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Epigraphical cones I

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Abstract. Up to orthogonal transformation, a solid closed convex cone K in the Euclidean space $\mathbb{R}^{n+1}$ is the epigraph of a nonnegative sublinear function $f : \mathbb{R}^n \to \mathbb{R}$. This work explores the link between the geometric properties of K and the analytic properties of f.

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1 Introduction

The purpose of this work is to survey the class of epigraphical cones and to derive new results concerning the geometric structure of these mathematical objects. A convex cone in the Euclidean space $\mathbb{R}^{n+1}$ is an *epigraphical cone* if it can be represented as epigraph

$$\text{epi}f = \{(x,t) \in \mathbb{R}^{n+1} : f(x) \leq t\}$$

of a nonnegative sublinear function $f : \mathbb{R}^n \to \mathbb{R}$. Sublinearity is understood as combination of subadditivity and positive homogeneity. If K is an epigraphical cone in $\mathbb{R}^{n+1}$, then its associated nonnegative sublinear function is given by

$$f_K(x) = \min\{t \in \mathbb{R} : (x,t) \in K\}.$$

An epigraphical cone is necessarily closed and *nontrivial*, i.e., different from the whole space and different from the zero cone. It is also *solid* in the sense that it has a nonempty topological interior. Conversely, up to orthogonal transformation, every nontrivial solid closed convex cone is an epigraphical cone (cf. [23, Proposition 2.8]).

Epigraphical cones are important enough to justify a preferential and exhaustive treatment. A few striking examples are appropriate to illustrate this point. In the sequel $\langle \cdot, \cdot \rangle$ indicates the usual inner product of $\mathbb{R}^d$, regardless of the dimension d.

Example 1.1. The most prominent example of epigraphical cone is

$$\mathcal{E}(Q) := \{(x,t) \in \mathbb{R}^{n+1} : \sqrt{\langle x, Qx \rangle} \leq t\}$$

with Q standing for a positive definite symmetric matrix of order n. One refers to $\mathcal{E}(Q)$ as the *elliptic cone* associated to Q. Elliptic cones have been studied under different angles in a number of references, including our own works [28, 30, 31] and those of Stern and Wolkowicz [44, 45]. See [5] for an application of elliptic cones in control theory. Elliptic cones can be defined also in Hilbert spaces (cf. [9]), but we stick to finite dimensionality.

Example 1.2. Another interesting example of epigraphical cone is the ℓ^p - cone

$$\mathcal{K}_p := \{(x,t) \in \mathbb{R}^{n+1} : \|x\|_p \leq t\}.$$

Here $p \in [1, \infty]$ and $\| \cdot \|_p$ denotes the ℓ^p - norm in $\mathbb{R}^n$. Basic information on ℓ^p - cones can be found in [4, 8, 18, 36].

Example 1.3. Finally, consider an epigraphical cone that is polyhedral. Such a cone has the particular structure

$$\mathcal{P}_C := \left\{ (x,t) \in \mathbb{R}^{n+1} : \max_{1 \leq i \leq m} \langle c_i, x \rangle \leq t \right\},$$

where $C = \{c_1, \ldots, c_m\}$ is a finite subset of $\mathbb{R}^n$ such that $0_n \in \mathrm{co}(C)$. The symbol 0_n stands for the zero vector of $\mathbb{R}^n$ and "co" refers to the convex hull operation. Note that $\mathcal{K}_1$ and $\mathcal{K}_\infty$ fit into this category of cones.

The above list of examples gives already a good idea on the kind of convex cones we are interested in. The class of epigraphical cones is wider than the class of top-heavy cones introduced by Fiedler and Haynsworth [12]. Recall that a *top-heavy cone* is understood as an epigraphical cone associated to a norm. Our definition of epigraphical cone slightly deviates from that of [23]. Indeed, for the sake of simplicity in the overall exposition we have decided not to work with extended-real-valued sublinear functions. The use of extended-real-valued sublinear functions would allow to consider, for instance, the class of $\mathcal{L}^p$-cones in the sense of Glineur [15, 16]. The organization of the paper is as follows:

- Section 2 develops the basic algebra of epigraphical cones.
- Section 3 provides some rules for computing the inradius of an epigraphical cone. The inradius is a coefficient that serves to measure the degree of solidity of the cone.
- Section 4 is about measuring the degree of pointedness of an epigraphical cone.
- Section 5 analyzes the angular structure of an epigraphical cone.

Other aspects concerning the theory of epigraphical cones are treated in the companion paper [42]. The emphasis in [42] is put in the study of properties that are valid up to orthogonal characterizations, which allows to consider a class of convex cones larger than the class of epigraphical cones. The notation that we use in both parts is standard or self-explanatory: $\mathrm{int}(\Omega)$, $\mathrm{bd}(\Omega)$, $\mathrm{cl}(\Omega)$ indicate respectively the interior, boundary, and closure of a set Ω. The unit sphere and the closed unit ball of $\mathbb{R}^d$ are denoted by $\mathbb{S}_d$ and $\mathbb{B}_d$, respectively.

2 The basic algebra of epigraphical cones

2.1 Representation of epi$f$ in terms of a convex base

Sometimes it is useful to write a sublinear function $f : \mathbb{R}^n \to \mathbb{R}$ as *support function*

$$x \in \mathbb{R}^n \mapsto f(x) = \Psi_C^*(x) := \max_{y \in C} \langle y, x\rangle$$

of a convex compact set C in $\mathbb{R}^n$. Hörmander's theorem asserts that C is unique and given by

$$C = \{y \in \mathbb{R}^n : \langle y, x\rangle \le f(x) \text{ for all } x \in \mathbb{R}^n\}.$$

In the parlance of convex analysis, this set corresponds to the subdifferential of f at 0_n. For this reason one writes $f = \Psi^*_{\partial f(0_n)}$. If the sublinear function f is nonnegative, then $\partial f(0_n)$ contains 0_n and admits the polar representation $\partial f(0_n) = B_f^\circ$ with

$$\begin{aligned} B_f &:= \{x \in \mathbb{R}^n : f(x) \le 1\} \\ B^\circ &:= \{y \in \mathbb{R}^n : \langle y, x\rangle \le 1 \text{ for all } x \in B\}. \end{aligned}$$

As shown in the next proposition, an epigraphical cone in $\mathbb{R}^{n+1}$ is the closed positive hull of a Cartesian product of the form $B \times \{1\}$ with B standing for a closed convex set in $\mathbb{R}^n$. Recall that a closed convex cone is *pointed* if it does not contain a line.

Proposition 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative sublinear function. Then*

$$\text{epi}f = \text{cl}\left[\bigcup_{t \ge 0} t\,(B_f \times \{1\})\right]. \tag{1}$$

Furthermore,

(a) The "upward" canonical vector $e_{n+1} = (0_n, 1)$ belongs to int (epi$f$)*.*

(b) If f vanishes only at 0_n, then the closure operation in (1) is superfluous and epi$f$ *is pointed.*

(c) epi$f$ *is polyhedral if and only if B_f is polyhedral (possibly the whole space $\mathbb{R}^n$).*

Proof. The representation formula (1) is stated already in [23]. Such equality yields in particular

$$\text{int}(B_f) \times \{1\} \subset \text{int}(\text{epi}f).$$

This proves (a) because $0_n \in \text{int}(B_f)$. That f vanishes only at 0_n is equivalent to saying that B_f is compact, in which case also $B_f \times \{1\}$ is compact and the closure operation in (1) is superfluous. The pointedness of epi$f$ follows by a simple inspection. The part (c) can be obtained by combining Corollaries 19.2.1 and 19.2.2 of Rockafellar's book [40]. □

Remark. The minimal hypothesis that warrants the pointedness of (1) is the sharpness of f. A nonnegative sublinear function $f : \mathbb{R}^n \to \mathbb{R}$ is called *sharp* if $f(x) = 0$ and $f(-x) = 0$ imply $x = 0_n$. This property is weaker than saying that f vanishes only at 0_n.

Recall that a *generator* (or extreme vector) of a pointed closed convex cone K is a nonzero vector $e \in K$ such that $z \in K$ and $e - z \in K$ imply that $z \in \mathbb{R}_+ e$. In such a case one refers to $\mathbb{R}_+ e$ as an extreme ray of K. The following intuitive result is part of the folklore on convex cones. It can be obtained as a consequence of Proposition 2.1 and [1, Theorem 1.48].

Corollary 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative sublinear function vanishing only at 0_n. Then $(x, 1)$ is a generator of* epi$f$ *if and only if x is an extreme point of B_f.*

2.2 Combining two epigraphical cones

Epigraphical cones can be combined in various ways in order to produce new epigraphical cones. For instance, the intersection and the Minkowski sum of two epigraphical cones are again epigraphical cones. More precisely,

$$\mathrm{epi} f_1 \cap \mathrm{epi} f_2 = \mathrm{epi}(f_1 \vee f_2) \tag{2}$$

$$\mathrm{epi} f_1 + \mathrm{epi} f_2 = \mathrm{epi}(f_1 \,\square\, f_2) \tag{3}$$

with $\vee$ and $\square$ standing, respectively, for the pointwise maximum and the infimal convolution operation. The formula (2) for the intersection is obvious. The formula (3) for the Minkowski sum follows from the infimal convolution theory as developed by Moreau [37] and Rockafellar [40].

Example 2.3. For each $x \in \mathbb{R}^n$, the function $p \in [1,\infty] \mapsto \|x\|_p$ is nonincreasing. One gets

$$\begin{aligned} \|\cdot\|_{p_1} \vee \|\cdot\|_{p_2} &= \|\cdot\|_{\min\{p_1,p_2\}} \\ \|\cdot\|_{p_1} \,\square\, \|\cdot\|_{p_2} &= \|\cdot\|_{\max\{p_1,p_2\}} \end{aligned}$$

for all $p_1, p_2 \in [1,\infty]$. Hence, $\mathcal{K}_{p_1} \cap \mathcal{K}_{p_2} = \mathcal{K}_{\min\{p_1,p_2\}}$ and $\mathcal{K}_{p_1} + \mathcal{K}_{p_2} = \mathcal{K}_{\max\{p_1,p_2\}}$.

The intersection of two elliptic cones is not necessarily an elliptic cone. However, one gets an epigraphical cone that has a special structure. The same remark applies to the Minkowski sum of two elliptic cones. The next proposition explains the details. In the sequel the symbol

$$E_Q := \left\{ y \in \mathbb{R}^n : \langle y, x\rangle \leq \sqrt{\langle x, Qx\rangle} \text{ for all } x \in \mathbb{R}^n \right\} \tag{4}$$

denotes the *ellipsoid* associated to a positive definite symmetric matrix Q. Clearly one has

$$\mathcal{E}(Q) = \mathrm{epi}(\Psi^*_{E_Q}).$$

According to the references [35, 41], two alternative characterizations for the ellipsoid (4) are

$$E_Q = Q^{1/2}(\mathbb{B}_n) = \left\{ y \in \mathbb{R}^n : \langle y, Q^{-1}y\rangle \leq 1 \right\},$$

where $Q^{1/2}$ denotes the symmetric square root of Q.

Proposition 2.4. *Let Q_1, Q_2 be positive definite symmetric matrices of order n. Then*

$$\begin{aligned} \mathcal{E}(Q_1) \cap \mathcal{E}(Q_2) &= \mathrm{epi}(f) \\ \mathcal{E}(Q_1) + \mathcal{E}(Q_2) &= \mathrm{epi}(g) \end{aligned}$$

with $f = \Psi^*_{\mathrm{co}(E_{Q_1} \cup E_{Q_2})}$ and $g = \Psi^*_{E_{Q_1} \cap E_{Q_2}}$.

Proof. This result is a direct consequence of the formulas (2)-(3). Note that

$$\begin{aligned} \Psi^*_{\mathrm{co}(E_{Q_1} \cup E_{Q_2})} &= \Psi^*_{E_{Q_1}} \vee \Psi^*_{E_{Q_2}} \\ \Psi^*_{E_{Q_1} \cap E_{Q_2}} &= \Psi^*_{E_{Q_1}} \,\square\, \Psi^*_{E_{Q_2}} \end{aligned}$$

and that $x \in \mathbb{R}^n \mapsto \Psi^*_{E_{Q_k}}(x) = \sqrt{\langle x, Q_k x\rangle}$ corresponds to the nonnegative sublinear function associated to the elliptic cone $\mathcal{E}(Q_k)$. □

There are also other ways of combining sets in a product space. For instance, if M_1, M_2 are sets in $\mathbb{R}^{n+1}$, then one can define their *vertical sum* and their *horizontal sum* respectively by

$$\begin{aligned} M_1 \oplus_{\mathrm{v}} M_2 &:= \big\{(x,t) \in \mathbb{R}^{n+1} : \exists\, r \in \mathbb{R} \ \text{ s.t. } (x,r) \in M_1 \text{ and } (x,\, t-r) \in M_2\big\}\,, \\ M_1 \oplus_{\mathrm{h}} M_2 &:= \big\{(x,t) \in \mathbb{R}^{n+1} : \exists\, u \in \mathbb{R}^n \text{ s.t. } (u,t) \in M_1 \text{ and } (x-u,t) \in M_2\big\}\,. \end{aligned}$$

These operations are mentioned in [40, Section 3], though under a different terminology. The next proposition shows that the vertical sum and the horizontal sum of two epigraphical cones are again epigraphical cones. Recall that

$$C_1 \triangle C_2 := \cup_{r\in[0,1]}(1-r)C_1 \cap rC_2$$

is the inverse sum of two sets C_1, C_2 in $\mathbb{R}^n$ and that

$$x \in \mathbb{R}^n \mapsto (f_1 \triangle f_2)(x) \ := \ \min_{u\in\mathbb{R}^n} \max\{f_1(x-u), f_2(u)\}$$

is the inverse sum of two nonnegative sublinear functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$. General information about inverse addition can be found in [41, 43].

Proposition 2.5. *Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ be nonnegative sublinear functions. Then the usual sum $f_1 + f_2 : \mathbb{R}^n \to \mathbb{R}$ and the inverse sum $f_1 \triangle f_2 : \mathbb{R}^n \to \mathbb{R}$ are nonnegative sublinear functions. Furthermore,*

$$\begin{aligned} \mathrm{epi} f_1 \oplus_{\mathrm{v}} \mathrm{epi} f_2 &= \mathrm{epi}(f_1 + f_2) \\ \mathrm{epi} f_1 \oplus_{\mathrm{h}} \mathrm{epi} f_2 &= \mathrm{epi}(f_1 \triangle f_2). \end{aligned}$$

Proof. From the definition of the vertical sum, one sees that

$$\begin{aligned} (x,t) \in \mathrm{epi} f_1 \oplus_{\mathrm{v}} \mathrm{epi} f_2 &\iff f_1(x) + f_2(x) \le t \\ &\iff (x,t) \in \mathrm{epi}(f_1 + f_2). \end{aligned}$$

Similarly, the definition of the horizontal sum shows that

$$\begin{aligned} (x,t) \in \mathrm{epi} f_1 \oplus_{\mathrm{h}} \mathrm{epi} f_2 &\iff \exists\, u \in \mathbb{R}^n \text{ s.t. } \max\{f_1(x-u), f_2(u)\} \le t \\ &\iff (x,t) \in \mathrm{epi}(f_1 \triangle f_2). \end{aligned}$$

It is clear that $f_1 + f_2$ and $f_1 \triangle f_2$ are nonnegative sublinear functions. In fact,

$$\begin{aligned} f_1 + f_2 &= \Psi^*_{\partial f_1(0_n)} + \Psi^*_{\partial f_2(0_n)} = \Psi^*_{\partial f_1(0_n) + \partial f_2(0_n)}\,, \\ f_1 \triangle f_2 &= \Psi^*_{\partial f_1(0_n)} \triangle \Psi^*_{\partial f_2(0_n)} = \Psi^*_{\partial f_1(0_n) \triangle \partial f_2(0_n)}\,, \end{aligned}$$

the last equality being a particular case of [41, Theorem 5.2]. □

Remark. A word of caution is in order. The vertical sum

$$\mathcal{K}_{p_1} \oplus_{\mathrm{v}} \mathcal{K}_{p_2} = \{(x,t) \in \mathbb{R}^{n+1} : \|x\|_{p_1} + \|x\|_{p_2} \le t\}$$

of two ℓ^p-cones is not necessarily an ℓ^p-cone, and the vertical sum

$$\mathcal{E}(Q_1) \oplus_{\mathrm{v}} \mathcal{E}(Q_2) = \{(x,t) \in \mathbb{R}^{n+1} : \sqrt{\langle x, Q_1 x\rangle} + \sqrt{\langle x, Q_2 x\rangle} \le t\}$$

of two elliptic cones is not necessarily an elliptic cone. The same remark applies to horizontal sums.

Summarizing, if K_1 and K_2 are epigraphical cones, then so are their Minkowski sum, their intersection, their vertical sum, and their horizontal sum. Furthermore,

$$\begin{aligned} f_{K_1+K_2} &= f_{K_1}\Box f_{K_2} \\ f_{K_1\cap K_2} &= f_{K_1}\vee f_{K_2} \\ f_{K_1\oplus_{\rm v}K_2} &= f_{K_1}+f_{K_2} \\ f_{K_1\oplus_{\rm h}K_2} &= f_{K_1}\triangle f_{K_2}. \end{aligned}$$

In the same vein one can derive composition rules for yet more elaborated operations like direct and inverse addition of order $p\in]1,\infty[$ (in the sense of [41, Definition 3.1]).

2.3 The dual of an epigraphical cone

Duality plays a conspicuous role throughout this work. Recall that the *dual cone* of a closed convex cone K in $\mathbb{R}^d$ is defined by

$$K^+ = \{v\in\mathbb{R}^d : \langle v,z\rangle \geq 0 \text{ for all } z\in K\}.$$

The first question that comes to mind is this: is the dual of an epigraphical cone yet another epigraphical cone? Before answering this question, we recall the concept of polarity for positive sublinear functions. That a sublinear function is *positive* means that it is nonnegative and vanishes only at the origin. Lyubich [36] refers to a positive sublinear function as a subnorm and to the epigraph of a subnorm as an hyperbolic cone. We do not adhere to this terminology because subnorms and hyperbolic cones are often used with a different meaning in the literature.

Recall that the *polar* $f^\circ:\mathbb{R}^n\to\mathbb{R}$ of a positive sublinear function $f:\mathbb{R}^n\to\mathbb{R}$ is given by

$$f^\circ(y)=\max_{x\neq 0_n}\frac{\langle y,x\rangle}{f(x)}\,.$$

The *skewed polar* of f is the function $y\in\mathbb{R}^n\mapsto f^\diamond(y)=f^\circ(-y)$. The definition of polarity is classical and can be found in many books. Skewed polarity is less standard, but it appears already in [36]. Of course, if f is even, then f° is also even and there is no distinction between polarity and skewed polarity. The next theorem is a particular case of [23, Lemma 2.10], see also [40, Theorem 14.4] or [36, Proposition 3.1].

Theorem 2.6. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a positive sublinear function. Then $f^\diamond:\mathbb{R}^n\to\mathbb{R}$ is a positive sublinear function and* $(\mathrm{epi}f)^+=\mathrm{epi}f^\diamond$.

So, under the hypotheses of the above theorem, the dual of epi$f$ is again an epigraphical cone. In practice, computing the dual of an epigraphical cone amounts to computing the polar of a positive sublinear function.

Example 2.7. Elliptic cones and ℓ^p - cones can both be embedded in the wider class of epigraphical cones of the form

$$\Gamma_{p,H} := \{(x,t)\in\mathbb{R}^{n+1} : \|Hx\|_p\leq t\} \tag{5}$$

with $p\in[1,\infty]$ and H denoting a nonsingular matrix of order n. The model (5) is quite flexible and encompasses a large variety of cones arising in applications. Note that $f(x)=\|Hx\|_p$ defines a positive sublinear function. Its polar is given by $f^\circ(y)=\|Gy\|_q$ with $p^{-1}+q^{-1}=1$ and $G=(H^{-1})^T$. One gets in this way the formula $(\Gamma_{p,H})^+=\Gamma_{q,G}$.

3 Degree of solidity of an epigraphical cone

The *inradius* of a nontrivial closed convex cone K in $\mathbb{R}^d$ is defined as the number

$$\rho(K) := \max_{x \in K \cap \mathbb{S}_d} \operatorname{dist}[x, \operatorname{bd}(K)] \tag{6}$$

with $\operatorname{dist}[x, C]$ denoting the distance from x to a set C. The concept of inradius has been discussed in detail in [22, 23] and also in earlier references like [6, 7, 11, 13, 14, 28, 32]. Note that $\rho(K)$ corresponds to the optimal value of the maximization problem

$$\begin{aligned} &\text{maximize } r \\ &\|x\| = 1 \\ &r \in [0, 1] \\ &x + r\mathbb{B}_d \subset K. \end{aligned} \tag{7}$$

Geometrically speaking, the optimization problem (7) is about finding a ball of largest radius centered at a unit vector and contained in K. This observation explains why the term (6) measures to which extent the cone K is solid. In fact, the function $K \mapsto \rho(K)$ is a solidity index in the axiomatic sense of [28].

The following lemma by Henrion and Seeger [23] tells how to compute the inradius of an epigraphical cone. This result is not absolutely general because the associated nonnegative sublinear function is required to be even.

Lemma 3.1. *If the nonnegative sublinear function $f : \mathbb{R}^n \to \mathbb{R}$ is even, then*

$$\rho(\operatorname{epi} f) = [1 + \beta_f^2]^{-1/2}$$

with

$$\beta_f := \max_{y \in \partial f(0_n)} \|y\| \; = \; \max_{\|x\|=1} f(x).$$

With this lemma at hand one can derive a number of new results. For instance, one can obtain a rule for computing the inradius of an epigraphical cone whose associated nonnegative sublinear function has the decomposable structure

$$f(x) = \mathcal{N}(\varphi(x_1, \ldots, x_m), \psi(x_{m+1}, \ldots, x_n)) \tag{8}$$

with $\mathcal{N}$ standing for a norm in $\mathbb{R}^2$ that is *monotonic* in the sense that

$$|a_1| \le |c_1| \quad \text{and} \quad |a_2| \le |c_2| \quad \text{imply} \quad \mathcal{N}(a_1, a_2) \le \mathcal{N}(c_1, c_2).$$

The next theorem involves the expression

$$\mathcal{N}^{\otimes}(b_1, b_2) := \max_{a_1^2 + a_2^2 = 1} \mathcal{N}(a_1 b_1, a_2 b_2),$$

which is yet another monotonic norm on $\mathbb{R}^2$, not to be confused with the dual norm of $\mathcal{N}$.

Theorem 3.2. *Let $\mathcal{N}$ be a monotonic norm in $\mathbb{R}^2$ and let $2 \leq m \leq n-1$. Suppose that the nonnegative sublinear functions $\varphi : \mathbb{R}^m \to \mathbb{R}$ and $\psi : \mathbb{R}^{n-m} \to \mathbb{R}$ are even. Then the inradius of the epigraphical cone*

$$K^{\mathcal{N}}_{\varphi,\psi} := \{(x,t) \in \mathbb{R}^{n+1} : \mathcal{N}(\varphi(x_1,\ldots,x_m), \psi(x_{m+1},\ldots,x_n)) \leq t\}$$

is given by

$$\rho(K^{\mathcal{N}}_{\varphi,\psi}) = \Big[1 + \big(\mathcal{N}^{\otimes}(\beta_\varphi,\beta_\psi)\big)^2\Big]^{-1/2}.$$

Proof. The function f given by (8) satisfies the hypotheses of Lemma 3.1. By splitting

$$x = (\underbrace{x_1,\ldots,x_m}_{u}, \underbrace{x_{m+1},\ldots,x_n}_{v}) \tag{9}$$

into two group of variables, one obtains

$$\begin{aligned} \beta_f &= \max_{\|u\|^2+\|v\|^2=1} \mathcal{N}(\varphi(u),\psi(v)) \\ &= \max_{\substack{a_1^2+a_2^2=1 \\ a_1\geq 0, a_2\geq 0}} \max_{\substack{\|u\|=a_1 \\ \|v\|=a_2}} \mathcal{N}(\varphi(u),\psi(v)). \end{aligned} \tag{10}$$

But

$$\max_{\substack{\|u\|=a_1 \\ \|v\|=a_2}} \mathcal{N}(\varphi(u),\psi(v)) = \mathcal{N}\left(\max_{\|u\|=a_1}\varphi(u), \max_{\|v\|=a_2}\psi(v)\right) = \mathcal{N}(a_1\beta_\varphi, a_2\beta_\psi),$$

where the first equality is due to the monotonicity of $\mathcal{N}$. That $\mathcal{N}$ is monotonic has further consequences: it implies that $\mathcal{N}(c_1,c_2) = \mathcal{N}(|c_1|,|c_2|)$ for all $(c_1,c_2) \in \mathbb{R}^2$. Hence, the constraints $a_1 \geq 0, a_2 \geq 0$ in (10) are superfluous. This shows that $\beta_f = \mathcal{N}^{\otimes}(\beta_\varphi,\beta_\psi)$ and completes the proof of the theorem. □

The example below illustrates how Theorem 3.2 works in practice.

Example 3.3. Consider the convex cone

$$K = \left\{(x,t) \in \mathbb{R}^{n+1} : \max_{1\leq k\leq m} |x_k| + \sum_{k=m+1}^{n} |x_k| \leq t\right\}.$$

Here φ is the ℓ^∞-norm on $\mathbb{R}^m$, ψ is the ℓ^1-norm on $\mathbb{R}^{n-m}$, and $\mathcal{N}(c_1,c_2) = |c_1| + |c_2|$. Hence,

$$\mathcal{N}^{\otimes}(b_1,b_2) = [b_1^2 + b_2^2]^{1/2}, \quad \beta_\varphi = 1, \quad \beta_\psi = \sqrt{n-m}.$$

One gets in this way $\rho(K) = [2 + n - m]^{-1/2}$.

The monotonicity of $\mathcal{N}$ is an essential assumption in Theorem 3.2. For instance, a norm like $\mathcal{N}(c_1,c_2) = |c_1| + |c_2 - c_1|$ would not be acceptable. On the other hand, Theorem 3.2 admits a more general formulation in which the vector $x \in \mathbb{R}^n$ is split into several portions, and not just into two portions as in (9).

The next proposition provides a rule for computing the inradius of an intersection of epigraphical cones.

Proposition 3.4. *If the nonnegative sublinear functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ are even, then*

$$\rho(\mathrm{epi} f_1 \cap \mathrm{epi} f_2) = \min\{\rho(\mathrm{epi} f_1),\ \rho(\mathrm{epi} f_2)\}.$$

Proof. It is enough to combine (2), Lemma 3.1, and the fact that $\beta_{f_1 \vee f_2} = \max\{\beta_{f_1}, \beta_{f_2}\}$. □

The next example illustrates the usefulness of Proposition 3.4. It concerns the intersection of two elliptic cones.

Example 3.5. The inradius of an elliptic cone $\mathcal{E}(Q)$ is related to the maximal eigenvalue of the corresponding matrix Q. Indeed, if one sets $f(x) = \sqrt{\langle x, Qx\rangle}$, then

$$\beta_f = \max_{\|x\|=1} \sqrt{\langle x, Qx\rangle} = \sqrt{\lambda_{\max}(Q)}\,,$$

and Lemma 3.1 yields

$$\rho(\mathcal{E}(Q)) = [1 + \lambda_{\max}(Q)]^{-1/2}. \tag{11}$$

If Q_1, Q_2 are two positive definite symmetric matrices of order n, then one gets

$$\rho(\mathcal{E}(Q_1) \cap \mathcal{E}(Q_2)) = [1 + \max\{\lambda_{\max}(Q_1), \lambda_{\max}(Q_2)\}]^{-1/2}\,.$$

Next we state a rule for computing the inradius of a Minkowski sum of epigraphical cones.

Proposition 3.6. *If the nonnegative sublinear functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ are even, then*

$$\rho(\mathrm{epi} f_1 + \mathrm{epi} f_2) = [1 + \beta^2]^{-1/2}$$

with β being the optimal value of the nonconvex optimization problem

$$\textit{maximize}\ \{\|y\| : y \in \partial f_1(0_n) \cap \partial f_2(0_n)\}. \tag{12}$$

If f_1 and f_2 are norms, then (12) can be reformulated in the simpler form

$$\begin{aligned} &\text{maximize} \quad \|y\| \\ &f_1^\circ(y) \le 1 \\ &f_2^\circ(y) \le 1. \end{aligned}$$

Proof. The combination of (3) and Lemma 3.1 yields

$$\rho(\mathrm{epi} f_1 + \mathrm{epi} f_2) = [1 + \|C\|^2]^{-1/2},$$

where $\|C\| := \max_{y \in C} \|y\|$ and

$$C = \partial(f_1 \Box f_2)(0_n) = \partial f_1(0_n) \cap \partial f_2(0_n).$$

Suppose now that f_1 and f_2 are norms. In such a case, f_1° and f_2° are norms as well, and

$$\partial f_k(0_n) = \{f_k \le 1\}^\circ = \{f_k^\circ \le 1\}$$

for $k \in \{1, 2\}$. □

Example 3.7. Let Q_1, Q_2 be positive definite symmetric matrices of order n. Then

$$\rho(\mathcal{E}(Q_1) + \mathcal{E}(Q_2)) = [1 + \chi]^{-1/2}$$

with χ standing for optimal value of the nonconvex quadratic optimization problem

$$\begin{aligned} &\text{maximize} \quad \|y\|^2 \\ &\langle y, Q_1^{-1} y\rangle \leq 1 \\ &\langle y, Q_2^{-1} y\rangle \leq 1. \end{aligned} \tag{13}$$

This follows from Proposition 3.6 and the fact that $f_k^\circ(y) = \left[\langle y, Q_k^{-1} y\rangle\right]^{1/2}$. The geometric interpretation of (13) is clear: one searches for a vector of largest norm that lies in the intersection of two ellipsoids. The optimization problem (13) is very interesting in itself and has been studied by a number of authors, see [3, 20, 21, 38, 39] and references therein.

Proposition 3.8. *If the nonnegative sublinear functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ are even, then*

$$\begin{aligned} \rho(\mathrm{epi} f_1 \oplus_{\mathrm{v}} \mathrm{epi} f_2) &= [1 + \beta_{\mathrm{v}}^2]^{-1/2} \\ \rho(\mathrm{epi} f_1 \oplus_{\mathrm{h}} \mathrm{epi} f_2) &= [1 + \beta_{\mathrm{h}}^2]^{-1/2} \end{aligned}$$

with

$$\begin{aligned} \beta_{\mathrm{v}} &= \max\{\|y\| : y \in \partial f_1(0_n) + \partial f_2(0_n)\} \\ \beta_{\mathrm{h}} &= \max\{\|y\| : y \in \partial f_1(0_n) \triangle \partial f_2(0_n)\}. \end{aligned}$$

Proof. It is a matter of combining Proposition 2.5 and Lemma 3.1. □

Remark. If f_1, f_2 are norms, then β_{v} is the optimal value of the nonconvex optimization problem

$$\begin{aligned} &\text{maximize} \quad \|u + v\| \\ &f_1^\circ(u) \leq 1 \\ &f_2^\circ(v) \leq 1. \end{aligned}$$

3.1 Intermezzo: a tale of multispectra

As mentioned before, the inradius of an elliptic cone $\mathcal{E}(Q)$ is related to the maximal eigenvalue of the corresponding matrix Q. We now explain how the formula (11) can be extended to the context of a vertical sum

$$\begin{aligned} K^{\mathcal{Q}} &:= \mathcal{E}(Q_1) \oplus_{\mathrm{v}} \ldots \oplus_{\mathrm{v}} \mathcal{E}(Q_N) \\ &= \left\{(x,t) \in \mathbb{R}^{n+1} : \sum_{k=1}^N \sqrt{\langle x, Q_k x\rangle} \leq t\right\} \end{aligned}$$

of finitely many elliptic cones. Here $\mathcal{Q} = \{Q_1, \ldots, Q_N\}$ is a collection of positive definite symmetric matrices of order n.

The next proposition provides a formula for computing $\rho(K^{\mathcal{Q}})$. Before stating such result in a proper manner, we need first to open a parenthesis and recall some facts on multispectra. The so-called Multivariate Eigenvalue Problem (MEP) consists in finding real numbers $\lambda_1, \dots, \lambda_N$ such that the linear system

$$\begin{bmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,N} \\ A_{2,1} & A_{2,2} & \dots & A_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N,1} & A_{N,2} & \dots & A_{N,N} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{bmatrix} = \begin{bmatrix} \lambda_1 \xi_1 \\ \lambda_2 \xi_2 \\ \vdots \\ \lambda_N \xi_N \end{bmatrix} \tag{14}$$

admits a solution $(\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$ satisfying the N-fold normalization condition

$$\|\xi_1\| = 1, \dots, \|\xi_N\| = 1. \tag{15}$$

The *multispectrum* of the block structured matrix

$$A = [A_{i,j}]_{i,j \in \{1,\dots,N\}} \tag{16}$$

is denoted by $\mathrm{msp}(A)$ and is defined as the set of N-tuples $(\lambda_1, \dots, \lambda_N)$ for which the system (14)-(15) is solvable.

A concrete MEP for a symmetric block structured matrix (16) was introduced for the first time by Hotelling [27]. The specific problem treated by Hotelling concerns the determination of canonical correlation coefficients for multivariate statistics. An iterative method for solving MEP's was proposed by Horst [25]. For additional information on theoretical aspects and algorithms for solving MEP's, the reader is conveyed to the references [2, 10, 19, 26]. The only thing one needs to know here about multispectra is the next lemma. Its proof is a simple matter of applying the technique of Lagrange multipliers and proceeding as in [2, Proposition 2.2].

Lemma 3.9 (Variational Principle for Multispectra)**.** *If the block structured matrix (16) is symmetric, then* $\mathrm{msp}(A)$ *is nonempty and the maximal value of the quadratic form*

$$q_A(\xi) = \left\langle \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_N \end{bmatrix}, A \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_N \end{bmatrix} \right\rangle$$

on the multisphere $(\mathbb{S}_n)^N$ *is equal to*

$$\max \left\{ \sum_{k=1}^N \lambda_k : (\lambda_1, \dots, \lambda_N) \in \mathrm{msp}(A) \right\}.$$

We now are ready to state:

Proposition 3.10. *Let* $\mathcal{Q} = \{Q_1, \dots, Q_N\}$ *be a finite collection of positive definite symmetric matrices of order* n*. Then*

$$\rho(K^{\mathcal{Q}}) = [1 + \chi_{\mathcal{Q}}]^{-1/2} \tag{17}$$

with

$$\chi_{\mathcal{Q}} = \max \left\{ \sum_{k=1}^N \lambda_k : (\lambda_1, \dots, \lambda_N) \in \mathrm{msp}(A^{\mathcal{Q}}) \right\} \tag{18}$$

and $A^{\mathcal{Q}}$ *denoting the symmetric block structured matrix whose* (i,j)*-block is given by* $Q_i^{1/2} Q_j^{1/2}$*.*

Proof. Proposition 3.8 extends to the vertical sum of N elliptic cones and yields (17) with $\chi_{\mathcal{Q}}$ denoting the optimal value of

$$\chi_{\mathcal{Q}} = \max\left\{ \left\| \sum_{k=1}^{N} \eta_k \right\|^2 : \langle \eta_k, Q_k^{-1}\eta_k\rangle \le 1 \text{ for all } k \in \{1,\ldots,N\} \right\}. \tag{19}$$

The inequality constraints in (19) are all are active at a solution. This fact and the change of variables $\xi_k = Q_k^{-1/2}\eta_k$ lead to the equivalent formulation

$$\chi_{\mathcal{Q}} = \max\left\{ \left\| \sum_{k=1}^{N} Q_k^{1/2}\xi_k \right\|^2 : \|\xi_1\| = 1,\ \ldots,\ \|\xi_N\| = 1 \right\}. \tag{20}$$

Note that

$$\left\| \sum_{k=1}^{N} Q_k^{1/2}\xi_k \right\|^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \langle \xi_i, Q_i^{1/2}Q_j^{1/2}\xi_j\rangle = \left\langle \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_N \end{bmatrix}, A^{\mathcal{Q}} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_N \end{bmatrix} \right\rangle$$

is the quadratic form associated to $A^{\mathcal{Q}}$. Finally, passing from (20) to (18) is a matter of applying Lemma 3.9. □

4 Degree of pointedness of an epigraphical cone

As explained in [28, 30], there are many ways of measuring the degree of pointedness of a nontrivial closed convex cone K in $\mathbb{R}^d$. One can use for instance the coefficient

$$\mu(K) := \min_{z \in \mathrm{co}(K \cap \mathbb{S}_d)} \|z\| \tag{21}$$

whose geometric meaning is clear: the minimization problem (21) is about finding the least norm element of the convex set $\mathrm{co}(K \cap \mathbb{S}_d)$. One refers to the number $\mu(K)$ as the *basic pointedness coefficient* of K. General information on the function $K \mapsto \mu(K)$ can be found scattered in a number of references [22, 23, 28, 32].

By definition, the basic pointedness coefficient of epi$f$ is given by

$$\mu(\mathrm{epi} f) = \min\left\{ \sqrt{\|x\|^2 + t^2} : (x,t) \in \mathrm{co}\left[\mathrm{epi} f \cap \mathbb{S}_{n+1}\right] \right\}. \tag{22}$$

Some simplification is achieved in (22) if the nonnegative sublinear function f is even. Indeed, under such assumption one obtains the simpler expression

$$\mu(\mathrm{epi} f) = \min\left\{ t : (0_n, t) \in \mathrm{co}\left[\mathrm{epi} f \cap \mathbb{S}_{n+1}\right] \right\}. \tag{23}$$

Obtaining an explicit characterization of the convex hull of epi$f \cap \mathbb{S}_{n+1}$ is however a hard task. The next result tells how to compute the minimal value (23) in practice. One can see Lemma 4.1 as a sort of dual version of Lemma 3.1.

Lemma 4.1. *If the nonnegative sublinear function $f : \mathbb{R}^n \to \mathbb{R}$ is even, then*

$$\mu(\mathrm{epi} f) = \frac{\alpha_f}{\sqrt{1 + \alpha_f^2}} \tag{24}$$

with

$$\alpha_f := \min_{\|x\|=1} f(x). \tag{25}$$

Proof. Consider first the case $\alpha_f = 0$, that is, $f(\bar{x}) = 0$ for some $\bar{x} \in \mathbb{S}_n$. Since f is even, one also has $f(-\bar{x}) = 0$. Hence, epi$f$ is not pointed because it contains the line $\mathbb{R}\,(\bar{x}, 0)$. In short, both sides of (24) are equal to zero. Consider now the case $\alpha_f \neq 0$, that is, f vanishes only at 0_n. By Proposition 2.1 one knows already that epi$f$ is pointed. Recall that f is also even. Thus, f and f° are norms. On the other hand, according to a duality result stated in [28, Proposition 6.3], one can write $\mu(K) = \rho(K^+)$ for any nontrivial closed convex cone K in $\mathbb{R}^d$. Hence, by combining Theorem 2.6 and Lemma 3.1 one gets

$$\mu(\mathrm{epi} f) = \rho((\mathrm{epi} f)^+) = [1 + \gamma_f^2]^{-1/2}$$

with $\gamma_f = \max_{\|y\|=1} f^\circ(y)$. The rest of the proof consists in showing that $\gamma_f = 1/\alpha_f$, but this equality is known or ought to be known. In fact, it follows from the general identity

$$\min_{x\neq 0} \frac{g(x)}{h(x)} = \min_{y\neq 0} \frac{h^\circ(y)}{g^\circ(y)} \tag{26}$$

that applies to any pair g, h of norms on $\mathbb{R}^n$. □

The next two corollaries are given only with a sketch of proof because everything is more or less the same as in Section 3.

Corollary 4.2. *If the nonnegative sublinear functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ are even, then*

$$\mu(\mathrm{epi} f_1 + \mathrm{epi} f_2) = \min\{\mu(\mathrm{epi} f_1),\, \mu(\mathrm{epi} f_2)\}. \tag{27}$$

Proof. If α_{f_1} or α_{f_2} is equal to 0, then epif_1 or epif_2 is not pointed. In such a case the Minkowski sum epif_1 + epif_2 is not pointed either. Hence, both sides of (27) are equal to 0. Suppose now that α_{f_1} and α_{f_2} are both different from 0. In such a case, f_1 and f_2 are norms. One gets

$$\begin{aligned} \mu(\mathrm{epi} f_1 + \mathrm{epi} f_2) &= \rho\left((\mathrm{epi} f_1 + \mathrm{epi} f_2)^+\right) \\ &= \rho\left(\mathrm{epi} f_1^\circ \cap \mathrm{epi} f_2^\circ\right) \\ &= \min\left\{\rho\left(\mathrm{epi} f_1^\circ\right), \rho\left(\mathrm{epi} f_2^\circ\right)\right\} \\ &= \min\left\{\mu\left(\mathrm{epi} f_1\right), \mu\left(\mathrm{epi} f_2\right)\right\}. \end{aligned}$$

We have omitted some easy intermediate steps. □

Corollary 4.3. *If the nonnegative sublinear functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ are even, then*

$$\mu(\mathrm{epi} f_1 \cap \mathrm{epi} f_2) = \frac{\alpha}{\sqrt{1+\alpha^2}}$$

with $\alpha = \min_{x\in\mathbb{S}_n} \max\{f_1(x), f_2(x)\}$. If f_1 and f_2 are norms, then one can also write

$$\mu(\mathrm{epi} f_1 \cap \mathrm{epi} f_2) = [1 + \beta^2]^{-1/2}$$

with β standing for the optimal value of the nonconvex optimization problem

$$\begin{aligned} &\text{maximize} \quad \|x\| \\ &f_1(x) \leq 1 \\ &f_2(x) \leq 1. \end{aligned}$$

Proof. The first formula is obtained by applying Lemma 4.1 to $f_1 \vee f_2$. Unfortunately, the number α is not always easy to evaluate. If f_1 and f_2 are norms, then one can proceed a bit further by using duality arguments. More precisely, one can write

$$\begin{aligned}\mu(\mathrm{epi} f_1 \cap \mathrm{epi} f_2) &= \rho\left((\mathrm{epi} f_1 \cap \mathrm{epi} f_2)^+\right) \\ &= \rho\left(\mathrm{epi} f_1^\circ + \mathrm{epi} f_2^\circ\right)\end{aligned}$$

and then one can apply Proposition 3.6 to the pair f_1°, f_2°. □

Example 4.4. Let Q_1, Q_2 be positive definite symmetric matrices of order n. Then

$$\mu(\mathcal{E}(Q_1) \cap \mathcal{E}(Q_2)) = [1 + \chi]^{-1/2}$$

with χ standing for optimal value of the nonconvex quadratic optimization problem

$$\begin{aligned}&\text{maximize} \quad \|x\|^2 \\ &\langle x, Q_1 x\rangle \leq 1 \\ &\langle x, Q_2 x\rangle \leq 1.\end{aligned}$$

5 Angular structure of an epigraphical cone

5.1 Maximal angle

The maximal angle of a nontrivial closed convex cone K in $\mathbb{R}^d$ is defined as the number

$$\theta_{\max}(K) := \max_{z,v \in K \cap \mathbb{S}_d} \arccos \langle z, v\rangle. \tag{28}$$

One of the reasons why the maximal angle of a convex cone is a mathematical tool of interest is that the coefficients

$$\begin{aligned}\gamma(K) &:= 1 - \frac{\theta_{\max}(K)}{\pi} \\ \nu(K) &:= \cos\left(\frac{\theta_{\max}(K)}{2}\right)\end{aligned}$$

also serve to measure the degree of pointedness of K. Both of them qualify as pointedness index in the axiomatic sense of [28]. The maximal angle function $K \mapsto \theta_{\max}(K)$ has however many other uses.

One says that (y, z) is an *antipodal pair* of K if y and z are unit vectors in K achieving the maximal angle of the cone, i.e.,

$$y, z \in K \cap \mathbb{S}_d \quad \text{and} \quad \arccos \langle z, v\rangle = \theta_{\max}(K).$$

Antipodality in convex cones has been extensively theorized in [29, 33, 34]. The value (28) has been computed in [17, 24, 31] for several particular convex cones arising in applications. The case of an epigraphical cone is discussed next.

Suppose for a moment that $f : \mathbb{R}^n \to \mathbb{R}$ is a positive sublinear function, so that B_f is compact. If one looks at the formula (1), then the first idea that comes to mind is that finding an antipodal pair of epi$f$ is somewhat related to the problem of finding two points in B_f that are at maximal distance. Unfortunately, solving the diameter maximization problem

$$\mathrm{diam}(B_f) = \max_{u,v\in B_f} \|u - v\| \tag{29}$$

is of no use, and, what is worse, it may lead to wrong conclusions. Since

$$(\mathrm{epi} f) \cap \mathbb{S}_{n+1} = \left\{ \frac{(x,1)}{\sqrt{1+\|x\|^2}} \;:\; x \in B_f \right\},$$

what we have to compute is

$$\cos\left[\theta_{\max}(\mathrm{epi} f)\right] = \min_{u,v\in B_f} \frac{\langle u,v\rangle + 1}{\sqrt{1+\|u\|^2}\,\sqrt{1+\|v\|^2}}\,.$$

This can be reformulated as

$$\sqrt{2}\sqrt{1-\cos\left[\theta_{\max}(\mathrm{epi} f)\right]} = \max_{u,v\in B_f} \widehat{d}\,(u,v) \tag{30}$$

with

$$\widehat{d}\,(u,v) := \left\| \frac{(u,1)}{\sqrt{1+\|u\|^2}} - \frac{(v,1)}{\sqrt{1+\|v\|^2}} \right\|.$$

One has to evaluate the "diameter" of B_f after all, but with respect to the metric $\widehat{d}$ and not with respect to the usual metric of $\mathbb{R}^n$. The next example shows that a solution to (29) is not necessarily a solution to (30).

Example 5.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the positive sublinear function whose set B_f is the triangle of vertices $a = (100, 0)$, $b = (0, -100)$, and $c = (-40, 1)$. The diameter of B_f is $100\sqrt{2}$ and this value is achieved with the pair (a, b). However, this pair does not solve the maximization problem (30) because

$$\widehat{d}\,(a,c) = \left\| \frac{(100,0,1)}{\sqrt{1+100^2}} - \frac{(-40,1,1)}{\sqrt{1+40^2+1}} \right\| \approx 2.0$$

is greater than

$$\widehat{d}\,(a,b) = \left\| \frac{(100,0,1)}{\sqrt{1+100^2}} - \frac{(0,100,1)}{\sqrt{1+100^2}} \right\| \approx 1.4.$$

Example 5.1 concerns a function f that is not even. The next theorem tells how to compute the maximal angle of epi$f$ when f is even.

Theorem 5.2. *Suppose that the nonnegative sublinear function $f : \mathbb{R}^n \to \mathbb{R}$ is even. Then*

$$\theta_{\max}(\mathrm{epi} f) = 2\arccos\left(\frac{\alpha_f}{\sqrt{1+\alpha_f^2}}\right) = \arccos\left(\frac{\alpha_f^2 - 1}{\alpha_f^2 + 1}\right). \tag{31}$$

The maximal angle of epi$f$ *is formed with the unit vectors*

$$\frac{1}{\sqrt{1+\alpha_f^2}}\,(x,\alpha_f) \quad \text{and} \quad \frac{1}{\sqrt{1+\alpha_f^2}}\,(-x,\alpha_f)\,, \tag{32}$$

where $x \in \mathbb{S}_n$ is any solution to the minimization problem (25).

Proof. The second equality in (31) follows from a general trigonometric identity, namely, the formula for the cosine of the half-angle. If $\alpha_f = 0$, then the three terms in (31) are equal to π and this angle is attained with the vectors $(x,0)$ and $(-x,0)$. Suppose then that $\alpha_f \neq 0$, in which case f is a norm. Let ϑ_f denote the maximal angle that a unit vector of epi$f$ forms with respect to the canonical vector e_{n+1}. Hence, $\cos\vartheta_f$ is equal to the optimal value of the minimization problem

$$\begin{aligned}&\text{minimize } \langle 0_n, u\rangle + 1t\\ &(u,t) \in \text{epi}f\\ &\|u\|^2 + t^2 = 1.\end{aligned}$$

Since this minimum is attained on the boundary of epi$f$, the constraint $(u,t) \in \text{epi}f$ can be converted into $f(u) = t$. By getting rid of the variable t, one obtains

$$\cos\vartheta_f = \min\{f(u) : \|u\|^2 + [f(u)]^2 = 1\}.$$

A positive homogeneity argument leads to

$$\begin{aligned}\cos\vartheta_f &= \min_{u\neq 0_n} \frac{f(u)}{\sqrt{\|u\|^2 + [f(u)]^2}}\\ &= \min_{u\neq 0_n} \left[1 + \left(\frac{\|u\|}{f(u)}\right)^2\right]^{-1/2}\\ &= \left[1 + \left(\min_{u\neq 0_n} \frac{f(u)}{\|u\|}\right)^{-2}\right]^{-1/2}.\end{aligned}$$

We have shown in this way that

$$\cos\vartheta_f = \left(1 + \alpha_f^{-2}\right)^{-1/2} = \alpha_f / \left(1 + \alpha_f^2\right)^{1/2}.$$

The evenness of f has not been used yet. This property is needed for proving the following claim:

$$\theta_{\max}(\text{epi}f) = 2\vartheta_f. \tag{33}$$

From the very definition of ϑ_f one sees that epi$f$ is contained in the revolution cone

$$\begin{aligned}\text{rev}(\vartheta_f) &= \left\{(u,t) \in \mathbb{R}^{n+1} : (\cos\vartheta_f)\sqrt{\|u\|^2 + t^2} \leq t\right\}\\ &= \{(x,t) \in \mathbb{R}^{n+1} : (\tan\vartheta)^{-1}\|x\| \ \leq t\}\end{aligned}$$

whose axis is generated by e_{n+1} and whose half-aperture angle is equal to ϑ_f. Hence,

$$\theta_{\max}(\text{epi}f) \leq \theta_{\max}(\text{rev}(\vartheta_f)) = 2\vartheta_f.$$

On the other hand, both vectors in (32) have unit length and belong to epi$f$. Therefore

$$\theta_{\max}(\text{epi}f) \geq \arccos\left(\frac{\langle x, -x\rangle + \alpha_f^2}{1 + \alpha_f^2}\right) = \arccos\left(\frac{\alpha_f^2 - 1}{\alpha_f^2 + 1}\right) = 2\theta_f.$$

This confirms the claim (33) and completes the proof. □

Remark. If f is not even, then the second vector in (32) does not belong necessarily to epi$f$. It is still possible to write

$$\theta_{\max}(\mathrm{epi}f) \leq 2\arccos\left(\frac{\alpha_f}{\sqrt{1+\alpha_f^2}}\right),$$

but this inequality may be very coarse. To see this one must construct an epigraphical cone whose associated sublinear function is not even, but highly skewed.

By combining Lemma 3.1 and Theorem 5.2 one gets the following by-product.

Corollary 5.3. *If the nonnegative sublinear function $f : \mathbb{R}^n \to \mathbb{R}$ is even, then $\nu(\mathrm{epi}f) = \mu(\mathrm{epi}f)$.*

5.2 Minimal angle

The maximal angle of a convex cone is one side of the story. The other side is the minimal angle. In fact, between the maximal and the minimal one there is a full collection of intermediate critical angles.

Definition 5.4. *Let K be a closed convex cone of $\mathbb{R}^d$ and let z, v be unit vectors in K.*

i) That (z, v) is a critical pair *of K means that $v - \langle z, v\rangle z \in K^+$ and $z - \langle z, v\rangle v \in K^+$.*

ii) The angle $\theta(z, v) = \arccos \langle z, v\rangle$ formed by a critical pair is called a critical angle*. A critical pair (u, v) and the corresponding critical angle $\theta(u, v)$ are said to be proper if u and v are not collinear.*

iii) The angular spectrum *of K, indicated with the symbol* asp(K)*, is the set of all proper critical angles of K. The smallest element of this set is denoted by $\theta_{\min}(K)$ and called the* minimal angle *of K.*

The criticality conditions formulated in (i) can be seen as first order optimality conditions for the angle maximization problem (28). General information concerning the theory of critical angles in convex cones can be found in the standard references [29, 33, 34].

The next proposition provides a formula for computing the minimal angle of an epigraphical cone associated to a norm.

Proposition 5.5. *Let f be a norm on $\mathbb{R}^n$. Then*

$$\theta_{\min}(\mathrm{epi}f) \;=\; 2\arcsin\left(\frac{1}{\sqrt{1+\beta_f^2}}\right) \;=\; \arcsin\left(\frac{2\beta_f}{1+\beta_f^2}\right). \tag{34}$$

Proof. In the present context the cone epi$f$ is solid and pointed. In view of a duality result for extremal angles established in [34, Theorem 3], one can write

$$\theta_{\min}(\mathrm{epi}f) = \pi - \theta_{\max}((\mathrm{epi}f)^+). \tag{35}$$

By exploiting (35) and Theorems 2.6 and 5.2, one obtains

$$\theta_{\min}(\mathrm{epi}f) \;=\; \pi - \theta_{\max}(\mathrm{epi}f^\circ) \;=\; \pi - 2\arccos\left(\frac{\delta_f}{\sqrt{1+\delta_f^2}}\right) \tag{36}$$

with $\delta_f = \min_{\|y\|=1} f^\circ(y)$. But the identity (26) shows that $\delta_f = 1/\beta_f$. By plugging this value in (36) and simplifying one arrives at the first equality in (34). The second equality in (34) is a general trigonometric identity. □

We state without proof two immediate corollaries.

Corollary 5.6. *If f is a norm on $\mathbb{R}^n$, then*

$$\rho(\mathrm{epi} f) = \sin\left(\frac{\theta_{\min}(\mathrm{epi} f)}{2}\right).$$

Corollary 5.7. *Let $\mathcal{N}$ be a monotonic norm in $\mathbb{R}^2$ and let $2 \leq m \leq n-1$. Let φ and ψ be norms on $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$, respectively. Then the minimal angle of the epigraphical cone $K^{\mathcal{N}}_{\varphi,\psi}$ is given by*

$$\rho(K^{\mathcal{N}}_{\varphi,\psi}) = 2 \arcsin\left(\left[1 + \left(\mathcal{N}^{\otimes}(\beta_\varphi, \beta_\psi)\right)^2\right]^{-1/2}\right).$$

Example 5.8. Consider the ℓ^p-cone $\mathcal{K}_p$ with $p \in [1,\infty]$. In this case $f = \|\cdot\|_p$ and the computation of α_f and β_f offers no difficulty. By applying Theorem 5.2 and Proposition 5.5 one gets

$$\cos\left(\frac{\theta_{\max}(\mathcal{K}_p)}{2}\right) = \begin{cases} \left(1 + n^{1-\frac{2}{p}}\right)^{-1/2} & \text{if } p \in [2,\infty] \\ 2^{-1/2} & \text{if } p \in [1,2] \end{cases}$$

and

$$\sin\left(\frac{\theta_{\min}(\mathcal{K}_p)}{2}\right) = \begin{cases} \left(1 + n^{\frac{2}{p}-1}\right)^{-1/2} & \text{if } p \in [1,2] \\ 2^{-1/2} & \text{if } p \in [2,\infty]. \end{cases}$$

In particular,

$$\begin{aligned} \theta_{\max}(\mathcal{K}_\infty) &= 2 \arccos\left(1/\sqrt{1+n}\,\right), \quad \theta_{\min}(\mathcal{K}_\infty) = \pi/2, \\ \theta_{\min}(\mathcal{K}_1) &= 2 \arcsin\left(1/\sqrt{1+n}\,\right), \quad \theta_{\max}(\mathcal{K}_1) = \pi/2. \end{aligned}$$

5.3 Critical angles

Identifying the critical angles of a convex cone is in general a difficult task. The next theorem tell how to construct critical pairs in an epigraphical cone associated to a norm. The key ingredient of the discussion is the concept of umbilical point for a positive sublinear function $f : \mathbb{R}^n \to \mathbb{R}$. From the very definition of polar function it follows that

$$f^\circ(y) f(x) \geq \langle y, x\rangle \quad \text{for all } x, y \in \mathbb{R}^n.$$

An *umbilical point* of f is a nonzero vector $u \in \mathbb{R}^n$ such that $f^\circ(u) f(u) = \|u\|^2$. Of course, any positive multiple of an umbilical point is an umbilical point. The *umbilical spectrum* of f is the set

$$\mathrm{usp}(f) := \left\{\frac{f(u)}{\|u\|} : u \text{ is an umbilical point of } f\right\}.$$

Each element of this set is called an *umbilical value* of f.

Example 5.9. Let Q be a positive definite symmetric matrix of order n. As pointed out in [34, Lemma 1], a nonzero vector $u \in \mathbb{R}^n$ satisfies the relation

$$\langle u, Q^{-1}u\rangle\langle u, Qu\rangle = \|u\|^4$$

if and only if u is an eigenvector of Q. Hence, the umbilical points of the norm $f = \Psi^*_{E(Q)}$ are the eigenvectors of Q, and

$$\begin{aligned}\operatorname{usp}(\Psi^*_{E(Q)}) &= \left\{\frac{\sqrt{\langle u, Qu\rangle}}{\|u\|} : u \text{ is an eigenvector of } Q\right\} \\ &= \left\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } Q\right\}.\end{aligned}$$

This norm admits at most n umbilical values.

The next proposition shows that the umbilical points of a positive sublinear function f are exactly the eigenvectors of the subdifferential map ∂f. A nonzero vector $u \in \mathbb{R}^n$ is called an eigenvector of a multivalued map $\mathcal{A} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ if there exists a scalar $\lambda \in \mathbb{R}$ such that $\lambda u \in \mathcal{A}(u)$. In such a case one refers to λ as an eigenvalue of $\mathcal{A}$ associated to the eigenvector u.

Proposition 5.10. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positive sublinear function. Then u is an umbilical point of f if and only if u is an eigenvector of ∂f.*

Proof. We suppose that $u \neq 0_n$, otherwise there is nothing to prove. That u is an umbilical point of f amounts to saying that u solves the maximization problem

$$f^\circ(u) = \max_{x \neq 0_n} \frac{\langle u, x\rangle}{f(x)}.$$

Equivalently, u solves the convex optimization problem

$$f^\circ(u) = \max_{f(x) \leq f(u)} \frac{\langle u, x\rangle}{f(u)}. \tag{37}$$

The so-called Slater qualification hypothesis holds for (37). Hence, for u to solve (37) it is necessary and sufficient that

$$\frac{u}{f(u)} \in \eta\, \partial f(u) \tag{38}$$

for some Karush-Kuhn-Tucker multiplier $\eta \geq 0$. Since the left-hand side of (38) is a nonzero vector, the multiplier η must be positive. Thus, (38) can be written in the form $\lambda u \in \partial f(u)$ with $\lambda = [\eta f(u)]^{-1}$. This proves the proposition. □

We now are ready to state:

Theorem 5.11. *Let f be a norm on $\mathbb{R}^n$. The vectors (x, t) and (y, s) form a proper critical pair of* epi$f$ *if and only if the following three conditions hold:*

(a) $y = -x$.

(b) $s = t = \sqrt{1 - \|x\|^2} = f(x)$.

(c) *x is an umbilical point of f.*

Proof. We start with the "if" part. Suppose that the system (a)–(c) is in force. We must prove that the vectors

$$(x,t) = (x, f(x)) \tag{39}$$
$$(y,s) = (-x, f(x)) \tag{40}$$

form a proper critical pair of epi$f$. These vectors have unit length because

$$[f(x)]^2 + \|x\|^2 = 1. \tag{41}$$

Clearly, (39) belongs to epi$f$. The vector (40) also belongs to epi$f$ because f is even. Since (39) and (40) are not collinear, their inner product

$$\lambda = \langle (x, f(x)), (-x, f(x)) \rangle \; = \; [f(x)]^2 - \|x\|^2 \; = \; 1 - 2\|x\|^2$$

belongs to the open interval $]-1,1[$. It remains to show that

$$(x, f(x)) - \lambda\,(-x, f(x)) \in (\mathrm{epi}f)^+ \tag{42}$$
$$(-x, f(x)) - \lambda\,(x, f(x)) \in (\mathrm{epi}f)^+. \tag{43}$$

Thanks to Theorem 2.6 and the fact that f is even, the system (42)-(43) reduces to

$$(1+\lambda) f^\circ(x) \leq (1-\lambda) f(x). \tag{44}$$

Since x is an umbilical point of f, there exists a positive scalar γ such that

$$f(x) = \gamma\|x\| \quad \text{ and } \quad f^\circ(x) = (1/\gamma)\|x\|. \tag{45}$$

In fact, γ is the umbilical value of f associated to x. Hence, the inequality (44) becomes

$$(1+\lambda) \leq \gamma^2(1-\lambda). \tag{46}$$

By combining (41) and (45) one gets $\|x\|^2 = (1+\gamma^2)^{-1}$. Hence,

$$\begin{aligned} 1+\lambda &= 2(1-\|x\|^2) \; = \; 2\gamma^2/(1+\gamma^2) \\ \gamma^2(1-\lambda) &= 2\gamma^2\|x\|^2 \; = \; 2\gamma^2/(1+\gamma^2). \end{aligned}$$

This proves that (46) holds in fact as an equality. We now prove the "only if" part. We assume that (x,t) and (y,s) form a critical pair of epi$f$. By criticality, we have

$$(x,t) - \lambda(y,s) \in (\mathrm{epi}f)^+ \tag{47}$$
$$(y,s) - \lambda(x,t) \in (\mathrm{epi}f)^+ \tag{48}$$

with $\lambda = \langle x, y\rangle + ts$. Since f is even, both $(-x,t)$ and $(-y,s)$ also belong to epi$f$. Multiplying the left hand side of (47) by $(-x,t)$, one gets

$$0 \leq \big\langle (-x,t), (x,t) - \lambda(y,s) \big\rangle = -\|x\|^2 + t^2 - \lambda(ts - \langle x,y\rangle).$$

By plugging the value of λ and rearranging, one obtains

$$0 \leq t^2(1-s^2) - \|x\|^2 + \langle x,y\rangle^2. \tag{49}$$

Recall that (x,t) and (y,s) are vectors of unit length, i.e.,

$$\|x\|^2 + t^2 = 1, \quad \|y\|^2 + s^2 = 1. \tag{50}$$

The combination of (49) and (50) produces

$$0 \leq (1 - \|x\|^2)\|y\|^2 - \|x\|^2 + \langle x, y\rangle^2,$$

and therefore

$$\|x\|^2 - \|y\|^2 \ \leq\ \langle x, y\rangle^2 - \|x\|^2\|y\|^2 \ \leq\ 0. \tag{51}$$

By the same token, multiplying the left hand side of (48) by $(-y, s)$ leads to

$$\|y\|^2 - \|x\|^2 \ \leq\ \langle x, y\rangle^2 - \|x\|^2\|y\|^2 \ \leq\ 0. \tag{52}$$

By combining (51) and (52) one gets

$$0 = \|x\|^2 - \|y\|^2 = \langle x, y\rangle^2 - \|x\|^2\|y\|^2.$$

Hence, $y = \pm x$ and $t = s$. The case $y = x$ must be ruled out because we are assuming properness of the critical pair $\{(x,t),(y,s)\}$. We conclude that $y = -x$, establishing (a). The first equality in (b) has also been proved. The second one is contained in (50), and the third one follows from the fact that (x,t) is necessarily in the boundary of epi$f$. Next we prove (c). By using (47) and the parts (a) and (b), one gets the inequality (44) and

$$\begin{aligned} 1 + \lambda &= 2[f(x)]^2, \\ 1 - \lambda &= 2\|x\|^2. \end{aligned}$$

These three relations together yield

$$2[f(x)]^2 f^\circ(x) \leq 2\|x\|^2 f(x),$$

that is, $f(x)f^\circ(x) \leq \|x\|^2$. This proves that x is an umbilical point of f. □

The proof of Theorem 5.11 is very much in the spirit of [34, Theorem 5], but the novelty of our approach is the introduction and use of the concept of umbilicity. As a complement to Theorem 5.11 we provide below a full description of the angular spectrum of epi$f$.

Theorem 5.12. *If f is a norm on $\mathbb{R}^n$, then*

$$\text{asp}(\text{epi}f) = \left\{ \arccos\left(\frac{\gamma^2 - 1}{\gamma^2 + 1}\right) : \gamma \in \text{usp}(f) \right\}. \tag{53}$$

Proof. Let γ be an umbilical value of f. Then $\gamma = f(u)/\|u\|$ for some umbilical point u of f. The "if" part of Theorem 5.11 shows that the vectors

$$(x,t) = \frac{1}{\sqrt{[f(u)]^2 + \|u\|^2}}\,(u, f(u)) \tag{54}$$

$$(y,s) = \frac{1}{\sqrt{[f(u)]^2 + \|u\|^2}}\,(-u, f(u)) \tag{55}$$

form a proper critical pair of epi$f$. Hence, the corresponding angle

$$\theta \;=\; \arccos\left(\frac{[f(u)]^2 - \|u\|^2}{[f(u)]^2 + \|u\|^2}\right) \;=\; \arccos\left(\frac{\gamma^2-1}{\gamma^2+1}\right) \tag{56}$$

is a proper critical angle of epi$f$. This proves the "$\supset$" part of the equality (53). Conversely, let θ be a proper critical angle of epi$f$. Suppose that θ is formed with the proper critical pair $\{(x,t),(y,s)\}$. Due to the "only if" part of Theorem 5.11, one must have (54)–(55) for some umbilical point u of f. Hence, θ has the form (56) with $\gamma = f(u)/\|u\|$. This completes the proof of (53). □

Remark. If f is a norm on $\mathbb{R}^n$, then the angular spectrum of epi$f$ can also be represented as

$$\mathrm{asp}(\mathrm{epi}f) = \left\{\arccos\left(\frac{f(u)-f^\circ(u)}{f(u)+f^\circ(u)}\right) : u \text{ umbilical point of } f\right\}.$$

Since $(f^\circ)^\circ = f$, the norms f and f° have the same umbilical points and

$$\mathrm{asp}(\mathrm{epi}f^\circ) = \left\{\arccos\left(\frac{f^\circ(u)-f(u)}{f^\circ(u)+f(u)}\right) : u \text{ umbilical point of } f\right\}.$$

Note that θ is a proper critical angle of epif° if and only if $\pi - \theta$ is a proper critical angle of epi$f$.

It is worthwhile to mention that (53) can be inverted so as to get a characterization for the umbilical spectrum of a norm:

$$\mathrm{usp}(f) = \left\{\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} : \theta \in \mathrm{asp}(\mathrm{epi}f)\right\}. \tag{57}$$

In practice, the relation (57) is less interesting than (53) because asp(epi$f$) is usually hard to compute. However, (57) has some theoretical relevance. For instance, since a polyhedral cone is known to have finitely many critical angles, one gets:

Corollary 5.13. *A polyhedral norm on $\mathbb{R}^n$ has finitely many umbilical values.*

6 By way of conclusion

This completes the first part of our work on epigraphical cones. The companion paper [42] presents additional material on this topic: smoothness and rotundity in epigraphical cones, facial analysis, etc. Applications of epigraphical cones in optimization theory are also considered. In particular, [42] provides rules for constructing barrier functions for epigraphical cones.

References

[1] C.D. Aliprantis and R. Tourky. *Cones and Duality.* American Mathematical Society, Providence, RI, 2007.

[2] A. Amri and A. Seeger. Spectral analysis of coupled linear complementarity problems. *Linear Algebra Appl.* 432 (2010), 2507–2523.

[3] D. Arzelier, D. Henrion, and S. Tarbouriech. LMI approximations for the radius of the intersection of ellipsoids: survey. *J. Optim. Theory Appl.* 108 (2001), 1–28.

[4] G.P. Barker and D. Carlson. Cones of diagonally dominant matrices. *Pacific J. Math.* 57 (1975), 15–32.

[5] R. Bhattacharya, J. Fung, R.M. Murray, and A. Tiwari. Ellipsoidal cones and rendezvous of multiple agents. 43rd IEEE Conference on Decision and Control, Nassau, Bahamas, Dec. 2004.

[6] A. Belloni and R.M. Freund. A geometric analysis of Renegar's condition number, and its interplay with conic curvature. *Math. Program.* 119 (2009), 95–107.

[7] A. Belloni and R.M. Freund. On the second-order feasibility cone: primal-dual representation and efficient projection. *SIAM J. Optim.* 19 (2008), 1073–1092.

[8] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge Univ. Press, Cambridge, 2004.

[9] G.G. Burton. Contracting the maximal points of an ordered convex set. *J. Convex Anal.* 10 (2003), 255–264.

[10] M.T. Chu and J.L. Watterson. On a multivariate eigenvalue problem. I. Algebraic theory and a power method. *SIAM J. Sci. Comput.* 14 (1993), 1089–1106.

[11] M. Epelman and R.M. Freund. A new condition measure, preconditioners, and relations between different measures of conditioning for conic linear systems. *SIAM J. Optim.* 12 (2002), 627–655.

[12] M. Fiedler and E. Haynsworth. Cones which are topheavy with respect to a norm. *Linear and Multilinear Algebra* 1 (1973), 203–211.

[13] R.M. Freund. On the primal-dual geometry of level sets in linear and conic optimization. *SIAM J. Optim.* 13 (2003), 1004–1013.

[14] R.M. Freund and J.R. Vera. Condition-based complexity of convex optimization in conic linear form via the ellipsoid algorithm. *SIAM J. Optim.* 10 (1999), 155–176.

[15] F. Glineur. Topics in convex optimization: interior-point methods, conic duality and approximations, PhD thesis, Faculté Polytechnique de Mons, Belgium, Dec. 2000.

[16] F. Glineur and T. Terlaky. Conic formulation for l_p-norm optimization. *J. Optim. Theory Appl.* 122 (2004), 285–307.

[17] D. Gourion and A. Seeger. Critical angles in polyhedral convex cones: numerical and statistical considerations. *Math. Program.* 123 (2010), 173–198.

[18] O. Güler. Barrier functions in interior point methods. *Math. Oper. Res.* 21 (1996), 860–885.

[19] M. Hanafi and J.M.F. Ten Berge. Global optimality of the successive Maxbet algorithm. *Psychometrika*, 68 (2003), 97–103.

[20] D. Henrion, M. Sebek, and S. Tarbouriech. Algebraic approach to robust controller design: a geometric interpretation. Proceedings of the American Control Conference, Philadelphia, Pennsylvania, pp. 2703–2707, 1998.

[21] D. Henrion, M. Sebek, and S. Tarbouriech. Rank-one LMI approach to simultaneous stabilization of linear systems. *Systems Control Lett.* 38 (1999), 79–89.

[22] R. Henrion and A. Seeger. On properties of different notions of centers for convex cones. *Set-Valued Var. Anal.* 18 (2010), 205–231.

[23] R. Henrion and A. Seeger. Inradius and circumradius of various convex cones arising in applications. *Set-Valued Var. Anal.* 18 (2010), 483–511.

[24] J.B. Hiriart-Urruty and A. Seeger. A variational approach to copositive matrices. *SIAM Review* 52 (2010), 593–629.

[25] P. Horst. Relations among m sets of measures. *Psychometrika*, 26 (1961), 129–149.

[26] P. Horst. *Factor Analysis of Data Matrices.* Holt, Rinehart and Winston, Inc., New York, 1965.

[27] H. Hotelling. The most predictable criterion. *J. Educ. Psychol.*, 26 (1935), 139-142.

[28] A. Iusem and A. Seeger. Axiomatization of the index of pointedness for closed convex cones. *Comput. Applied Math.* 24 (2005), 245–283.

[29] A. Iusem and A. Seeger. On pairs of vectors achieving the maximal angle of a convex cone. *Math. Programm.* 104 (2005), 501–523.

[30] A. Iusem and A. Seeger. Measuring the degree of pointedness of a closed convex cone: a metric approach. *Math. Nachrichten* 279 (2006), 599–618.

[31] A. Iusem and A. Seeger. Angular analysis of two classes of non-polyhedral convex cones: the point of view of optimization theory. *Comput. Appl. Math.* 26 (2007), 191–214.

[32] A. Iusem and A. Seeger. Normality and modulability indices. Part I: Convex cones in normed spaces. *J. Math. Anal. Appl.* 338 (2008), 365–391.

[33] A. Iusem and A. Seeger. Antipodal pairs, critical pairs, and Nash angular equilibria in convex cones. *Optim. Meth. Software*, 23 (2008), 73–93.

[34] A. Iusem and A. Seeger. Searching for critical angles in a convex cone. *Math. Program.* 120 (2009), 3–25.

[35] S.N. Lee and M.H. Shih. A volume problem for an n-dimensional ellipsoid intersecting with a hyperplane. *Linear Algebra Appl.* 132 (1990), 93-102.

[36] Y. Lyubich. Perron-Frobenius theory for finite-dimensional spaces with a hyperbolic cone. *Linear Algebra Appl.* 220 (1995), 283–309.

[37] J.J. Moreau. *Fonctionelles Convexes, Equations aux derivées partielles.* Seminaire J. Leray, Collège de France, 1966.

[38] A. Nemirovski, C. Roos, and T. Terlaky. On maximization of quadratic form over intersection of ellipsoids with common center. *Math. Program.* 86 (1999), 463–473.

[39] G. P. Papavassilopoulos and M. G. Safonov. The diameter of an intersection of ellipsoids and BMI robust synthesis. Proceedings of the IFAC Symposium on Robust Control Design, Rio de Janeiro, Brazil, pp. 313-317, 1994.

[40] R.T. Rockafellar. *Convex Analysis*. Princeton Univ. Press, Princeton, 1970.

[41] A. Seeger. Direct and inverse addition in convex analysis and applications. *J. Math. Anal. Appl.* 148 (1990), 317–349.

[42] A. Seeger. Epigraphical cones II. *J. Convex Anal.*, in press.

[43] A. Seeger and M. Volle. On a convolution operation obtained by adding level sets: classical and new results. *RAIRO Rech. Opér.* 29 (1995), 131–154.

[44] R.J. Stern and H. Wolkowicz. Exponential nonnegativity on the ice cream cone. *SIAM J. Matrix Anal. Appl.* 12 (1991), 160–165.

[45] R.J. Stern and H. Wolkowicz. Invariant ellipsoidal cones. *Linear Algebra Appl.* 150 (1991), 81–106.