ERROR BOUNDS FOR VECTOR-VALUED FUNCTIONS: NECESSARY AND SUFFICIENT CONDITIONS

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Abstract. In this paper, we attempt to extend the definition and existing local error bound criteria to vector-valued functions, or more generally, to functions taking values in a normed linear space. Some new derivative-like objects (slopes and subdifferentials) are introduced and a general classification scheme of error bound criteria is presented.

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1. Introduction

In variational analysis, the term "*error bounds*" usually refers to the following property. Given an (extended) real-valued function f on a set X, consider its lower level set

$$S(f) := \{x \in X|\ f(x) \le 0\} \quad - \tag{1.1}$$

the set of all solutions of the inequality $f(x) \le 0$. If a point x is not a solution, that is, $f(x) > 0$, then it can be important to have an estimate of its distance from the set (1.1) (assuming that X is a metric space) in terms of the value $f(x)$. If a linear estimate is possible, that is, there exists a constant $\gamma > 0$ such that

$$d(x, S(f)) \le \gamma f^+(x) \tag{1.2}$$

for all $x \in X$, then f possesses the (linear) error bound property or the error bound property holds for f. Here the denotation $f^+(x) = \max(f(x), 0)$ is used. Hence (1.2) is satisfied automatically for all $x \in S(f)$.

If $\bar{x} \in S(f)$ (usually it is assumed that $f(\bar{x}) = 0$) and (1.2) is required to hold (with some $\gamma > 0$) for all x near $\bar{x}$, then we have the definition of the local (near $\bar{x}$) error bound property.

Error bounds play a key role in variational analysis. They are of great importance for optimality conditions, subdifferential calculus, stability and sensitivity issues, convergence of numerical methods, etc. For the summary of the theory of error bounds and its various applications to sensitivity analysis, convergence analysis of algorithms, penalty function methods in mathematical programming the reader is referred to the survey papers by Azé [2], Lewis & Pang [21], Pang [29], as well as the book by Auslender & Teboule [1].

Numerous characterizations of the error bound property have been established in terms of various derivative-like objects either in the primal space (directional derivatives, slopes, etc.) or in the dual space (subdifferentials, normal cones) [3,4,7–15,17,19,20,24–30,34–36].

In the present paper, we attempt to extend (1.2) as well as local error bound criteria to vector-valued functions f, defined on a metric space X and taking values in a normed linear space Y. The presentation, terminology and notation follow that of [11]. Some new derivative-like objects (slopes and subdifferentials), which can be of independent interest, are introduced and a general classification scheme of error bound criteria is presented. It is illustrated in Fig. 3 – 8.

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The plan of the paper is as follows. In Section 2, we introduce an abstract ordering operation and define an extension of (1.1) and a nonnegative real-valued function f_y^+ whose role is to replace f^+ in (1.2) in the case f takes values in a normed linear space. Note that, unlike the scalar case, an additional parameter y is required now. The function f_y^+ can be viewed as a *scalarizing* function for the vector minimization problem defined by f. In Sections 3 and 4 various kinds of slopes, directional derivatives and subdifferentials for vector-valued function f are defined via the scalarizing function f_y^+. In Section 3, we discuss primal space error bound criteria in terms of slopes. Section 4 is devoted to the criteria in terms of directional derivatives and subdifferentials. In the final Section 5, three special cases are considered: error bounds when either the image or the pre-image is finite dimensional and in the convex case.

Our basic notation is standard, see [23, 31]. Depending on the context, X is either a metric or a normed space. Y is always a normed space. If X is a normed space, then its topological dual is denoted X^* while $\langle \cdot, \cdot \rangle$ denotes the bilinear form defining the pairing between the two spaces. The closed unit balls in a normed space and its dual are denoted $\mathbb{B}$ and $\mathbb{B}^*$ respectively. $B_\delta(x)$ denotes the closed ball with radius δ and center x. If A is a set in metric or normed space, then $\operatorname{cl} A$, $\operatorname{int} A$, and $\operatorname{bd} A$ denote its closure, interior, and boundary respectively; $d(x, A) = \inf_{a \in A} \|x - a\|$ is the point-to-set distance. We also use the denotation $\alpha^+ = \max(\alpha, 0)$.

2. Minimality and error bounds

In this section, an ordering operation in a normed linear space is discussed and the error bound property is defined.

2.1. **Minimality.** Let Y be a normed linear space. Suppose that for each $y \in Y$ a subset $V_y \subset Y$ is given with the property $y \in V_y$. We are going to consider an abstract "order" operation in Y defined by the collection of sets $\{V_y\}$: we say that v is dominated by y in Y if $v \in V_y$.

Of course, this operation does not possess in general typical order properties. It can get more natural, for example, if a closed cone $C \subset Y$ is given and V_y is defined as one of the following (for (2.3) C must be assumed pointed):

$$\{v \in Y |\ y - v \in C\}, \tag{2.1}$$

$$\{v \in Y |\ v - y \notin C\} \cup \{y\}, \tag{2.2}$$

$$\{v \in Y |\ v - y \notin \operatorname{int} C\}. \tag{2.3}$$

Now let X be a metric space and $f : X \to Y$. Denote

$$S_y(f) := \{u \in X |\ f(u) \in V_y\} \quad -$$

the y-sublevel set of f (with respect to the order defined by the collection of sets $\{V_y\}$). We will also use the following nonnegative real-valued function

$$f_y^+(u) := d(f(u), V_y), \quad u \in X. \tag{2.4}$$

If V_y is closed, then $f_y^+(u) = 0$ if and only if $u \in S_y(f)$.

We say that x is a local V_y-*minimal* point of f if

$$f_y^+(x) \le f_y^+(u) \quad \text{for all } u \text{ near } x. \tag{2.5}$$

The definition depends on the choice of y. Condition (2.5) is obviously satisfied for any $x \in S_y(f)$.

If V_y is given by (2.1), i.e., $V_y = y - C$, then $f_y^+(u) = d(y - f(u), C)$ and the function f_y^+ is nondecreasing in the sense that for any $x_1, x_2 \in X$ such that $f(x_2) - f(x_1) \in C$ it holds $f_y^+(x_1) \le f_y^+(x_2)$. Indeed,

$$\begin{aligned} f_y^+(x_1) = d(y - f(x_1), C) &\le d(y - f(x_2), C) + d(f(x_2) - f(x_1), C) \\ &= d(y - f(x_2), C) = f_y^+(x_2). \end{aligned}$$

By this property, if C is a pointed cone and x is a *strict local V_y-minimal* point of f, i.e.,

$$f_y^+(x) < f_y^+(u) \quad \text{for all } u \text{ near } x,\ u \neq x,$$

then x is locally minimal with respect to the ordering relation generated by C, i.e., there exists a neighborhood U_x of x such that

$$(f(U_x) - f(x)) \cap (-C) = \{0\}. \tag{2.6}$$

If $Y = \mathbb{R}$, we will always assume the natural ordering: $V_y := \{v \in \mathbb{R} |\ v \le y\}$. This corresponds to setting $C = \mathbb{R}_+$ in any of the definitions in (2.1)–(2.3). We will also consider the usual distance in $\mathbb{R}$: $d(v_1, v_2) = |v_1 - v_2|$. Then

$$S_y(f) = \{u \in X |\ f(u) \le y\},$$
$$f_y^+(u) = (f(u) - y)^+.$$

If $y < f(x)$, then (2.5) means that x is a point of local minimum of f in the usual sense and does not depend on y (as long as $y < f(x)$). If $y \ge f(x)$, then x is automatically a *V_y-minimal* point of f.

If $Y = \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, $f = (f_1, f_2, \ldots, f_n) : X \to \mathbb{R}^n$, and $V_y = \{(v_1, v_2, \ldots, v_n) \in \mathbb{R}^n |\ v_i \le y_i,\ i = 1, 2, \ldots, n\}$, then

$$S_y(f) = \{u \in X |\ f_i(u) \le y_i,\ i = 1, 2, \ldots, n\} \tag{2.7}$$

and, for any norm $\|\cdot\|$ in $\mathbb{R}^n$, we have

$$f_y^+(u) = \|(f_1(u) - y_1)^+, (f_2(u) - y_2)^+, \ldots, (f_n(u) - y_n)^+\|. \tag{2.8}$$

The local minimality condition (2.5) is satisfied if

$$f_i(x) \le f_i(u) \quad \text{for all } i \text{ such that } f_i(x) > y_i \text{ and all } u \text{ near } x. \tag{2.9}$$

The opposite statement is not true in general.

Example 1. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined as $f(u) = (u, 1 - u)$. Then the range of f is the set $\{(v_1, v_2) \in \mathbb{R}^2 |\ v_1 + v_2 = 1\}$. Let $y = 0 \in \mathbb{R}^2$ and $V_0 = \mathbb{R}^2_- := \{(v_1, v_2) \in \mathbb{R}^2 |\ v_1 \le 0, v_2 \le 0\}$. Suppose that $\mathbb{R}^2$ is equipped with the norm $\|v_1, v_2\| = |v_1| + |v_2|$. Then

$$f_0^+(u) = \begin{cases} 1 - u, & \text{if } u \le 0, \\ 1, & \text{if } 0 < u \le 1, \\ u, & \text{if } u > 1. \end{cases}$$

Hence f_0^+ attains its minimum value 1 on the set $[0, 1]$. At the same time, for any $x \in [0, 1]$, condition (2.9) is violated.

Consider, for instance, the max-type norm on $\mathbb{R}^n$: $\|y\| = \max_i |y_i|$. Then

$$f_y^+(u) = \max_{1 \le i \le n} (f_i(u) - y_i)^+ = \left(\max_{1 \le i \le n} (f_i(u) - y_i) \right)^+.$$

The local minimality in the sense of (2.6) means that there is no $u \in U_x$ such that $f_i(u) \le f_i(x)$ for all $1 \le i \le n$ and $f(u) \neq f(x)$. In other words, for every $u \in U_x$, either we have $f(u) = f(x)$ or there exists an index i, $1 \le i \le n$, such that $f_i(u) > f_i(x)$.

Hence for any $y \in \mathbb{R}^n$ and all $u \in U_x$, we have either $f(u) = f(x)$ or $f_y^+(x) < f_y^+(u)$. In consequence, for any $y \in \mathbb{R}^n$

$$f_y^+(x) \leq f_y^+(u) \text{ for all } u \text{ close to } x.$$

which shows that x is a local V_y-minimal point of f.

2.2. **Error bounds.** Let $\bar{x} \in X$ and $\bar{y} := f(\bar{x})$. Consider the $\bar{y}$-sublevel set $S_{\bar{y}}(f)$ and the function $f_{\bar{y}}^+$. We say that f satisfies the (local) *error bound* property at $\bar{x}$ if there exists a $\gamma > 0$ and a $\delta > 0$ such that

$$d(x, S_{\bar{y}}(f)) \leq \gamma f_{\bar{y}}^+(x), \quad \forall x \in B_\delta(\bar{x}). \tag{2.10}$$

If $\bar{x}$ is a locally minimal solution in the sense of (2.6), then (2.10) means that $\bar{x}$ is a *local weak sharp solution* [5, Definition 8.2.3] to f.

The error bound property can be equivalently defined in terms of the *error bound modulus* of f at $\bar{x}$:

$$\mathrm{Er}\, f(\bar{x}) := \liminf_{x \to \bar{x}} \left[\frac{f_{\bar{y}}^+(x)}{d(x, S_{\bar{y}}(f))} \right]_\infty, \tag{2.11}$$

namely, the error bound property holds for f at $\bar{x}$ if and only if $\mathrm{Er}\, f(\bar{x}) > 0$.

The notation $[\cdot/\cdot]_\infty$ in (2.11) is used for the extended division operation which differs from the conventional one by the additional property $[0/0]_\infty = \infty$. This allows one to incorporate implicitly the requirement $x \notin \mathrm{cl}\, S_{\bar{y}}(f)$ in definition (2.11).

The error bound property can be characterized in terms of certain derivative-like objects.

3. Criteria in terms of slopes

In this section, several primal space error bound criteria in terms of various kinds of slopes are established. The slopes are defined via the scalarizing function f_y^+. A different approach to defining slopes directly in terms of the original vector function and then deducing error bound criteria based on the application of the vector variational principle [6] will be considered elsewhere.

3.1. **Slopes.** Recall that in the case $f : X \to \mathbb{R}_\infty$, the *slope* of f at $x \in X$ (with $f(x) < \infty$) is defined as (see, for instance, [16])

$$|\nabla f|(x) := \limsup_{u \to x} \frac{(f(x) - f(u))^+}{d(u, x)}. \tag{3.1}$$

In our general setting $f : X \to Y$, we define the *slope* of f at $x \in X$ relative to $y \in Y$ as

$$|\nabla f|_y(x) := |\nabla f_y^+|(x) = \limsup_{u \to x} \frac{(f_y^+(x) - f_y^+(u))^+}{d(u, x)}. \tag{3.2}$$

Equivalently

$$|\nabla f|_y(x) = \begin{cases} \limsup\limits_{u \to x} \dfrac{f_y^+(x) - f_y^+(u)}{d(u, x)}, & \text{if } x \text{ is not a local } V_y\text{-minimal point of } f, \\ 0, & \text{otherwise.} \end{cases}$$

If V_y is given by (2.1) with C being a convex cone, then $c + C \subset C$ for any $c \in C$, and consequently

$$f_y^+(x) - f_y^+(u) = \sup_{c \in C} [d(0, f(x) - y + C) - d(0, f(u) - y + c)] \leq d(f(u) - f(x), C).$$

Hence $|\nabla f|_y(x) \le |\nabla f|^1_y(x)$, where

$$(3.3)\quad |\nabla f|^1_y(x) := \begin{cases} \limsup\limits_{u\to x} \dfrac{d(f(u)-f(x),C)}{d(u,x)}, & \text{if } x \text{ is not a local } y\text{-minimal point of } f,\\ 0, & \text{otherwise.}\end{cases}$$

If C is a pointed cone, then the upper limit in the above formula admits the following equivalent representation:

$$\limsup_{u\to x} \frac{d(f(u)-f(x),C)}{d(u,x)} = \inf_{\varepsilon>0}\inf_{r>0}\left\{r \middle|\ \frac{d(f(u)-f(x),C)}{d(u,x)} < r,\ \forall u\in B_\varepsilon(x)\setminus\{x\}\right\}$$
$$= \inf_{\varepsilon>0}\inf_{r>0}\{r|\ \forall u\in B_\varepsilon(x)\setminus\{x\}\ \exists v\in \mathbb{B}_Y \text{ such that } rd(u,x)v\notin f(x)-f(u)-C\}.$$

If, additionally, $\operatorname{int} C\neq\emptyset$, then we have $\mathbb{B}_Y\subset e+C$ for some $e\in -\operatorname{int} C$ and the latter formula gives

$$\limsup_{u\to x} \frac{d(f(u)-f(x),C)}{d(u,x)} \le \inf_{\varepsilon>0}\inf_{r>0}\{r|\ rd(u,x)e\notin f(x)-f(u)-C,\ \forall u\in B_\varepsilon(x)\setminus\{x\}\}.$$

In view of this, we have $|\nabla f|^1_y(x)\le |\nabla f|^1_y(x,e)$, where

$$|\nabla f|^1_y(x,e) := \begin{cases} \inf\limits_{\varepsilon>0}\inf\limits_{r>0}\{r|\ rd(u,x)e\notin f(x)-f(u)-C, & x \text{ is not a local } y\text{-minimal}\\ \qquad\qquad \forall u\in B_\varepsilon(x)\setminus\{x\}\}, & \text{point of } f,\\ 0, & \text{otherwise.}\end{cases}$$

There are obvious similarities between definitions (3.1) and (3.2). Note also an important difference: the latter one depends on an additional parameter $y\in Y$. If $Y=\mathbb{R}$ and $y<f(x)$, then y vanishes and (3.2) reduces to (3.1).

Proposition 3.1. *Let $Y=\mathbb{R}$. Then*

$$|\nabla f|_y(x) = \begin{cases} |\nabla f|(x), & \text{if } y<f(x),\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Under the assumptions, $f_y^+(u)=(f(u)-y)^+$. In the case $y<f(x)$, one obviously has $f_y^+(x)=f(x)-y$. If f is lower semicontinuous at x, then also $f_y^+(u)=f(u)-y$ for all $u\in X$ near x. Hence $f_y^+(x)-f_y(u)=f(x)-f(u)$, and consequently, $|\nabla f_y^+|(x)=|\nabla f|(x)$. If f is not lower semicontinuous at x, then there exists a sequence $x_k\to x$ such that $f(x_k)\to\alpha<f(x)$. Then, by definition (3.1), $|\nabla f|(x)=\infty$. At the same time, $f_y^+(x_k)\to(\alpha-y)^+<f_y^+(x)$. Hence f_y^+ is not lower semicontinuous at x either and $|\nabla f_y^+|(x)=\infty$.

In the case $y\ge f(x)$, one has $f_y^+(x)=0$. It follows that x is a point of minimum of f_y^+, and consequently, $|\nabla f_y^+|(x)=0$. □

It is easy to check that in the case $Y=\mathbb{R}$ and $y<f(x)$, (3.3) also reduces to (3.1).

It always holds $|\nabla f|_y(x)\ge 0$ while the equality $|\nabla f|_y(x)=0$ means that x is a *stationary* point of f_y^+. If the slope $|\nabla f|_y(x)$ is strictly positive, it characterizes quantitatively the *descent rate* of f_y^+ at x. If V_y is given by (2.1), then obviously $|\nabla f|^1_y(x)\ge 0$. Moreover, if C is a closed convex pointed cone with nonempty interior, f is directionally differentiable at x, and x is not its local V_y-minimal point, then the equality $|\nabla f|^1_y(x)=0$ means that x is a *stationary* point of f in the sense of Smale [32,33], i.e., for any direction $p\in X$ we have $f'(x;p)\notin -\operatorname{int} C$. In this context the following proposition holds.

Proposition 3.2. *Let C be a closed pointed cone,* $\operatorname{int} C\neq\emptyset$ *and $f: X\to Y$ be directionally differentiable at $x\in X$. If x is not a local y-minimal point of f, then*

$$f'(x;p)+\|p\||\nabla f|^1_y(x)e\notin -\operatorname{int} C \quad \text{for all } p\in X,$$

where $e \in C$ and $f'(x;p)$ is the directional derivative of f at x in the direction p.

Proof. Suppose on the contrary that

$$f'(x;p) + |\nabla f|^1_y(x)e + r\mathbb{B}_Y \subset -C$$

for some $p \in X$, $\|p\| = 1$, and $r > 0$. Hence, there exists an $\varepsilon > 0$ such that

$$f(x+tp) - f(x) + t(|\nabla f|^1_y(x) + r\kappa/2)e \in -C \quad \text{for all } t \in (0,\varepsilon),$$

where $\kappa > 0$ and $\kappa e \in \mathbb{B}_Y$. Consequently

$$f(u) - f(x) + t(|\nabla f|^1_y(x) + r\kappa/2)e \in -C \quad \text{for some } u \in B_\varepsilon(x),$$

contradictory to the definition of $|\nabla f|^1_y(x)$. □

3.2. **Strict slopes.** Given a fixed point $\bar{x} \in X$ (with $\bar{y} = f(\bar{x})$) one can use the collection of slopes (3.2) computed at nearby points to define a more robust object – the *strict outer slope* of f at $\bar{x}$:

$$\overline{|\nabla f|}^{>}(\bar{x}) = \liminf_{x\to\bar{x},\, f^+_{\bar{y}}(x)\downarrow 0} |\nabla f|_{\bar{y}}(x). \tag{3.4}$$

The word "strict" reflects the fact that slopes at nearby points contribute to the definition (3.4) making it an analogue of the strict derivative. The word "outer" is used to emphasize that only points outside the set $S_{\bar{y}}(f)$ are taken into account.

If $Y = \mathbb{R}$, then, due to Proposition 3.1, definition (3.4) takes the form

$$\overline{|\nabla f|}^{>}(\bar{x}) = \liminf_{x\to\bar{x},\, f(x)\downarrow f(\bar{x})} |\nabla f|(x) \tag{3.5}$$

and coincides with the strict outer slope defined in [11]. On the other hand, one can apply (3.5) to the scalar function (2.4) (with $y = \bar{y}$). This leads to an equivalent representation of (3.4):

$$\overline{|\nabla f|}^{>}(\bar{x}) = \overline{|\nabla f^+_{\bar{y}}|}^{>}(\bar{x}).$$

The last constant provides the exact lower estimate of the "uniform" descent rate of $f^+_{\bar{y}}$ near $\bar{x}$.

The strict outer slope (3.4) is the limit of usual slopes $|\nabla f|_{\bar{y}}(x)$ which themselves are limits and do not take into account how close to $\bar{x}$ the point x is. This can be important when characterizing error bounds. In view of this observation, the next definition can be of interest.

The *uniform strict slope* of f at $\bar{x}$:

$$\overline{|\nabla f|}^{\diamond}(\bar{x}) := \liminf_{x\to\bar{x},\, f^+_{\bar{y}}(x)\downarrow 0} \sup_{u\neq x} \frac{(f^+_{\bar{y}}(x) - f^+_{\bar{y}}(u))^+}{d(u,x)}. \tag{3.6}$$

It is easy to check that (3.6) coincides with the middle uniform strict slope [11] of $f^+_{\bar{y}}$ at $\bar{x}$. The following representation is straightforward:

$$\overline{|\nabla f|}^{\diamond}(\bar{x}) = \liminf_{x\to\bar{x},\, f^+_{\bar{y}}(x)\downarrow 0} \max\left[\sup_{0<d(u,x)<d(x,S_{\bar{y}}(f))} \frac{(f^+_{\bar{y}}(x) - f^+_{\bar{y}}(u))^+}{d(u,x)}, \frac{f^+_{\bar{y}}(x)}{d(x,S_{\bar{y}}(f))}\right].$$

It implies, in particular, the lower estimate:

$${}^{-}\overline{|\nabla f|}^{\diamond}(\bar{x}) \le \overline{|\nabla f|}^{\diamond}(\bar{x}), \tag{3.7}$$

where

$${}^{-}\overline{|\nabla f|}^{\diamond}(\bar{x}) := \liminf_{x\to\bar{x},\, f^+_{\bar{y}}(x)\downarrow 0} \sup_{0<d(u,x)<d(x,S_{\bar{y}}(f))} \frac{(f^+_{\bar{y}}(x) - f^+_{\bar{y}}(u))^+}{d(u,x)} \tag{3.8}$$

is the *lower uniform strict slope* of f at $\bar{x}$. If f (or just $f_{\bar{y}}^{+}$) is Lipschitz continuous near $\bar{x}$, then (3.7) holds as equality (if $\dim X < \infty$, it is sufficient to assume that f is simply continuous – Proposition 5.1 (i)). In general, inequality (3.7) can be strict.

Example 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$f(x_1, x_2) = \begin{cases} x_1 + x_2, & \text{if } x_1 > 0, x_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

We are assuming that $\mathbb{R}^2$ is equipped with the Euclidean norm. Then $d(f(x), V_0) = f(x)$. We are going to show that ${}^{-}\overline{|\nabla f|}^{\diamond}(0) = \sqrt{2}$ while $\overline{|\nabla f|}^{\diamond}(0) = 2$.

Indeed, let $x = (x_1, x_2)$ with $x_1 > 0$ and $x_2 > 0$. Then $d(x, S_0(f)) = \min(x_1, x_2)$. Obviously,

$$\sup_{0<d(u,x)<d(x,S_0(f))} \frac{(f(x) - f(u))_+}{d(u, x)} = |\nabla f|(x) = \sup_{\|(v_1,v_2)\|=1} (v_1 + v_2) = \sqrt{2},$$

$$\sup_{u \neq x} \frac{(f(x) - f(u))_+}{d(u, x)} = \frac{x_1 + x_2}{\min(x_1, x_2)}.$$

The last expression attains its minimum value 2 when $x_1 = x_2$. It follows that ${}^{-}\overline{|\nabla f|}^{\diamond}(0) = \sqrt{2}$ and $\overline{|\nabla f|}^{\diamond}(0) = 2$.

Note also the inequality

$$\overline{|\nabla f|}^{>}(\bar{x}) \le {}^{-}\overline{|\nabla f|}^{\diamond}(\bar{x}), \tag{3.9}$$

which follows from definitions (3.2), (3.4), and (3.8).

3.3. **Main criterion.** The next theorem provides the relationship between the error bound modulus and the uniform strict outer slope.

Theorem 3.3. *Let X be complete and $f_{\bar{y}}^{+}$ be lower semicontinuous. Then* $\operatorname{Er} f(\bar{x}) = \overline{|\nabla f|}^{\diamond}(\bar{x})$.

Proof. Let $0 < \gamma < \operatorname{Er} f(\bar{x})$. We are going to show that $\overline{|\nabla f|}^{\diamond}(\bar{x}) \ge \gamma$. By (2.11), there is a $\delta > 0$ such that

$$\frac{f_{\bar{y}}^{+}(x)}{d(x, S_{\bar{y}}(f))} > \gamma. \tag{3.10}$$

for any $x \in B_\delta(\bar{x}) \setminus S_{\bar{y}}(f)$. Take any $x_k \in B_{1/k}(\bar{x})$ with $0 < f_{\bar{y}}^{+}(x_k) \le 1/k$. Then, by (3.10), for any $k > \delta^{-1}$, one can find a $w_k \in S_{\bar{y}}$ such that

$$\frac{f_{\bar{y}}^{+}(x_k) - f_{\bar{y}}^{+}(w_k)}{d(x_k, w_k)} = \frac{f_{\bar{y}}^{+}(x_k)}{d(x_k, w_k)} > \gamma.$$

It follows from definition (3.6) that $\overline{|\nabla f|}^{\diamond}(\bar{x}) \ge \gamma$.

Let $\gamma > \operatorname{Er} f(\bar{x})$. Then for any $\delta > 0$ there is an $x \in B_{\delta \min(1/2, \gamma^{-1})}(\bar{x})$ such that

$$0 < f_{\bar{y}}^{+}(x) < \gamma d(x, S_{\bar{y}}(f)).$$

Applying to $f_{\bar{x}}^{+}$ the Ekeland variational principle with $\varepsilon = f_{\bar{y}}^{+}(x)$ and an arbitrary $\lambda \in (\gamma^{-1}\varepsilon, d(x, S_{\bar{y}}(f)))$, one can find a w such that $f_{\bar{y}}^{+}(w) \le f_{\bar{y}}^{+}(x)$, $d(w, x) \le \lambda$ and

$$f_{\bar{y}}^{+}(u) + (\varepsilon/\lambda) d(u, w) \ge f_{\bar{y}}^{+}(w), \qquad \forall u \in X. \tag{3.11}$$

Obviously, $d(w, \bar{x}) \le 2d(x, \bar{x}) \le \delta$ and $f_{\bar{y}}^{+}(w) < \gamma d(x, S_{\bar{y}}(f)) \le \delta$. Besides, $f(w) \notin V_{\bar{y}}$ since $d(w, x) \le \lambda < d(x, S_{\bar{y}}(f))$. It follows from (3.11) that

$$\frac{f_{\bar{y}}^{+}(w) - f_{\bar{y}}^{+}(u)}{d(u, w)} \le \varepsilon/\lambda < \gamma, \qquad \forall u \in X.$$

This implies the inequality $\overline{|\nabla f|}^{\diamond}(\bar{x}) \le \gamma$, and consequently $\overline{|\nabla f|}^{\diamond}(\bar{x}) \le \operatorname{Er} f(\bar{x})$. □

Remark 1. The assumptions that X is complete and $f_{\bar{x}}^{+}$ is lower semicontinuous in Theorem 3.3 were used in the proof of inequality $\overline{|\nabla f|}^{\diamond}(\bar{x}) \le \operatorname{Er} f(\bar{x})$. The opposite inequality $\operatorname{Er} f(\bar{x}) \le \overline{|\nabla f|}^{\diamond}(\bar{x})$ is always true and can be strict, for instance, if the assumption of lower semicontinuity is dropped.

Example 3 ([11]). Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows (Fig. 1):

$$f(x) = \begin{cases} -3x, & \text{if } x \le 0, \\ 3x - \dfrac{1}{2^i}, & \text{if } \dfrac{1}{2^{i+1}} < x \le \dfrac{1}{2^i},\ i = 0, 1, \ldots, \\ 2x, & \text{if } x > 1. \end{cases}$$

Obviously, $\operatorname{Er} f(0) = 1$ while $\overline{|\nabla f|}^{\diamond}(0) = 3$.

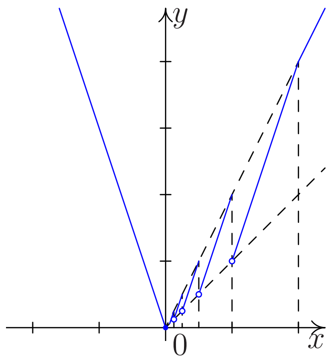

FIGURE 1. Example 3

Remark 2. When $Y = \mathbb{R}$, Theorem 3.3 improves [11, Theorem 2] and goes in line with [18, Corollary 4.3] by Kummer.

3.4. **Criteria in terms of slopes.** Under the assumptions of Theorem 3.3, constants (3.4), (3.6), and (3.8) produce criteria for the error bound property of f at $\bar{x}$.

UC1. $\overline{|\nabla f|}^{\diamond}(\bar{x}) > 0$.

UC2. ${}^{-}\overline{|\nabla f|}^{\diamond}(\bar{x}) > 0$.

C1. $\overline{|\nabla f|}^{>}(\bar{x}) > 0$.

(Uniform) criteria UC1 and UC2 are sufficient while criterion C1 is necessary and sufficient. Due to (3.7) and (3.9), it holds C1 ⇒ UC2 ⇒ UC1.

Another sufficient criterion

C2. $|\nabla f|^{0}(\bar{x}) > 0$

can be formulated using the next nonnegative constant:

$$|\nabla f|^{0}(\bar{x}) := \liminf_{x \to \bar{x}} \frac{f_{\bar{y}}^{+}(x)}{d(x, \bar{x})}. \tag{3.12}$$

Comparing this definition with (3.2), one can easily establish the next relation:

$$|\nabla f|_{\bar{y}}(\bar{x}) = (-|\nabla f|^{0}(\bar{x}))_{+},$$

which shows that when $|\nabla f|^{0}(\bar{x}) = 0$, this constant does not provide any new information. At the same time, when $|\nabla f|^{0}(\bar{x}) > 0$, the other constant (3.2) equals 0 and cannot be used for characterization of error bounds while criterion C2 involving (3.12) can be useful.

Proposition 3.4. *If* $|\nabla f|^0(\bar{x}) > 0$*, then* $|\nabla f|^0(\bar{x}) = \mathrm{Er}\, f(\bar{x})$*.*

Proof. If $|\nabla f|^0(\bar{x}) > 0$, then $\bar{x}$ is an isolated point in $S_{\bar{y}}(f)$, and consequently $d(x, S_{\bar{y}}(f)) = d(x, \bar{x})$ for all x near $\bar{x}$. □

The short proof above shows that when $|\nabla f|^0(\bar{x}) > 0$ the situation with error bounds is pretty simple. We formulate criterion C2 explicitly to make the picture of the existing criteria complete and simplify the comparison between the different criteria.

Note that criteria C1 and C2 are independent. For the function f in Example 2, one obviously has $\overline{|\nabla f|}^{>}(0) = {}^{-}\overline{|\nabla f|}^{\diamond}(0) = \sqrt{2}$ while $|\nabla f|^0(0) = 0$. At the same time, for the function in the next example the situation is opposite.

Example 4 ([11,17]). Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows (Fig. 2):

$$f(x) = \begin{cases} -x, & \text{if } x \le 0, \\ \dfrac{1}{i}, & \text{if } \dfrac{1}{i+1} < x \le \dfrac{1}{i},\ i = 1, 2, \ldots, \\ x, & \text{if } x > 1. \end{cases}$$

Obviously $|\nabla f|(x) = 0$ for any $x \in (0,1)$, and consequently $\overline{|\nabla f|}^{>}(0) = 0$. At the same time, $d(f(x), V_0) = f(x)$ and $|\nabla f|^0(0) = 1$.

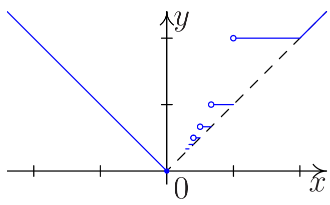

FIGURE 2. Example 4

The relationships among the primal space error bound criteria UC1, UC2, C1, C2 are illustrated in Fig. 3. It is assumed that the space X is complete and the function $f^+_{\bar{y}}$ is lower semicontinuous.

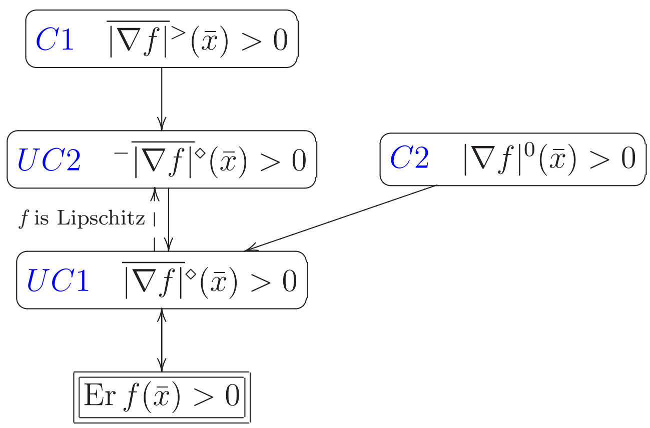

FIGURE 3. Criteria in terms of slopes

4. Criteria in terms of directional derivatives and subdifferentials

In this section, X is assumed a normed linear space. The slopes considered above have corresponding to them directional derivatives and subdifferentials. The criteria in terms of slopes established above can be translated into the corresponding criteria in terms of directional derivatives and subdifferentials.

4.1. **Directional derivatives.** Given $x, y, h \in X$, the *lower Dini derivative* of f at x in direction h relative to y is defined as

$$f'_y(x;h) := \liminf_{t \downarrow 0,\, z \to h} \frac{f_y^+(x+tz) - f_y^+(x)}{t}. \tag{4.1}$$

Note that (4.1) is actually the lower Dini derivative of f_y^+.

Proposition 4.1. (i) $|\nabla f|_y(x) \geq (-\inf_{\|h\|=1} f'_y(x;h))^+$ *for all* $x, y \in X$.

(ii) *If* $\dim X < \infty$, *then (i) holds as equality and the infimum in the right-hand side is attained.*

Proof. (i) Let $x, y \in X$, $\|h\| = 1$. By (4.1), there exist sequences $t_k \downarrow 0$ and $z_k \to h$ such that

$$\frac{f_y^+(x+t_k z_k) - f_y^+(x)}{t_k} \to f'_y(x;h).$$

Put $x_k := x + t_k z_k$. Then $d(x_k, x) = t_k \|z_k\| \to 0$ and

$$\frac{f_y^+(x) - f_y^+(x_k)}{d(x_k, x)} \to -f'_y(x;h).$$

The conclusion follows from definition (3.2).

(ii) Let $\dim X < \infty$ and $x, y \in X$. Taking into account (i), we need to prove the opposite inequality. If $|\nabla f|_y(x) = 0$, the required inequality holds trivially. Suppose $|\nabla f|_y(x) > 0$. By definition (3.2), there exists a sequence $x_k \to x$ such that

$$\frac{f_y^+(x) - f_y^+(x_k)}{d(x_k, x)} \to |\nabla f|_y(x).$$

Put $t_k := d(x_k, x)$, $z_k := t_k^{-1}(x_k - x)$. Without loss of generality, $z_k \to h$, $\|h\| = 1$. Then

$$f'_y(x;h) \leq \lim_{k \to \infty} \frac{f_y^+(x+t_k z_k) - f_y^+(x)}{t_k} = -|\nabla f|_y(x).$$

$\square$

The next statement provides estimates for the strict outer slope (3.4).

Corollary 4.2. (i) $\overline{|\nabla f|}^{>}(\bar{x}) \geq (-\limsup_{x \to \bar{x},\, f_{\bar{y}}^+(x) \downarrow 0} \inf_{\|h\|=1} f'_{\bar{y}}(x;h))^+$.

(ii) *If* $\dim X < \infty$, *then (i) holds as equality and the infimum in the right-hand side is attained.*

The first assertion of Corollary 4.2 implies a similar estimate using the *strict outer Dini derivative* of f at $\bar{x}$ in direction h:

$$\overline{f'}^{>}(\bar{x};h) = \limsup_{x \to \bar{x},\, f_{\bar{y}}^+(x) \downarrow 0} f'_{\bar{y}}(x;h). \tag{4.2}$$

The slopes considered above characterize descent rates of f. If negative, the lower Dini derivative (4.1) characterizes the descent rate at the given point in the given direction while the strict outer Dini derivative (4.2), if negative, characterizes the "worst" descent rate in the given direction among all points near $\bar{x}$ lying outside $V_{\bar{y}}$.

Corollary 4.3. $\overline{|\nabla f|}^{>}(\bar{x}) \geq (-\inf_{\|h\|=1} \overline{f'}^{>}(\bar{x};h))^+$.

When $x = \bar{x}$ and $y = \bar{y} = f(\bar{x})$ the lower Dini derivative (4.1) takes the form

$$f'_{\bar{y}}(\bar{x}; h) = \liminf_{t\downarrow 0,\, z\to h} \frac{f^+_{\bar{y}}(\bar{x} + tz)}{t} \tag{4.3}$$

and corresponds to constant (3.12).

Proposition 4.4. (i) $|\nabla f|^0(\bar{x}) \le \inf\limits_{\|h\|=1} f'_{\bar{y}}(\bar{x}; h)$.

(ii) *If* $\dim X < \infty$*, then (i) holds as equality and the infimum in the right-hand side is attained.*

The uniform strict slopes (3.6) and (3.8) require different type of directional derivatives. Given $h \in X$, two *uniform strict Dini derivatives* of f at $\bar{x}$ in direction h are defined as

$$\overline{f'}^{\diamond}(\bar{x}; h) := \lim_{\varepsilon\downarrow 0} \limsup_{x\to\bar{x},\, f^+_{\bar{y}}(x)\downarrow 0} \inf_{\|z-h\|<\varepsilon,\, t>0} \frac{f^+_{\bar{y}}(x + tz) - f^+_{\bar{y}}(x)}{t}, \tag{4.4}$$

$$\overline{f'}^{\vartriangle}(\bar{x}; h) := \lim_{\varepsilon\downarrow 0} \limsup_{x\to\bar{x},\, f^+_{\bar{y}}(x)\downarrow 0} \inf_{\substack{\|z-h\|<\varepsilon,\, t>0\\ t\|z\|<f^+_{\bar{y}}(x)}} \frac{f^+_{\bar{y}}(x + tz) - f^+_{\bar{y}}(x)}{t}. \tag{4.5}$$

Proposition 4.5. (i) $\overline{f'}^{\diamond}(\bar{x}; h) \le \overline{f'}^{\vartriangle}(\bar{x}; h) \le \overline{f'}^{>}(\bar{x}; h)$ *for any* $h \in X$*;*

(ii) $\overline{|\nabla f|}^{\diamond}(\bar{x}) \ge (-\inf\limits_{\|h\|=1} \overline{f'}^{\diamond}(\bar{x}; h))^+$*;*

(iii) ${}^{-}\overline{|\nabla f|}^{\diamond}(\bar{x}) \ge (-\inf\limits_{\|h\|=1} \overline{f'}^{\vartriangle}(\bar{x}; h))^+$.

Proof. (i) The inequalities are obvious.

(ii) Let $\|h\| = 1$ and $\varepsilon \in (0, 1)$. By (3.6),

$$\begin{aligned}\overline{|\nabla f|}^{\diamond}(\bar{x}) &\ge \liminf_{x\to\bar{x},\, f^+_{\bar{y}}(x)\downarrow 0} \sup_{\|z-h\|<\varepsilon,\, t>0} \frac{(f^+_{\bar{y}}(x) - f^+_{\bar{y}}(x + tz))^+}{t\|z\|} \\ &\ge (1+\varepsilon)^{-1} \left[- \limsup_{x\to\bar{x},\, f^+_{\bar{y}}(x)\downarrow 0} \inf_{\|z-h\|<\varepsilon,\, t>0} \frac{f^+_{\bar{y}}(x + tz) - f^+_{\bar{y}}(x)}{t} \right]^+.\end{aligned}$$

Passing to the limit as $\varepsilon \downarrow 0$ in the right-hand side of the above inequality, we obtain $\overline{|\nabla f|}^{\diamond}(\bar{x}) \ge (-\overline{f'}^{\diamond}(\bar{x}; h))^+$. This inequality must hold for all $h \in X$ with $\|h\| = 1$. The conclusion follows.

(iii) The proof repeats that of (ii) with appropriate changes caused by the differences between definitions (3.6) and (3.8). □

Using directional derivatives (4.2), (4.4), and (4.5) and taking into account Corollary 4.3 and Proposition 4.5, we can formulate another set of sufficient criteria for the error bound property of f at $\bar{x}$.

UCD1. $\overline{f'}^{\diamond}(\bar{x}; h) < 0$ *for some* $h \in X$.

UCD2. $\overline{f'}^{\vartriangle}(\bar{x}; h) < 0$ *for some* $h \in X$.

CD1. $\overline{f'}^{>}(\bar{x}; h) < 0$ *for some* $h \in X$.

The relationships among various primal space criteria are illustrated in Fig. 4. It is assumed that the space X is Banach and the function $f^+_{\bar{y}}$ is lower semicontinuous.

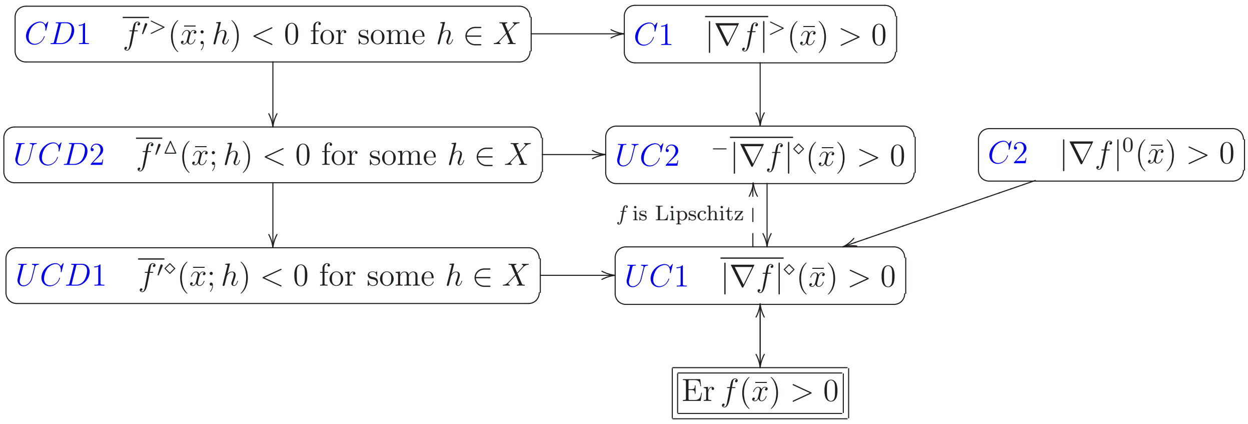

FIGURE 4. Primal space criteria

4.2. **Subdifferentials.** In this subsection, we discuss subdifferential error bounds criteria corresponding to the conditions formulated in Section 3 in terms of (primal space) slopes.

Consider first a subdifferential operator ∂ defined on the class of extended real-valued functions and satisfying the following conditions (axioms):

(A1) For any $f : X \to \mathbb{R}_\infty$ and any $x \in X$, the subdifferential $\partial f(x)$ is a (possibly empty) subset of X^*.

(A2) If f is convex, then ∂f coincides with the subdifferential of f in the sense of convex analysis.

(A3) If $f(u) = g(u)$ for all u near x, then $\partial f(x) = \partial g(x)$.

The majority of known subdifferentials satisfy conditions (A1)–(A3).

In the general case of a function $f : X \to Y$ between normed linear spaces, we define the *subdifferential* and *subdifferential slope* of f at x relative to $y \in Y$ as

$$\partial_y f(x) := \partial(f_y^+)(x) \text{ and}$$

$$|\partial f|_y(x) := \inf\{\|x^*\| :\ x^* \in \partial_y f(x)\}. \tag{4.6}$$

respectively. Here the convention $\inf \emptyset = +\infty$ is in use.

Subdifferential slopes (4.6) are the main building blocks when defining the *strict outer subdifferential slope* of f at $\bar{x}$:

$$\overline{|\partial f|}^{>}(\bar{x}) := \liminf_{x\to\bar{x},\ f_{\bar{y}}^+(x)\downarrow 0} |\partial f|_{\bar{y}}(x). \tag{4.7}$$

If ∂ is the Fréchet subdifferential operator, we will use notations $\partial_y^F f(x)$, $|\partial^F f|_y(x)$, and $\overline{|\partial^F f|}^{>}(\bar{x})$ respectively. Thus

$$\partial_y^F f(x) = \left\{ x^* \in X^* |\ \liminf_{u\to x} \frac{f_y^+(u) - f_y^+(x) - \langle x^*, u-x\rangle}{\|u-x\|} \geq 0 \right\}. \tag{4.8}$$

This is a convex subset of X^*. If $Y = \mathbb{R}$ and $y < f(x)$, then it coincides with the usual Fréchet subdifferential of f at x and does not depend on y. The corresponding to (4.8) subdifferential slopes $|\partial^F f|_y(x)$ and $\overline{|\partial^F f|}^{>}(\bar{x})$ represent dual space counterparts of the primal space slopes (3.2) and (3.4) respectively. The relationship between the constants is straightforward.

Proposition 4.6. (i) *$|\nabla f|_y(x) \leq |\partial^F f|_y(x)$;*

(ii) *$\overline{|\nabla f|}^{>}(\bar{x}) \leq \overline{|\partial^F f|}^{>}(\bar{x})$.*

The inequality in Proposition 4.6 (i) can be strict rather often (for example, if $\partial^F_y f(x) = \emptyset$). If f^+_y is convex, then the two constants coincide (see [11, Theorem 5]).

Let us now impose another condition on the subdifferential operator ∂.

(A4) If x is a point of local minimum of $f+g$, where $f: X \to \mathbb{R}_\infty$ is lower semicontinuous and $g: X \to \mathbb{R}$ is convex and Lipschitz continuous, then for any $\varepsilon > 0$ there exist $x_1, x_2 \in x + \varepsilon\mathbb{B}$, $x_1^* \in \partial f(x_1)$, $x_2^* \in \partial g(x_2)$ such that $|f(x_1) - f(x)| < \varepsilon$, $|g(x_2) - g(x)| < \varepsilon$, and $\|x_1^* + x_2^*\| < \varepsilon$.

Obviously, inequality $|g(x_2) - g(x)| < \varepsilon$ in the above condition can be omitted.

The typical examples of subdifferentials satisfying conditions (A1)–(A4) are Rockafellar-Clarke and Ioffe subdifferentials in Banach spaces and Fréchet subdifferentials in Asplund spaces.

Proposition 4.7. *Let $f^+_{\bar y}$ be lower semicontinuous.*

(i) *If the subdifferential operator ∂ satisfies conditions (A1)–(A4), then $\overline{|\nabla f|}^>(\bar x) \ge \overline{|\partial f|}^>(\bar x)$.*

(ii) *If X is Asplund, then $\overline{|\nabla f|}^>(\bar x) = \overline{|\partial^F f|}^>(\bar x)$.*

Proof. (i) If $\overline{|\nabla f|}^>(\bar x) = \infty$, the assertion is trivial. Take any $\gamma > \overline{|\nabla f|}^>(\bar x)$. We are going to show that $\overline{|\partial f|}^>(\bar x) \le \gamma$. By definition (3.4), for any $\beta \in (\overline{|\nabla f|}^>(\bar x), \gamma)$ and any $\delta > 0$ there is an $x \in B_{\delta/2}(\bar x)$ such that $0 < f^+_{\bar y}(x) \le \delta/2$ and $|\nabla f|_{\bar y}(x) < \beta$. By definition (3.2), x is a local minimum point of the function $u \mapsto f^+_{\bar y}(u) + \beta\|u - x\|$. By (A4) and (A2), this implies the existence of a point $w \in B_{\delta/2}(x)$ with $0 < f^+_{\bar y}(w) \le f^+_{\bar y}(x) + \delta/2$ and an element $x^* \in \partial f^+_{\bar y}(w)$ with $\|x^*\| < \gamma$. The inequality $\overline{|\partial f|}^>(\bar x) \le \gamma$ follows from definition (4.7).

(ii) Since in Asplund spaces Fréchet subdifferentials satisfy conditions (A1)–(A4), the conclusion follows from (i) and Proposition 4.6 (ii). ☐

As a direct consequence of Proposition 4.7 we have the following statement.

Proposition 4.8. *If X is Asplund and $f^+_{\bar y}$ is lower semicontinuous, then $\overline{|\partial^F f|}^>(\bar x) \ge \overline{|\partial f|}^>(\bar x)$ for any subdifferential operator ∂ satisfying conditions (A1)–(A4).*

Thanks to Proposition 4.6, the following sufficient (under appropriate assumptions) subdifferential criteria can be used for characterizing the error bound property.

DC1. $\overline{|\partial f|}^>(\bar x) > 0$.

Consider now the Fréchet subdifferential of f at $\bar x$ relative to $\bar y = f(\bar x)$:

$$\partial^F_{\bar y} f(\bar x) = \left\{ x^* \in X^* | \ \liminf_{x \to \bar x} \frac{f^+_{\bar y}(x) - \langle x^*, x - \bar x\rangle}{\|x - \bar x\|} \ge 0 \right\}.$$

Following [11] we say that a subset $G \subset \partial^F_{\bar y} f(\bar x)$ is a *regular set of subgradients* of f at $\bar x$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$f^+_{\bar y}(x) - \sup_{x^* \in G} \langle x^*, x - \bar x\rangle + \varepsilon\|x - \bar x\| \ge 0, \quad \forall x \in B_\delta(\bar x).$$

The set $\partial^F_{\bar y} f(\bar x)$ itself does not have to be a regular set of subgradients – see [11, Example 10].

A regular set of subgradients is defined not uniquely. The typical examples are

- any finite subset of $\partial^F_{\bar y} f(\bar x)$;
- any subset of a regular set of subgradients;
- the set of all $x^* \in X^*$ such that

$$f^+_{\bar y}(x) \ge \langle x^*, x - \bar x\rangle \quad \text{for all } x \text{ near } \bar x;$$

- in particular, the subdifferential of $f_{\bar y}^+$ at $\bar x$ if $f_{\bar y}^+$ is convex.

Regular sets of subgradients are needed to define the *internal subdifferential slope* of f at $\bar x$:

$$|\partial f|^0(\bar x) = \sup\{r \ge 0|\ r\mathbb{B}^* \text{ is a regular set of subgradients of } f \text{ at } \bar x\}. \tag{4.9}$$

In other words,

$$|\partial f|^0(\bar x) = \sup\left\{ r \ge 0|\ \liminf_{x\to\bar x} \frac{f_{\bar y}^+(x) - \sup\limits_{x^*\in r\mathbb{B}^*}\langle x^*, x-\bar x\rangle}{\|x-\bar x\|} \ge 0 \right\}.$$

Since $\sup_{x^*\in r\mathbb{B}^*}\langle x^*, x-\bar x\rangle = r\|x-\bar x\|$, it follows immediately that $|\partial f|^0(\bar x)$ coincides with the primal space slope defined by (3.12).

Proposition 4.9. $|\partial f|^0(\bar x) = |\nabla f|^0(\bar x)$.

The next proposition is another consequence of definition (4.9).

Proposition 4.10. $|\partial f|^0(\bar x) \le \sup\{r \ge 0:\ r\mathbb{B}^* \subset \partial_{\bar y}^F f(\bar x)\}$.

In infinite dimensions the inequality in Proposition 4.10 can be strict [11, Example 11]. Inequality $|\partial f|^0(\bar x) > 0$ obviously implies inclusion $0 \in \operatorname{int} \partial_{\bar y}^F f(\bar x)$.

Thanks to Proposition 4.9 and condition C2, we can formulate another sufficient subdifferential criteria for the error bound property:

DC2. $|\partial f|^0(\bar x) > 0$.

This criterion is equivalent to C2 and independent of DC1. Note that inclusion $0 \in \operatorname{int} \partial_{\bar y}^F f(\bar x)$ is in general not sufficient.

The following nonlocal modification of the Fréchet subdifferential (4.8), depending on two parameters $\alpha \ge 0$ and $\varepsilon \ge 0$, can be of interest:

$$\partial^\diamond_{y,\alpha,\varepsilon} f(x) = \left\{ x^* \in X^*|\ \inf_{0<\|u-x\|\le\alpha} \frac{f_y^+(u) - f_y^+(x) - \langle x^*, u-x\rangle}{\|u-x\|} \ge -\varepsilon \right\}. \tag{4.10}$$

We are going to call (4.10) the *uniform* (α,ε)*-subdifferential* of f at x relative to y. Obviously it is a convex set in X^*.

Using uniform (α,ε)-subdifferentials (4.10) one can define the *uniform strict subdifferential slope* of f at $\bar x$ – a subdifferential counterpart of the uniform strict slope (3.6):

$$\overline{|\partial f|}^\diamond(\bar x) = \liminf_{\substack{x\to\bar x,\ \varepsilon\downarrow 0,\ f_{\bar y}^+(x)\downarrow 0\\ \alpha - d(x,S_{\bar y}(f))\downarrow 0}} \inf\{\|x^*\|:\ x^* \in \partial^\diamond_{\bar y,\alpha,\varepsilon} f(x)\}. \tag{4.11}$$

Proposition 4.11. (i) $\overline{|\nabla f|}^\diamond(\bar x) \le \overline{|\partial f|}^\diamond(\bar x)$.

(ii) *Suppose that the following uniformity condition holds true for f:*

(UC) *There is a $\delta > 0$ and a function $o: \mathbb{R}_+ \to \mathbb{R}$ such that $\lim_{t\downarrow 0} o(t)/t = 0$ and for any $x \in B_\delta(\bar x)$ with $0 < f_{\bar y}^+(x) \le \delta$ and any $x^* \in \partial_{\bar y}^F f(x)$ it holds*

$$f_{\bar y}^+(u) - f_{\bar y}^+(x) - \langle x^*, u-x\rangle + o(\|u-x\|) \ge 0, \quad \forall u \in X. \tag{4.12}$$

Then $\overline{|\partial f|}^\diamond(\bar x) \le \overline{|\partial^F f|}^>(\bar x)$.

(iii) *If X is Asplund, $f_{\bar y}^+$ is lower semicontinuous near $\bar x$, and uniformity condition (UC) is satisfied then*

$$\operatorname{Er} f(\bar x) = \overline{|\partial^F f|}^>(\bar x) = \overline{|\nabla f|}^>(\bar x) = {}^-\overline{|\nabla f|}^\diamond(\bar x) = \overline{|\nabla f|}^\diamond(\bar x) = \overline{|\partial f|}^\diamond(\bar x) \ge |\partial f|^0(\bar x).$$

Proof. (i) If $x^* \in \partial^{\diamond}_{\bar y,\alpha,\varepsilon} f(x)$ for some $x \notin S_{\bar y}(f)$, $\alpha > d(x, S_{\bar y}(f))$, and $\varepsilon > 0$, then, by (4.10),

$$\sup_{u \neq x} \frac{f^+_{\bar y}(x) - f^+_{\bar y}(u)}{\|u - x\|} = \sup_{0<\|u-x\|\le\alpha} \frac{f^+_{\bar y}(x) - f^+_{\bar y}(u)}{\|u - x\|} \le \|x^*\| + \varepsilon.$$

The conclusion follows from definitions (3.6) and (4.11).

(ii) If $\overline{|\partial^F f|}^{>}(\bar x) = \infty$, the inequality holds trivially. Let $\overline{|\partial^F f|}^{>}(\bar x) < \gamma < \infty$ and $\varepsilon > 0$. By definition (4.7), for any $\delta \in (0, \varepsilon)$ there is an $x \in X$ with $\|x - \bar x\| < \delta$, $0 < f^+_{\bar y}(x) < \delta$ and an $x^* \in \partial^F_{\bar y} f(x)$ with $\|x^*\| < \gamma$. Without loss of generality we can take $\delta > 0$ such that (4.12) holds true and $o(t)/t < \varepsilon$ if $0 < t < \delta$. Then

$$\inf_{0<\|u-x\|\le\delta} \frac{f^+_{\bar y}(u) - f^+_{\bar y}(x) - \langle x^*, u - x\rangle}{\|u - x\|} \ge -\frac{o(\|u - x\|)}{\|u - x\|} > -\varepsilon.$$

Thus, $x^* \in \partial^{\diamond}_{\bar y,\delta,\varepsilon} f(x)$, $d(x, S_{\bar y}(f)) \le \|x - \bar x\| < \delta$, and consequently $\overline{|\partial f|}^{\diamond}(\bar x) < \gamma$.

(iii) follows from (i) and (ii), conditions (3.7) and (3.9), Propositions 3.4, 4.6 (ii), 4.7 (ii), and 4.9, and Theorem 3.3. □

Thanks to Proposition 4.11 (i), the uniform strict subdifferential slope can be used to formulate a necessary condition for the error bound property.

UDC1. $\overline{|\partial f|}^{\diamond}(\bar x) > 0$.

When uniformity condition (UC) is imposed, Fréchet subdifferentials (4.8) gain uniformity properties and, thanks to Proposition 4.11, sufficient condition DC1 in terms of the Fréchet strict outer subdifferential slope becomes also necessary.

The relationships among the subdifferential and primal space error bound criteria on Banach and Asplund spaces are illustrated in Fig. 5 and Fig. 6 respectively. $f^+_{\bar y}$ is assumed lower semicontinuous.

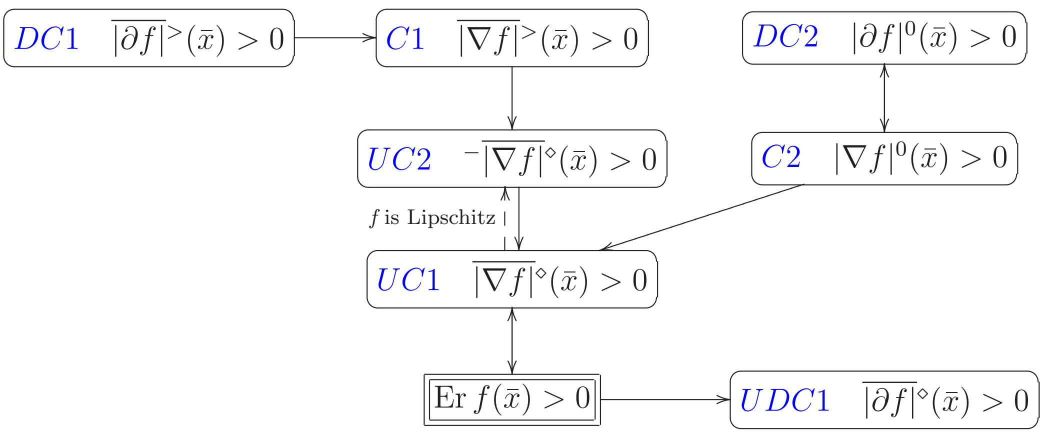

FIGURE 5. Subdifferential and primal space criteria in Banach spaces

5. Finite dimensional and convex cases

In this section, three special cases are considered: error bounds when either X or Y is finite dimensional and in the convex case.

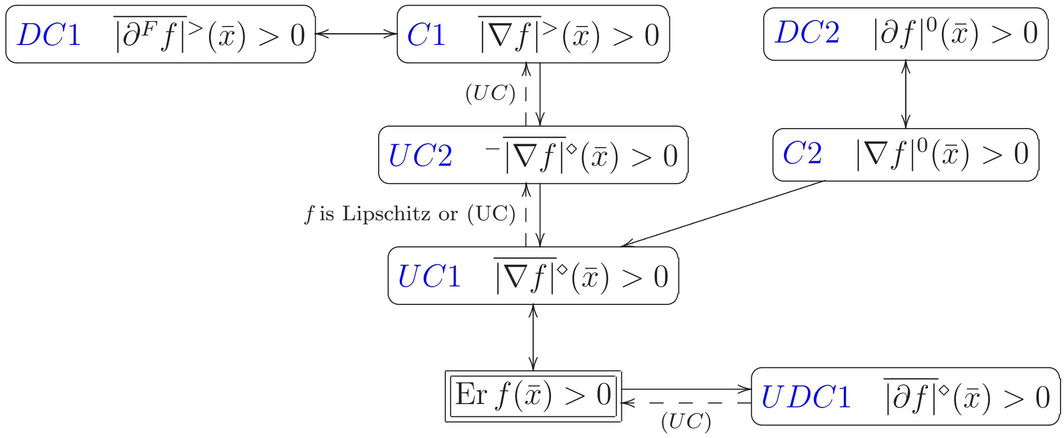

FIGURE 6. Subdifferential and primal space criteria in Asplund spaces

5.1. X **is finite dimensional.** Let $\dim X < \infty$ and $f_{\bar{y}}^{+}$ be lower semicontinuous. Many relations and estimates from previous sections can be simplified and sharpened. In particular, the following limiting subdifferentials can be used:

$$\overline{\partial}^{>} f(\bar{x}) = \operatorname*{Lim\,sup}_{x \to \bar{x},\ f_{\bar{y}}^{+}(x) \downarrow 0} \partial_{\bar{y}}^{F} f(x), \tag{5.1}$$

$$\overline{\partial}^{\diamond} f(\bar{x}) = \operatorname*{Lim\,sup}_{\substack{x \to \bar{x},\ \varepsilon \downarrow 0,\ f_{\bar{y}}^{+}(x) \downarrow 0 \\ \alpha - d(x, S_{\bar{y}}(f)) \downarrow 0}} \partial_{\bar{y},\alpha,\varepsilon}^{\diamond} f(x). \tag{5.2}$$

In the above definitions, $\operatorname{Lim\,sup}$ denotes the outer limit [31] operation for sets: each of the sets (5.1) and (5.2) is the set of all limits of elements of appropriate subdifferentials.

Sets (5.1) and (5.2) are called the *limiting outer subdifferential* [17] and *uniform limiting subdifferential* [11] of f at $\bar{x}$ respectively.

Proposition 5.1. (i) *If f is continuous near $\bar{x}$, then* $^{-}\overline{|\nabla f|}^{\diamond}(\bar{x}) = \overline{|\nabla f|}^{\diamond}(\bar{x})$

(ii) $\overline{|\partial^F f|}^{>}(\bar{x}) = \inf\{\|x^*\| : \ x^* \in \overline{\partial}^{>} f(\bar{x})\}$.

(iii) $|\partial f|^0(\bar{x}) = \sup\{r \geq 0 : \ r\mathbb{B}^* \in \partial_{\bar{y}}^{F} f(\bar{x})\}$.

(iv) $\overline{|\partial f|}^{\diamond}(\bar{x}) = \inf\{\|x^*\| : \ x^* \in \overline{\partial}^{\diamond} f(\bar{x})\}$.

Proof. Assertion (i) follows from [11, Propopsition 11]. Assertions (ii) and (iv) are consequences of definitions (4.7), (4.11), (5.1), and (5.2). Assertion (iii) follows from definition (4.9) and [11, Propopsition 13]. □

Thanks to Proposition 5.1 (ii)–(iv) one can formulate the finite dimensional versions of criteria DC1, DC2, and UDC1.

SD1. $0 \notin \overline{\partial}^{>} f(\bar{x})$.

SD2. $0 \in \operatorname{int} \partial_{\bar{y}}^{F} f(\bar{x})$.

USD1. $0 \notin \overline{\partial}^{\diamond} f(\bar{x})$.

Criteria SD2 and SD1 are in general independent. They can be combined to form a weaker sufficient criterion

$$0 \notin \overline{\partial}^{>} f(\bar{x}) \setminus \operatorname{int} \partial_{\bar{y}}^{F} f(\bar{x}).$$

If $f_{\bar{y}}^{+}$ is semismooth at $\bar{x}$ [22], then $\operatorname{int} \partial_{\bar{y}}^{F} f(\bar{x}) \cap \overline{\partial}^{>} f(\bar{x}) = \emptyset$ [11, Proposition 14], and consequently SD2 ⇒ SD1.

The relationships among the error bound criteria for a lower semicontinuous function on a finite dimensional space are illustrated in Fig. 7.

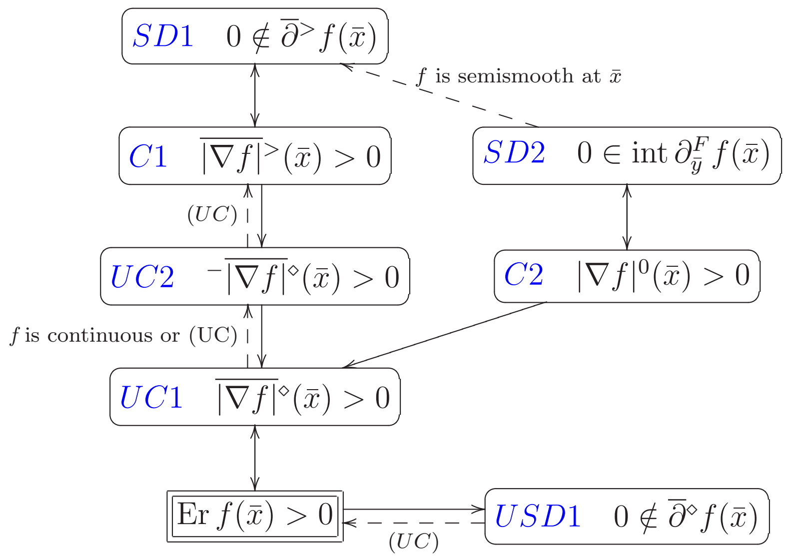

FIGURE 7. Finite dimensional case

5.2. **Convex Case.** In this subsection, X is a general Banach space and $f^+_{\bar{y}}$ is convex lower semicontinuous.

In the convex case, the Fréchet subdifferential (4.8) coincides with the subdifferential of f^+_y at x in the sense of convex analysis and uniformity condition (UC) is satisfied automatically. We will omit superscript F in all denotations where Fréchet subdifferentials are involved. A number of constants considered in the preceding sections coincide while some definitions get simpler.

Proposition 5.2. (i) $\operatorname{Er} f(\bar{x}) = {}^{-}\overline{|\nabla f|}^{\diamond}(\bar{x}) = \overline{|\nabla f|}^{\diamond}(\bar{x}) = \overline{|\nabla f|}^{>}(\bar{x}) = \overline{|\partial f|}^{>}(\bar{x}) = \overline{|\partial f|}^{\diamond}(\bar{x})$.

(ii) $|\partial f|^0(\bar{x}) = \sup\{r \geq 0 : \; r\mathbb{B}^* \in \partial_{\bar{y}} f(\bar{x})\}$.

(iii) $0 \in \operatorname{int} \partial_{\bar{y}} f(\bar{x})$ *if and only if* $0 \notin \operatorname{bd} \partial_{\bar{y}} f(\bar{x})$.

Proof. (i) and (ii) follow from [11, Theorem 5 and Proposition 15]. $\partial_{\bar{y}} f(\bar{x})$ always contains 0 since $f^+_{\bar{y}}(\bar{x}) = 0$. This observation proves (iii). □

Thanks to Proposition 5.2, we have another subdifferential condition for characterizing the error bound property.

SD3. $0 \notin \operatorname{bd} \partial_{\bar{y}} f(\bar{x})$.

The relationships among the error bound criteria for a convex lower semicontinuous function on a Banach space are illustrated in Fig. 8.

5.3. Y **is finite dimensional.** Let $Y = \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, $f = (f_1, f_2, \ldots, f_n) : X \to \mathbb{R}^n$, and $V_y = \{(v_1, v_2, \ldots, v_n) \in \mathbb{R}^n | \; v_i \leq y_i, \; i = 1, 2, \ldots, n\}$. Then $S_y(f)$ and f^+_y take the form (2.7) and (2.8) respectively.

In this subsection, we suppose that Y is equipped with the maximum type norm: $\|y\| = \max(|y_1|, |y_2|, \ldots, |y_n|)$. Then (2.8) can be rewritten as

$$f^+_y(u) = \max_{1 \leq i \leq n} (f_i(u) - y_i)^+ = \left[\max_{1 \leq i \leq n} (f_i(u) - y_i)\right]^+. \tag{5.3}$$

Denote by $I_y(u)$ the set of indexes, for which the maximum in $\max_{1 \leq i \leq n}(f_i(u) - y_i)$ is attained.

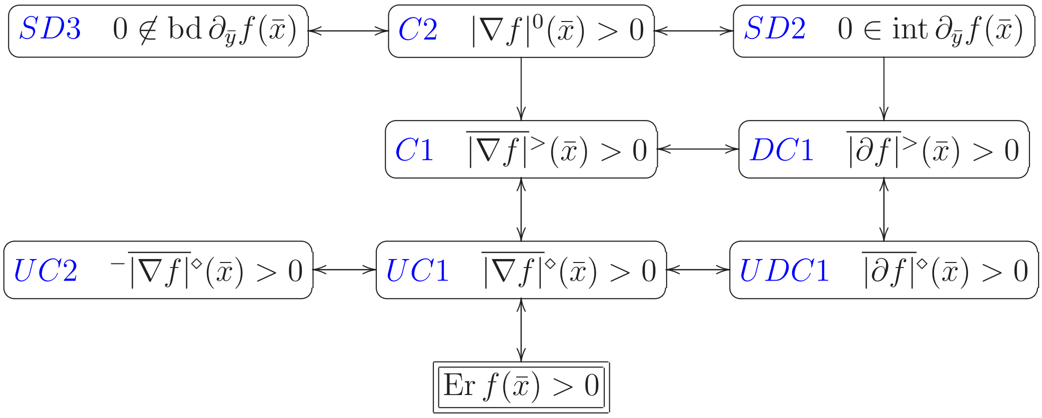

FIGURE 8. Convex case

Let $\bar x \in X$ and $\bar y = f(\bar x)$. The error bound modulus (2.11) of f at $\bar x$ takes the form:

$$\mathrm{Er}\, f(\bar x) = \liminf_{\substack{x \to \bar x\\ x \notin S_{\bar y}(f)}} \max_{1 \le i \le n} \frac{f_i(x) - y_i}{d(x, S_{\bar y}(f))}. \tag{5.4}$$

As it was discussed in Section 3, error bounds can be characterized in terms of slopes. If $f_i(x) \le y_i$ for all $i = 1, 2, \ldots, n$, then, in accordance with (5.3), $f_y^+(x) = 0$ and consequently (due to (3.2)) $|\nabla f|_y(x) = 0$. Otherwise,

$$|\nabla f|_y(x) = \limsup_{u \to x} \min_{i \in I_y(x)} \frac{(f_i(x) - f_i(u))^+}{d(u, x)} = \left| \nabla \left(\max_{i \in I_y(x)} f_i \right) \right| (x), \tag{5.5}$$

provided the functions f_i are lower semicontinuous at x for $i \in I_y(x)$ and upper semicontinuous at x for $i \notin I_y(x)$. The strict outer slope of f at $\bar x$ (3.4) can be computed as

$$\overline{|\nabla f|}^{>}(\bar x) = \liminf_{\substack{x \to \bar x\\ \max\limits_{1 \le i \le n} (f_i(x) - \bar y_i) \downarrow 0}} |\nabla f|_{\bar y}(x), \tag{5.6}$$

where $\bar y_i = f_i(\bar x)$, $\bar y = (\bar y_1, \bar y_2, \ldots, \bar y_n)$, while for the uniform strict slope we have the following formula:

$$\overline{|\nabla f|}^{\diamond}(\bar x) = \liminf_{\substack{x \to \bar x\\ \max\limits_{1 \le i \le n} (f_i(x) - \bar y_i) \downarrow 0}} \sup_{u \ne x} \min_{i \in I_y(x)} \frac{(f_i(x) - f_i(u))^+}{d(u, x)}, \tag{5.7}$$

provided all functions f_i are continuous near $\bar x$. At the same time, constant (3.12) admits the next representation:

$$|\nabla f|^{0}(\bar x) = \liminf_{x \to \bar x} \max_{1 \le i \le n} \frac{(f_i(x) - \bar y_i)^+}{d(x, \bar x)}. \tag{5.8}$$

Application of Theorem 3.3 and Proposition 3.4 leads to the following equivalent representations of the error bound modulus (5.4).

Proposition 5.3. (i) *If X is complete and f_i, $i = 1, 2, \ldots, n$, are continuous, then*

$$\mathrm{Er}\, f(\bar x) = \liminf_{\substack{x \to \bar x\\ \max\limits_{1 \le i \le n} (f_i(x) - \bar y_i) \downarrow 0}} \sup_{u \ne x} \min_{i \in I_y(x)} \frac{(f_i(x) - f_i(u))^+}{d(u, x)} \ge \overline{|\nabla f|}^{>}(\bar x).$$

(ii) *If $|\nabla f|^{0}(\bar x) > 0$, then*

$$\mathrm{Er}\, f(\bar x) = \liminf_{x \to \bar x} \max_{1 \le i \le n} \frac{(f_i(x) - \bar y_i)^+}{d(x, \bar x)}.$$

It is also possible to characterize error bounds using directional derivatives. Given $x, y, h \in X$, suppose the functions f_i are lower semicontinuous at x for $i \in I_y(x)$ and upper semicontinuous at x for $i \notin I_y(x)$. Then for the lower Dini derivative (4.1) of f at x in direction h relative to y we have the following representations: $f'_y(x;h) = 0$ if $f_i(x) < y_i$ for all $i = 1, 2, \ldots, n$; otherwise

$$f'_y(x;h) = \liminf_{t\downarrow 0,\, z\to h} \max_{i\in I_y(x)} \frac{f_i(x+tz) - f_i(x)}{t} = \left(\max_{i\in I_y(x)} f_i\right)'(x;h). \tag{5.9}$$

If all f_i are continuous in a neighborhood of $\bar{x}$, then obviously $I_{\bar{y}}(x) \subset I_{\bar{y}}(\bar{x})$ for all x near $\bar{x}$ and we have the following representations for the strict outer Dini derivative (4.2) and uniform strict Dini (4.4) derivative of f at $\bar{x}$ in direction h:

$$\overline{f'}^{>}(\bar{x};h) = \limsup_{\substack{x\to\bar{x} \\ \max\limits_{1\le i\le n}(f_i(x)-\bar{y}_i)\downarrow 0}} \left(\max_{i\in I_{\bar{y}}(\bar{x})} f_i\right)'(x;h), \tag{5.10}$$

$$\overline{f'}^{\diamond}(\bar{x};h) = \lim_{\varepsilon\downarrow 0} \limsup_{\substack{x\to\bar{x} \\ \max\limits_{1\le i\le n}(f_i(x)-\bar{y}_i)\downarrow 0}} \inf_{\|z-h\|<\varepsilon,\, t>0} \max_{1\le i\le n} \frac{f_i(x+tz) - f_i(x)}{t}. \tag{5.11}$$

Constants (5.6), (5.7), and (5.8) as well as directional derivatives (5.10) and (5.11) give rise to the sufficient error bound criteria C1, UC1, C2, CD1 and UCD1 respectively.

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