \circ \rho_2 \circ \cdots \circ \rho_{T-1}(Z). \end{aligned} \tag{4.11}$$

We obtain the following result.

Proposition 4.1 *For risk measure $\rho(Z) := \sup_{Q\in\mathfrak{M}} \mathbb{E}_Q[Z]$ and the corresponding composite risk measure $\bar{\varrho} = \rho_1 \circ \rho_2 \circ \cdots \circ \rho_{T-1}$ the following inequality holds*

$$\rho(Z) \le \bar{\varrho}(Z), \ \ \forall Z \in \mathcal{Z}. \tag{4.12}$$

It follows that the optimal value of the minimax problem (4.1) (risk averse problem (4.2)) is less than or equal to the optimal value of the corresponding problem (4.8). As the following example shows the inequality (4.12) can be strict even in the case of stagewise independence.

Example 1 Let $T = 3$ and $\mathfrak{M} := \mathcal{M}_2 \times \mathcal{M}_3$, with set $\mathcal{M}_2 := \{P\}$ being a singleton and $\mathcal{M}_3 := \{\Delta(\xi) : \xi \in \Xi\}$ being a set of probability measures formed by measures of unit mass at a single point $\xi \in \Xi$. Then for $Z = Z(\xi_2, \xi_3)$,

$$\rho(Z) = \sup_{Q_2\in\mathcal{M}_2, Q_3\in\mathcal{M}_3} \mathbb{E}_{Q_2\times Q_3}[Z(\xi_2,\xi_3)] = \sup_{\xi_3\in\Xi} \mathbb{E}_P[Z(\xi_2,\xi_3)], \tag{4.13}$$

and

$$\bar{\varrho}(Z) = \sup_{Q_2\in\mathcal{M}_2} \mathbb{E}_{Q_2}\left[\sup_{Q_3\in\mathcal{M}_3} \mathbb{E}_{Q_3}[Z(\xi_2,\xi_3)]\right] = \mathbb{E}_P\left\{\sup_{\xi_3\in\Xi} Z(\xi_2,\xi_3)\right\}. \tag{4.14}$$

In (4.13) and (4.14) the expectations are taken with respect to the probability distribution P of ξ_2. As it is well known in stochastic programming the inequality

$$\sup_{\xi_3\in\Xi} \mathbb{E}_P[Z(\xi_2,\xi_3)] \le \mathbb{E}_P\left\{\sup_{\xi_3\in\Xi} Z(\xi_2,\xi_3)\right\} \tag{4.15}$$

can be strict. Suppose, for example, that the set Ξ is finite. Then the maximum of $Z(\xi_2, \xi_3)$ over $\xi_3 \in \Xi$ is attained at a point $\bar{\xi}_3 = \bar{\xi}_3(\xi_2)$ depending on ξ_2. Consequently the right hand side of (4.15) is equal to $\mathbb{E}_P[Z(\xi_2, \bar{\xi}_3(\xi_2))]$, and can be strictly bigger than the left hand side unless $\bar{\xi}_3(\cdot)$ is constant. Therefore, the inequality (4.15) can be strict if the set Ξ contains more than one point.

Proposition 4.2 *Let $\rho(Z) := \sup_{Q\in\mathfrak{M}} \mathbb{E}_Q[Z]$ and suppose that the stagewise independence holds, i.e., the set $\mathfrak{M}$ is given in the form (4.10). Then $\rho(\cdot) = \bar{\varrho}(\cdot)$ if the interchageability property*

$$\mathbb{E}_{Q_2\times\cdots\times Q_t}\left\{\sup_{Q_{t+1}\in\mathcal{M}_{t+1}} \mathbb{E}_{Q_{t+1}}\left[Z_{t+1}(\xi_{[t]},\xi_{t+1})\right]\right\} = \sup_{Q_{t+1}\in\mathcal{M}_{t+1}} \mathbb{E}_{Q_2\times\cdots\times Q_{t+1}}\left[Z_{t+1}(\xi_{[t]},\xi_{t+1})\right]. \tag{4.16}$$

holds for all $Z \in \mathcal{Z}$ and $t = 2, ..., T-1$.

Proof. Let the set $\mathfrak{M}$ be given in the form (4.10). Then equation (4.6) takes the form

$$[\rho_t(Z_{t+1})](\xi_{[t]}) = \sup_{Q_{t+1}\in\mathcal{M}_{t+1}} \mathbb{E}_{Q_{t+1}}\left[Z_{t+1}(\xi_{[t]},\xi_{t+1})\right], \tag{4.17}$$

where the expectation $\mathbb{E}_{Q_{t+1}}\left[Z_{t+1}(\xi_{[t]},\xi_{t+1})\right]$ is taken with respect to the distribution Q_{t+1} of ξ_{t+1} for fixed $\xi_{[t]}$. Suppose that condition (4.16) holds. Then

$$\begin{aligned}
\rho(Z) &= \sup_{Q_2\in\mathcal{M}_2,\dots,Q_T\in\mathcal{M}_T} \mathbb{E}_{Q_2}\Big[\cdots \mathbb{E}_{Q_{T-1}}\big[\mathbb{E}_{Q_T}[Z(\xi_1,...,\xi_T)]\big]\cdots\Big]\\
&= \sup_{Q_2\in\mathcal{M}_2}\cdots\sup_{Q_T\in\mathcal{M}_T} \mathbb{E}_{Q_2}\Big[\cdots \mathbb{E}_{Q_{T-1}}\big[\mathbb{E}_{Q_T}[Z(\xi_1,...,\xi_T)]\big]\cdots\Big]\\
&= \sup_{Q_2\in\mathcal{M}_2}\cdots\sup_{Q_{T-1}\in\mathcal{M}_{T-1}} \mathbb{E}_{Q_2}\Big[\cdots \mathbb{E}_{Q_{T-1}}\big[\sup_{Q_T\in\mathcal{M}_T}\mathbb{E}_{Q_T}[Z(\xi_1,...,\xi_T)]\big]\cdots\Big]\\
&= \sup_{Q_2\in\mathcal{M}_2} \mathbb{E}_{Q_2}\Big[\cdots \sup_{Q_{T-1}\in\mathcal{M}_{T-1}}\mathbb{E}_{Q_{T-1}}\big[\sup_{Q_T\in\mathcal{M}_T}\mathbb{E}_{Q_T}[Z(\xi_1,...,\xi_T)]\big]\cdots\Big],
\end{aligned} \tag{4.18}$$

and hence $\rho(Z) = \bar{\varrho}(Z)$. ∎

5 Applications and Examples

In this section we discuss applications and examples of the general approach outlined in sections 3 and 4. Let us start with the following example corresponding to robust formulation of multistage programming. Let $\mathfrak{M}$ be the set of *all* probability measures on $(\Xi, \mathcal{B})$. Then for computing the maximum in $\rho(\cdot) = \sup_{Q\in\mathfrak{M}} \mathbb{E}_Q[\,\cdot\,]$ it suffices to perform the maximization with respect to measures of mass one at a point of the set Ξ, and hence the minimax formulation (4.1) can be written as

$$\begin{array}{cll} \operatorname*{Min}\limits_{x_1,x_2(\cdot),\dots,x_T(\cdot)} & \sup\limits_{(\xi_2,\dots,\xi_T)\in\Xi} & \big\{F_1(x_1) + F_2(x_2(\xi_{[2]}),\xi_2) + \ldots + F_T\left(x_T(\xi_{[T]}),\xi_T\right)\big\} \\ \text{s.t.} & & x_1 \in \mathcal{X}_1,\ x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}),\xi_t),\ t = 2,\dots,T. \end{array} \tag{5.1}$$

Here the interchageability property (4.16) holds and hence $\rho(\cdot)$ coincides with the corresponding composite risk measure $\bar{\varrho}(\cdot)$, the minimax formulation is equivalent to the corresponding nested formulation, and thus formulation (5.1) is time consistent. It could be noted that in this example there is no reference probability measure with respect to which all measures $Q \in \mathfrak{M}$ are absolutely continuous. Therefore strictly speaking the above $\rho(\cdot)$ is not a risk measure as it was defined in section 3. In order to reformulate this in terms of risk measures we may replace "sup" in (5.1) with the "ess sup" operator (recall that $\operatorname{ess\,sup}(\cdot)$ can be interpreted as $\mathsf{AV@R}_0(\cdot)$ risk measure, see (3.14)). For $\rho := \mathsf{AV@R}_0$ it holds that $\rho(\cdot) = \bar{\varrho}(\cdot)$ as well, and the minimax formulation is equivalent to the corresponding nested formulation. All that is discussed in detail in [21].

Let now $\rho(\cdot) := \mathsf{AV@R}_\alpha(\cdot)$ with $\mathcal{Z} := L_1(\Xi, \mathcal{B}, P)$ and $\alpha \in (0,1)$. In that case ρ is not equal to the corresponding composite risk measure $\bar{\varrho} = \rho_1 \circ \rho_2 \circ \cdots \circ \rho_{T-1}$. Note that the associated mapping $\rho_{t|\xi_{[t]}}$ (see (4.6)) is not the same here[4] as $\mathsf{AV@R}_{\alpha|\xi_{[t]}}(\cdot)$. Suppose, for example, that $T = 3$ and the stagewise independence holds, i.e., the set $\mathfrak{M}$ is of the form (4.10). Then for $Z = Z(\xi_2, \xi_3)$,

$$\rho_{2|\xi_2}(Z) = \sup_{\zeta\in\mathfrak{A}} \mathbb{E}_{|\xi_2}[Z(\xi_2,\xi_3)\zeta(\xi_2,\xi_3)],$$

where $\mathfrak{A} = \{\zeta(\xi_2,\xi_3) : 0 \preceq \zeta(\xi_2,\xi_3) \preceq \alpha^{-1},\ \mathbb{E}[\zeta] = 1\}$. Consider the set $\mathfrak{A}'$ formed by densities $\zeta \in \mathfrak{A}$ which are functions of ξ_3 alone, i.e., $\mathfrak{A}' = \{\zeta(\xi_3) : 0 \preceq \zeta(\xi_3) \preceq \alpha^{-1},\ \mathbb{E}[\zeta] = 1\}$. Then

$$\mathsf{AV@R}_{\alpha|\xi_2}(Z) = \sup_{\zeta\in\mathfrak{A}'} \mathbb{E}_{|\xi_2}[Z(\xi_2,\xi_3)\zeta(\xi_3)].$$

Since $\mathfrak{A}'$ is a (strict) subset of $\mathfrak{A}$, it follows that $\rho_{2|\xi_2}(Z) \geq \mathsf{AV@R}_{\alpha|\xi_2}(Z)$, and the inequality can be strict. Let, for example, ξ_2 and ξ_3 have uniform distributions on [0,1]. Then for a given $\xi_2 \in [0,1]$ and for $Z \succeq 0$, by taking $\zeta \in \mathfrak{A}$ such that $\zeta(\xi_2,\xi_3) = \alpha^{-1}$ for all $\xi_3 \in [0,1]$, we have that $\rho_{2|\xi_2}(Z) = \alpha^{-1}\mathbb{E}_{|\xi_2}[Z(\xi_2,\xi_3)]$.

The corresponding multistage problem (4.2) can be written as

$$\begin{array}{cl} \operatorname*{Min}\limits_{z,x_1,x_2(\cdot),\dots,x_T(\cdot)} & \mathbb{E}\left\{z + \alpha^{-1}\big[F_1(x_1) + F_2(x_2(\xi_{[2]}),\xi_2) + \ldots + F_T\left(x_T(\xi_{[T]}),\xi_T\right) - z\big]_+\right\} \\ \text{s.t.} & x_1 \in \mathcal{X}_1,\ x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}),\xi_t),\ t = 2,\dots,T. \end{array} \tag{5.2}$$

If the multistage problem is linear and the number of scenarios (realizations of the data process) is finite, then it is possible to write problem (5.2) as a large linear programming problem. As far as dynamic equations are concerned let us observe that at the last stage $t = T$ we would need to solve problem conditional on z and decisions up to stage $t = T - 1$. Therefore dynamic equations

[4] The author is indebted to Dan Iancu for pointing this out.

cannot be written in an obvious way and formulation (5.2) is not time consistent. The corresponding nested formulation, of course, is time consistent. It is interesting to observe that in extreme cases of $\alpha = 1$ (when $\rho(\cdot) = \mathbb{E}(\cdot)$) and $\alpha = 0$ (when $\rho(\cdot) = \operatorname{ess\,sup}(\cdot)$) the minimax and nested formulations are equivalent.

As another example consider the problem of moments in a multistage setting (see, e.g., [10] and references therein for a discussion of the problem of moments). Let $\Xi_t \subset \mathbb{R}^{d_t}$, $b_t \in \mathbb{R}^{q_t}$ and $\psi_t : \Xi_t \to \mathbb{R}^{q_t}$ be a measurable mapping, $t = 2, ..., T$. Define $\mathcal{M}_t$ to be the set of probability measures Q_t on $(\Xi_t, \mathcal{B}_t)$ satisfying the following moment conditions

$$\mathbb{E}_{Q_t}[\psi_t(\xi_t)] = b_t, \;\; t = 2, ..., T, \tag{5.3}$$

and let $\mathfrak{M}$ be the set of the form (4.10) of products of these measures . By this setting the stagewise independence condition holds here.

In this example the minimax and nested formulations are not necessarily equivalent. In order to see this consider the following instance. Let $T = 3$ and the set Ξ_2 be finite. Then the moment constraints (5.3) take a form of linear equations for the respective probabilities associated with points of the set Ξ_2. By an appropriate choice the moment constraints define a unique probability measure on Ξ_2. If furthermore the set $\mathcal{M}_3$ consists of all probability measures on $\Xi_3 \subset \mathbb{R}^{d_3}$, then this becomes a case considered in Example 1. This shows that the corresponding inequality (4.12) can be strict in this example.

For the respective nested formulation we can write the dynamic programming equations in the form (2.10) with

$$\mathcal{V}_{t+1}(x_t) = \sup_{Q_{t+1} \in \mathcal{M}_{t+1}} \mathbb{E}_{Q_{t+1}}\left[V_{t+1}(x_t, \xi_{t+1})\right]. \tag{5.4}$$

It can be noted that by the Richter - Rogosinski Theorem (cf., [14]) the maximum in the right hand side of (5.4) is attained at a probability measure supported on at most $1 + q_{t+1}$ points.

In the next section we discuss the classical inventory problem with moment constraints (see, e.g., [23] for a thorough discussion of the inventory model).

5.1 Inventory Model

5.1.1 Static Case

Let us start by setting the problem in a static case. Suppose that a company has to decide about order quantity x of a certain product to satisfy demand d. The cost of ordering is $c > 0$ per unit. If the demand d is larger than x, then the company makes an additional order for the unit price $b \geq 0$. The cost of this is equal to $b(d - x)$ if $d > x$, and is zero otherwise. On the other hand, if $d < x$, then holding cost of $h(x - d) \geq 0$ is incurred. The total cost is then equal to[5]

$$F(x, d) = cx + b[d - x]_+ + h[x - d]_+ = \max\big\{(c - b)x + bd, (c + h)x - hd\big\}. \tag{5.5}$$

We assume that $b > c$, i.e., the back order penalty cost is *larger* than the ordering cost. The objective is to minimize the total cost $F(x, d)$, with x being the decision variable.

One has to make a decision before knowing realization of the demand d, so we model the demand as a random variable D. Suppose that we have a partial information about probability distribution of D. That is, we can specify a family $\mathfrak{M}$ of probability measures on $\mathbb{R}_+$ and consider the following worst case distribution problem

$$\operatorname*{Min}_{x \geq 0} \left\{ \phi(x) := \sup_{Q \in \mathfrak{M}} \mathbb{E}_Q[F(x, D)] \right\}. \tag{5.6}$$

[5]For a number $a \in \mathbb{R}$, $[a]_+$ denotes the maximum $\max\{a, 0\}$.

The above problem, with the set $\mathfrak{M}$ defined by first and second order moments of the demand D, was studied in the pioneering paper by Scarf [18].

Suppose that range of the demand is known, i.e., it is known that $D \in [l, u]$. If there is no other information about distribution of D, then we can take $\mathfrak{M}$ to be the set of all probability distributions supported on the interval $[l, u]$. In that case the maximum of $\mathbb{E}_Q[F(x, D)]$ over $Q \in \mathfrak{M}$ is attained at measure of mass one at a point of $[l, u]$, and the respective optimal solution of the minimax problem is (e.g., [20, p.5])

$$x^* = \frac{hl + bu}{h + b}. \tag{5.7}$$

Suppose, further, that mean $\mu = \mathbb{E}[D]$ of the demand is known, and hence let $\mathfrak{M}$ be the set of probability distributions supported on an interval $[l, u] \subset \mathbb{R}_+$ and having mean μ.

Proposition 5.1 *Suppose that $\mathfrak{M}$ is the set of probability distributions supported on an interval $[l, u] \subset \mathbb{R}_+$ and having mean μ. Then problem* (5.6) *has the following optimal solution*

$$\bar{x} = \begin{cases} l & \text{if } \frac{b-c}{b+h} < \frac{u-\mu}{u-l}, \\ u & \text{if } \frac{b-c}{b+h} > \frac{u-\mu}{u-l}. \end{cases} \tag{5.8}$$

If $\frac{b-c}{b+h} = \frac{u-\mu}{u-l}$, then the set of optimal solutions of (5.6) *coincides with the interval* $[l, u]$.

Proof. Since the function $F(x, d)$ is convex in d, we have by the following Lemma 5.1 that for any x the worst probability measure in (5.6) is the measure supported on points l and u with respective probabilities $(u - \mu)/(u - l)$ and $(\mu - l)/(u - l)$. Therefore problem (5.6) is reduced to the classical Newsvendor Problem problem with the respective cdf of the demand:

$$H(t) = \begin{cases} 0 & \text{if } t < l, \\ \frac{u-\mu}{u-l} & \text{if } l \le t < u, \\ 1 & \text{if } u \le t. \end{cases}$$

The optimal solution of the Newsvendor Problem is $\bar{x} = H^{-1}\left(\frac{b-c}{b+h}\right)$ (e.g., [20, p.3]), and hence (5.8) follows. ▮

Lemma 5.1 *Consider points $l < u$ and $\mu \in [l, u]$, a convex function $g : \mathbb{R} \to \mathbb{R}$ and the set $\mathfrak{M}$ of probability measures on the interval $[l, u]$ having mean μ. Then the problem*

$$\operatorname*{Max}_{Q \in \mathfrak{M}} \mathbb{E}_Q[g(D)] \tag{5.9}$$

attains its optimal solution at probability measure supported on points l and u with respective probabilities $(u - \mu)/(u - l)$ and $(\mu - l)/(u - l)$.

Proof. By the Richter - Rogosinski Theorem we have that maximum in (5.9) is attained at a probability measure supported on two points of the interval $[l, u]$.

Let us observe that for any $c' \le c$, $d' \ge d$, $p \in [0, 1]$ and $p' \in [0, 1]$ such that

$$(1 - p)c + pd = (1 - p')c' + p'd',$$

it follows by convexity of $g(t)$ that

$$(1 - p)g(c) + pg(d) \le (1 - p')g(c') + p'g(d'). \tag{5.10}$$

Indeed, suppose for the moment that $c = c'$. Moreover, by making change of variables $t \to t - c$ and replacing $g(\cdot)$ with $g(\cdot) - g(c)$, we can assume without loss of generality that $c = 0$ and $g(c) = 0$. By convexity of $g(\cdot)$ we have that for $\tau \in (0, 1)$ the inequality $g(\tau d') \le \tau g(d')$ holds. Taking $\tau = p'/p$ and noting that $p'/p = d/d'$, we obtain that $pg(d) \le p'g(d')$. This proves (5.10) in case of $c = c'$. The other case where $d = d'$ can be verified in a similar way.

Since the left hand side of (5.10) is equal to the expectation of $g(D)$ with respect to the probability measure supported on the points c and d with respective probabilities $1 - p$ and p such that $(1 - p)c + pd = \mu$, and the right hand side of (5.10) is equal to the expectation of $g(D)$ with respect to the probability measure supported on the points c' and d' such that $(1 - p')c' + p'd' = \mu$, it follows that the optimal value of problem (5.9) is attained at a probability measure supported on the points l and u. The corresponding probabilities can be computed in a straightforward way from the equation $(1 - p)l + pu = \mu$. ∎

5.1.2 Multistage Case

Consider the following multistage worst distribution formulation of inventory model

$$\begin{array}{ll} \underset{x_t \ge y_t}{\text{Min}} \; \underset{Q \in \mathfrak{M}}{\sup} & \mathbb{E}_Q \left[\sum_{t=1}^{T} c_t(x_t - y_t) + \psi_t(x_t, D_t) \right] \\ \text{s.t.} & y_{t+1} = x_t - d_t, \; t = 1, ..., T - 1. \end{array} \tag{5.11}$$

Here y_1 is a given initial inventory level, c_t, b_t, h_t are the ordering, backorder penalty, and holding costs per unit, respectively, at time t, and

$$\psi_t(x_t, d_t) := b_t[d_t - x_t]_+ + h_t[x_t - d_t]_+.$$

We assume that $b_t > c_t > 0$ and $h_t \ge 0$, $t = 1, ..., T$, and that $\mathfrak{M}$ is a set of probability measures (distributions) of the demand process vector $(D_1, ..., D_T) \in \mathbb{R}^T_+$. The minimization in (5.11) is performed over (nonanticipative) policies of the form $x_1, x_2(d_{[1]}), ..., x_T(d_{[T-1]})$ satisfying the feasibility constraints of (5.11) for almost every realization $(d_1, ..., d_T)$ of the demand process. As before $d_{[t]} := (d_1, ..., d_t)$ denotes history of the process up to time t.

Suppose that the distribution of $(D_1, ..., D_T)$ is supported on the set $\Xi = \Xi_1 \times \cdots \times \Xi_T$, given by the direct product of (finite) intervals $\Xi_t := [l_t, u_t] \subset \mathbb{R}_+$, and we know respective means $\mu_t = \mathbb{E}[D_t]$. That is, let $\mathcal{M}_t$ be the set of probability distributions supported on the interval $[l_t, u_t]$ and having mean $\mu_t \in [l_t, u_t]$, $t = 1, ..., T$, and let $\mathfrak{M}$ be the corresponding set of product measures of the form (4.10).

For the nested formulation the corresponding cost-to-go functions are given by the following dynamic equations, $t = T, ..., 2$,

$$V_t(y_t) = \inf_{x_t \ge y_t} \left\{ c_t(x_t - y_t) + \sup_{Q_t \in \mathcal{M}_t} \mathbb{E}_{Q_t} \left[\psi_t(x_t, D_t) + V_{t+1}(x_t - D_t) \right] \right\}, \tag{5.12}$$

where $V_{T+1}(\cdot) \equiv 0$. It is straightforward to verify by induction that the functions $V_t(\cdot)$ are convex, and hence by Lemma 5.1 we have that the maximum in (5.12), over probability measures $Q_t \in \mathcal{M}_t$, is attained at the probability measure

$$Q_t^* = p_t \Delta(l_t) + (1 - p_t)\Delta(u_t)$$

supported on points l_t and u_t with respective probabilities $p_t = (u_t - \mu_t)/(u_t - l_t)$ and $1 - p_t = (\mu_t - l_t)/(u_t - l_t)$. Therefore the respective problem is reduced to the corresponding problem with single probability distribution $Q^* = Q_1^* \times \cdots \times Q_T^*$ of the demand process with the random variables

D_t, $t = 1, ..., T$, being independent of each other. That is, the problem is reduced to the risk neutral case with the demand process having finite number $N = 2^T$ scenarios.

Here the minimax and nested formulations are equivalent. Indeed, the optimal value of the nested formulation is always greater than or equal to the optimal value of the minimax formulation. Here the opposite inequality also holds, this can be seen by setting $Q = Q^*$ in (5.11).

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