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Epigraphical cones II

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Abstract. This is the second part of a work devoted to the theory of epigraphical cones and their applications. A convex cone K in the Euclidean space $\mathbb{R}^{n+1}$ is an epigraphical cone if it can be represented as epigraph of a nonnegative sublinear function $f : \mathbb{R}^n \to \mathbb{R}$. We explore the link between the geometric properties of K and the analytic properties of f.

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1 Introduction

This is the second part of a work initiated in [21] and whose aim is to survey the class of epigraphical cones. A convex cone in the Euclidean space $\mathbb{R}^{n+1}$ is an *epigraphical cone* if it can be represented as epigraph

$$\text{epi}f = \{(x,t) \in \mathbb{R}^{n+1} : f(x) \le t\}$$

of a nonnegative sublinear function $f : \mathbb{R}^n \to \mathbb{R}$. An epigraphical cone is always closed and *nontrivial*, i.e., different from the whole space and different from the zero cone. It is also *solid* in the sense that it has a nonempty topological interior. If K is an epigraphical cone in $\mathbb{R}^{n+1}$, then its associated nonnegative sublinear function is given by

$$f_K(x) = \min\{t \in \mathbb{R} : (x,t) \in K\}.$$

Any geometric statement on K can be formulated in terms of a corresponding analytic property of f_K. The reference [21] provides various examples of interesting epigraphical cones and explains how to combine them in order to produce new epigraphical cones. The next lemma is a bridge for passing from the class of epigraphical cones to the wider class of solid closed convex cones. We start by recalling a useful definition.

Definition 1.1. *Let $\mathcal{O}_d$ denote the group of orthogonal matrices of order d. Two convex cones K_1, K_2 in the Euclidean space $\mathbb{R}^d$ are* orthogonally equivalent *if there exists $U \in \mathcal{O}_d$ such that $K_2 = U(K_1)$.*

Lemma 1.2. *Let K be a nontrivial closed convex cone in $\mathbb{R}^{n+1}$. Then K is solid if and only if there exist $U \in \mathcal{O}_{n+1}$ and a nonnegative sublinear function $f : \mathbb{R}^n \to \mathbb{R}$ such that*

$$K = U(\text{epi} f). \tag{1}$$

Proof. That a solid cone K admits the representation (1) has been proven in [10, Proposition 2.8]. Conversely, that a cone of the form (1) is solid is a consequence of [21, Proposition 2.1] and the fact that solidity is a property that is invariant under orthogonal transformations. □

Example 1.3. The nonnegative orthant $\mathbb{R}^{n+1}_+$ is a solid closed convex cone in $\mathbb{R}^{n+1}$. It is not an epigraphical cone, but it is orthogonally equivalent to an epigraphical cone. Consider the orthogonal matrix

$$U = [u_1, \ldots, u_n, u_{n+1}]$$

constructed as follows: the columns of the submatrix $\tilde{U} = [u_1, \ldots, u_n]$ form an orthonormal basis of the linear subspace

$$\mathbb{L}_n = \{v \in \mathbb{R}^{n+1} : v_1 + \ldots + v_{n+1} = 0\}$$

and $u_{n+1} = (n+1)^{-1/2}\,(1, \ldots, 1)^T$ is a unit vector orthogonal to $\mathbb{L}_n$. One can check that

$$\mathbb{R}^{n+1}_+ = U(\text{epi} f),$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is the nonnegative sublinear function given by

$$f(x) = \max_{1 \le i \le n+1} -\sqrt{n+1}\,(\tilde{U}x)_i$$

and $(\tilde{U}x)_i$ denotes the i-th component of $\tilde{U}x$. One sees that f is nonnegative because $\tilde{U}x \in \mathbb{L}_n$ for all $x \in \mathbb{R}^n$. The set

$$\text{epi} f = \{(x,t) \in \mathbb{R}^{n+1} : (\tilde{U}x)_i + (n+1)^{-1/2} t \ge 0 \text{ for all } i = 1, \ldots, n+1\}$$

looks more involved than $\mathbb{R}^{n+1}_+$, but it has the merit of being an epigraphical cone.

The equality (1) expresses the fact that K is orthogonally equivalent to an epigraphical cone. The interest of having such a representation formula for K is clear: when it comes to study the structure of K, everything boils down to examining the function f. This observation is the leading motivation behind the present work. The organization of the paper is as follows:

- Section 2 concerns the facial structure of an epigraphical cone.
- Section 3 establishes a few results about rotundity and smoothness of epigraphical cones.
- Section 4 exploits the theory of epigraphical cones for obtaining some approximation results involving Painlevé-Kuratowski limits.
- Section 5 provides rules for computing the characteristic function of an epigraphical cone. The concept of characteristic function of a cone is understood in the sense of Vinberg [24].
- Section 6 deals with the application of epigraphical cones in optimization theory.

We keep the same notation and terminology as in [21]. In particular, $\text{int}(\Omega)$, $\text{bd}(\Omega)$, $\text{cl}(\Omega)$ indicate respectively the interior, boundary, and closure of a set Ω. The unit sphere and the closed unit ball of $\mathbb{R}^d$ are denoted by $\mathbb{S}_d$ and $\mathbb{B}_d$, respectively. However, we deviate from the general spirit of [21]. The emphasis now is put in the study of properties that are valid up to orthogonal characterizations, which allows to consider a class of convex cones larger than the class of epigraphical cones.

2 Facial analysis of epigraphical cones

A *face* of a nonempty closed convex set Ω in an Euclidean space is a nonempty subset F of Ω satisfying the following property:

$$(1-\lambda)x + \lambda y \in F \text{ with } x, y \in \Omega \text{ and } \lambda \in]0,1[\quad \Longrightarrow \quad x, y \in F.$$

A face is necessarily closed and convex. If Ω is a closed convex cone, then so is every face of Ω. The next theorem tells how to identify the faces of an epigraphical cone. For the sake of clarity in the exposition, we assume that the associated nonnegative sublinear function $f : \mathbb{R}^n \to \mathbb{R}$ is *positive*, i.e., it vanishes only at the origin 0_n. In such a case

$$B_f := \{x \in \mathbb{R}^n : f(x) \leq 1\}$$

is a compact convex set.

Theorem 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positive sublinear function and $p \in \{0, 1, \ldots, n\}$. Then*

$$F \mapsto \Psi(F) = \mathbb{R}_+ (F \times \{1\})$$

is a bijection between the set of p - dimensional faces of B_f and the set of $(p+1)$ - dimensional faces of epi$f$.

Proof. Let F be a face of B_f. We claim that $\Psi(F)$ is a face of epi$f$. Pick (x, t) and (y, s) in epi$f$ and $\lambda \in]0, 1[$ such that

$$(1-\lambda)(x,t) + \lambda(y,s) \in \Psi(F). \tag{2}$$

One may suppose that $t \neq 0$, otherwise $(x, t) = 0_{n+1}$ and we are done. Similarly, one may suppose that $s \neq 0$. The condition (2) says that

$$\begin{aligned} (1-\lambda)x + \lambda y &= \alpha z \\ (1-\lambda)t + \lambda s &= \alpha \end{aligned}$$

with $\alpha \geq 0$ and $z \in F$. Hence, $\alpha \neq 0$ and

$$z = \left[\frac{(1-\lambda)t}{(1-\lambda)t + \lambda s}\right] \frac{x}{t} + \left[\frac{\lambda s}{(1-\lambda)t + \lambda s}\right] \frac{y}{s}$$

is a convex combination of two vectors in B_f. Since F is a face of B_f, it follows that x/t and y/s are in F. This proves that (x, t) and (y, s) are in $\Psi(F)$ and completes the proof of our claim. Conversely, let G be a nonzero face of epi$f$. There exists a compact convex set F in $\mathbb{R}^n$ such that $\Psi(F) = G$. Such F is unique and given by

$$F = \{x \in \mathbb{R}^n : (x, 1) \in G\}. \tag{3}$$

We claim that (3) is a face of B_f. Pick $x, y \in B_f$ and $\lambda \in]0, 1[$ such that $(1-\lambda)x + \lambda y \in F$. Hence,

$$(1-\lambda)(x, 1) + \lambda(y, 1) \in G.$$

Since G is a face of epi$f$, it follows that $(x, 1)$ and $(y, 1)$ are in G. Therefore, x and y are in F. This proves that F is a face of B_f. For completing the proof of the theorem we check that

$$\dim[\Psi(F)] = \dim(F) + 1. \tag{4}$$

The relation (4) is probably known since it holds for any nonempty compact convex set F in $\mathbb{R}^n$, and not just for a face of B_f. Let p be the dimension of F. Then,

$$F \subset u_0 + \{u_1, \ldots, u_p\}$$

for suitable vectors $\{u_k\}_{k=0}^p$ in $\mathbb{R}^n$. Hence, any element of $\Psi(F)$ can be expressed in the form

$$(x,t) \ = \ t\left(u_0 + \sum_{k=1}^{p} \lambda_k u_k, 1\right) \ = \ t\,(u_0, 1) + \sum_{k=1}^{p} t\lambda_k\,(u_k, 0)$$

with $t \geq 0$ and $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$. Thus,

$$\Psi(F) \subset \mathrm{span}\{(u_0, 1)\,, (u_1, 0)\,, \ldots, (u_p, 0)\}.$$

This shows that the dimension of $\Psi(F)$ is at most $p+1$. On the other hand, since $\dim(F) = p$, it is possible to find vectors $\{v_0, v_1, \ldots, v_p\}$ in F such that $\{(v_0, 1), (v_1, 1), \ldots, (v_p, 1)\}$ are linearly independent. Hence, $\Psi(F)$ contains $p+1$ linearly independent vectors, and therefore its dimension is at least $p+1$. This completes the proof of (4). □

Remark. The case $p = 0$ is of special interest because it concerns the identification of the extreme rays of an epigraphical cone. This theme has been treated already in [21, Corollary 2.2]. The case $p = n$ yields the well known formula

$$\mathrm{epi} f = \mathbb{R}_+ \,(B_f \times \{1\}) \tag{5}$$

for the epigraph of a positive sublinear function.

By combining Theorem 2.1 and the next lemma one can identify the faces of a proper cone, not necessarily an epigraphical one. As many authors do, we say that a closed convex cone is *proper* if it solid and pointed.

Lemma 2.2. *For a closed convex cone K in $\mathbb{R}^{n+1}$ the following statements are equivalent:*

(a) K is proper.

(b) There exists $v \in \mathrm{int}(K)$ such that $\langle v, z\rangle > 0$ for all nonzero $z \in K$.

(c) K is orthogonally equivalent to the epigraph of a positive sublinear function $f : \mathbb{R}^n \to \mathbb{R}$.

Proof. (a) ⇒ (b). Let K^+ denote the dual cone of K. Since

$$K \cap -K^+ = \{0_{n+1}\}\,, \tag{6}$$

the convex cone $P = K + K^+$ is closed. We claim that P is pointed. To see this, take $a \in P \cap -P$ and write $a = b_1 + c_1 = -(b_2 + c_2)$ with $b_1, b_2 \in K$ and $c_1, c_2 \in K^+$. Hence,

$$K \ni (b_1 + b_2) \ = \ -(c_1 + c_2) \in -K^+.$$

In view of (6), one gets $b_1 = -b_2$ and $c_1 = -c_2$. Since K and K^+ are pointed, it follows that $b_1 = b_2 = c_1 = c_2 = 0_{n+1}$. This yields $a = 0_{n+1}$ and proves the pointedness of P. As a consequence,

$$P^+ = (K + K^+)^+ = K^+ \cap K$$

is solid. To see that (b) holds, one just needs to pick any v from the interior of P^+.
(b) $\Rightarrow$ (c). Let $v \in \mathrm{int}(K)$ be such that

$$\langle v, z\rangle > 0 \quad \text{for all } z \in K\backslash\{0_{n+1}\}. \tag{7}$$

Without loss of generality one may assume that $\|v\| = 1$. Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis for the linear subspace

$$v^\perp = \{z \in \mathbb{R}^{n+1} : \langle v, z\rangle = 0\}.$$

Then $U = [u_1, \ldots, u_n, v] \in \mathcal{O}_{n+1}$ and

$$U^T v = e_{n+1} := (0_n, 1)$$

belongs to the interior of $Q = U^T(K)$. In view of (7), the closed convex cone Q is contained in the half-space $\mathbb{R}^n \times \mathbb{R}_+$. Hence, Q is the epigraph of the nonnegative sublinear function

$$x \in \mathbb{R}^n \mapsto f_Q(x) = \min\{t \in \mathbb{R} : (x,t) \in Q\}.$$

It remains to check that f_Q is positive. Suppose, on the contrary, that f_Q vanishes at some nonzero vector x. Hence, the linear combination $z = \sum_{i=1}^n x_i u_i$ is a nonzero vector in K such that

$$\langle v, z\rangle = \sum_{i=1}^n x_i \langle v, u_i\rangle = 0,$$

contradicting (7). Summarizing, we have shown that K is orthogonally equivalent to the epigraph of the positive sublinear function f_Q.
(c) $\Rightarrow$ (a). It follows from [21, Proposition 2.1] and the fact that properness is invariant under orthogonal transformations. $\square$

3 Rotundity and smoothness

Rotundity and smoothness are fundamental properties concerning the unit ball of a normed space. Such notions can be extended to general convex bodies and even to proper cones. The next definition can be found for instance in [23].

Definition 3.1. *A proper cone K in an Euclidean space is* rotund *if every face of K, other than K itself and $\{0\}$, is a half-line (called an* extreme ray*).*

Rotund cones are often times referred to as strictly convex cones because they are characterized by the strict convexity condition

$$z, v \in K \text{ not collinear},\ \lambda \in]0,1[\quad \Longrightarrow \quad (1-\lambda)z + \lambda v \in \mathrm{int}(K).$$

Rotund cones play an important role in mathematical economics [18] and other fields [8, 20]. A nice example of rotund cone is the elliptic cone

$$\mathcal{E}(Q) := \{(x,t) \in \mathbb{R}^{n+1} : \sqrt{\langle x, Qx\rangle} \leq t\}$$

associated to a positive definite symmetric matrix Q of order n. Also the ℓ^p-cone

$$\mathcal{K}_p \quad = \quad \{(x,t) \in \mathbb{R}^{n+1} : \|x\|_p \leq t\}$$

is rotund if one takes $p \in]1, \infty[$. By contrast, the choices $p = 1$ or $p = \infty$ lead to proper cones that are not rotund. All this can be explained in a clear-cut manner with the help of the next proposition.

Proposition 3.2. *For a positive sublinear function $f : \mathbb{R}^n \to \mathbb{R}$ the following statements are equivalent:*

(a) $\mathrm{epi} f$ *is rotund.*

(b) f *is rotund in the sense that* $f(x) = 1, f(y) = 1,\ x \neq y$ *imply* $f(x + y) < 2$.

(c) The boundary of B_f *contains no line-segment.*

Proof. (a) $\Leftrightarrow$ (c) is a consequence of Theorem 2.1 and (c) $\Leftrightarrow$ (b) is obvious. □

Example 3.3. The intersection of two elliptic cones is a rotund epigraphical cone. Indeed, if Q_1 and Q_2 are positive definite symmetric matrices of order n, then the intersection $\mathcal{E}(Q_1) \cap \mathcal{E}(Q_2)$ is an epigraphical cone whose associated nonnegative sublinear function

$$x \in \mathbb{R}^n \mapsto f(x) = \max\left\{ \sqrt{\langle x, Q_1 x\rangle}, \sqrt{\langle x, Q_2 x\rangle} \right\}$$

is positive and rotund.

The next corollary is helpful when it comes to check the rotundity of a proper cone that is not necessarily an epigraphical cone.

Corollary 3.4. *A proper cone K in $\mathbb{R}^{n+1}$ is rotund if and only if it is orthogonally equivalent to the epigraph of a rotund positive sublinear function $f : \mathbb{R}^n \to \mathbb{R}$.*

Proof. It is a matter of combining Lemma 2.2 and Proposition 3.2. A key observation is that the notion of rotundity for proper cones is invariant under orthogonal transformations. □

Remark. The notion of local uniform rotundity is one of the main topics in renorming theory [5, 15]. It has been traditionally considered only as a matter for norms. However, the definition can be easily generalized for positive sublinear functions. This opens the way to the definition of local uniform rotundity for proper cones. For instance, one may declare a proper cone K in $\mathbb{R}^{n+1}$ to be locally uniformly rotund if it is orthogonally equivalent to the epigraph of a locally uniformly rotund positive sublinear function $f : \mathbb{R}^n \to \mathbb{R}$.

There is no universally accepted definition of smoothness for proper cones. The notion of smoothness that we adopt here is not that of [14], but one that is dual to rotundity.

Definition 3.5. *A proper cone K in an Euclidean space is* smooth *if its dual cone K^+ is rotund.*

The most bothering aspect of Definition 3.5 is the need of knowing K^+, a cone that is not always directly available. Anyhow, the next proposition clarifies the geometric meaning of smoothness. As one can see, it is possible to check whether an epigraphical cone is smooth without evaluating its dual cone.

Proposition 3.6. *For a positive sublinear function $f : \mathbb{R}^n \to \mathbb{R}$ the following statements are equivalent:*

(a) $\mathrm{epi} f$ *is smooth.*

(b) f *is smooth, i.e., differentiable on* $\mathbb{R}^n \backslash \{0_n\}$.

(c) Every boundary point of B_f admits exactly one supporting hyperplane.

Proof. (a) $\Rightarrow$ (b). By assumption, the dual cone of epi$f$ is rotund. But

$$(\mathrm{epi} f)^+ = \mathrm{epi} f^\diamond \tag{8}$$

with $f^\diamond : \mathbb{R}^n \to \mathbb{R}$ denoting the skewed polar of f (cf. [21, Section 2.3]). Recall that $f^\diamond(y) = f^\circ(-y)$, where f° is the usual polar function. By Proposition 3.2, the positive sublinear function $f^\diamond$ is rotund. Hence, f° is rotund. By a classical duality argument,

$$x \in \mathbb{R}^n \to f(x) \;=\; (f^\circ)^\circ(x) \;=\; \max_{y \neq 0_n} \frac{\langle y, x\rangle}{f^\circ(y)}$$

is then differentiable on $\mathbb{R}^n \backslash \{0_n\}$. In fact, by using Danskin's differentiability theorem [4] one sees that the gradient of f at a given point $x \in \mathbb{R}^n \backslash \{0_n\}$ is the unique solution to the maximization problem

$$f(x) = \max_{y \in B_{f^\circ}} \langle y, x\rangle.$$

(b) $\Rightarrow$ (c). This implication is easy and well known.
(c) $\Rightarrow$ (a). The theory of convex bodies asserts that if each boundary point of B_f has a unique supporting hyperplane, then the boundary of the polar set

$$(B_f)^\circ = \{y \in \mathbb{R}^n : \langle y, x\rangle \leq 1 \;\text{ for all } x \in B_f\}$$

contains no line-segment. Since $(B_f)^\circ = B_{f^\circ}$, it follows that f° is rotund. A posteriori, also $f^\diamond$ is rotund. By combining (8) and Proposition 3.2, one sees that epi$f$ is smooth. □

Example 3.7. The Minkowski sum of two elliptic cones is a smooth epigraphical cone. Indeed, if Q_1 and Q_2 are positive definite symmetric matrices of order n, then $\mathcal{E}(Q_1)+\mathcal{E}(Q_2)$ is an epigraphical cone whose associated nonnegative sublinear function

$$x \in \mathbb{R}^n \mapsto f(x) = \min_{u+v=x} \left\{ \sqrt{\langle u, Q_1 u\rangle} + \sqrt{\langle v, Q_2 v\rangle} \right\}$$

is positive and smooth. Also the vertical sum

$$\mathcal{E}(Q_1) \oplus_{\mathrm{v}} \mathcal{E}(Q_2) = \left\{ (x,t) \in \mathbb{R}^{n+1} : \sqrt{\langle x, Q_1 x\rangle} + \sqrt{\langle x, Q_2 x\rangle} \leq t \right\}$$

is a smooth epigraphical cone, but the intersection $\mathcal{E}(Q_1) \cap \mathcal{E}(Q_2)$ may not be smooth.

The next corollary fully characterizes the class of smooth cones.

Corollary 3.8. *A proper cone K in $\mathbb{R}^{n+1}$ is smooth if and only if it is orthogonally equivalent to the epigraph of a smooth positive sublinear function $f : \mathbb{R}^n \to \mathbb{R}$.*

Proof. Combine Lemma 2.2, Proposition 3.6, and the fact that the notion of smoothness for proper cones is invariant under orthogonal transformations. □

4 Approximation results

An epigraphical cone is not necessarily pointed, but it can always be approximated in the Painlevé-Kuratowski sense by a sequence of pointed epigraphical cones. This is the idea behind the formulation of the next lemma. In the sequel, the notation $\lim_{k\to\infty} C_k$ stands for the Painlevé-Kuratowski limit of a sequence $\{C_k\}_{k\in\mathbb{N}}$ of nonempty sets in an Euclidean space. The definition and main properties of Painlevé-Kuratowski limits can be consulted in standard books on set convergence [1, 19]. Since we are working in a finite dimensional setting, convergence in the Painlevé-Kuratowski sense is equivalent to convergence with respect to the uniform metric

$$\varrho(K_1, K_2) := \max_{\|z\|=1} |\mathrm{dist}[z, K_1] - \mathrm{dist}[z, K_2]|\,,$$

or with respect to any other equivalent metric for that matter (cf. [12]).

Lemma 4.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative sublinear function. Then*

$$\mathrm{epi} f \;=\; \lim_{k\to\infty} \mathrm{epi} f_k \;=\; \mathrm{cl}\left[\bigcup_{k=1}^{\infty} \mathrm{epi} f_k\right] \tag{9}$$

where $f_k : \mathbb{R}^n \to \mathbb{R}$ is the positive sublinear function given by $f_k(x) = \max\{f(x), k^{-1}\|x\|\}$.

Proof. That f_k is a positive sublinear function is clear. The sequence $\{f_k\}_{k\geq 1}$ is pointwisely nonincreasing and

$$f(x) = \lim_{k\to\infty} f_k(x) = \inf_{k\geq 1} f_k(x)$$

for all $x \in \mathbb{R}^n$. This implies the Painlevé-Kuratowski convergence of the sequence $\{\mathrm{epi} f_k\}_{k\in\mathbb{N}}$ and the formulas stated in (9). □

The geometric interpretation of (9) is as follows: what we are doing is to approximate $\mathrm{epi} f$ by another epigraphical cone that is smaller, namely $\mathrm{epi} f_k = (\mathrm{epi} f) \cap \mathcal{R}_k$. Here

$$\mathcal{R}_k := \{(x, t) \in \mathbb{R}^{n+1} : \|x\| \leq kt\}$$

is a revolution cone whose axis is generated by e_{n+1}. The half-aperture angle of $\mathcal{R}_k$ is equal to

$$\theta_k = \arccos\left(1/\sqrt{1 + k^2}\right),$$

i.e., a positive number smaller than $\pi/2$. Note that $\mathrm{epi} f_k$ is pointed because $\mathcal{R}_k$ is pointed.

The next approximation result can be found in [22]. We give here a shorter proof that is based on the theory of epigraphical cones.

Theorem 4.2. *Let K be a closed convex cone in an Euclidean space X. Suppose that K is not a sublinear space. Then*

- *(a) There exists an upward monotonic sequence $\{Q_k\}_{k\in\mathbb{N}}$ of pointed closed convex cones in X such that $K = \lim_{k\to\infty} Q_k$.*
- *(b) There exists a downward monotonic sequence $\{P_k\}_{k\in\mathbb{N}}$ of solid closed convex cones in X such that $K = \lim_{k\to\infty} P_k$.*

Proof. Since X can be taken as the linear space spanned by K, there is no loss of generality in assuming that K is solid. To avoid trivialities we assume also that $\dim X \geq 2$. If one sets $n = \dim X - 1$, then one can identify X with $\mathbb{R}^{n+1}$. In view of [10, Proposition 2.8], there exist $U \in \mathcal{O}_{n+1}$ and a nonnegative sublinear function $f : \mathbb{R}^n \to \mathbb{R}$ such that $K = U(\text{epi} f)$. Hence, one can approximate K by means of the pointed closed convex cone

$$Q_k = U(\text{epi} f_k)$$

with f_k as in Lemma 4.1. Note that $Q_k \subset Q_{k+1}$ for all $k \in \mathbb{N}$, and

$$\lim_{k\to\infty} Q_k \;=\; U\left(\lim_{k\to\infty} \text{epi} f_k\right) \;=\; U(\text{epi} f) \;=\; K.$$

Consider now the part (b). Since K^+ is not a linear subspace either, there exists an upward monotonic sequence $\{W_k\}_{k\in\mathbb{N}}$ of pointed closed convex cones in X such that $K^+ = \lim_{k\to\infty} W_k$. If one sets $P_k = W_k^+$, then $\{P_k\}_{k\in\mathbb{N}}$ is a downward monotonic sequence of solid closed convex cones in X such that

$$\lim_{k\to\infty} P_k \;=\; \left(\lim_{k\to\infty} W_k\right)^+ \;=\; (K^+)^+ \;=\; K,$$

the leftmost equality being a consequence of the Walkup-Wets isometry theorem [25]. □

Another approximation result in the same spirit concerns the possibility of approximating a closed convex cone by a rotund cone or by a smooth cone.

Theorem 4.3. *Let K be a closed convex cone in an Euclidean space X. Suppose that K is not a sublinear space. Then*

(a) $K = \lim_{k\to\infty} R_k$ for some upward monotonic sequence $\{R_k\}_{k\in\mathbb{N}}$ of rotund cones in X.

(b) $K = \lim_{k\to\infty} S_k$ for some downward monotonic sequence $\{S_k\}_{k\in\mathbb{N}}$ of smooth cones in X.

Proof. Without loss of generality one can assume that K is solid and that $\dim X \geq 2$. As before, we set $n = \dim X - 1$ and identify X with $\mathbb{R}^{n+1}$. Then we write

$$\begin{aligned} K &= U(\text{epi} f) \\ R_k &= U(\text{epi}\, g_k), \end{aligned}$$

where U and f are as in the proof of Theorem 4.2, and

$$g_k(x) = f(x) + k^{-1}\|x\|.$$

Note that $R_k \subset R_{k+1}$ for all $k \in \mathbb{N}$, and

$$\lim_{k\to\infty} R_k \;=\; U\left(\lim_{k\to\infty} \text{epi}\, g_k\right) \;=\; U(\text{epi} f) \;=\; K.$$

Each R_k is pointed because the sublinear function g_k is positive. Since K is solid, R_k is solid (hence, proper) for all k large enough (cf. [11, Corollary 5.2]). It remains to check that R_k is rotund, but this is a consequence of Proposition 3.2 and the fact that g_k is rotund. The part (b) is obtained from (a) by relying on duality arguments. □

Theorem 4.4 says that any closed convex cone can be approximated in the Painlevé-Kuratowski sense by a polyhedral cone.

Theorem 4.4. *Any closed convex cone in an Euclidean space X can be written as Painlevé-Kuratowski limit of a sequence of polyhedral cones in X.*

Proof. Let K be a closed convex cone in X. We suppose that K is not a linear subspace, otherwise we are done. Since one can take X as the linear space spanned by K, there is no loss of generality in assuming that K is solid. As in Theorem 4.2, one assumes that $\dim X \geq 2$, one sets $n = \dim X - 1$, and one identifies X with $\mathbb{R}^{n+1}$. In view of Theorem 4.2, for all $k \in \mathbb{N}$ there exists a pointed closed convex cone Q_k in $\mathbb{R}^{n+1}$ such that

$$\varrho(Q_k, K) \leq 1/k.$$

We represent $Q_k = U_k(\mathrm{epi} f_k)$ in terms of a suitable matrix $U_k \in \mathcal{O}_{n+1}$ and a corresponding positive sublinear function $f_k : \mathbb{R}^n \to \mathbb{R}$. As in (5), one has

$$\mathrm{epi} f_k = \mathbb{R}_+ \left(B_{f_k} \times \{1\} \right)$$

Note that B_{f_k} is a nonempty compact convex set in $\mathbb{R}^n$. Thanks to [17, Theorem 2.1], there exists a polytope Ω_k in $\mathbb{R}^n$ such that

$$\mathrm{haus}(\Omega_k, B_{f_k}) \leq 1/k$$

with "haus" standing for the classical Pompeiu-Hausdorff metric. Since $0_n \in \mathrm{int}(B_{f_k})$, one may suppose that $0_n \in \mathrm{int}(\Omega_k)$. Consider now the polyhedral cone

$$P_k = \mathbb{R}_+ \left(\Omega_k \times \{1\} \right).$$

Thanks to [12, Proposition 6.3], one has the estimate

$$\varrho(P_k, \mathrm{epi} f_k) \leq 2\, \mathrm{haus}(\Omega_k, B_{f_k}).$$

Observe that $U_k(P_k)$ is a polyhedral cone and

$$\begin{aligned} \varrho(U_k(P_k), K) &\leq \varrho(U_k(P_k), Q_k) + \varrho(Q_k, K) \\ &= \varrho\left(U_k(P_k), U_k(\mathrm{epi} f_k)\right) + \varrho(Q_k, K) \\ &= \varrho(P_k, \mathrm{epi} f_k) + \varrho(Q_k, K) \\ &\leq 2(1/k) + (1/k). \end{aligned}$$

This shows that $K = \lim_{k \to \infty} U_k(P_k)$ and completes the proof of the theorem. □

5 Characteristic function of an epigraphical cone

If K is a solid closed convex cone in $\mathbb{R}^d$, then its *characteristic function* $\Phi_K : \mathrm{int}(K) \to \mathbb{R}$ is defined by the d-dimensional integral

$$\Phi_K(z) := \int_{K^+} e^{-\langle z, v \rangle} dv.$$

Such a definition of characteristic function is discussed in the book [7, Chapter 1] and in many other places [6, 9, 24]. Some authors refer to Φ_K as the Vinberg (or Koszul-Vinberg) characteristic function of K.

It is clear that Φ_K is positively homogeneous of degree $-d$. As shown in [7, Proposition I.3.2], a fundamental property of Φ_K is that of behaving as *barrier function* for the cone K. This means that

$$\lim_{k\to\infty} \Phi_K(z_k) = \infty \tag{10}$$

for any sequence $\{z_k\}_{k\in\mathbb{N}}$ in $\mathrm{int}(K)$ converging to a point on the boundary of K. As can be seen from the next lemma, there is also a converse statement: the condition (10) forces $\{z_k\}_{k\in\mathbb{N}}$ to approach the boundary of K.

Lemma 5.1. *Let K is a solid closed convex cone in $\mathbb{R}^d$. Then for all $z \in \mathrm{int}(K)$, one has*

$$\Phi_K(z) \leq \left(\frac{1}{\mathrm{dist}[z, \mathrm{bd}(K)]}\right)^d \int_{K^+} e^{-\|v\|} dv.$$

Proof. The point z is at positive distance from $\mathrm{bd}(K)$. If r is such distance, then $z + r\mathbb{B}_d \subset K$. This inclusion amounts to saying that $r\,\|v\| \leq \langle z, v\rangle$ for all $v \in K^+$. Hence

$$\Phi_K(z) \leq \int_{K^+} e^{-r\|v\|} dv.$$

By using a positive homogeneity argument and a suitable change of variables, the term r moves out of the integral as a factor r^{-d}. □

We now address the question of computing the characteristic function of an epigraphical cone. In the sequel one uses the notation

$$\Xi_{f^\circ}(x, s) := \int_{\{f^\circ \leq s\}} e^{\langle x,y\rangle} dy, \tag{11}$$

where integration is carried out over the sublevel set $\{f^\circ \leq s\} = \{y \in \mathbb{R}^n : f^\circ(y) \leq s\}$.

Theorem 5.2. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a positive sublinear function, then*

$$\Phi_{\mathrm{epi} f}(x, t) = \frac{1}{t}\int_{\mathbb{R}^n} e^{\langle x,y\rangle - t f^\circ(y)} dy \tag{12}$$

$$= \int_0^\infty e^{-ts}\, \Xi_{f^\circ}(x, s) ds \tag{13}$$

for all $(x, t) \in \mathrm{int}(\mathrm{epi} f)$.

Proof. Let (x, t) be in the interior of E_f. Then

$$\Phi_{\mathrm{epi} f}(x, t) = \int_{\mathrm{epi} f^\diamond} e^{-(\langle x,w\rangle + ts)} dw ds \tag{14}$$

$$= \int_{\mathbb{R}^n} \left[\int_{f^\diamond(w)}^\infty e^{-\langle x,w\rangle - ts} ds\right] dw \tag{15}$$

$$= \int_{\mathbb{R}^n} \left[e^{-\langle x,w\rangle}\, \frac{e^{-t f^\diamond(w)}}{t}\right] dw,$$

where (14) is a consequence of (8), and (15) is due to Fubini's integration theorem. The change of variables $y = -w$ leads finally to the formula (12). In order to obtain (13) one integrates (14) in a different order. One has

$$\begin{aligned}\Phi_{\mathrm{epi} f}(x,t) &= \int_0^\infty \left[\int_{\{f^\diamond \le s\}} e^{-(\langle x,w\rangle + ts)} dw\right] ds \\ &= \int_0^\infty e^{-ts} \left[\int_{\{f^\diamond \le s\}} e^{-\langle x,w\rangle} dw\right] ds.\end{aligned}$$

It suffices now to observe that the inner integral is equal to $\Xi_{f^\circ}(x,s)$. □

As one can see from (13), the function $\Phi_{\mathrm{epi} f}(x,\,\cdot\,)$ is nothing but the standard Laplace transform of $\Xi_{f^\circ}(x,\,\cdot\,)$. It is not always easy to compute the multidimensional integral (11), not to mention its Laplace transform, but there are some particular cases were this can be done explicitly. The next example is inspired from [9, Section 7.4].

Example 5.3. Consider the positive sublinear function $f(x) = \|x\|_1$, in which case $f^\circ(y) = \|y\|_\infty$. One start by computing

$$\Xi_{f^\circ}(x,s) \;=\; \int_{\|y\|_\infty \le s} e^{\langle x,y\rangle} dy \;=\; \prod_{i=1}^n \int_{-s}^{s} e^{-x_i y_i} dy_i \;=\; \prod_{i=1}^n \sigma(x_i,s).$$

Here

$$\sigma(\tau,s) = \begin{cases} \left(e^{\tau s} - e^{-\tau s}\right)/\tau & \text{if} \quad \tau \neq 0 \\ 2s & \text{if} \quad \tau = 0. \end{cases}$$

Then one needs to evaluate the Laplace transform at t of the function $\prod_{i=1}^n \sigma(x_i,\cdot)$. Of course, one supposes that $\|x\|_1 < t$. For each $i \in \{1,\ldots,n\}$, one has to distinguish between the cases $x_i \neq 0$ and $x_i = 0$. Consider for instance the configuration $x_1 > 0, \ldots, x_n > 0$. Since,

$$\prod_{i=1}^n \left(e^{x_i s} - e^{-x_i s}\right) = \sum_{\varepsilon_i = \pm 1} \left(\prod_{i=1}^n \varepsilon_i\right) e^{\sum_{i=1}^n s\varepsilon_i x_i},$$

one ends up with

$$\Phi_{\mathrm{epi} f}(x,t) = \left[\prod_{i=1}^n x_i\right]^{-1} \sum_{\varepsilon_i = \pm 1} \frac{\prod_{i=1}^n \varepsilon_i}{t - \sum_{i=1}^n \varepsilon_i x_i}\,.$$

5.1 Moment-generating function techniques

Since f° is positively homogeneous, a simple change of variables in the integral (12) leads to

$$\Phi_{\mathrm{epi} f}(x,t) = \frac{1}{t^{n+1}} \int_{\mathbb{R}^n} e^{\langle t^{-1}x,\,y\rangle - f^\circ(y)} dy\,,$$

which in turn implies that

$$\Phi_{\mathrm{epi} f}(x,t) = \frac{1}{t^{n+1}}\, \Phi_{\mathrm{epi} f}\left(\frac{x}{t}, 1\right). \tag{16}$$

In other words, one can always restrict the attention to the particular case $t = 1$. By the way, the relation (16) is consistent with the fact that $\Phi_{\mathrm{epi} f}$ is positively homogeneous of degree $-(n+1)$.

Example 5.4. Let $f(x) = \|x\|_\infty$, in which case $f^\circ(y) = \|y\|_1$. A matter of computation shows that

$$\Phi_{\mathrm{epi}f}(x,1) \;=\; \int_{\mathbb{R}^n} e^{\langle x,\,y\rangle} e^{-\|y\|_1}\, dy \;=\; \prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{x_i\tau - |\tau|}\, d\tau \;=\; \prod_{i=1}^{n} \frac{2}{1-x_i^2}$$

whenever $\|x\|_\infty < 1$. In view of (16), one obtains

$$\Phi_{\mathrm{epi}f}(x,1) = \frac{2^n t^{n-1}}{\prod_{i=1}^{n}(t^2 - x_i^2)}$$

whenever $\|x\|_\infty < t$, recovering in this way a formula stated in [9, Lemma 7.3].

It is worthwhile to mention that

$$x \in \mathrm{int}(B_f) \mapsto M_f(x) \;:=\; \Phi_{\mathrm{epi}f}(x,1) \;=\; \int_{\mathbb{R}^n} e^{\langle x,\,y\rangle} e^{-f^\circ(y)}\, dy$$

can be seen as the moment-generating function of an n-dimensional random vector distributed according to the density function $y \mapsto e^{-f^\circ(y)}$. Strictly speaking, e^{-f° is a density function only up to a positive normalization factor. A density functions of the form ce^{-g}, for some constant c and some norm g, generalizes the classical Laplace density function. This way of perceiving M_f leads to a number of analytic results that are well known in probability theory. The next corollary is mentioned just by way of illustration.

Corollary 5.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positive sublinear function. Then*

(a) M_f is strictly logconvex and infinitely often differentiable on $\mathrm{int}(B_f)$.

(b) If f is even, then so is M_f.

Remark. Partial differentiation of $\Phi_{\mathrm{epi}f}$ with respect to x leads to

$$\nabla_x \Phi_{\mathrm{epi}f}(x,t) = \frac{1}{t} \int_{\mathbb{R}^n} y\, e^{\langle x,y\rangle - t f^\circ(y)} dy$$

for all $(x,t) \in \mathrm{int}(\mathrm{epi}f)$. In particular,

$$\nabla M_f(x) = \int_{\mathbb{R}^n} y\, e^{\langle x,y\rangle - f^\circ(y)} dy$$

for all $x \in \mathrm{int}(B_f)$. The above gradient has a special meaning if one sets $x = 0_n$. Indeed, the term

$$\nabla M_f(0_n) = \int_{\mathbb{R}^n} y\, e^{-f^\circ(y)} dy$$

corresponds to a mathematical expectation.

The next proposition provides a formula for computing the characteristic function of an elliptic cone. The moment-generating function

$$x \in \mathrm{int}\,(\mathbb{B}_n) \mapsto \mathcal{M}(x) \;=\; \int_{\mathbb{R}^n} e^{\langle x,\,y\rangle} e^{-\|y\|}\, dy$$

associated to $e^{-\|\cdot\|}$ is something intrinsic to the Euclidean space $\mathbb{R}^n$ and can be computed once and for all. One has

$$\mathcal{M}(x) = \frac{\kappa_n}{[1 - \|x\|^2]^{(n+1)/2}},$$

where the constant κ_n is given by

$$\kappa_n = \mathcal{M}(0_n) = \int_{\mathbb{R}^n} e^{-\|y\|}\, dy = \frac{\pi^{\frac{n}{2}}\, n!}{\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)}$$

with Γ standing for the Euler gamma function.

Proposition 5.6. *Let Q be a positive definite symmetric matrix of order n. Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis of eigenvectors of Q and let $\{\lambda_1, \ldots, \lambda_n\}$ be the corresponding eigenvalues. Then, for all $(x,t) \in \mathrm{int}[\mathcal{E}(Q)]$, one has*

$$\Phi_{\mathcal{E}(Q)}(x,t) = \frac{1}{t^{n+1}} \sqrt{\det Q}\, \mathcal{M}\left(\frac{L^T x}{t}\right) \tag{17}$$

$$= \frac{\kappa_n \sqrt{\det Q}}{[t^2 - \|L^T x\|^2]^{(n+1)/2}} \tag{18}$$

with L standing for the matrix of order n whose j-th column is the vector $\sqrt{\lambda_j}\, u_j$.

Proof. The polar of $f(x) = \sqrt{\langle x, Qx\rangle}$ is given by $f^\circ(y) = \sqrt{\langle y, Q^{-1}y\rangle}$. Hence,

$$\Phi_{\mathcal{E}(Q)}(x,1) = \int_{\mathbb{R}^n} e^{\langle x,y\rangle} e^{-\sqrt{\langle y, Q^{-1}y\rangle}}\, dy.$$

By using the spectral decomposition

$$Q = UDU^T = \sum_{j=1}^{n} \lambda_j\, u_j u_j^T$$

and the orthogonal transformation $\eta = U^T y$, one gets

$$\Phi_{\mathcal{E}(Q)}(x,1) = \int_{\mathbb{R}^n} e^{\langle U^T x, \eta\rangle} e^{-\sqrt{\langle \eta, D^{-1}\eta\rangle}}\, d\eta.$$

Finally, the change of variables $\xi_j = \eta_j / \sqrt{\lambda_j}$ leads to

$$\Phi_{\mathcal{E}(Q)}(x,1) = \sqrt{\lambda_1 \cdots \lambda_n} \int_{\mathbb{R}^n} e^{\langle L^T x,\, \xi\rangle} e^{-\|\xi\|}\, d\xi.$$

This completes the proof of the case $t = 1$. The formula (17) is then obtained by using the homogenization mechanism (16). □

Example 5.7. By taking $Q = I_n$ as the identity matrix of order n one recovers the well known formula (cf. [7, 9])

$$\Phi_\Lambda(x,t) = \frac{\kappa_n}{[t^2 - \|x\|^2]^{(n+1)/2}}$$

for the characteristic function of the $(n+1)$ - dimensional Lorentz cone

$$\Lambda := \{(x,t) \in \mathbb{R}^{n+1} : \|x\| \leq t\}.$$

More generally, the choice $Q = (\tan\vartheta)^{-2} I_n$ leads to the expression

$$\Phi_{\mathrm{rev}(\vartheta)}(x,t) = \frac{\kappa_n \tan\vartheta}{[(t\tan\vartheta)^2 - \|x\|^2]^{(n+1)/2}}$$

for the characteristic function of a revolution cone

$$\mathrm{rev}(\vartheta) := \{(x,t) \in \mathbb{R}^{n+1} : (\tan\vartheta)^{-1}\|x\| \ \leq t\}$$

with half-aperture angle equal to ϑ.

6 Epigraphical conic programming

An epigraphical conic program (ECP) is an optimization problem of the form

$$\begin{aligned} &\text{minimize } \langle c, z\rangle \\ &A_k z - b_k \in \mathrm{epi} f \quad \text{for all } k \in \{1,\ldots,N\} \end{aligned} \tag{19}$$

with $f : \mathbb{R}^n \to \mathbb{R}$ standing for a positive sublinear function. The decision variable z is a vector in some Euclidean space, say $\mathbb{R}^d$. The matrices $A_1,\ldots,A_N$ and the vectors $c, b_1,\ldots,b_N$ are known and have appropriate dimensions.

A dual problem to (19) can be constructed by using the standard Lagrangean formalism. If one introduces a Lagrangean function L of the form

$$L(z,\lambda_1,\ldots,\lambda_N) = \langle c, z\rangle + \sum_{k=1}^{N} \langle \lambda_k, b_k - A_k z\rangle,$$

then the dual problem to (19) consists in maximizing the term

$$\inf_{z\in\mathbb{R}^d} L(z,\lambda_1,\ldots,\lambda_N)$$

with respect to the Karush-Kuhn-Tucker multipliers $\lambda_1,\ldots,\lambda_N \in (\mathrm{epi} f)^+$. If one keeps (8) in mind, then a matter of simplification shows that the dual problem can be written in the form

$$\begin{aligned} &\text{maximize} \quad \sum_{k=1}^{N} \langle b_k, \lambda_k\rangle \\ &\lambda_1,\ldots,\lambda_N \in \mathrm{epi} f^{\diamond} \\ &\sum_{k=1}^{N} A_k^T \lambda_k = c. \end{aligned} \tag{20}$$

Let v^{primal} and v^{dual} denote the optimal values of (19) and (20), respectively. The general duality theory for convex optimization problems leads to the following result.

Lemma 6.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positive sublinear function. Then the equality*

$$v^{\text{primal}} = v^{\text{dual}}$$

holds under any of the following two qualification hypotheses:

i) There exists $\tilde{z} \in \mathbb{R}^d$ such that $A_k \tilde{z} - b_k \in \operatorname{int}(\operatorname{epi} f)$ for all $k \in \{1, \ldots, N\}$.

ii) f is polyhedral and (19) is feasible.

Proof. Under the assumption (ii), one can convert (19) and (20) into a pair of dual linear programs, one of which is feasible. Hence, the other problem is feasible as well, and both problems have the same optimal value. Under the Slater type qualification hypothesis (i), the dual problem (20) admits a solution and both problems have the same optimal value. This can be shown by using one of the many existing min-max theorems for convex-concave Lagrangeans, see for instance [16, Corollary 4.2]. □

6.1 First application: finding a Chebishev center

Consider a finite set $\Omega = \{\omega_1, \ldots, \omega_N\}$ of distinct points in $\mathbb{R}^n$. One wishes to find a vector $x \in \mathbb{R}^n$ that is regarded as the "center" of Ω. There are different ways of formalizing the concept of center. The Chebishev strategy consists in minimizing the maximal deviation function

$$x \in \mathbb{R}^n \mapsto \delta_{\max}(x) := \max_{1 \leq k \leq N} f(x - \omega_k)\,,$$

where f is a given norm on $\mathbb{R}^n$. By definition, a *Chebishev center* of Ω is a solution to the minimization problem

$$r_{\text{cheb}} = \min_{x \in \mathbb{R}^n} \delta_{\max}(x).$$

The number r_{cheb} is called the *Chebishev radius* of Ω. Since $\delta_{\max}$ is continuous and has bounded level sets, the existence of a Chebishev center is automatically guaranteed. On the other hand, minimizing $\delta_{\max}$ is clearly equivalent to solving

$$\begin{aligned}
&\text{minimize } t \\
&(x,t) \in \mathbb{R}^{n+1} \\
&f(x - \omega_k) \leq t \ \text{ for all } k \in \{1, \ldots, N\}.
\end{aligned}$$

This can be written of course in the conic programming format

$$\begin{aligned}
&\text{minimize } t \\
&(x,t) \in (\omega_k, 0) + \operatorname{epi} f \ \text{ for all } k \in \{1, \ldots, N\}.
\end{aligned} \tag{21}$$

Since (21) is a particular instance of the general ECP, one obtains the following conclusion.

Corollary 6.2. *Let f be a norm on $\mathbb{R}^n$. Then the Chebishev radius of Ω is equal to the optimal value of the maximization problem*

$$\begin{aligned}
&\textit{maximize } \ \sum_{k=1}^{N} \langle \omega_k, y_k \rangle \\
&(y_k, s_k) \in \operatorname{epi} f^{\circ} \ \textit{ for all } k \in \{1, \ldots, N\} \\
&\sum_{k=1}^{N} y_k = 0_n, \ \sum_{k=1}^{N} s_k = 1,
\end{aligned} \tag{22}$$

where the decision variables are the vectors $y_1, \ldots, y_N \in \mathbb{R}^n$ and the scalars $s_1, \ldots, s_N$.

Proof. For obtaining the dual problem (22), one just needs to work out the general model (20) with

$$z = \begin{bmatrix} x \\ t \end{bmatrix}, \quad c = \begin{bmatrix} 0_n \\ 1 \end{bmatrix}, \quad b_k = \begin{bmatrix} \omega_k \\ 0 \end{bmatrix}, \quad A_k = I_{n+1}.$$

For the sake of matrix calculus we are writing the vectors of $\mathbb{R}^{n+1}$ in column notation. The KKT multipliers are the $(n+1)$-dimensional vectors

$$\lambda_1 = \begin{bmatrix} y_1 \\ s_1 \end{bmatrix}, \ldots, \lambda_N = \begin{bmatrix} y_N \\ s_N \end{bmatrix},$$

which we identify with the pairs $(y_1, s_1), \ldots, (y_N, s_N)$. Note that the Slater type qualification condition (i) mentioned in Lemma 6.1 holds automatically in the present context. Indeed, if one picks any $\tilde{x} \in \mathbb{R}^n$ and $\tilde{t}$ bigger than $\max_{1 \le k \le N} f(\tilde{x} - \omega_k)$, then

$$A_k \tilde{z} - b_k = \begin{bmatrix} \tilde{x} \\ \tilde{t} \end{bmatrix} - \begin{bmatrix} \omega_k \\ 0 \end{bmatrix} \in \operatorname{int}(\operatorname{epi} f)$$

for all $k \in \{1, \ldots, N\}$. □

The maximization problem (22) is perhaps better understood if one writes it in the form

$$\begin{aligned} &\text{maximize} \quad \sum_{k=1}^N s_k \langle \omega_k, u_k \rangle \\ &s_k \ge 0, \ f^\circ(u_k) \le 1 \ \text{ for all } k \in \{1, \ldots, N\} \\ &\sum_{k=1}^N s_k u_k = 0_n, \ \sum_{k=1}^N s_k = 1, \end{aligned}$$

where the maximum is taken over all the representations of 0_n as convex combination of N vectors in the ball B_{f° associated to the polar norm f°.

6.2 Second application: finding a Fermat center

Instead of the maximal deviation function $\delta_{\max}$ one could use the total deviation function

$$x \in \mathbb{R}^n \mapsto \delta_{\text{sum}}(x) := \sum_{k=1}^N f(x - \omega_k).$$

The existence of a minimum of δ_{sum} is not problematic either. By definition, a *Fermat center* of Ω is a solution to the minimization problem

$$r_{\text{fermat}} = \min_{x \in \mathbb{R}^n} \delta_{\text{sum}}(x). \tag{23}$$

The above minimal value is called the *Fermat radius* of Ω. Following [3], we write the unconstrained problem (23) in the form

$$\begin{aligned} &\text{minimize} \ \sum_{k=1}^N \tau_k \\ &f(x - \omega_k) \le \tau_k \ \text{ for all } k \in \{1, \ldots, N\}, \end{aligned}$$

where the decision variables are the components of $x \in \mathbb{R}^n$ and the auxiliary variables $\tau_1, \ldots, \tau_N$. The latter problem can be reformulated as

$$\begin{aligned} &\text{minimize } \sum_{k=1}^{N} \tau_k \\ &(x, \tau_k) \in (\omega_k, 0) + \text{epi} f \ \text{ for all } k \in \{1, \ldots, N\}, \end{aligned} \tag{24}$$

so we are dealing with yet another particular case of the general ECP.

Corollary 6.3. *Let f be a norm on $\mathbb{R}^n$. Then the Fermat radius of Ω is equal to the optimal value of the maximization problem*

$$\begin{aligned} &\textit{maximize} \quad \sum_{k=1}^{N} \langle \omega_k, y_k \rangle \\ &f^\circ(y_k) \leq 1 \ \textit{ for all } k \in \{1, \ldots, N\} \\ &\sum_{k=1}^{N} y_k = 0_n \, . \end{aligned} \tag{25}$$

Proof. We apply again Lemma 6.1, but this time we use

$$z = \begin{bmatrix} x \\ \tau \end{bmatrix}, \quad c = \begin{bmatrix} 0_n \\ 1_N \end{bmatrix}, \quad b_k = \begin{bmatrix} \omega_k \\ 0 \end{bmatrix}, \quad A_k = \begin{bmatrix} I_n & O_{n,N} \\ 0_n^T & e_{k,N}^T \end{bmatrix}.$$

Here τ is the column vector whose components are $\tau_1, \ldots, \tau_N$, the symbol $O_{n,N}$ indicates the zero matrix of size $n \times N$, and $e_{k,N}$ is the k-th canonical vector of $\mathbb{R}^N$. The equality constraint in (20) becomes

$$\sum_{k=1}^{N} \begin{bmatrix} I_n & 0_n \\ O_{N,n} & e_{k,N} \end{bmatrix} \begin{bmatrix} y_k \\ s_k \end{bmatrix} = \begin{bmatrix} 0_n \\ 1_N \end{bmatrix},$$

which after simplification yields $\sum_{k=1}^{N} y_k = 0_n$ and $s_k = 1$ for all $k \in \{1, \ldots, N\}$. This explains the form (25) that we are getting for the dual problem associated to (24). The Slater type qualification condition (i) mentioned in Lemma 6.1 is again in force. To see this, pick any $\tilde{x} \in \mathbb{R}^n$ and then let $\tilde{\tau}_1 > f(\tilde{x} - \omega_1), \ldots, \tilde{\tau}_N > f(\tilde{x} - \omega_N)$. In such a case,

$$A_k \tilde{z} - b_k = \begin{bmatrix} \tilde{x} \\ \tilde{\tau}_k \end{bmatrix} - \begin{bmatrix} \omega_k \\ 0 \end{bmatrix} \in \text{int}\,(\text{epi} f)$$

for all $k \in \{1, \ldots, N\}$. □

6.3 Final comments

The nice survey by Boyd et al. [3] on second-order conic programming (SOCP) focuses the attention on the optimization model

$$\begin{aligned} &\text{minimize } \langle w, x \rangle \\ &\|A_k x + b_k\| \leq \langle c_k, x \rangle + d_k \ \text{ for all } k \in \{1, \ldots, N\}. \end{aligned} \tag{26}$$

This is of course a particular instance of the general model (19).

SOCP are typically solved by interior point methods (cf. [2, 3, 13]). Such methods can be adapted to the epigraphic conic program (19), provided one has a suitable barrier function for the set epi$f$. It is here where the Vinberg characteristic function $\Phi_{\mathrm{epi}f}$ enters into action. It would lead us to far to describe all the technicalities of the interior point method applied to (19), but this is certainly something that deserves an exhaustive treatment in a more specialized paper.

References

[1] J.P. Aubin and H. Frankowska. *Set-Valued Analysis.* Birkhäuser, Boston, MA, 1990.

[2] F. Alizadeh and D. Goldfarb. Second-order cone programming. *Math. Program.* 95 (2003), 3–51.

[3] S. Boyd, H. Lebret, M.S. Lobo, and L. Vandenberghe. Applications of second-order cone programming. *Linear Algebra Appl.* 284 (1998), 193–228.

[4] J.M. Danskin. The theory of max-min, with applications. *SIAM J. Appl. Math.* 14 (1966), 641–664.

[5] R. Deville, G. Godefroy, and V. Zizler. *Smoothness and Renormings in Banach Spaces.* John Wiley and Sons, New York, 1993.

[6] L. Faybusovich. Self-concordant barriers for cones generated by Chebyshev systems. *SIAM J. Optim.* 12 (2002), 770–781.

[7] J. Faraut and A. Korányi. *Analysis on Symmetric Cones.* Oxford University Press, New York, 1994.

[8] P. Gritzmann, V. Klee, and B.S. Tam. Cross-positive matrices revisited. *Linear Algebra Appl.* 223/224 (1995), 285–305.

[9] O. Güler. Barrier functions in interior point methods. *Math. Oper. Res.* 21 (1996), 860–885.

[10] R. Henrion and A. Seeger. Inradius and circumradius of various convex cones arising in applications. *Set-Valued Var. Anal.* 18 (2010), 483–511.

[11] A. Iusem and A. Seeger. Pointedness, connectedness, and convergence results in the space of closed convex cones. *J. Convex Anal.* 11 (2004), 267–284.

[12] A. Iusem and A. Seeger. Distances between closed convex cones: old and new results. *J. Convex Anal.* 17 (2010), 1033–1055.

[13] Y.J. Kuo and H.D. Mittelmann. Interior point methods for second-order cone programming and OR applications. *Comput. Optim. Appl.* 28 (2004), 255–285.

[14] S. Kuriki and A. Takemura. Weights of chi-square distribution for smooth or piecewise smooth cone alternatives. *Ann. Statist.* 25 (1997), 2368–2387.

[15] A. Moltó, J. Orihuela, S. Troyanski, and M. Valdivia. *A Nonlinear Transfer Technique for Renorming*, Springer-Verlag, Berlin, 2009.

[16] M. Moussaoui and M. Volle. Subdifferentiability and inf-sup theorems. *Positivity* 3 (1999), 345–355.

[17] P.E. Ney and S.M. Robinson. Polyhedral approximation of convex sets with an application to large deviation probability theory. *J. Convex Anal.* 2 (1995), 229–240.

[18] R. Radner. Paths of economic growth that are optimal with regard only to final states: a turnpike theorem. *Review Econ. Studies* 28 (1961), 98–104.

[19] R.T. Rockafellar and R.J.B. Wets. *Variational Analysis.* Springer, Berlin, 1998.

[20] H. Schneider and B.T. Tam. On the invariant faces associated with a cone-preserving map. *Trans. Amer. Math. Soc.* 353 (2001), 209–245.

[21] A. Seeger. Epigraphical cones I. *J. Convex Anal.*, in press.

[22] A. Seeger and M. Torki. On eigenvalues induced by a cone constraint. *Linear Algebra Appl.* 372 (2003), 181–206.

[23] R.J. Stern and H. Wolkowicz. Invariant ellipsoidal cones. *Linear Algebra Appl.* 150 (1991), 81–106.

[24] E.B. Vinberg. The theory of homogeneous convex cones. *Trans. Moskow. Math.* 12 (1963), 340–403.

[25] D.W. Walkup and R.J.B. Wets. Continuity of some convex-cone-valued mappings. *Proc. Amer. Math. Soc.* 18 (1967), 229–235.