Multiobjective DC Programming with Infinite Convex Constraints *

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Abstract

In this paper new results are established in multiobjective DC programming with infinite convex constraints (*MOPIC* for abbr.) that are defined on Banach space (finite or infinite) with objectives given as the difference of convex functions subject to infinite convex constraints. This problem can also be called multiobjective DC semi-infinite and infinite programming, where decision variables run over finite-dimensional and infinite-dimensional spaces, respectively. This problem has not been studied yet at present. Necessary and sufficient optimality conditions for weak Pareto-optimality are introduced. Also, we seek a connection between multiobjective linear infinite programming and *MOPIC*. The Wolfe and Mond-Weir

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dual problems are presented and the weak, strong and strict converse duality theorems are also derived. The results obtained are new in both semi-infinite and infinite frameworks.

Keywords: Multiobjective DC programming with infinite convex constraints, optimality, duality, saddle point, (weak) Pareto-optimality.

1 Introduction

Infinite programs are generally defined on infinite dimensional spaces of decision variables and contain, among other constraints, infinitely many inequality constraints. These problems reduce to semi-infinite programs in the case of finite dimensional spaces of decision variables. This paper concerns a class of multiobjective programs in Banach spaces with objectives given as the difference of two convex functions and constraints described by an arbitrary (possibly infinite) number of convex inequalities. We refer to such problems as to multiobjective DC programming with infinite convex constraints or multiobjective DC infinite programming, where the abbreviation "DC" signifies the difference of convex functions.

Let X be a Banach space, and let T be an arbitrary (maybe infinite) index set. We mainly focus in this paper on a special class of multiobjective DC programming with infinite convex constraints ($MOPIC$ for abbr.):

$$min\{\ f(x) = \phi(x) - \psi(x):\ s.t.\ g_t(x) \leq 0,\ t \in T,\ x \in S\} \tag{1.1}$$

where $f = (f_1, \cdots, f_m)^T$, $f_j = \phi_j - \psi_j$, $j \in I := \{1, \cdots, m\}$, the functions ϕ_j, ψ_j and g_t in (1.1) defined on X with their values in the extended real line $\bar{R} := R \cup \{\infty\}$ are proper, lower semi-continuous (l.s.c.), and convex; the set $S \subset X$ is closed and convex. This problem can also be called multiobjective DC semi-infinite and infinite programming, where decision variables run over finite-dimensional and infinite-dimensional spaces, respectively.

In practice, many problems can be formulated as the above multiobjective semi-infinite programming. For example, consider a stochastic multiobjective linear programming

$$min\left\{Cx:\ s.t.\ \ a_j^T x - b_j \geq 0,\ \forall j \in J\right\}$$

where $C \in R^{m\times n}$, J is finite index set, a_j and b_j are uncertain. It is only known that vector $(a_j,\ b_j)$ may vary in a set $T_j \subset R^{n+1}$. If a pessimistic model is considered to find a solution x which is feasible for all possible data vectors. Then a multiobjective semi-infinite programming is obtained,

$$min\left\{Cx:\ s.t.\ \ a^T x - b \geq 0,\ \forall (a,b) \in T := \cup_{j\in J} T_j\right\}.$$

Note that in the case f is a scalar function (maybe not a DC function), problem (1.1) reduces to semi-infinite programming which has been extensively studied from theoretical and numerical points [1-6]. Semi-infinite programming has many applications, e.g., pollution control models, the popular semi-definite programming, optimal experimental design in regression and portfolio optimization problems. We refer the readers to the review by L*ó*pez and Still [6]. If f is a DC function, then (1.1) reduces DC programming with infinite convex constraints [1]. In [1], a necessary optimality condition is proposed by applying the generalized Farkas's Lemma [2] and the sufficient optimality condition is also derived for convex programming with infinite convex constraints. As $X = R^n$, f and g_t are all convex, Shapiro [3] considers the duality and optimality conditions for convex semi-infinite programming.

When T is finite, then (1.1) reduces to multiobjective programming [7-10]. Some necessary and sufficient optimality conditions for the weakly efficient solutions of vector optimization problems with finite equality and inequality constraints are consider in [7] by using two kinds of constraints qualifications in terms of the MP subdifferential due to Ye [11]. Bot and Wanka [10] present a duality theory by conjugacy approach for a multiobjective optimization problem with a convex objective function and a finite many DC constraints.

Caristi *et al* [12] obtain the optimality criteria of Kuhn-Tucker type under generalized invexity conditions for differentiable semi-infinite multiobjective programming with a set of restrictions indexed in a compact. As an important and classical example of (1.1) multiobjective programming with cone constraints [13,14] which duality and optimality conditions have been extensively studied. However, there are no research results about multiobjective DC programming with infinite convex constraints. So our research results in this paper is original.

In Section 2, some notations and preliminary results are presented. Section 3 establishes the necessary and sufficient optimality conditions. The infinite multiobjecitive linear programming approximation is proposed in Section 4. The relations of the optimality solutions between the original problem and the approximation problem are also discussed, and some saddle-point rules are derived. Section 5 gives the Wolfe and Mond-Weir dual problems and the weak, strong and strict converse duality theorems.

2 Notations and preliminary results

We first give the concept of optimality for (1.1), i.e., (weak) Pareto optimality or (weak) efficiency. Define the feasible solution set to $MOPIC$ by

$$\Omega := S \cap \{x \in X |\ g_t(x) \leq 0 \ \ \forall t \in T\}. \tag{2.1}$$

Definition 2.1 *Given a point $x^* \in X$, x^* is said to be locally (weak efficient) Pareto efficient solution of MOPIC if and only if there exists a neighborhood $V \subset \Omega$ such that there does not exist $x \in V$ such that ($f(x) < f(x^*)$) $f(x) \leq f(x^*)$, and $f(x) \neq f(x^*)$, where we define any $u,\ v \in R^m$,*

$$u \leq v \Longleftrightarrow v - u \in R^m_+ \Longleftrightarrow v_j - u_j \geq 0,\ j \in I;$$

$$u < v \Longleftrightarrow v - u \in R^m_{++} \Longleftrightarrow v_j - u_j > 0,\ j \in I.$$

For a Banach space X, we denote the topologically *dual* space X^* equipped with *weak** *topology* w^*, where $< \cdot, \cdot >$ stands for the canonical pairing between X and X^*. The *weak** closure of a set in the dual space (i.e., its closure in the *weak** topology) is denoted by cl*.

The indicator function of a set $S \subset X$ is defined as $\delta(x; S) = 0$ if $x \in S$ and $\delta(x; S) = +\infty$. The normal cone to a convex set S at $\bar{x} \in S$ is defined by

$$N(\bar{x}; S) := \partial\delta(\bar{x}; S) = \{x^* \in X^* |\ < x^*, x - \bar{x} > \leq 0,\quad \forall x \in S\}.$$

Let $\phi : X \to \bar{R}$ are proper l.s.c., and convex function. The *domain* and the *epigraph* of ϕ are defined as follows: $dom\phi := \{x \in X |\ \phi(x) \leq \infty\}$ and $epi\phi := \{(x, \nu) \in X \times R |\ \nu \geq \phi(x)\}$. The conjugate function of ϕ, $\phi^* : X^* \to \bar{R}$, is defined for all $v \in X^*$ by $\phi^*(v) := sup\{< v, x > -\phi(x) |\ x \in dom\phi\}$. It follows from [15] that if $\bar{x} \in dom\phi$, the *epigraph of the conjugate function* is defined by

$$epi\phi^* = \bigcup_{\epsilon \geq 0} \{(x^*, < x^*, \bar{x} > +\epsilon - \phi(\bar{x})) |\ x^* \in \partial_\epsilon\phi(\bar{x})\}$$

where, for a given $\epsilon \geq 0$, the $\epsilon-$subdifferential of ϕ at $\bar{x} \in dom\phi$, $\partial_\epsilon\phi(\bar{x})$, is defined by the possibly empty weak* closed convex set,

$$\partial_\epsilon\phi(\bar{x}) = \{x^* \in X^* |\ < x^*, x - \bar{x} > \leq \phi(x) - \phi(\bar{x}) + \epsilon\ \forall x \in X\},\ \epsilon \geq 0,$$

and $\partial_\epsilon\phi(\bar{x}) := \emptyset$ for $\bar{x} \overline{\in} dom\phi$. If $\epsilon = 0$, then $\partial_\epsilon\phi(\bar{x})$ collapses to $\partial\phi(\bar{x})$, the usual classic subdifferential of ϕ at $\bar{x}$ in the sense of convex analysis.

Finally, we recall some results on the refined epigraphical and subdifferential rules for convex function which are useful for our study in the next sections and have been recently established in [16].

Lemma 2.2 *Suppose that $\phi_i : X \to \bar{R}$, $i = 1, 2$ is l.s.c. and convex. If $dom\phi_1 \cap dom\phi_2 \neq \emptyset$, then we have the following conditions are equivalent:*
(i) The set $epi\phi_1^* + epi\phi_2^*$ is weak* closed in $X^* \times R$.
(ii) The refined conjugate epigraphical rule holds: $epi(\phi_1 + \phi_2)^* = epi\phi_1^* + epi\phi_2^*$.

The above lemma implies that if the equivalent conditions hold, then the following subdifferential sum rule is satisfied, $\partial(\phi_1 + \phi_2)(\bar{x}) = \partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x})$. In the following part of this paper we assume that the equivalent conditions of Lemma 2.2 hold.

Invexity hypothesis on functions are also needed in this paper for deriving some results. The DC version of the concept of an invex function is introduced here. The other definition of invex functions can be found in [17,18].

Definition 2.3 *Consider the DC function $f(x) = \phi(x) - \psi(x) : X \to R$ and the given function $\beta : X \times X \to X$. Then the function f is generalized invex at $\bar{x} \in X$ with function β if*

$$f(x) - f(\bar{x}) \geq \max_{v^* \in \partial\phi(\bar{x})} < v^* - w^*,\ \beta(x, \bar{x}) >,\ \forall w^* \in \partial\psi_j(\bar{x})$$

3 Optimality Conditions

Further let P be the product space of $\lambda = \{\lambda_t |\ t \in T\}$ with $\lambda_t \in R$ for all $t \in T$, let $\widetilde{P}$ be the collection of $\lambda \in P$ such that $\lambda_t \neq 0$ for finitely many $t \in T$, and define the positive cone in $\widetilde{P}$ by

$$\widetilde{P}_+ := \{\lambda \in \tilde{P} |\ \lambda_t \geq 0\ \forall t \in T\}. \tag{3.1}$$

Denote the index set of nonzero part of λ as $Np(\lambda) := \{t \in T |\ \lambda_t \neq 0\}$. Then define

$$u\lambda := \sum_{t \in T} u_t \lambda_t = \sum_{t \in Np(\lambda)} u_t \lambda_t$$

for $u \in P$ and $\lambda \in \widetilde{P}$.

In some of the results of this paper we use a generalized closedness qualification condition: (ϕ_j, g_t, S) satisfies the generalized closedness qualification condition, GCQC in brief, if the set

$$\sum_{j=1}^{m} epi\ \phi_j^* + cone\ \left\{ \bigcup_{t \in T} epi\ g_t^* \right\} + epi\ \delta^*(\cdot\ ; S)$$

is weak* closed in the space $X^* \times R$. This qualification condition helps us derive the related results and is a generalization of the closedness qualification condition, which is presented in optimization literature [1]. In this paper we also assume that optimal solution $\bar{x}$ of (1.1) is limited to $\Omega \cap dom\phi_j,\ \forall j \in I$.

The following theorem presents the necessary optimality condition for $MOPIC$. First we define the active constraint multipliers set as follows

$$A(\bar{x}) := \left\{ \lambda \in \widetilde{P}_+ |\ \lambda_t g_t(\bar{x}) = 0\ \forall t \in Np(\lambda) \right\}. \tag{3.2}$$

Theorem 3.1 ***(Necessary optimality condition for MOPIC)*** *Assume that $\bar{x}$ is a locally weak Pareto efficient solution (WPES) to MOPIC, then there exist $\xi := (\xi_1, \cdots, \xi_m)^T \in R^m_+$ such that*

$$\sum_{j\in I}\xi_j\partial\psi_j(\bar{x}) \subset \sum_{j\in I}\xi_j\partial\phi_j(\bar{x}) + \bigcup_{\lambda\in A(\bar{x})}\left\{\sum_{t\in Np(\lambda)}\lambda_t\partial g_t(\bar{x})\right\} + N(\bar{x};S), \ \sum_{j\in I}\xi_j = 1. \quad (3.3)$$

Proof. Since $\bar{x} \in \Omega \cap dom\phi_j$, $\forall j \in I$, then there are two possible cases regarding $\bar{x}$: $\forall j \in I$, $\bar{x}\overline{\in} dom\psi_j$ or there is at least one $\bar{j} \in I$ such that $\bar{x} \in dom\psi_{\bar{j}}$, where $\overline{\in}$ denotes that $\bar{x}$ not in $dom\psi_j$. If $\forall j \in I$, $\bar{x}\overline{\in} dom\psi_j$, then we have $\partial\phi_j(\bar{x}) = \emptyset$, and hence (3.3) holds directly. For the second case, we can assume that $\bar{x} \in dom\psi_j$, $\forall j \in I$. According to the definition of weak Pareto efficient, we have there exists at least one index $j_0 \in I$ such that

$$f_{j_0}(x) \geq f_{j_0}(\bar{x}), \quad \forall x \in \Omega \bigcap B_\epsilon(\bar{x}), \quad (3.4)$$

where $B_\epsilon(\bar{x}) = \{x : \ \parallel x - \bar{x} \parallel < \epsilon\}$. If let

$$F_\xi(x) = \sum_{j\in I}\xi_j f_j(x),$$

then (3.4) means that there exists some $\bar{\xi} \in R^m_+$ with $\sum_{j\in I}\bar{\xi}_j = 1$, $\bar{x}$ is the local optimal solution to the following DC infinite program,

$$min \ \ \{F_{\bar{\xi}}(x) : \ x \in \Omega\}. \quad (3.5)$$

This is obvious by defining $\bar{\xi}_j = \frac{1}{|I_1|}$, for $j \in I_1$ and $\bar{\xi}_j = 0$, for $j \in I_2$, where $I_1 = \{j \in I : \ f_j(x) \geq f_{j_0}(\bar{x}), \ \forall x \in \Omega \bigcap B_\epsilon(\bar{x})\}$, $I_2 = I_1/I$. It is obvious that index set I_1 is nonempty and $\bar{x}$ is a local optimal solution to (3.5) with $\bar{\xi}$ is defined as above. Given subgradient $w^*_j \in X^*$ of ψ_j such that $\psi_j(x) - \psi_j(\bar{x}) \geq < w^*_j, \ x - \bar{x} >$, for all $x \in X$, then $\bar{x}$ is also a local minimizer to the convex infinite program:

$$min \ \ \left\{\bar{F}(x) := \sum_{j\in I}\left(\phi_j(x) - < w^*_j, \ x - \bar{x} > -\psi_j(\bar{x})\right) : \ x \in \Omega\right\}. \quad (3.6)$$

From the convexity of the above problem (3.6), the local minimizer $\bar{x}$ is a global solution to this problem, i.e., $\bar{F}(\bar{x}) \leq \bar{F}(x)$, $\forall x \in \Omega$. This together with Theorem 1 of [1] and subdifferential sum rule we have the conclusion (3.3). ■

According to the proof of Theorem 3.1 and Theorem 2 of [1], we can give the necessary and sufficient optimality condition for convex multiobjective infinite programming.

Theorem 3.2 ***(Necessary and Sufficient Optimality Conditions for Convex Multiobjective Infinite Programming)****Let f_j, and g_t be all convex. Then $\bar{x}$ is a (WPES) to MOPIC if and only if there exist $\xi := (\xi_1, \cdots, \xi_m)^T \in R_+^m$ such that*

$$0 \in \sum_{j\in I} \xi_j \partial f_j(\bar{x}) + \bigcup_{\lambda\in A(\bar{x})} \left\{ \sum_{t\in Np(\lambda)} \lambda_t \partial g_t(\bar{x}) \right\} + N(\bar{x}; S), \ \ \sum_{j\in I} \xi_j = 1.$$

Theorem 3.2 gives the necessary and sufficient optimality conditions for convex multiobjective infinite programming. In the following theorem we will discuss the sufficient optimality condition for MOPIC. From the above two theorems without loss generality, in the following theorems and proofs of this paper, we can assume that any optimal solution $\bar{x}$ of (1.1) satisfies $\bar{x} \in dom\psi_j, \ \forall j \in I$.

Theorem 3.3 ***(Sufficient Optimality Condition for MOPIC)*** *Suppose that f_j, g_t and $\delta(\cdot; S)$ are all generalized invex at $\bar{x}$ with function β, $\forall j \in I$, $\forall t \in T$. If there are $0 \neq \xi \in R_+^m$ and $\lambda \in \widetilde{P}_+$ such that (3.3) holds, then $\bar{x}$ is a locally weak Pareto efficient solution to MOPIC.*

Proof. We will prove this theorem by contradiction, i.e., suppose that $\bar{x}$ is not the weak Pareto efficient solution and there is $x^{'} \in \Omega$ such that

$$f(x^{'}) < f(\bar{x}), \ \lambda_t g_t(x^{'}) \leq \lambda_t g_t(\bar{x}), \ \forall \lambda \in A(\bar{x}). \tag{3.7}$$

It follows from the conditions of this theorem that for all $x \in \Omega$

$$f_j(x) - f_j(\bar{x}) \geq \max_{v_j^*\in\partial\phi_j(\bar{x})} < v_j^* - w_j^*, \ \beta(x, \bar{x}) >, \ w_j^* \in \partial\psi_j(\bar{x}), \ j \in I \tag{3.8}$$

$$g_t(x) - g_t(\bar{x}) \geq \max_{\pi_t^*\in\partial g_t(\bar{x})} < \pi_t^*, \ \beta(x, \bar{x}) >, \quad t \in T \tag{3.9}$$

$$0 = \delta(x; S) - \delta(\bar{x}; S) \geq \max_{\mu^*\in N(\bar{x};S)} < \mu^*, \ \beta(x, \bar{x}) > . \tag{3.10}$$

(3.7-3.9) means that

$$\max_{v_j^*\in\partial\phi_j(\bar{x})} < v_j^* - w_j^*, \ \beta(x^{'}, \bar{x}) >< 0, \ w_j^* \in \partial\psi_j(\bar{x}), \ j \in I \tag{3.11}$$

$$\max_{\pi_t^*\in\partial g_t(\bar{x})} < \lambda_t\pi_t^*, \ \beta(x^{'}, \bar{x}) >\leq \lambda_t(g_t(x) - g_t(\bar{x})), \quad \forall \lambda \in A(\bar{x}). \tag{3.12}$$

Since the positivity of ξ and $\lambda \in A(\bar{x})$, (3.10-3.12) implies that

$$\sum_{j\in I} \xi_j \max_{v_j^*\in\partial\phi_j(\bar{x})} < v_j^* - w_j^*, \ \beta(x^{'}, \bar{x}) > + \max_{\mu^*\in N(\bar{x};\Omega)} < \mu^*, \ \beta(x^{'}, \bar{x}) >$$
$$+ \sum_{t\in Np(\lambda)} \max_{\pi_t^*\in\partial g_t(\bar{x})} < \lambda_t\pi_t^*, \ \beta(x^{'}, \bar{x}) >< 0. \tag{3.13}$$

This contradicts with (3.3), therefore the theorem is valid. ■

4 Mltiobjective Infinite Linear Optimization Approximation

Consider the following multiobjective infinite linear optimization, (in brief, ILMP), for any $(v_1^*, \cdots, v_m^*) \in \prod_{j \in I} \partial \phi_j(\bar{x})$ and $(w_1^*, \cdots, w_m^*) \in \prod_{j \in I} \partial \psi_j(\bar{x})$

$$min \ \{\bar{f}(x): \ g_t(\bar{x}) + < \pi_t^*, \ x - \bar{x} > \leq 0, \ \delta(\bar{x}; S) + < \mu^*, \ x - \bar{x} > \leq 0, \\ \forall \pi_t^* \in \partial g_t(\bar{x}), \ \forall t \in T, \ \forall \mu^* \in N(\bar{x}; S)\} \tag{4.1}$$

where $\bar{f} = (\bar{f}_1, \cdots, \bar{f}_m)$ and $\bar{f}_j(x) = f_j(\bar{x}) + < v_j^* - w_j^*, \ x - \bar{x} >, \ j \in I$. The following results derive some relationships between (1.1) and (4.1). The feasible domain of (4.1) is denoted by $\bar{\Omega}$, i.e., $\bar{\Omega} := \{x \in X: \ g_t(\bar{x}) + < \pi_t^*, \ x - \bar{x} > \leq 0, \ \delta(\bar{x}; S) + < \mu^*, \ x - \bar{x} > \leq 0, \ \forall \pi_t^* \in \partial g_t(\bar{x}), \ \forall t \in T, \ \forall \mu^* \in N(\bar{x}; S)\}$. It is obvious $\Omega \subset \bar{\Omega}$.

Theorem 4.1 *If $\bar{x}$ is a WPES (or PES) to (1.1), then it is also a WPES (or PES) to (4.1).*

Proof. Theorem 3.1 implies that there is $\xi \in R_+^m$ such that (3.3) holds. If $\bar{x}$ is not a WPES to (4.1), i.e., $\exists x' \in \bar{\Omega}$, s.t., $\bar{f}(x') < \bar{f}(\bar{x})$, this means that for all $(v_1^*, \cdots, v_m^*) \in \prod_{j \in I} \partial \phi_j(\bar{x})$ and $(w_1^*, \cdots, w_m^*) \in \prod_{j \in I} \partial \psi_j(\bar{x})$,

$$\sum_{j \in I} \xi_j < v_j^* - w_j^*, \ x' - \bar{x} >< 0, \ \sum_{j \in I} \xi_j = 1. \tag{4.2}$$

Since x' is feasible to (4.1) and $\bar{x}$ is a WPES to (1.1), then for any $\lambda \in A(\bar{x})$ we have

$$\lambda_t < \pi_t^*, x' - \bar{x} > \leq -\lambda_t g_t(\bar{x}) = 0, \ \forall \pi_t^* \in \partial g_t(\bar{x}) \tag{4.3}$$

$$< \mu^*, \ x' - \bar{x} > \leq -\delta(\bar{x}; S) = 0, \ \forall \mu^* \in N(\bar{x}; \Omega) \tag{4.4}$$

(4.2)-(4.4) means that

$$\sum_{j \in I} \xi_j \max_{v_j^* \in \partial \phi_j(\bar{x})} < v_j^* - w_j^*, \ x' - \bar{x} - > + \max_{\mu^* \in N(\bar{x}; S)} < \mu^*, \ x' - \bar{x} > \\ + \bigcup_{\lambda \in A(\bar{x})} \left\{ \sum_{t \in Np(\lambda)} \max_{\pi_t^* \in \partial g_t(\bar{x})} < \lambda_t \pi_t^*, \ x' - \bar{x} > \right\} < 0. \tag{4.5}$$

This contradicts with (3.3). The contradiction implies $\bar{x}$ is also a WPES to (4.1).

If $\bar{x}$ is a PES to (1.1), we will prove that it is also a PES to (4.1). Suppose that $\bar{x}$ is not a PES to (4.1), i.e., $\exists x' \in \bar{\Omega}$, s.t., $\bar{f}(x') \leq \bar{f}(\bar{x})$, but $\bar{f}(x') \neq \bar{f}(\bar{x})$, this means that there is some index $j_0 \in I$ such that $\bar{f}_{j_0}(x') < \bar{f}_{j_0}(\bar{x})$. Therefore similar to the above proof we have that $\bar{x}$ is also a PES to (4.1). ■

Theorem 4.2 *Suppose that the functions $f_j,\ j \in I$ are all generalized invex at $\bar{x}$ with function $\beta(x, \bar{x}) := x - \bar{x}$. If $\bar{x}$ is a PES (or WPES) to (4.1), then it is also a PES (or WPES) to (1.1).*

Proof. We will prove the results by contradiction. Suppose that $\bar{x}$ is a PES to (4.1). If $\bar{x}$ is not a PES to (1.1), then without loss of generality we can assume that there exist $x^{'} \in \Omega$ and some $j_0 \in I$ such that

$$f_j(x^{'}) \leq f_j(\bar{x}),\ j \neq j_0, \quad f_{j_0}(x^{'}) < f_{j_0}(\bar{x}). \tag{4.6}$$

Since $f_j,\ j \in I$ are all invex at $\bar{x}$ with function $\beta(x, \bar{x})$ and $\beta(\bar{x}, \bar{x}) = 0$, then it follows from (4.6) that for any $v_j^* \in \phi_j(\bar{x})$ and $w_j^* \in \psi_j(\bar{x})$

$$\begin{aligned} &< v_j^* - w_j^*, \beta(x^{'}, \bar{x}) > \leq 0 = < v_j^* - w_j^*, \beta(x^{'}, \bar{x}) >,\ j \neq j_0 \\ &< v_{j_0}^* - w_{j_0}^*, \beta(x^{'}, \bar{x}) > < 0 = < v_{j_0}^* - w_{j_0}^*, \beta(x^{'}, \bar{x}) > . \end{aligned}$$

The above two inequalities implies that $\bar{x}$ is not a PES to (4.1). Therefore $\bar{x}$ is a PES to (1.1). Similar to the above proof, we can derive that if $\bar{x}$ is a WPES to (4.1), then it is also a WPES to (1.1). ■

Suppose that f_j, g_t and $\delta(\cdot; \Omega)$ are all generalized invex at $\bar{x}$ with function β, $\forall j \in I$, $\forall t \in T$. Then we can give the following multiobjective infinite optimization approximation to (1.1) with $x - \bar{x}$ in (4.1) replaced with $\beta(x, \bar{x})$, for any $(v_1^*, \cdots, v_m^*) \in \prod_{j \in I} \partial\phi_j(\bar{x})$ and $(w_1^*, \cdots, w_m^*) \in \prod_{j \in I} \partial\psi_j(\bar{x})$

$$\begin{aligned} min\ \{\bar{f}(x) :\ & g_t(\bar{x}) + < \pi_t^*,\ \beta(x, \bar{x}) > \leq 0,\ \delta(\bar{x}; S) + < \mu^*,\ \beta(x, \bar{x}) > \leq 0, \\ & \forall \pi_t^* \in \partial g_t(\bar{x}),\ \forall t \in T,\ \forall \mu^* \in N(\bar{x}; S)\} \end{aligned} \tag{4.7}$$

where $\bar{f} = (\bar{f}_1, \cdots, \bar{f}_m)$ and $\bar{f}_j(x) = f_j(\bar{x}) + < v_j^* - w_j^*,\ \beta(x, \bar{x}) >,\ j \in I$. Similar to Theorem 4.1 and 4.2, we can obtain the corresponding relationships between (1.1) and (4.7) under the corresponding conditions.

In the following of this section, we will discuss the connection between the saddle point of (4.7) and the WPES (or PES) of (1.1). First we present the definitions of Lagrangian function of (4.7) and its saddle point.

Definition 4.3 *Given $\bar{x} \in \Omega$, $\lambda \in A(x)$ and $w_j^* \in \partial\psi_j(\bar{x})$, $j \in I$, let*

$$\begin{aligned} L_j(x, \tau, \lambda) := & \max_{v_j^* \in \partial\phi_j(\bar{x})} < v_j^* - w_j^*,\ \beta(x, \bar{x}) > + \tau \max_{\mu^* \in N(\bar{x}; S)} < \mu^*,\ \beta(x, \bar{x}) > \\ & + \sum_{t \in Np(\lambda)} \left(\lambda_t g_t(\bar{x}) + \max_{\pi_t^* \in \partial g_t(\bar{x})} < \lambda_t \pi_t^*,\ \beta(x, \bar{x}) > \right). \end{aligned} \tag{4.8}$$

We say that function $L(x, \tau, \lambda) := (L_1(x, \tau, \lambda), \cdots, L_m(x, \tau, \lambda))$ is the $\beta-$Lagrangian function to (4.7).

Definition 4.4 *Given $\bar{x} \in \Omega$, $\bar{\tau} \geq 0$ and $\bar{\lambda}_t \geq 0$, for all $t \in T$, we say that $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a saddle point to $\beta-$Lagrangian function $L(x, \tau, \lambda)$, if the following two conditions hold:*

$$(i)\ L(\bar{x}, \tau, \lambda) \leq L(\bar{x}, \bar{\tau}, \bar{\lambda}), \quad \forall \tau \geq 0,\ \forall \lambda_t \geq 0,\ t \in T; \ (ii)\ L(x, \bar{\tau}, \bar{\lambda}) \overline{<} L(\bar{x}, \bar{\tau}, \bar{\lambda}), \quad \forall x \in \Omega.$$

Theorem 4.5 *If $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a saddle point to $\beta-$Lagrangian function $L(x, \tau, \lambda)$, then the following conclusions hold:*

(i) If functions $f_j,\ j \in I$ are all generalized invex at $\bar{x}$ with function $\beta(x, \bar{x})$ and $\beta(\bar{x}, \bar{x}) = 0$, then $\bar{x}$ is a WPES to (1.1).

(ii) If functions $f_j,\ j \in I$ are all strictly generalized invex at $\bar{x}$ with function $\beta(x, \bar{x})$ and $\beta(\bar{x}, \bar{x}) = 0$, then $\bar{x}$ is a PES to (1.1).

Proof.(i) As $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a saddle point to $\beta-$Lagrangian function L and $\beta(\bar{x}, \bar{x}) = 0$, then for any $w_j^* \in \partial\psi_j(\bar{x})$, $j \in I$ it follows from (i) of Definition 4.4 that $\lambda g(\bar{x}) \leq \bar{\lambda} g(\bar{x})$, $\forall \lambda_t \geq 0,\ t \in T$. If let $\lambda = 0$, then we have $\bar{\lambda} g(\bar{x}) \geq 0$. This together with $\bar{x} \in \Omega$ and $\lambda \geq 0$ means that $\bar{\lambda} g(\bar{x}) = 0$

Assume that $\bar{x}$ is not a WPES to (1.1), i.e., $\exists x' \in \Omega$, s.t., $f(x') < f(\bar{x})$. Since functions $f_j,\ j \in I$ are all generalized invex at $\bar{x}$ with function $\beta(x, \bar{x})$ and $\beta(\bar{x}, \bar{x}) = 0$, we have that for any $w_j^* \in \partial\psi_j(\bar{x})$,

$$\max_{v_j^* \in \partial\phi_j(\bar{x})} < v_j^* - w_j^*,\ \beta(x', \bar{x}) >< 0 = \max_{v_j^* \in \partial\phi_j(\bar{x})} < v_j^* - w_j^*,\ \beta(\bar{x}, \bar{x}) >,\ \forall j \in I. \tag{4.9}$$

This together with $\bar{\lambda} g(\bar{x}) = 0$ and $\bar{\lambda} g(x') \leq 0$ implies that for any $j \in I$

$$\max_{v_j^* \in \partial\phi_j(\bar{x})} < v_j^* - w_j^*,\ \beta(x', \bar{x}) > + \bar{\lambda} g(x) + \bar{\tau} \max_{\mu^* \in N(\bar{x};S)} < \mu^*,\ \beta(x', \bar{x}) >$$
$$< \max_{v_j^* \in \partial\phi_j(\bar{x})} < v_j^* - w_j^*,\ \beta(\bar{x}, \bar{x}) > + \bar{\tau} \max_{\mu^* \in N(\bar{x};S)} < \mu^*,\ \beta(x', \bar{x}) > + \bar{\lambda} g(\bar{x}).$$

This means $L(x, \bar{\tau}, \bar{\lambda}) < L(\bar{x}, \bar{\tau}, \bar{\lambda})$, which contradicts with (ii) of Definition 4.4. Therefore $\bar{x}$ is a WPES of (1.1).

(ii) Similar to the above proof, we have that if functions $f_j,\ j \in I$ are all strictly generalized invex at $\bar{x}$ with function $\beta(x, \bar{x})$ and $\beta(\bar{x}, \bar{x}) = 0$, then $\bar{x}$ is a PES to (1.1). ∎

The above theorem implies that the saddle point of L is a WPES or PES to (1.1) under corresponding conditions. But if $\bar{x}$ is a WPES or PES to (4.1), we can not obtain the conclusion that there are $\bar{\tau} \geq 0$ and $\bar{\lambda} = (\bar{\lambda}_t)_{t \in T}$ with $\bar{\lambda}_t \geq 0$, $\forall t \in T$, such that $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a saddle point of $L(x, \tau, \lambda)$. In the left of this section we will give the multiobjective infinite convex approximation programming to $MOPIC$

Suppose that f_j, $\forall j \in I$ are all generalized invex at $\bar{x}$ with function β. Then we can give the following multiobjective infinite convex optimization approximation as follows, for any $(v_1^*, \cdots, v_m^*) \in \prod_{j\in I} \partial\phi_j(\bar{x})$ and $(w_1^*, \cdots, w_m^*) \in \prod_{j\in I} \partial\psi_j(\bar{x})$

$$min\ \{\bar{f}(x) :\ g_t(x) \le 0,\ \forall t \in T,\ \delta(x;S) = 0\} \tag{4.10}$$

where $\bar{f} = (\bar{f}_1, \cdots, \bar{f}_m)$ and $\bar{f}_j(x) = f_j(\bar{x}) + < v_j^* - w_j^*,\ \beta(x, \bar{x}) >,\ j \in I.$ Similar to Theorem 4.1 and 4.2, we can obtain the corresponding relationships between WPES (or PES) of (1.1) and WPES (or PES) of (4.10) under the corresponding conditions. We can also define Lagrangian function of (4.10) and its saddle point.

Definition 4.6 *Given $\bar{x} \in \Omega$, $\lambda \in \widetilde{P}_+$ and $w_j^* \in \partial\psi_j(\bar{x})$, $j \in I$, let*

$$\widehat{L}_j(x, \lambda) := \max_{v_j^* \in \partial\phi_j(\bar{x})} < v_j^* - w_j^*,\ \beta(x, \bar{x}) > +\delta(x;S) + \sum_{t \in Np(\lambda)} \lambda_t g_t(x). \tag{4.11}$$

We say that function $\widehat{L}(x, \lambda) := (\widehat{L}_1(x, \lambda), \cdots, \widehat{L}_m(x, \lambda))$ is the $\beta-$Lagrangian function to (4.10).

Definition 4.7 *Given $\bar{x} \in \Omega$, $\bar{\tau}$ and $\bar{\lambda}_t \ge 0$, for all $t \in T$, we say that $(\bar{x}, \bar{\lambda})$ is a saddle point to $\beta-$Lagrangian function $\widehat{L}(x, \lambda)$, if the following two conditions hold:*

$$(i)\ \widehat{L}(\bar{x}, \lambda) \le \widehat{L}(\bar{x}, \bar{\lambda}),\quad \forall \tau,\ \forall \lambda_t \ge 0,\ t \in T;\ (ii)\ \widehat{L}(x, \bar{\lambda}) \overline{<} \widehat{L}(\bar{x}, \bar{\lambda}),\quad \forall x \in \Omega.$$

If $(\bar{x}, \bar{\lambda})$ is a saddle point to $\beta-$Lagrangian function $\widehat{L}(x, \lambda)$, then similar to Theorem 4.5, we can derive the following conclusions that (i) if functions $f_j,\ j \in I$ are all generalized invex at $\bar{x}$ with function $\beta(x, \bar{x})$ and $\beta(\bar{x}, \bar{x}) = 0$, then $\bar{x}$ is a WPES to (1.1); (ii) if functions $f_j,\ j \in I$ are all strictly generalized invex at $\bar{x}$ with function $\beta(x, \bar{x})$ and $\beta(\bar{x}, \bar{x}) = 0$, then $\bar{x}$ is a PES to (1.1).

Theorem 4.8 *Assume that functions $\delta(\cdot;S)$ and $g_t,\ t \in T$ are all generalized invex at $\bar{x}$ with function $\beta(x, \bar{x})$ and $\beta(\bar{x}, \bar{x}) = 0$. If $\bar{x}$ is a WPES to (4.10), then there are multipliers $\bar{\lambda} = (\bar{\lambda}_t)_{t\in T}$ with $\bar{\lambda}_t \ge 0$, $\forall t \in T$, such that $(\bar{x}, \bar{\lambda})$ is a saddle point of $\widehat{L}(x, \lambda)$.*

Proof. According to Theorem 3.1, there are multipliers $\bar{\xi}$ and $\bar{\lambda}$ such that (3.3) holds. Since $\delta(\cdot;S)$ and $g_t,\ t \in T$ are all generalized invex at $\bar{x}$ with function $\beta(x, \bar{x})$, so $\forall t \in T$

$$g_t(x) - g_t(\bar{x}) \ge \max_{\pi_t^* \in \partial g_t(\bar{x})} < \pi_t^*, \beta(x, \bar{x}) >;\ \delta(x;S) \ge \max_{\mu^* \in N(\bar{x};S)} < \mu^*, \beta(x, \bar{x}) > .$$

This combining with $\bar{\lambda}_t \ge 0,\ \forall t \in T$ means that

$$\begin{aligned} &\sum_{t\in Np(\bar{\lambda})} \bar{\lambda}_t[g_t(x) - g_t(\bar{x})] + \delta(x;S) \\ &\ge \max_{\mu^* \in N(\bar{x};S)} < \mu^*, \beta(x, \bar{x}) > + \sum_{t\in Np(\bar{\lambda})} \max_{\pi_t^* \in \partial g_t(\bar{x})} < \bar{\lambda}_t \pi_t^*, \beta(x, \bar{x}) > . \end{aligned} \tag{4.12}$$

So from $\beta(\bar{x},\bar{x})=0$, (3.3) and (4.12), we have that

$$\sum_{j=1}^{m}\bar{\xi}_j \max_{v_j^*\in\partial\phi_j(\bar{x})} < v_j^*-w_j^*,\ \beta(x,\bar{x})> +\delta(x;S)+\sum_{t\in Np(\bar{\lambda})}\bar{\lambda}_t g_t(x)$$
$$\geq \sum_{j=1}^{m}\bar{\xi}_j \max_{v_j^*\in\partial\phi_j(\bar{x})} < v_j^*-w_j^*,\ \beta(\bar{x},\bar{x})> +\delta(\bar{x};S)+\sum_{t\in Np(\bar{\lambda})}\bar{\lambda}_t g_t(\bar{x}). \tag{4.13}$$

This with $\sum_{j=1}^{m}\bar{\xi}_j = 1$ implies that $\bar{\xi}^T\widehat{L}(x,\bar{\lambda}) \geq \bar{\xi}^T\widehat{L}(\bar{x},\bar{\lambda}),\ \forall x\in\Omega$, which means that $\widehat{L}(x,\bar{\lambda})\overline{<}\widehat{L}(\bar{x},\bar{\lambda}),\quad \forall x\in\Omega$, i.e., (ii) of Definition 4.7 holds. It is obvious that (i) of Definition 4.7 holds. Therefore we have that the conclusion of this theorem is true. ■

5 Duality

In section, we consider Wolfe type dual as well as Mond-Weir type dual to $MOPIC$ and establish the weak, strong, and strict duality theorems, respectively.

5.1 Wolfe type dual

Consider the Wolfe type dual to $MOPIC$ as follows,

$$\min \quad F(u)=\left(f_j(u)+\sum_{t\in Np(\lambda)}\lambda_t g_t(u)\right)_{j\in I}$$

$$\text{s.t.}\quad 0\in\sum_{j\in I}\xi_j(\partial\phi_j(u)-w_j^*)+\sum_{t\in Np(\lambda)}\lambda_t\partial g_t(u)+N(u;S), \tag{5.1}$$

$$w_j^*\in\partial\psi_j(u),\ \lambda\in\widetilde{P}_+,\ \xi\in R_+^m,\ \sum_{j\in I}\xi_j=1,\ u\in S.$$

The above problem (5.1) is denoted by $WMOPID$. If for any $w_j^*\in\partial\psi_j(u)$, there exist $\lambda\in\widetilde{P}_+,\ \xi\in R_+^m,\ \sum_{j\in I}\xi_j=1$ such that $0\in\sum_{j\in I}\xi_j(\partial\phi_j(u)-w_j^*)+\sum_{t\in Np(\lambda)}\lambda_t\partial g_t(u)+N(u;S)$, then we say that (u,ξ,λ) is feasible to $WMOPID$.

Theorem 5.1 ***(Weak Duality)*** *Assume that x and (u,ξ,λ) is feasible to $MOPIC$ and $WMOPID$, respectively. If $f_j,\ \forall j\in I$ are all strictly generalized invex at u with function β and functions $\delta(\cdot;S),\ g_t,\ \forall t\in T$ are all generalized invex at u with function β, then there exist some $\lambda\in\widetilde{P}_+$ and at least one index $j_0\in I$, such that*

$$f_{j_0}(x)>f_{j_0}(u)+\sum_{t\in Np(\lambda)}\lambda_t g_t(u). \tag{5.2}$$

Proof. We will prove the theorem by contradiction, i.e., suppose that for any $\lambda \in \widetilde{P}_+,\ \xi \in R^m_+,\ \sum\limits_{j\in I}\xi_j = 1,\ f_j(x) \le f_j(u) + \sum\limits_{t\in Np(\lambda)}\lambda_t g_t(u),\ \forall j \in I$. This implies that

$$\xi^T(f(x) - f(u)) - \sum_{t\in Np(\lambda)} \lambda_t g_t(u) \le 0. \tag{5.3}$$

It is obvious that from the conditions of this theorem

$$f_j(x) - f_j(u) > \max_{v_j^*\in\partial\phi_j(u)} < v_j^* - w_j^*,\ \beta(x,u) >$$

$$-g_t(u) \ge g_t(x) - g_t(u) \ge \max_{\pi_t^*\in\partial g_t(u)} < \pi_t^*,\ \beta(x,u) > \tag{5.4}$$

$$0 \ge \max_{\mu^*\in N(u;S)} < \mu^*,\ \beta(x,u) > .$$

(5.4) means that $\sum\limits_{j\in I}\xi_j < v_j^* - w_j^*,\ \beta(x,u) > + \sum\limits_{t\in Np(\lambda)} \max\limits_{\lambda_t\pi_t^*\in\partial g_t(u)} < \pi_t^*,\ \beta(x,u) > + \max\limits_{\mu^*\in N(u;S)} < \mu^*,\ \beta(x,u) >< \xi^T(f(x) - f(u)) - \sum\limits_{t\in Np(\lambda)}\lambda_t g_t(u)$. Then according to (5.3), we have that $\sum\limits_{j\in I}\xi_j < v_j^* - w_j^*,\ \beta(x,u) > + \sum\limits_{t\in Np(\lambda)} \max\limits_{\lambda_t\pi_t^*\in\partial g_t(u)} < \pi_t^*,\ \beta(x,u) > + \max\limits_{\mu^*\in N(u;S)} < \mu^*,\ \beta(x,u) >< 0$. This contradicts with the first constraint of $WMOPID$. Therefore we have the conclusion of this theorem. ■

Theorem 5.2 ***(Strong Duality)*** *If $\bar{x}$ is a WPES to (1.1), then there exist $\bar{\lambda} \in \widetilde{P}_+,\ \bar{\xi} \in R^m_+,\ \sum\limits_{j\in I}\bar{\xi}_j = 1$ such that $(\bar{x}, \bar{\xi}, \bar{\lambda})$ is feasible to $WMOPID$. Furthermore if $f_j,\ \forall j \in I$ are all strictly generalized invex at u with function β and $\delta(\cdot; S)$, $g_t,\ \forall t \in T$ are all generalized invex at u with function β, then $(\bar{x}, \bar{\xi}, \bar{\lambda})$ is a PES to $WMOPID$.*

Proof. From Theorem 3.1, we have that if $\bar{x}$ is a WPES to (1.1), then there exist $\bar{\lambda} \in \widetilde{P}_+,\ \bar{\xi} \in R^m_+,\ \sum\limits_{j\in I}\bar{\xi}_j = 1$ such that $0 \in \sum\limits_{j\in I}\xi_j(\partial\phi_j(\bar{x}) - w_j^*) + \sum\limits_{t\in Np(\bar{\lambda})}\bar{\lambda}_t\partial g_t(\bar{x}) + N(\bar{x}; S)$, that is $(\bar{x}, \bar{\xi}, \bar{\lambda})$ is feasible to $WMOPID$.

If $(\bar{x}, \bar{\xi}, \bar{\lambda})$ is not a PES to $WMOPID$, that is, there is a feasible solution (u, ξ, λ) of $WMOPID$ such that $f_j(\bar{x}) + \sum\limits_{t\in Np(\bar{\lambda})}\bar{\lambda}_t g_t(\bar{x}) \le f_j(u) + \sum\limits_{t\in Np(\lambda)}\lambda_t g_t(u),\ \forall j \in I$. Since $\bar{x}$ is feasible to (1.1) and $\bar{\lambda} \ge 0$, then $\bar{\lambda}_t g_t(\bar{x}) \le 0$. So we have that $f_j(\bar{x}) \le f_j(u) + \sum\limits_{t\in Np(\lambda)}\lambda_t g_t(u), \forall j \in I$. This contradicts with (5.2). Therefore $(\bar{x}, \bar{\xi}, \bar{\lambda})$ is also a PES to $WMOPID$. ■

Theorem 5.3 ***(Strictly Converse Duality)****Suppose that x and (u, ξ, λ) are feasible to $MOPIC$ and $WMOPID$, respectively, and suppose that the following inequality holds,*

$$\xi^T f(x) \le \xi^T f(u) + \sum_{t\in Np(\lambda)} \lambda_t g_t(u) \tag{5.5}$$

If $f_j,\ \forall j \in I$ are all strictly generalized invex at u with function β and functions $\delta(\cdot;S)$, $g_t,\ \forall t \in T$ are all generalized invex at u with function β, then we have that $x = u$.

Proof. Since (u,ξ,λ) is feasible to $WMOPID$, then for any $w_j^* \in \partial\psi_j(u)$, there exist $\lambda \in \widetilde{P}_+,\ \xi \in R_+^m,\ \sum_{j\in I}\xi_j = 1$ such that $0 \in \sum_{j\in I}\xi_j(\partial\phi_j(u) - w_j^*) + \sum_{t\in Np(\lambda)}\lambda_t\partial g_t(u) + N(u;S)$. If $x \neq u$, then similar to the proof of the Weak Duality Theorem, we have that $\sum_{j\in I}\xi_j < v_j^* - w_j^*,\ \beta(x,u) > + \sum_{t\in Np(\lambda)} \max_{\lambda_t\pi_t^*\in\partial g_t(u)} < \pi_t^*,\ \beta(x,u) > + \max_{\mu^*\in N(u;S)} < \mu^*,\ \beta(x,u) >< 0$. This contradicts with the first constraint of $WMOPID$ which implies that $x = u$. ■

From the above theorem, if $x = u$, from (5.5) we have that $\sum_{t\in Np(\lambda)}\lambda_t g_t(x) \geq 0$. Since x is feasible to (1.1) and $\lambda \in \widetilde{P}_+$, then $\sum_{t\in Np(\lambda)}\lambda_t g_t(x) \geq 0$. Therefore $\sum_{t\in Np(\lambda)}\lambda_t g_t(x) = 0$, which implies that the objective function values of $MOPIC$ and $WMOPID$ are equal.

5.2 Mond-Weir type dual

Give the Mond-Weir type dual to $MOPIC$ denoted by $MWID$ as follows,

$$\begin{aligned} \min \quad & f(u) = (f_1(u),\cdots,f_m(u))^T \\ \text{s.t.} \quad & 0 \in \sum_{j\in I}\xi_j(\partial\phi_j(u) - w_j^*) + \sum_{t\in Np(\lambda)}\lambda_t\partial g_t(u) + N(u;S), \\ & w_j^* \in \partial\psi_j(u),\ \lambda \in \widetilde{P}_+,\ \sum_{j\in I}\xi_j = 1, \\ & \sum_{t\in Np(\lambda)}\lambda_t g_t(u) \geq 0,\ u \in S. \end{aligned} \tag{5.6}$$

Theorem 5.4 ***(Weak Duality)*** *Let x and (u,ξ,λ) is feasible to $MOPIC$ and $MWID$, respectively. Let $f_j,\ \forall j \in I$ are all strictly generalized invex at u with function β and $\delta(\cdot;S)$, $g_t,\ \forall t \in T$ are all generalized invex at u with function β, then we have that the following inequality does not hold, $f(x) \leq f(u)$.*

Proof. Since x and (u,ξ,λ) is feasible to $MOPIC$ and $MWID$, respectively, then we have that $g_t(x) \leq 0,\ \forall t \in T$ and there exist $\lambda \in \widetilde{P}_+,\ \xi \in R_+^m,\ \sum_{j\in I}\xi_j = 1$ such that $\sum_{t\in Np(\lambda)}\lambda_t g_t(u) \geq 0$ and $0 \in \sum_{j\in I}\xi_j(\partial\phi_j(u) - w_j^*) + \sum_{t\in Np(\lambda)}\lambda_t\partial g_t(u) + N(u;S),\ u \in S$. The last inequality implies that there are $v_j^* \in \partial\phi_j(u)$, $\pi_t^* \in \partial g_t(u)$ and $\mu^* \in N(u;S)$, such that

$$0 = \sum_{j\in I}\xi_j(v_j^* - w_j^*) + \sum_{t\in Np(\lambda)}\pi_t^* + \mu^*. \tag{5.7}$$

Since $f_j,\ \forall j \in I$, $\delta(\cdot; S)$, and $g_t,\ \forall t \in T$ are all generalized invex at u with function β, then

$$\sum_{j\in I} \xi_j \left(f_j(x) - f_j(u)\right) > \sum_{j\in I} \xi_j < v_j^* - w_j^*,\ \beta(x,u) >, \tag{5.8}$$

$$\sum_{t\in Np(\lambda)} \lambda_t(g_t(x) - g_t(u)) \geq \sum_{t\in Np(\lambda)} \lambda_t < \pi_t^*,\ \beta(x,u) >, \tag{5.9}$$

$$0 = \delta(x;S) - \delta(u;S) \geq < \mu^*,\ \beta(x,u) > . \tag{5.10}$$

(5.7)-(5.10) means that $\sum_{j\in I} \xi_j \left(f_j(x) - f_j(u)\right) + \sum_{t\in Np(\lambda)} \lambda_t(g_t(x) - g_t(u)) > 0$, which combining with $\sum_{t\in Np(\lambda)} \lambda_t g_t(u) \geq 0$ and $g_t(x) \leq 0,\ \forall t \in T$ implies that $\xi^T f(x) > \xi^T f(u)$. Therefore there at least exists one idex $j_0 \in I$ such that $f_{j_0}(x) > f_{j_0}(u)$, i.e., inequality $f(x) \leq f(u)$ does not hold. ■

Theorem 5.5 ***(Strong Duality)****If $\bar{x}$ is a WPES to (1.1), then there exist $\bar{\lambda} \in \widetilde{P}_+,\ \bar{\xi} \in R_+^m,\ \sum_{j\in I} \bar{\xi}_j = 1$ such that $(\bar{x}, \bar{\xi}, \bar{\lambda})$ is feasible to MWID. Furthermore under the same conditions of Theorem 10, we have that $(\bar{x}, \bar{\xi}, \bar{\lambda})$ is a PES to MWID.*

Proof. It follows from Theorem 3.1 that $\bar{\lambda} \in A(\bar{x})$ and then $\bar{\lambda}_t g_t(\bar{x}) = 0,\ \forall t \in Np(\bar{\lambda})$, that is $\sum_{t\in Np(\bar{\lambda})} \bar{\lambda}_t g_t(\bar{x}) = 0$ which implies that $(\bar{x}, \bar{\xi}, \bar{\lambda})$ is feasible to $MWID$. Since Theorem 5.4 means that for any feasible solution to $MWID$, the following inequality can not hold, $f(\bar{x}) \leq f(u)$. Therefore $\bar{x}$ is also a PES to $MWID$. ■

Theorem 5.6 ***(Strictly Converse Duality)****Suppose that x and (u, ξ, λ) are feasible to $MOPIC$ and $MWID$, respectively. Suppose that the following inequality holds, $\sum_{j\in I} \xi_j f_j(x) \leq \xi_j f_j(u),\ \forall \xi \in R_+^m$. Then under the conditions of Theorem 5.3, we have that $x = u$.*

Proof. Similar to the proof of Theorem 5.4, the following inequality holds,

$$\sum_{j\in I} \xi_j \left(f_j(x) - f_j(u)\right) + \sum_{t\in Np(\lambda)} \lambda_t(g_t(x) - g_t(u)) > 0.$$

This with $g_t(x) \leq 0$ and $\sum_{t\in Np(\lambda)} \lambda_t g_t(u) \geq 0$ implies that $\sum_{j\in I} \xi_j \left(f_j(x) - f_j(u)\right) > 0$, which contradicts with $\sum_{j\in I} \xi_j f_j(x) \leq \xi_j f_j(u),\ \forall \xi \in R_+^m$. Therefore we have that $x = u$. ■

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