On the relation between concavity cuts and the surrogate dual for convex maximization problems

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Abstract

In this note we establish a relation between two bounds for convex maximization problems, the one based on a concavity cut, and the surrogate dual bound. Both bounds have been known in the literature for a few decades but, to the authors' knowledge, the relation between them has not been previously observed in the literature.

KEYWORDS: convex maximization, concavity cut, surrogate dual.

1 Introduction

Dual bounds for nonconvex problems can sometimes be related to other bounds for the same problems. The best known example of such relations is represented by the Lagrangian bound. Let us consider the nonconvex problem

$$\begin{aligned} \max \quad & f(\mathbf{x}) \\ & h_i(\mathbf{x}) \leq 0 \quad i = 1, \ldots, m \\ & \mathbf{x} \in X \end{aligned} \tag{1}$$

where $X \neq \emptyset$ is a convex and compact set, while f, h_i, $i = 1, \ldots, m$, are lower semicontinuous functions over X. For $\boldsymbol{\lambda} \in \mathbb{R}^m_+$, we define the Lagrangian function

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x}).$$

Obviously, for any $\boldsymbol{\lambda} \geq \mathbf{0}$,

$$g(\boldsymbol{\lambda}) = \max_{\mathbf{x} \in X} L(\mathbf{x}, \boldsymbol{\lambda})$$

is an upper bound for (1) and the best of such bounds corresponds to the solution of so called Lagrangian dual

$$\begin{aligned} &\min \;\; g(\boldsymbol{\lambda}) \\ &\boldsymbol{\lambda} \geq \mathbf{0} \end{aligned} \tag{2}$$

Under some regularity conditions, it has been proved (see [2, 4]) that if functions h_i's are affine ones, then, the Lagrangian dual bound is equal to the bound

$$\begin{aligned} &\max \;\; conc_{f,X}(\mathbf{x}) \\ &h_i(\mathbf{x}) \leq 0 \qquad\qquad i = 1, \ldots, m \\ &\mathbf{x} \in X \end{aligned}$$

obtained by substituting the objective function f with its concave envelope $conc_{f,X}$ over X. In [3] it is shown that strict dominance of the Lagrangian bound with respect to the concave envelope one holds as soon as we drop the assumption of affine functions h_i's.

Another dual bound is the surrogate one. This is usually more complicated to compute with respect to the Lagrangian bound. However, it can be studied for some classes of nonconvex problems. In particular, here we consider the convex maximization problem

$$\begin{aligned} \max \quad & f(\mathbf{x}) \\ & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned} \tag{3}$$

where f is a convex function, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m_+$, and the feasible region

$$P = \{\mathbf{x} \in \mathbb{R}^n_+ \;:\; \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$$

is a nonempty polytope; we also assume that f is continuous in the feasible set. Note that $\mathbf{b} \geq \mathbf{0}$ implies that the origin is a vertex of P and we further assume that the origin is *not* the unique point in P.

In this paper we would like to relate the surrogate dual bound for convex maximization problems with a well known bound based on concavity cuts. Although both the surrogate dual bound and the bound based on concavity cuts have been known in the literature for a few decades, to the authors' knowledge, the relation between them has never been remarked.

In Section 2 we will recall the notion of concavity cuts. In Section 3 we will discuss the surrogate dual. Finally, in Section 4 we will show the relation between the two bounds.

2 γ-extensions and concavity cuts

For convex maximization problems, γ-concavity cuts have been first introduced in [8] and employed within conical partitions algorithms (see, e.g., [5, 6, 7]). Given a polyhedral cone C lying in the nonnegative orthant and vertexed at $\mathbf{x}_0 \in P$, i.e.,

$$C = \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{x} = \mathbf{x}_0 + \mathbf{Q}\mathbf{y} \geq \mathbf{0}, \quad \mathbf{y} \in \mathbb{R}^n_+\},$$

where $\mathbf{Q}$ is an invertible $n \times n$ matrix whose columns are the generating rays of C, the subproblem over C is defined as follows

$$\begin{aligned} \max \quad & f(\mathbf{x}) \\ & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \in C \end{aligned} \tag{4}$$

Since by the change of variable $\mathbf{x} = \mathbf{x}_0 + \mathbf{Q}\mathbf{y}$ we can always rewrite (4) as (3), in what follows we will always refer to (3). Now, let us consider a value $\gamma > f(\mathbf{0})$, e.g., $\gamma = LB + \varepsilon$, for some $\varepsilon > 0$, where LB is some known lower bound for problem (3). Note that we can always take $LB \geq f(\mathbf{0})$ since $\mathbf{0}$ is a feasible solution in view of $\mathbf{b} \in \mathbb{R}^m_+$. For each $j = 1, \ldots, n$, let us denote by

$$s_j = s_j(\gamma) = \max\{\lambda \ : \ f(\lambda \mathbf{e}_j) \leq \gamma\} > 0, \tag{5}$$

where $\mathbf{e}_j$ is the vector whose components are all equal to zero, except the j-th one which is equal to 1. Point $s_j\mathbf{e}_j$ is called γ-*extension* over the j-th axis and we notice that it might happen that $s_j = +\infty$. It can be proved (see, e.g., [6]) that the optimal value of problem (3) is not larger than γ if the optimal value of the following linear problem

$$\begin{aligned} \max \quad & [\textstyle\sum_{j\,:\,s_j<\infty} \frac{x_j}{s_j} - 1] \\ & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{6}$$

is not larger than 0. Moreover, the following inequality, called γ-*concavity cut*, does not remove from the feasible region P any point with function value larger than γ

$$\sum_{j\,:\,s_j<\infty} \frac{x_j}{s_j} \leq 1.$$

3 Surrogate dual

Let us assume, w.l.o.g., that $f(\mathbf{0}) = 0$. Given $\mathbf{u} = (u_1, \ldots, u_m) \geq \mathbf{0}$, let us define the function

$$\begin{aligned} S(\mathbf{u}) = \max \quad & f(\mathbf{x}) \\ & \mathbf{u}^T\mathbf{A}\mathbf{x} \leq \mathbf{u}^T\mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{7}$$

In (7) we are substituting the original constraints $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ with the single *surrogate constraint* $\mathbf{u}^T\mathbf{A}\mathbf{x} \leq \mathbf{u}^T\mathbf{b}$. Since

$$\mathbf{u} \geq \mathbf{0}, \quad \mathbf{A}\mathbf{x} \leq \mathbf{b} \quad \Rightarrow \quad \mathbf{u}^T\mathbf{A}\mathbf{x} \leq \mathbf{u}^T\mathbf{b},$$

for any $\mathbf{u} \geq \mathbf{0}$ the optimal value of (7) is an upper bound for (3). The *surrogate dual* returns the best possible of such bounds

$$\min_{\mathbf{u}\geq\mathbf{0}} S(\mathbf{u}).$$

Given some value $\gamma > LB$, we would like to establish whether

$$\min_{\mathbf{u}\geq\mathbf{0}} S(\mathbf{u}) \leq \gamma.$$

When, for every $\gamma > LB$, every γ–extension is finite, the answer to this question is relatively simple and can be found following [1]. We report here an extension of the theorem and its proof.

Theorem 1 *Assume that, for every finite choice of* $\gamma > LB$*, the quantity* $s_j = s_j(\gamma)$ *defined in (5) is finite. Then, for each* $\gamma > LB$

$$\min_{\mathbf{u}\geq\mathbf{0}} S(\mathbf{u}) \leq \gamma.$$

if and only if the following linear system admits at least a solution

$$\begin{aligned} \mathbf{u}^T\mathbf{b} &= 1 \\ \mathbf{u}^T A_j &\geq 1/s_j(\gamma) \qquad\qquad \forall j \\ \mathbf{u} &\geq 0 \end{aligned} \tag{8}$$

where A_j *is the* j*–th column of* $\mathbf{A}$*.*

Proof. Let $\mathbf{u} \geq 0$ be given and assume that $\exists\, j : \mathbf{u}^T A_j \leq 0$. In this case, the surrogate feasible set is unbounded. In particular, as (7) is a convex maximization problem, it will be unbounded if and only if it is unbounded along an extreme ray of the feasible region. The extreme rays of the polyhedron

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{u}^T\mathbf{A}\mathbf{x} \leq \mathbf{u}^T\mathbf{b}, \mathbf{x} \geq 0\} \tag{9}$$

are:

1. $\mathbf{e}_j$, for all j such that $\mathbf{u}^T A_j \leq 0$
2. $\mathbf{e}_j - \frac{\mathbf{u}^T A_j}{\mathbf{u}^T A_i}\mathbf{e}_i$, where $\mathbf{u}^T A_i > 0$ and $\mathbf{u}^T A_j < 0$.

As it has been assumed that $s_j(\gamma) < \infty\, \forall\, \gamma$, then $f(\lambda \mathbf{e}_j) \to \infty$ as $\lambda \to \infty$ and $S(\mathbf{u})$ is unbounded along the ray $\mathbf{e}_j$.

As we wish to minimize $S(\mathbf{u})$ it is thus required that $\mathbf{u}$ is chosen so that the problem is not unbounded and thus the choice of $\mathbf{u}$ can be restricted to those vectors which satisfy $\mathbf{u}^T\mathbf{A} > 0$.

We can also assume that $\mathbf{u}^T\mathbf{b} > 0$. In fact, if $\mathbf{u}^T\mathbf{b} = 0$, as we are now assuming $\mathbf{u}^T\mathbf{A} > 0$, $\mathbf{0}$ would be the unique feasible solution to (7) and, thus, to the original problem (3), which, by assumption, can not hold.

So we can restrict the analysis to the case $\mathbf{u}^T\mathbf{b} > 0$ and, thanks to the homogeneity of the constraints, we let $\mathbf{u}^T\mathbf{b} = 1$.

If, as we are now assuming, $S(\mathbf{u})$ is not unbounded, it must have an optimal solution at a vertex. One vertex is the origin, while all the others lie on axes $\mathbf{e}_j$ for which $\mathbf{u}^T A_j > 0$. Any vertex can be represented as

$$V_j = \delta_j \mathbf{e}_j$$

where

$$\delta_j = \frac{\mathbf{u}^T \mathbf{b}}{\mathbf{u}^T A_j}$$

when $\mathbf{u}^T A_j > 0$. Thus

$$S(\mathbf{u}) = \max \left\{ f(\mathbf{0}), \max_{j=1,\ldots,n:\mathbf{u}^T A_j > 0} f\left(\frac{\mathbf{u}^T \mathbf{b}}{\mathbf{u}^T A_j} \mathbf{e}_j \right) \right\}$$

Thus $S(\mathbf{u}) \leq \gamma$ if and only if

$$f\left(\frac{\mathbf{u}^T \mathbf{b}}{\mathbf{u}^T A_j} \mathbf{e}_j \right) \leq \gamma \qquad \forall\, j : \mathbf{u}^T A_j > 0$$

or, equivalently, if and only if

$$\frac{\mathbf{u}^T \mathbf{b}}{\mathbf{u}^T A_j} \leq s_j(\gamma) \qquad \forall\, j : \mathbf{u}^T A_j > 0.$$

Consequently, $S(\mathbf{u}) \leq \gamma$ has a solution if and only if the following system

$$\begin{aligned} \mathbf{u}^T \mathbf{b} &= 1 \\ \mathbf{u}^T A_j &\geq 1/s_j(\gamma) && \forall\, j \\ \mathbf{u} &\geq 0 \end{aligned}$$

is feasible.

□

Notice that in the above proof an important role is played by the assumption that $s_j < \infty$. In the next section we will see that when this assumption is satisfied, a strong relationship exists between bounds derived by concavity cuts and those obtained through the surrogate dual. However, as we will show later, when this assumption is relaxed, the equivalence between the two bounds does not hold any more (in fact, we will see that the surrogate dual bound dominates the bound based on concavity cuts).

4 The relation between concavity cuts and surrogate dual

In this section we will study the strict relation between γ-concavity cuts and the surrogate dual. To the authors' knowledge this relation was not previously observed in the literature.

Theorem 2 *If $s_j(\gamma) < \infty$ for all $\gamma > LB$, then, given $\gamma > LB$, there exists* $\mathbf{u}$*:*

$$\begin{aligned} \mathbf{u}^T b &= 1 \\ \mathbf{u}^T A_j &\geq 1/s_j \qquad \forall j \in 1,\ldots,n \\ \mathbf{u} &\geq 0 \end{aligned}$$

if and only if

$$\begin{aligned} \max \sum_{j=1}^{n} \frac{x_j}{s_j} - 1 \\ \mathbf{Ax} \leq \mathbf{b} \\ \mathbf{x} \geq 0 \end{aligned}$$

is non positive.

Proof. In order to establish whether the linear system (8) admits a solution, we can, obviously, solve this linear program

$$\begin{aligned} \min \quad & 0 \\ & \textstyle\sum_{i=1}^{m} u_i b_i = 1 \\ & \textstyle\sum_{i=1}^{m} u_i a_{ij} \geq \frac{1}{s_j} \quad j = 1,\ldots,n \\ & u_i \geq 0 \qquad\qquad i = 1,\ldots,m \end{aligned}$$

Its dual is the following problem

$$\begin{aligned} \max \quad & \textstyle\sum_{j=1}^{n} \frac{\eta_j}{s_j} + \mu \\ & \textstyle\sum_{j=1}^{n} a_{ij}\eta_j + \mu b_i \leq 0 \quad i = 1,\ldots,m \\ & \eta_j \geq 0 \qquad\qquad j = 1,\ldots,n \end{aligned} \tag{10}$$

Note that feasible solutions $(\boldsymbol{\eta}, \mu)$ of the problem above either with $\mu > 0$ or with $\mu = 0$ and $\boldsymbol{\eta} \neq \mathbf{0}$ would imply that the recession cone of P contains a vector $\boldsymbol{\eta} \neq \mathbf{0}$, thus contradicting the fact that P is a polytope. Therefore, we can restrict our attention to feasible solutions with $\mu < 0$. For $\mu < 0$, let us rewrite (10) as follows

$$\begin{aligned} \max \quad & -\mu[\textstyle\sum_{j=1}^{n} -\frac{\eta_j}{\mu s_j} - 1] \\ & -\mu[\textstyle\sum_{j=1}^{n} -a_{ij}\frac{\eta_j}{\mu} - b_i] \leq 0 \quad i = 1,\ldots,m \\ & \eta_j \geq 0 \qquad\qquad j = 1,\ldots,n \end{aligned}$$

After the change of variables $\xi_j = -\eta_j/\mu$, for $\mu < 0$, the problem above will have optimal value equal to 0 (and, consequently, the linear system (8) will admit a solution) if and only if

$$\begin{array}{ll} \max & [\sum_{j=1}^n \frac{\xi_j}{s_j} - 1] \\ & \sum_{j=1}^n a_{ij}\xi_j \leq b_i \quad i = 1, \ldots, m \\ & \xi_j \geq 0 \qquad\qquad\quad j = 1, \ldots, n \end{array}$$

has negative optimal value. But the problem above is exactly problem (6), thus proving the relation between the γ-concavity bound and the surrogate dual bound. □

The case of unbounded γ–extensions

In the above proofs an important role was played by the finiteness assumption of every s_j. This assumption is crucial for proving the equivalence between surrogate bounds and γ–concavity cuts. The following example shows that when γ–extensions are unbounded, equivalence no longer holds. Consider the problem

$$\begin{array}{ll} \max & \max\{0; x_1 - x_2 - 1\} \\ & x_1 - x_2 \leq 1 \\ & x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

The objective function $f(\mathbf{x}) = \max\{0; x_1 - x_2 - 1\}$ is null at every feasible point; thus we choose $\gamma = 0$ and we wish to check whether a feasible point with $f(\mathbf{x}) > 0$ exixts.

The γ–extensions in this case are easily seen to be

$$s_1 = s_1(0) = 1$$
$$s_2 = s_2(0) = \infty$$

thus the bound based on γ–concavity cut in this case is obtained from

$$\begin{array}{ll} \max & x_1 - 1 \\ & x_1 - x_2 \leq 1 \\ & x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

which is attained at $x_1 = 2, x_1 = 1$ with function value 1. Thus adding the concavity cut $x_1 \leq 1$ is not sufficient to show that no feasible point exists with function value greater than 0.

If on the other side we consider the surrogate dual

$$
\begin{aligned}
S(\mathbf{u}) = \max \quad & \max\{0; x_1 - x_2 - 1\} \\
& u_1 x_1 + (u_2 - u_1) x_2 \leq u_1 + u_2 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

it is easy to see that it is possible to choose u_1, u_2 so that $S(\mathbf{u}) = 0$ (a possible choice is $u_1 = 1, u_2 = 0$). Then, thanks to the surrogate bound, we can prove that no point with positive function value exists in this case. Thus, in this case, the surrogate dual bound dominates the one obtained from γ–concavity cuts.

The question now arises about the general relationship between the two types of cuts. In what follows we prove that the surrogate bound always dominates the concavity–based one; strict dominance is possible only when at least one of the γ–extensions is unbounded.

In fact we can prove the following:

Theorem 3 *If the following linear system*

$$
\begin{array}{llll}
\mathbf{b}^T\mathbf{u} & = 1 & & \\
\mathbf{u}^T A_j & \geq 1/s_j & \forall\, j & : s_j(\gamma) < \infty \\
\mathbf{u}^T A_j & \geq 0 & \forall\, j & : s_j(\gamma) = \infty \\
\mathbf{u} & \geq \mathbf{0} & &
\end{array}
\tag{11}
$$

admits a feasible solution $\bar{\mathbf{u}}$, *then* $S(\bar{\mathbf{u}}) \leq \gamma$.

Proof. Let $\mathcal{F} := \{j : s_j(\gamma) = \infty\}$ be the set of indices of coordinate axes along which the objective function does not exceed γ. Consider the surrogate problem

$$
\begin{aligned}
S(\bar{\mathbf{u}}) = \max \quad & f(\mathbf{x}) \\
& \bar{\mathbf{u}}^T \mathbf{A} \mathbf{x} \leq \bar{\mathbf{u}}^T \mathbf{b} = 1 \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}
\tag{12}
$$

Let

$$
\mathcal{F}_{\bar{\mathbf{u}}} = \{j \in \mathcal{F} \; : \; \bar{\mathbf{u}}^T A_j = 0\}.
$$

The feasible region $X_{\bar{\mathbf{u}}}$ of (12) has vertices $\mathbf{v}_0 = \mathbf{0}$ and

$$\mathbf{v}_j = \frac{1}{\bar{\mathbf{u}}^T A_j}\mathbf{e}_j \quad \forall\ j \in \{1, \ldots, n\} \setminus \mathcal{F}_{\bar{\mathbf{u}}}.$$

Since

$$\frac{1}{\bar{\mathbf{u}}^T A_j} \leq s_j(\gamma), \quad \forall\ j \in \{1, \ldots, n\} \setminus \mathcal{F}_{\bar{\mathbf{u}}}$$

and $f(\mathbf{0}) \leq \gamma$, we can conclude that $f(\mathbf{v}_j) \leq \gamma$ holds at all vertices of $X_{\bar{\mathbf{u}}}$. Then, if we are able to prove that the optimal value of problem (12) is attained at a vertex of $X_{\bar{\mathbf{u}}}$, we are done. The extreme rays $\mathbf{r}_j$ of $X_{\bar{\mathbf{u}}}$ are the directions of the axes $\mathbf{e}_j$, $j \in \mathcal{F}_{\bar{\mathbf{u}}}$. Now, let

$$I_{\bar{\mathbf{u}}} = \{0, \ldots, n\} \setminus \mathcal{F}_{\bar{\mathbf{u}}},$$

By the well known Minkovski's theorem, each point $\mathbf{x} \in X_{\bar{\mathbf{u}}}$ can be represented as follows

$$\mathbf{x} = \sum_{j \in I_{\bar{\mathbf{u}}}} \lambda_j \mathbf{v}_j + \sum_{j \in \mathcal{F}_{\bar{\mathbf{u}}}} \mu_j \mathbf{r}_j \tag{13}$$

with

$$\lambda_j \geq 0, \quad j \in I_{\bar{\mathbf{u}}}, \quad \sum_{j \in I_{\bar{\mathbf{u}}}} \lambda_j = 1, \quad \mu_j \geq 0,\ j \in \mathcal{F}_{\bar{\mathbf{u}}}.$$

Let

$$K_{\mathbf{x}} = \sum_{j \in \mathcal{F}_{\bar{\mathbf{u}}}} \mu_j.$$

If $K_{\mathbf{x}} = 0$, then, by convexity of f

$$f(\mathbf{x}) \leq \sum_{j \in I_{\bar{\mathbf{u}}}} \lambda_j f(\mathbf{v}_j) \leq \max_{j \in I_{\bar{\mathbf{u}}}} f(\mathbf{v}_j) \leq \gamma$$

and the result is proved. If $K_{\mathbf{x}} > 0$, then take $\alpha > 0$ and rewrite (13) as follows

$$\mathbf{x} = \sum_{j \in I_{\bar{\mathbf{u}}}} (1-\alpha)\lambda_j \frac{\mathbf{v}_j}{1-\alpha} + \sum_{j \in \mathcal{F}_{\bar{\mathbf{u}}}} \alpha \frac{\mu_j}{K_{\mathbf{x}}} \frac{K_{\mathbf{x}} \mathbf{r}_j}{\alpha}$$

Noting that

$$\sum_{j \in I_{\bar{\mathbf{u}}}} \underbrace{(1-\alpha)\lambda_j}_{\geq 0} + \sum_{j \in \mathcal{F}_{\bar{\mathbf{u}}}} \underbrace{\alpha \frac{\mu_j}{K_{\mathbf{x}}}}_{\geq 0} = 1$$

and that

$$f\left(\frac{K_{\mathbf{x}}\mathbf{r}_j}{\alpha}\right) \leq \gamma, \quad j \in \mathcal{F}_{\bar{\mathbf{u}}},$$

by convexity of f we have

$$\begin{array}{rl} f(\mathbf{x}) & \leq \sum_{j\in I_{\bar{\mathbf{u}}}} (1-\alpha)\lambda_j f\left(\frac{\mathbf{v}_j}{1-\alpha}\right) + \sum_{j\in\mathcal{F}_{\bar{\mathbf{u}}}} \alpha \frac{\mu_j}{K_{\mathbf{x}}} f\left(\frac{K_{\mathbf{x}}\mathbf{r}_j}{\alpha}\right) \leq \\ & \sum_{j\in I_{\bar{\mathbf{u}}}} (1-\alpha)\lambda_j f\left(\frac{\mathbf{v}_j}{1-\alpha}\right) + \alpha\gamma. \end{array}$$

Letting $\alpha \to 0$, by continuity of f, we end up with

$$f(\mathbf{x}) \leq \max_{j\in I_{\bar{\mathbf{u}}}} f(\mathbf{v}_j) \leq \gamma,$$

as we wanted to prove. □

Since, by following the same proof as in Section 4, the dual of the feasibility problem (11) turns out to be the bound based on the γ–concavity cut, we can conclude that the surrogate dual bound is always at least as good as the one obtained via concavity cuts and, in some cases, it is strictly preferable.

In the proof of Theorem 1 we noticed that the extreme rays of the polyhedron (9) are $\mathbf{e}_j$, for all j such that $\mathbf{u}^T A_j \leq 0$, and

$$\mathbf{e}_j - \frac{\mathbf{u}^T A_j}{\mathbf{u}^T A_i}\mathbf{e}_i,$$

where $\mathbf{u}^T A_i > 0$ and $\mathbf{u}^T A_j < 0$. Now, let

$$K = \{j \;:\; s_j < +\infty\}.$$

Let us assume that, given the directions

$$\mathbf{e}_j + \delta\mathbf{e}_i, \quad i \in K, \; j \notin K,$$

the range $[0, \delta_{ij}]$ for the δ values such that

$$f(\lambda(\mathbf{e}_j + \delta\mathbf{e}_i)) \leq \gamma \quad \forall\, \lambda \geq 0$$

is known. Then, we can prove the following theorem.

Theorem 4 *It holds that* $\min_{\mathbf{u}\geq 0}\ S(\mathbf{u}) \leq \gamma$ *if and only if the optimal value of the following linear program*

$$\begin{array}{lll} \max & \sum_{i\in K} \frac{x_i}{s_i} - 1 & \\ & \sum_{i\in K} x_i A_i + \sum_{i\in K,\ j\notin K} z_{ij}(A_j + \delta_{ij} A_i) \leq \mathbf{b} & \\ & x_i, z_{ij} \geq 0 & i \in K,\ j \notin K \end{array} \tag{14}$$

is not larger than 0.

Proof. It holds that $\min_{\mathbf{u}\geq 0}\ S(\mathbf{u}) \leq \gamma$ if and only if either the following system admits a solution

$$\begin{array}{ll} \mathbf{u}^T\mathbf{b} > 0 & \\ \mathbf{u}^T A_j \geq \frac{\mathbf{u}^T\mathbf{b}}{s_j} & j \in K \\ \frac{-\mathbf{u}^T A_j}{\mathbf{u}^T A_i} \leq \delta_{ij} & i \in K,\ j \notin K \\ \mathbf{u} \geq \mathbf{0} & \end{array} \tag{15}$$

or the following system admits a solution

$$\begin{array}{ll} \mathbf{u}^T\mathbf{b} = 0 & \\ \mathbf{u}^T A_j > 0 & j \in K \\ \frac{-\mathbf{u}^T A_j}{\mathbf{u}^T A_i} \leq \delta_{ij} & i \in K,\ j \notin K \\ \mathbf{u} \geq \mathbf{0} & \end{array} \tag{16}$$

We consider the two systems separately. By homogeneity, system (15) is equivalent to

$$\begin{array}{ll} \mathbf{u}^T\mathbf{b} = 1 & \\ \mathbf{u}^T A_j \geq 1/s_j & j \in K \\ \mathbf{u}^T(A_j + \delta_{ij} A_i) \geq 0 & i \in K,\ j \notin K \\ \mathbf{u} \geq \mathbf{0} & \end{array}$$

After adding the objective min 0 to this linear system, by the same proof of Theorem 2, the dual of the resulting linear program can be rewritten as

(14). Again by homogeneity, system (16) is equivalent to

$$\begin{aligned}
&\mathbf{u}^T \mathbf{b} = 0 && \\
&\mathbf{u}^T A_j \geq 1 && j \in K \\
&\mathbf{u}^T (A_j + \delta_{ij} A_i) \geq 0 && i \in K,\ j \notin K \\
&\mathbf{u} \geq \mathbf{0} &&
\end{aligned}$$

After adding the objective min 0 to this linear system, again by the same proof of Theorem 2, the dual of the resulting linear program can be rewritten as

$$\begin{aligned}
\max \quad & \textstyle\sum_{i \in K} x_i \\
& \textstyle\sum_{i \in K} x_i A_i + \sum_{i \in K,\ j \notin K} z_{ij} (A_j + \delta_{ij} A_i) \leq \mathbf{b} \\
& x_i, z_{ij} \geq 0 \qquad\qquad i \in K,\ j \notin K
\end{aligned}$$

Therefore, $\min_{\mathbf{u} \geq 0} S(\mathbf{u}) \leq \gamma$ holds if and only if either the optimal value of the above linear program is equal to 0 or the optimal value of linear program (14) is not larger than 0. But by noting that if the optimal value of the above linear program is equal to 0, then the optimal value of (14) is not larger than 0, we can conclude that $\min_{\mathbf{u} \geq 0} S(\mathbf{u}) \leq \gamma$ holds if and only if the optimal value of (14) is not larger than 0. □

Conclusions

We have shown in this paper that for convex maximization problems over a polytope, the surrogate dual bound is always at least as strong as the bound based on the γ–concavity cut. Moreover, we have shown that if some assumptions hold for the objective function which imply that f diverges to infinity along coordinate axes, then the two bounds are equivalent.

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