VARIATIONAL CONVERGENCE OF BIFUNCTIONS: MOTIVATING APPLICATIONS *

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Abstract. It's shown that a number of variational problems can be cast as finding the maxinf-points (or minsup-points) of bivariate functions, coveniently abbreviated to bifunctions. These variational problems include: linear and nonlinear complementarity problems, fixed points, variational inequalities, inclusions, non-cooperative games, Walras and Nash equilibrium problems. One can then appeal to the theory of lopsided convergence for bifunctions to derive a variety of stability results for each one of these variational problems.

Keywords: lop-convergence, lopsided convergence, minsup-points, maxinf-points, Ky Fan Functions, variational inequalities, Nash and Walras equilibrium points, fixed points, inequality systems, inclusion systems.

AMS Classification: 91B02, 91B26, 90C31, 49F40

Date: January 28, 2011

*Research supported in part by grants of the National Science Foundation, Army Research Office (ARO) W911NS1010246 and Fondap-Matematicas Aplicadas, Universidad de Chile

1 Variational convergence of bifunctions

The analysis of the properties of *bifunctions*, equivalently *bivariate functions*, plays a key role in "Variational Analysis," for example, in the analysis of the Lagrangians associated with an optimization problem, of the Hamiltonians associated with Calculus of Variations and Optimal Control problems, the reward functions associated with cooperative or non-cooperative games, and so on. In this articles, and some related ones, we deal with the stability of the solutions of a wide collection of problems that can be re-cast as finding the maxinf-points of such bifunctions.

More explicitly: given a bifunction $F : C \times D \to I\!R$, we are interested in finding a point, say $\bar{x} \in C$, that maximizes with respect to the (first) x-variable the infimum of F, $\inf_{y \in D} F(\cdot, y)$, with respect to the (second) y-variable. We refer to such a point $\bar{x}$ as a *maxinf-point*. In some particular situations, for example when the bifunction is concave-convex, such a point can be a saddle point, but in many other situation it's just a maxinf-point, or a minsup-point when minimizing with respect to the first variable the supremum of F with respect to the second variable. To study the stability, and the existence, of such points, and the sensitivity of their associated values, one is led to introduce and analyze convergence notion(s) for bifunctions that in turn will guarantee the convergence either of their saddle points [6, 4, 9] or just, of their maxinf-points.

This paper is devoted to a collection of motivating examples. They include include geometric variational inequalities, fixed points problems, linear and nonlinear complementarity problems, Nash equilibrium for non-cooperative games and Walras economic equilibrium problems. Each one of these examples deserves a much more comprehensive analysis than what is possible to include in this 'introductory' article. We shall limit ourselves to illustrating how the general approach is applicable in these various instances.

The major tool is the notion of *lopsided convergence*, introduced in [5] but revisited, and more extensively analyzed, in [20] so that a wider class of applications could be handled. In fact, in [20], the accent is on finite-valued bifunctions that are only defined on a product set, although the connection to the earlier work [5] for the extended real-valued functions framework was also brought to the fore.

2 Examples

In this section, we introduce a number of 'equilibrium' problems that will be considered in this article and show that solving such a problem is equivalent to finding the maxinf-points of an associated bifunction.

2.1 Linear complementarity problems

The *linear complementarity problem (LCP)* can be formulated as follows,

$$\text{find } z \geq 0 \text{ so that } \; Mz + q = w \geq 0 \quad \text{and} \quad w \perp z,$$

where M a $n \times n$-matrix and $q \in I\!R^n$. Let's associate the bifunction

$$K(z, v) = \langle Mz + q, v - z \rangle \quad \text{defined on} \quad I\!R^n_+ \times I\!R^n_+$$

with our complementarity problem.

2.1 Proposition *(maxinf-points and solutions of LCP). $\hat z$ solves the linear complementarity problem if and only if*

$$\hat z \in \mathrm{argmax}_{z\geq 0}\,[\,\inf_{v\geq 0} K(z,v)\,] \quad \textit{and} \quad K(\hat z,\cdot)\geq 0.$$

Proof. Suppose $\hat z$ solve the LCP. Since $\inf_{v\geq 0} K(x,v) = -\infty$ unless $Mz+q\geq 0$, a condition satisfied by $\hat z$, $v=0$ yields the minimum. Hence, $0=\max_{z\geq 0} -\langle Mz+q,z\rangle$ and this maximum is attained by $\hat z$ with $K(\hat z,v)=0+\langle M\hat z+q,v\rangle\geq 0$ for all $v\geq 0$. On the other hand, when $\hat z$ is a maxinf-point of K on $\mathbb{R}^n_+\times\mathbb{R}^n_+$ and $K(\hat z,\cdot)\geq 0$, it implies (i) $\hat z\geq 0$, (ii) $K(\hat z,0)\geq 0$ which yields $\langle M\hat z+q,\hat z\rangle\leq 0$, and (iii) $0\leq -\langle M\hat z+q,\hat z\rangle+\inf_{v\geq 0}\langle M\hat z+q,v\rangle$ implies $M\hat z+q\geq 0$. Combining (ii) and (iii), one obtains $\langle M\hat z+q,\hat z\rangle=0$, and thus $\hat z$ also solves the linear complementarity problem. □

Stability analysis of the solutions to linear complementarity problems can thus be undertaken in terms of the stability properties of the maxinf-points of the corresponding bifunctions. More specifically, given a sequence of linear complementarity problems,

$$\text{find } z\geq 0 \text{ so that } M^\nu z+q^\nu = w\geq 0 \quad\text{and}\quad w\perp z,$$

where M^ν and q^ν converge 'appropriately' to M and q, one can define a sequence of approximating bifunctions

$$K^\nu(z,v)=\langle M^\nu z+q^\nu, v-z\rangle \quad\text{defined on}\quad \mathbb{R}^n_+\times\mathbb{R}^n_+$$

and recast the convergence of the solutions of the approximating linear complementarity problems in terms of the convergence of the maxinf-points of the bifunctions K^ν.

A similar argument, would work when dealing with a *nonlinear complementarity problem (NCP)*,

$$\text{find } z\geq 0 \text{ so that } H(z)\geq 0 \quad\text{and}\quad z\perp H(z)$$

where $H:\mathbb{R}^n\times\mathbb{R}^n$ is a vector-valued function, and the associated bifunction $K:\mathbb{R}^n_+\times\mathbb{R}^n_+$ is defined by $K(z,v)=\langle H(z),v-z\rangle$. However, one can also view such a problem, including the linear complementarity problem, as a special case of our next family of examples.

2.2 Variational inequalities

Let's consider the following *variational inequality (V.I.)*: find $\bar u\in C$, a non-empty, convex subset of $\mathbb{R}^n$, such that

$$\bar u\in C,\quad -G(\bar u)\in N_C(\bar u)$$

with

- G a function from C into $\mathbb{R}^n$, usually, but not necessarily, continuous,
- $N_C(\bar u)=\big\{z\in\mathbb{R}^n\,\big|\,\langle z,u-\bar u\rangle\leq 0,\ \forall u\in C\big\}$ the *normal cone* to C at $\bar u$.

With $\big\{C^\nu\subset\mathbb{R}^n,\ \nu\in\mathbb{N}\big\}$ a sequence of convex sets converging to C and $\big\{G^\nu : C^\nu\to\mathbb{R}^n,\ \nu\in\mathbb{N}\big\}$ a sequence of continuous functions converging (appropriately) to G, we like to find conditions under which one can refer to the variational inequalities: find

$$u\in C^\nu \text{ such that } -G^\nu(u)\in N_{C^\nu}(u),$$

as approximating V.I.'s. In particular, one would like to be able to assert that the solutions of the approximating variational inequalities converge to the solution(s) of the limit one.

As, in the case of the linear complementarity problem, the approach that we follow is to reformulate the problem in the following terms: Define the bifunctions

$$K(u,v) = \langle G(u), v-u \rangle \quad \text{on} \quad \operatorname{dom} K = C \times C$$

and, for $\nu \in I\!N$,

$$K^\nu(u,v) = \langle G^\nu(u), v-u \rangle \quad \text{on} \quad \operatorname{dom} K^\nu = C^\nu \times C^\nu.$$

and, as the next proposition shows, the solutions of the variational inequality can be identified with the maxinf-points of the corresponding bifunction.

2.2 Proposition (identifying V.I.-solutions as maxinf-points). *Consider the V.I.,*

$$u \in C \ \text{so that} \ -G(u) \in N_C(u).$$

and the associated bifunction $K : C \times C \to I\!R$ with

$$K(u,v) = \langle G(u), v-u \rangle.$$

Then, $\bar{u}$ is a solution of the variational inequality if and only if it's a maxinf-point of K, i.e., $\bar{u} \in \operatorname{argmax}_C g$ where $g(u) = \inf_{v \in C} [\, K(u,v) \,]$ on $C^\dagger$, and $K(\bar{u}, \cdot) \geq 0$.

Proof. Observe that $\bar{u} \in C$ is a solution of the variational inequality then $-G(\bar{u}) \in N_C(\bar{u})$ implies $K(\bar{u}, \cdot) \geq 0$ that, in turn, implies $g(\bar{u}) \geq 0$. On the other hand, by definition, for all $u \in C$, $g(u) \leq \langle G(u), u-u \rangle = 0$. Consequently, $g(\bar{u}) = 0$ and hence, $\bar{u}$ maximizes g on C, or still, it's a maxinf-point of K.

Conversely, if $\bar{u}$ maximizes g on C and $K(\bar{u}, \cdot) \geq 0$, then $\langle -G(\bar{u}), v - \bar{u} \rangle \leq 0$ for all $v \in C$, i.e., it's a solution of the variational inequality. □

Of course, the same argument applies to the approximating variational inequalities and their corresponding bifunctions. Convergence of the solution(s) of the variational inequalities can thus be formulated in terms of the convergence of the *maxinf-points* of the bifunctions K^ν to the maxinf-points of K. The main issue will be to identify the appropriate convergence notion for bifunctions that will yield the convergence of these maxinf-points.

To see how a nonlinear complementarity problem,

$$\text{find} \ \ z \geq 0 \ \text{so that} \ \ H(z) \geq 0, \ \ \langle H(z), z \rangle = 0$$

can be formulated as a variational inequality,

$$\text{find} \ \ z \geq 0 \ \text{so that} \ \ \langle -H(z), v - z \rangle \leq 0, \quad \forall\, v \geq 0,$$

can be argued as follows:

† The function g has been extensively studied in the variational inequalities literature under the name of *gap function*, a specific instance of a *merit function* for a variational inequality, cf. [13, §1.5.3]

- if $\bar{z}$ solves NCP then $\langle H(\bar{z}), \bar{z}\rangle = 0$, $\bar{z} \geq 0$ and $H(\bar{z}) \geq 0$ that imply $\langle H(\bar{z}), \bar{z}\rangle = 0 \geq \langle -H(\bar{z}), v\rangle$ for all $v \geq 0$, i.e., $\hat{z}$ solves the V.I.,
- if $\bar{z}$ solves V.I., $\bar{z} \geq 0$ and one has $\langle H(\bar{z}), \bar{z}\rangle \geq \langle H(\bar{z}), v\rangle$ for all $v \geq 0$. Hence $H(\bar{z}) \geq 0$ since otherwise there is no solution ($\to -\infty$). Also, $\langle H(\bar{z}), 0\rangle = 0 \geq \langle H(\bar{z}), \bar{z}\rangle$ and hence $H(\bar{z}) \perp \bar{z}$, i.e., $\bar{z}$ solves the NCP.

Later, it will be shown that also *fixed point problems* also can be dealt within this framework.

2.3 Non-cooperative Games

Consider a game with a finite collection of players $\mathcal{A}$: For each player $a \in \mathcal{A}$, $C_a \subset I\!R^n$ denotes the set of available strategies and the return from choosing $x_a \in C_a$ is $u_a(x_a, x_{-a})$ where x_{-a} is the vector of the strategies selected by the remaining players: $\mathcal{A} \setminus \{a\}$;

$$\forall\, \tilde{a} \in \mathcal{A} : \quad u_{\tilde{a}} : C_{\tilde{a}} \times \prod_{a \in \mathcal{A}\setminus \tilde{a}} C_a \to I\!R.$$

The strategies $x^* = (x^*_a,\, a \in \mathcal{A})$ determine a *Nash equilibrium point* of this game, when

$$\forall\, a \in \mathcal{A} : \quad x^*_a \in \mathrm{argmax}_{x_a \in C_a}\, u_a(x_a, x^*_{-a});$$

for further reference, let's denote this game by $\mathcal{G} = \big\{(C_a, u_a) \,\big|\, a \in \mathcal{A}\big\}$.

For all $a \in \mathcal{A}$, $\big\{C^\nu_a,\, \nu \in I\!N\big\}$ a sequence of sets converging to C_a and $\big\{u^\nu_a, \nu \in I\!N\big\}$ a sequence of payoff functions converging in an appropriate sense to u_a, we are looking for conditions that will allow us to assert that the games $\mathcal{G}^\nu = \big\{(C^\nu_a, u^\nu_a) \,\big|\, a \in \mathcal{A}\big\}$ approximate $\mathcal{G}$, i.e., that the Nash equilibrium points of the games $\mathcal{G}^\nu$ approximate those of the game $\mathcal{G}$.

Our approach to obtain existence and continuity results again relies on setting up a bifunction N, to which one refers as the *Nikaido-Isoda function*, defined as follows: $N : C \times C \to I\!R$ where $C = \prod_{a\in\mathcal{A}}$ and

$$N(x, y) = \sum_{a\in\mathcal{A}} \Big(u_a(x_a, x_{-a}) - u_a(y_a, x_{-a})\Big).$$

2.3 Proposition (Nash equilibrium points as maxinf-points). *The strategies $x^* = (x^*_a,\, a \in \mathcal{A})$, with $x^*_a \in C_a$ for all $a \in \mathcal{A}$, determine a Nash equilibrium point if and only if*

$$\sum_{a\in\mathcal{A}} \Big(u_a(x^*_a, x^*_{-a}) - u_a(y_a, x^*_{-a})\Big) \geq 0 \quad \textit{for all} \quad y_a \in C_a,$$

or equivalently, if and only if

$$x^* \ \ \textit{is a maxinf-point of} \ \ N \ \ \textit{and} \ \ \inf_{y\in C} N(x^*, y) \geq 0,$$

where $x = (x_a,\, a \in \mathcal{A}), y = (y_a,\, a \in \mathcal{A})$.

Proof. If $x^* = (x^*_a,\, a \in \mathcal{A})$ is a Nash equilibrium point, for all $a \in \mathcal{A}$,

$$u_a(x^*_a, x^*_{-a}) - u_a(y_a, x^*_{-a}) \geq 0, \quad \forall\, y_a \in C_a,$$

and consequently the sum over $a \in \mathcal{A}$ must also be nonnegative. On the other hand, if

$$\sum_{a \in \mathcal{A}} \Big(u_a(x^*_a, x^*_{-a}) - u_a(y_a, x^*_{-a}) \Big) \geq 0 \;\; \forall\, y_a \in C_a,$$

it implies, in particular, that given any player $a \in \mathcal{A}$, for all $y_a \in C_a$,

$$u_a(x^*_a, x^*_{-a}) - u_a(y_a, x^*_{-a}) + \sum_{a' \in \mathcal{A} \setminus a} \Big(u_{a'}(x^*_{a'}, x^*_{-a'}) - u_{a'}(x^*_{a'}, x^*_{-a'}) \Big) \geq 0.$$

The second term in this sum is 0, and thus $x^*_a \in \operatorname{argmax}_{x_a \in C_a} u_a(x_a, x^*_{-a})$, i.e., $x^* = (x^*_a,\, a \in \mathcal{A})$ is a Nash equilibrium point.

Turning to the second identity involving N, observe that for all $x \in C$, $\inf_y N(x,y) \leq 0$. Indeed, if for all $a \in \mathcal{A}$, $x_a \in C_a$,

$$\inf_{y \in C} N(x,y) = \sum_{a \in \mathcal{A}} \Big[u_a(x_a, x_{-a}) - \sup_{y_a \in C_a} u_a(y_a, x_{-a}) \Big].$$

Clearly,

$$\forall\, a \in \mathcal{A}: \quad u_a(x_a, x_{-a}) - \sup_{y_a} u_a(y_a, x_{-a}) \leq 0,$$

and hence $\inf_y N(x,y) \leq 0$. But since for $x^* = (x^*_a,\, a \in \mathcal{A})$, a Nash equilibrium point, $\inf_y N(x^*, y) = N(x^*, x^*) = 0$, it follows that

$$x^* \in \operatorname{argmax}_x \big[\inf_y N(x,y) \big].$$

Conversely, if $x^* = (x^*_a,\, a \in \mathcal{A}) \in C$ is a maxinf-point of N such that $\inf_{y \in C} N(x^*, y) \geq 0$, it means

$$\sum_{a \in \mathcal{A}} \Big(u_a(x^*_a, x^*_{-a}) - u_a(y_a, x^*_{-a}) \Big) \geq 0 \quad \text{for all} \quad y_a \in C_a,$$

i.e., x^* is a Nash equilibrium point as follows from our first assertion. □

One defines similarly the Nikaido-Isoda functions N^ν associated with the games $\mathcal{G}^\nu$, and in view of Proposition 2.3, $x^\nu = (x^\nu_a,\, a \in \mathcal{A})$ is a Nash equilibrium point for $\mathcal{G}^\nu$ if and only if it's a maxinf-point of N^ν.

Here again, the question of the convergence of Nash equilibrium points can be formulated in terms of the convergence of the maxinf-points of the (bivariate) Nikaido-Isoda functions N^ν to those of the function N.

2.4 Walras economic equilibrium

Our last example is a classical equilibrium problem in Economics. Here, we deal only with the *Pure Exchange model* or equivalently a Walras barter problem, but one can easily extend the result to the case when the economy also includes producers [3]. The economy is described by

$$\mathcal{E} = \big\{ (u_a, C_a, e_a),\, a \in \mathcal{A} \big\},$$

where

$\mathcal{A}$: the finite set of agents;
$e_a \in I\!R^n$: agent's $a \in \mathcal{A}$ *endowment*, a bundle of goods to be traded;
$C_a \subset I\!R^n_+$ non-empty, convex set identifying agent's a *survival set*,
$u_a : C_a \to I\!R$, agent's a *utility function*.

Trading takes place at a per-unit market price p_j for good j, $j = 1, \dots, n$. The bundle of goods agent a could acquire is thus limited to those satisfying $\langle p, x\rangle \leq \langle p, e_a\rangle$. It's assumed that agents act as utility maximizers. Thus, given $p \in I\!R^n_+$, each agent $a \in \mathcal{A}$ will end up with its consumption demand

$$c_a(p) = \mathrm{argmax}_{x \in C_a} \left\{ u_a(x) \,\middle|\, \langle p, x\rangle \leq \langle p, e_a\rangle \right\};$$

our assumptions will imply that $c_a(p)$ is well-defined. Note that $c_a(p) = c_a(\alpha p)$ for any positive scalar α, i.e., the agents demand functions are homogeneous of degree 0 with respect to prices. So, we may as well assume that $p \in \Delta$ with $\Delta = \left\{ p \in I\!R^n_+ \,\middle|\, \sum_{j=1}^n p_j = 1 \right\}$, the *price simplex*.

This economy $\mathcal{E}$ is operational only if for each good, total supply exceeds total demand, i.e., if

$$\sum_{a \in \mathcal{A}} s_a(p) = s(p) \geq 0 \quad \text{where} \quad s_a(p) = e_a - c_a(p);$$

this is the *market clearing* condition. The function $s : I\!R^n_+ \to I\!R^n$ is called the *excess supply function*. A price vector $\bar{p} \in \Delta$ so that $s(\bar{p}) \geq 0$ is called an *equilibrium price* (Walras, 1874). The existence of an equilibrium price isn't clear cut, albeit well-known since the mid-50's [11]. Here, our predominant objective is the analysis of the sensitivity of equilibrium prices to perturbations of the economy:

- perturbations of the agents' utilities,
- perturbations of the agents' endowments.

For this purpose, one introduces the bifunction $W : \Delta \times \Delta \to I\!R$, to which one refers as the *Walrasian*, defined by

$$W(p, q) = \langle q, s(p)\rangle.$$

Eventually, existence and stability results will be deduced from the general results applied to the function W. At this point, we limit ourselves to identifying the maxinf-points of the Walrasian with the equilibrium prices of $\mathcal{E}$.

2.4 Proposition (Walras equilibirum prices and maxinf-points). *A price vector $\bar{p}$ is an equilibrium point for the economy $\mathcal{E}$ if and only if it's a maxinf-point of the associated Walrasian with $W(\bar{p}, \cdot) \geq 0$ on Δ.*

Proof. Suppose $\bar{p}$ is an equilibrium price. This means $\bar{p} \in \Delta$ and $s(\bar{p}) \geq 0$. Then, any p with $s(p) \not\geq 0$ can't be a maxinf-point since for any such $p \in \Delta$,

$$\inf_{q \in \Delta} \langle q, s(p)\rangle < \inf_{q \in \Delta} \langle q, s(\bar{p})\rangle.$$

Thus, any maxinf-point p of the Walrasian must have $s(p) \geq 0$, and then $W(p, q)$ is necessarily non-negative for all $q \in \Delta$.

Conversely, if $\bar{p}$ is a maxinf-point of the Walrasian with $W(\bar{p}, \cdot) \geq 0$, it follows that for all unit vectors $e^j = (0, \dots, 1, \dots, 0)$, the jth entry is 1, $\langle e^j, s(\bar{p})\rangle \geq 0$, and this implies that $s(\bar{p}) \geq 0$. □

3 Lopsided convergence

To derive stability results for such variational problems, that we placed under the general heading of 'equilibrium problems,' we rely on *lopsided convergence*, or *lop-convergence* for short. In this section, we review the basic definitions and results from [20] that will be used in the sequel. Precisely, to deal with these applications, we had to revisit the definition of convergence for *bifunctions*. The (almost) classical framework, going back to Rockafellar's work on saddle functions [24, Sections 33-39], was to consider bifunctions that are extended real-valued and defined everywhere on the product of two linear spaces, here $\mathbb{R}^n$ and $\mathbb{R}^m$. It turns out that this paradigm comes with unnecessary technical obstacles on the theoretical side and renders it awkward when dealing with the applications being considered. This has led us to a 'parallel' framework where the bifunctions —and also functions in the univariate case— are real-valued and only defined on the product of subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$. The definition of lop-convergence is necessarily one-sided, in that one is either interested in the convergence of maxinf-points or minsup-points but not both; in general, the maxinf-points are not minsup-points and vice-versa. Here, as in [20], definitions and results are stated for the 'maxinf'-case. We conclude the section with an application of lopsided convergence to obtain an extension of Ky Fan's Inequality [14] to situations when the domain of definition is not necessarily compact; for proofs and further analysis refer to [20].

Thus, henceforth, the term *bifunction* is reserved for *finite-valued bivariate functions* defined on the product of two non-empty subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$. One writes,

$$fv\text{-biv}(\mathbb{R}^{n+m}) = \big\{F\colon C\times D\to\mathbb{R}\,\big|\,\emptyset\neq C\subset\mathbb{R}^n,\ \emptyset\neq D\subset\mathbb{R}^m\big\}$$

to denote this class of bifunctions and $\bar{x}$ as a *maxinf-point* of $F\in fv\text{-biv}(\mathbb{R}^{n+m})$ if

$$\bar{x}\in\operatorname*{argmax}_{x\in C}\big[\inf_{y\in D}F(x,y)\big].$$

3.1 Definition (lop-convergence, fv-biv). *A sequence in* fv-biv$(\mathbb{R}^{n+m})$, $\big\{F^\nu : C^\nu\times D^\nu\to\mathbb{R}\big\}_{\nu\in N}$ lop-converges, *or* converges lopsided, *to a function* $F: C\times D\to\mathbb{R}$, *also in* fv-biv$(\mathbb{R}^{n+m})$, *if*

(a) *for all* $y\in D$ *and all* $(x^\nu\in C^\nu)\to x$, *there exists* $(y^\nu\in D^\nu)\to y$ *such that*

$$\operatorname{limsup}_\nu F^\nu(x^\nu,y^\nu)\le F(x,y)\ \text{ when }\ x\in C\quad\text{ and }\quad F^\nu(x^\nu,y^\nu)\to-\infty\ \text{ when }\ x\notin C;$$

(b) *for all* $x\in C$, *there exists* $(x^\nu\in C^\nu)\to x$ *such that given any* $(y^\nu\in D^\nu)\to y$,

$$\operatorname{liminf}_\nu F^\nu(x^\nu,y^\nu)\ge F(x,y)\ \text{ when }\ y\in D\quad\text{ and }\quad F^\nu(x^\nu,y^\nu)\to\infty\ \text{ when }\ y\notin D.$$

Lop-convergence is ancillary tight *when (b) is strengthened to*

(b-t) *(b) holds and for any* $\varepsilon>0$ *one can find a compact set* B_ε, *possibly depending on* $\{x^\nu\to x\}$, *such that for all* ν *sufficiently large,*

$$\inf_{D^\nu\cap B_\varepsilon}F^\nu(x^\nu,\cdot)\le\inf_{D^\nu}F^\nu(x^\nu,\cdot)+\varepsilon.$$

Finally, it's said to be tight *if it's ancillary-tight and (a) is strengthened to*

(a-t) *(a) holds and for all* $\varepsilon>0$ *there is a compact set* A_ε *such that for all* ν *large enough,*

$$\sup_{x\in C^\nu\cap A_\varepsilon}\inf_{y\in D^\nu}F^\nu(x,y)\ge\sup_{x\in C^\nu}\inf_{y\in D^\nu}F^\nu(x,y)-\varepsilon,$$

3.2 Theorem (convergence of maxinf-points; [20, Theorem 4]). *When the bifunctions* $\{F^\nu\}_{\nu\in N}$ *lop-converge ancillary tightly to* F*, all in* fv-biv$(\mathbb{R}^{n+m})$ *and* $\varepsilon_\nu \searrow 0$ *then* any cluster point $\bar{x}$ of a sequence of ε_ν-maxinf-points of the bifunctions F^ν is a maxinf-point of the limit function F.

If the convergence is actually (fully) tight and $\sup_x \inf_y F(x,y)$ *is finite, then* [20, Theorem 5] $\sup_x \inf_y F^\nu(x,y) \to \sup_x \inf_y F(x,y)$ *and if* $\bar{x}$ *is a maxinf-point of* F*, one can always find sequences* $\left\{\varepsilon^\nu \searrow 0,\ x^\nu \in \varepsilon^\nu\text{-argmax}(\inf_y F^\nu(\cdot,y)\right\}_{\nu\in N}$ *such that* $x^\nu \to \bar{x}$.

Conversely, if such sequences exist, then $\sup_x \inf_y F^\nu(x,y) \to \inf_y F(\bar{x},y)$.

Proof. This statement sharpens the conclusions of [20, Theorem 4], namely, instead of just asserting the convergence of the maxinf-points, under ancillary tightness, it now claims the convergence of ε_ν-maxinf points. The argument is similar, it again relies on [20, Theorem 3] to conclude that the inf-projection g^ν of the bifunctions F^ν hypo-converge to the inf-projection g of F, and one needs to show that in the case of hypo-convergence for any sequence $\varepsilon_\nu \searrow 0$, the outer limit of the ε- argmin g^ν is contained in argmin g. From that point on, the proof becomes identical to that of [25, Theorem 7.31(b)]. □

It might be useful at this point to emphasize the role played by ancillary tightness and thus, indirectly, by [25, Theorem 7.31]. Let's begin with a simple example where $f^\nu \xrightarrow{e} f$, $\inf f \in \mathbb{R}$, $\operatorname{argmin} f \neq \emptyset$ but $\inf f^\nu \not\to \inf f$ precisely because for all $\varepsilon > 0$ one can't find a compact set B_ε and an index set $N_\varepsilon \in \mathcal{N}_\infty$ such that $\inf_{B_\varepsilon} f^\nu \le \inf f^\nu + \varepsilon$ for all $\nu \in N_\varepsilon$. Let

$$f^\nu(x) = \begin{cases} 0 & \text{when } x \neq 0, x \neq \nu \\ -1 & \text{for } x = 0 \\ -\nu & \text{for } x = \nu. \end{cases}$$

Clearly, $f^\nu \xrightarrow{e} f$ with $f(0) = -1$ and $f(x) = 0$ when $x \neq 0$. $\inf f$ is finite and attained at $x = 0$, but $\inf f^\nu \searrow -\infty \neq -1$! And it's obvious that there is no compact set and an index set with the desired properties. Note that example also works with defining $f^\nu(\nu) = -2$, for example.

In particular, this implies that the existence of an argmin point doesn't allows us to conclude that the 'required' compact sets exist. In the case of bifuncions this translates as follows: *the existence of a maxinf-point for the limit problem doesn't guarantee ancillary tightness, as one might have expected or hoped for*. The example shows that one must be concerned about other maxinf-points that disappear at (converge to) the horizon.

A bifunction $F\colon C \times C \to \mathbb{R}$ in fv-biv$(\mathbb{R}^{2n})$ with C a non-empty convex subset of $\mathbb{R}^n$ such that

(a) $\forall\, y \in C$: $x \mapsto F(x,y)$ is usc on C,

(b) $\forall\, x \in C$: $y \mapsto F(x,y)$ is convex on C.

is said to be a *Ky Fan function*. Note that the set C is not required to be compact. However,

3.3 Lemma (Ky Fan's Inequality; [14], [7, Theorem 6.3.5]). *Suppose* $F : C \times C \to \mathbb{R}$ *is a Ky Fan function with* C *compact and such that* $F(x,x) \ge 0$ *(on* $C \times C$*). Then, the set of maxinf-points of* F *is a nonempty subset of* C*. Moreover, for every maxinf-point* $\bar{x}$ *of* C*,* $F(\bar{x},\cdot) \ge 0$ *on* C.

When, the domain is not compact, one can nevertheless obtain the existence of maxinf-points by relying on [20, Theorem 8] which shows that the lop-limit of a sequence of Ky Fan functions is also a Ky Fan function. Consequently,

3.4 Corollary (extension of Ky Fan's Inequality; [20, Theorem 9]). *If F is a Ky Fan function defined on $C \times C$ with $\emptyset \neq C$ convex and one can find sequences of compact convex sets $\{C^\nu \subset \mathbb{R}^n\}$ and (finite-valued) Ky Fan functions $\{F^\nu : C^\nu \times C^\nu \to \mathbb{R}\}_{\nu \in N}$ lop-converging ancillary tightly to F, then every cluster point $\bar{x}$ of any sequence $\{x^\nu,\, \nu \in \mathbb{N}\}$ of maxinf-points of the F^ν is a maxinf-point of F.*

3.5 Remark (a word of caution). *It should be kept in mind that all one can expect from this general approach applied to specific instances is that it can, mostly, come up with sufficient conditions. For example, in §6 when dealing with the convergence of fixed points, we place ourselves in a 'standard' environment, i.e., the limit problem is to find $\bar{x}$ in a compact, convex set $C \subset \mathbb{R}^n$ that is a fixed point of a continuous function $G : C \to C$ and this limit problem is being approximated by a sequence of problems having similar characteristics. Of course, one can trivially find examples of mappings G, not continuous, that map C into C but nonetheless have a fixed point and the approximating sequence would just consist of problems reproducing the limit one. In such a situation, one might even be able to prove lop-convergence, but the conditions of Theorem 6.2 wouldn't apply, for example. The same type of observation could be made in the case of variational inequalities (§5) as well as to any other class of variational problems analyzed here.*

4 Complementarity Problems

Let's now return to the linear complementarity problem,

$$\text{find } z \geq 0 \text{ so that } \; Mz + q \geq 0, \quad (Mz + q) \perp z,$$

and a sequence of 'approximating' (truncated) linear complementarity problems,

$$\text{find } z \in [0, r^\nu] \text{ so that } \; M^\nu z + q^\nu \geq 0, \quad (M^\nu z + q^\nu) \perp z,$$

where $M^\nu \to M$, $q^\nu \to q$ and for $j = 1, \ldots, n$, $0 < r^\nu_j \to \infty$ as $\nu \to \infty$. We associate the bifunction

$$K(z, v) = \langle Mz + q, v - z \rangle, \quad K\colon \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$$

with the linear complementarity problem and the bifunctions

$$K^\nu(z, v) = \langle M^\nu z + q^\nu, v - z \rangle, \quad K^\nu\colon [0, r^\nu] \times \mathbb{R}^n_+ \to \mathbb{R}$$

with the 'approximating' problems. We used the word 'approximating' with quotes, because these are genuine approximating problems only if it can be shown that these approximating problems actually generate approximating solutions of the given linear complementarity problem. In view of our analysis in §2.1, in particular Proposition 2.1, and the results reviewed in §3, one could validate the term *approximating* by finding conditions under which the bifunctions K^ν lop-converge to K.

4.1 Theorem (lopsided convergence of LCP). *As long as $M^\nu \to M$, $q^\nu \to q$ and $r^\nu \nearrow \infty$, the bifunctions K^ν lop-converge to K. If in addition, $P = \{z \geq 0 \,|\, Mz + q \geq 0\}$ is included in the inner limit, $\operatorname{Liminf}_{\nu\to\infty} P^\nu$, of the polyhedral sets $P^\nu = \{z \in [0, r^\nu] \,|\, M^\nu z + q^\nu \geq 0\}$, then the sequence $\{K^\nu,\ \nu \in \mathbb{N}\}$ lop-converges ancillary tightly to K which means that any cluster point of the solutions of the sequence of 'truncated' linear complementarity problems is a solution of the limiting linear complementarity problems.*

Proof. For any sequence $\{z^\nu \in [0, r^\nu]\}_{\nu\in\mathbb{N}}$ converging to $z \geq 0$ and any $v \geq 0$, $\langle M^\nu z^\nu + q^\nu, v - z^\nu\rangle \to \langle Mz + q, v - z\rangle$ yields condition 3.1(a); the case $z^\nu \to z$ with $x \notin \mathbb{R}^n_+$ doesn't have to be considered since no sequence $\{z^\nu \in [0, r^\nu]\}_{\nu\in\mathbb{N}}$ can converge to a point $z \notin \mathbb{R}^n_+$. Finally, given $z \in \mathbb{R}^n_+$, eventually $z \in [0, r^\nu]$ for all ν sufficiently large, say $\nu \geq \nu_z$, since the entries of the vector r^ν all converge to ∞. So, given $z \geq 0$, define the sequence $z^\nu \to z$ to be any point $z^\nu \in [0, r^\nu]$ for $\nu < \nu_z$ and $z^\nu = z$ when $\nu > \nu_z$. Clearly, such a sequence converges to z and whatever be the sequence $v^\nu \geq 0 \to v \geq 0$, one has $\langle M^\nu z^\nu + q^\nu, v^\nu\rangle \to \langle Mz + q, v\rangle$, i.e., condition 3.1(b) is also satisfied.

Ancillary tightness requires in addition that given any sequence $z^\nu \in [0, r^\nu] \to z$, for all $\varepsilon > 0$, one can find a compact set B_ε and an index ν_ε such that for $\nu > \nu_\varepsilon$,

$$\inf_{v\in\mathbb{R}^n_+\cap B_\varepsilon} \langle M^\nu z^\nu + q^\nu, v\rangle \leq \inf_{v\in\mathbb{R}^n_+} \langle M^\nu z^\nu + q^\nu, v\rangle + \varepsilon;$$

note that B_ε isn't necessarily a rectangle of the type $[0, r^\nu]$. This condition can only be satisfied if the right-hand sides stay bounded, this requires that the (limiting) linear program,

$$\min \langle Mz + q, v\rangle \ \text{so that} \ \ v \geq 0,$$

and the approximating ones, for ν sufficiently large,

$$\min \langle M^\nu z^\nu + q^\nu, v\rangle \ \text{so that} \ \ v \geq 0,$$

be bounded, i.e., their optimal solutions is $v = 0$ with optimal value 0. This will only occur if $Mz + q$ and $M^\nu z + q^\nu$, for ν sufficiently large, are both non-negative. So, to satisfy condition 3.1(b-t), it's necessary that for all $\bar z \in P = \{z \in \mathbb{R}^n_+ \,|\, Mz + q \geq 0\}$, one can find a sequence $z^\nu \in P^\nu = \{z \in [0, r^\nu] \,|\, M^\nu z + q^\nu \geq 0\}$ converging to $\bar z$. This means precisely that P must be included in the inner limit of the polyhedral sets P^ν, cf. [25, Definition 4.1]. □

One can seek conditions that will guarantee: the inner-limit of the polyhedral sets P^ν is included in P. As it turns out, as long as the sets P^ν are non-empty for ν sufficiently large, P always includes the outer limit, $\limsup_{\nu\to\infty} P^\nu$, when $M^\nu \to M$, $q^\nu \to q$ and the entries of $r^\nu \to \infty$. This follows almost immediately from the definition [27, Proposition 1]. Thus, we are essentially requiring that $P = \lim_{\nu\to\infty} P^\nu$, i.e., P is actually the limit of the polyhedral sets P^ν. A substantial literature, surveyed and complemented in [27, 22], has been devoted to this issue. Hence, we won't deal here with all specific instances that are of particular interest in the theory underlying the linear complementarity problem, cf. [10], that would lead us too far astray from the main theme of this paper. Let's just record a couple of sufficient conditions that will serve as examples.

Let's suppose the approximating problems are formulated without upper bounds on z, i.e.,

$$\text{find } z \in \mathbb{R}^n_+ \ \text{so that} \ \ M^\nu z + q^\nu \geq 0, \quad (M^\nu z + q^\nu) \perp z,$$

with associated bifunctions

$$K^\nu(z,v) = \langle M^\nu z + q^\nu, v - z\rangle, \quad K^\nu\colon I\!R^n_+ \times I\!R^n_+ \to I\!R.$$

Let $\operatorname{pos} A$ denote the *positive hull* of the columns of the matrix A. Then, ancillary tightness of the collection of bifunctions $\{K^\nu,\, \nu \in I\!N\}$ can be characterized as follows:

4.2 Corollary (continuity of polyhedral set-valued mappings). *With $P = \{z \geq 0 \,|\, Mz + q \geq 0\}$, $P^\nu = \{z \geq 0 \,|\, M^\nu z + q^\nu \geq 0\}$, then $P = \operatorname{Lim}_{\nu\to\infty} P^\nu$ if and only if*

$$\operatorname{pos}\begin{pmatrix} M^\top & I & 0 \\ -q & 0 & -1 \end{pmatrix} \supset \operatorname*{Limsup}_{\nu\to\infty} \operatorname{pos}\begin{pmatrix} (M^\nu)^\top & I & 0 \\ -q^\nu & 0 & -1 \end{pmatrix}.$$

where q and q^ν are the row-versions of these vectors. Thus, under this last condition, the bifunctions K^ν are ancillary tight, and consequently, any cluster point of solutions of the approximating LCP's is a solution of the limiting linear complementarity problem.

Proof. It suffices to appeal to [27, Theorem 3].[‡].

References [26, 22] provide a number of sufficient conditions. Situations that are more amenable to immediate verification can be found in [27], for example:

4.3 Corollary (non-empty interior criterion, [27, Corollary 7]). *Suppose that for ν sufficiently large, the polyhedral set P^ν and P have non-empty interior and no row of the matrix $[M, q]$ is identically 0, then $P^\nu \to P$.*

Of course, one could consider other approximation schemes than the one discusses so far. One could rely on truncations: for example, for a sequence $0 < r^\nu \nearrow \infty$,

$$\text{find } z \in [0, r^\nu] \text{ so that } M^\nu z + q^\nu \in [0, r^\nu], \quad (M^\nu z + q^\nu) \perp z,$$

with corresponding bifunctions

$$K^\nu_\square(z,v) = \langle M^\nu z + q^\nu, v - z\rangle, \quad K^\nu_\square\colon [0, r^\nu] \times [0, r^\nu] \to I\!R.$$

Since $[0.r^\nu]$ is compact convex, it's known that problems of this type always have a solution, see [19], for example.

4.4 Theorem (lop-convergence of LCP, variant). *The bifunctions $K^\nu_\square$ lop-converge to K when $M^\nu \to M$, $q^\nu \to q$ and $r^\nu \nearrow \infty$. If in addition, $\operatorname{Liminf}_\nu P^\nu_\square \supset P = I\!R^n_+ \cap \{z \,|\, Mz + q \geq 0\}$ where*

$$P^\nu_\square = [0, r^\nu] \cap \{z \,|\, M^\nu z + q^\nu \geq 0\},$$

then lop-convergence is ancillary tight, and this means that any cluster point of a sequence of solutions of the approximating problems is a solution of the (given) limit problem.

[‡] for a slightly sharper but more involved condition, cf. [22, Corollary 4.6]

Proof. The same arguments as those used to obtain lop-convergence and ancillary tightness in Theorem 4.1 also work here; except that now the sequence $v^\nu \to v$ is such that $v^\nu \in [0, r^\nu]$. $\square$

One could also rely on the truncation used by Gowda and Pang [16], and more recently by Flores-Bazán and López [15], in their stability analysis of the solutions of linear complementarity problems. Namely, let $0 < d \in I\!R^n$ and $\alpha_\nu \nearrow \infty$, a sequence of positive scalars. The approximating problems are

$$\text{find } z \in \triangle^\nu = \{z \in I\!R^n_+ \,|\, \langle d, z\rangle \leq \alpha_\nu\} \text{ so that } M^\nu z + q^\nu \geq 0, \quad (M^\nu z + q^\nu) \perp z,$$

with

$$P^\nu_\triangle = \triangle^\nu \cap \{z \,|\, M^\nu z + q^\nu \geq 0, \ \}$$

and associated bifunctions

$$K^\nu_\triangle(z, v) = \langle M^\nu z + q^\nu, v - z\rangle, \qquad K^\nu_\triangle : \triangle^\nu \times \triangle^\nu \to I\!R.$$

Again, since $\triangle^\nu$ is compact convex, it's known that problems of this type always have at least one solution.

4.5 Theorem (lop-convergence of LCP, another variant). *The bifunctions $K^\nu_\triangle$ lop-converge to K when $M^\nu \to M$, $q^\nu \to q$ and $\alpha^\nu \nearrow \infty$. If in addition,* $\text{Liminf}_\nu P^\nu_\triangle \supset P = \triangle^\nu \cap \{z \,|\, Mz + q \geq 0\}$ *then lop-convergence is ancillary tight, and it means that any cluster point of a sequence of solutions of the approximating problems is a solution of the (given) limit problem.*

Proof. Similar to that proof of Theorem 4.4 except that the sequence $v^\nu \to v$ now has $v^\nu \in \triangle^\nu$. $\square$

Existence of solutions to the truncated LCP problems is well-known. Here, one could derive it directly from Ky Fan's Inequality 3.3 since the sets $[0, r^\nu]$, as well as $\triangle^\nu$, are non-empty, compact and convex and the functions $K^\nu_\square$ and $K^\nu_\triangle$ are Ky Fan functions, cf. §3. But even ancillary tight lop-convergence of these functions doesn't settle the question of the existence of solutions to limit LCP. This can be illustrated by the following simple example.

4.6 Example (ancillary tightness and existence of solutions). *Let*

$$M^\nu = \begin{bmatrix} 0 & \nu^{-1} \\ 1 & 1 \end{bmatrix}, \qquad M = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \qquad q^\nu = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = q, \qquad r^\nu = \begin{pmatrix} \nu \\ \nu \end{pmatrix},$$

then $P^\nu_\square \to P = \emptyset$ and, consequently, the lop-convergence of the bifunctions $K^\nu_\square$ to K is ancillary tight, but clearly the solutions $z^\nu = (\nu, 0)$ of the truncated problem don't converge to a solution of the (limit) linear complementarity problem that happens to have no solution; Gowda and Pang [16] and Flores-Bazán and López [15] go through a detailed analysis that allows then to guarantee the existence of solutions for matrices M that fall in very specific classes.

Deriving 'simple' verifiable conditions that enable us to assert that the sequence is tight requires a challenging study of quadratic forms that goes beyond the scope of this article.

5 Convergence of variational inequalities

We return to variational inequalities as introduced in §2.1, i.e., let $\emptyset \neq C \subset I\!R^n$, convex, compact, $G : C \to I\!R^n$ a, not necessarily continuous, function. The problem is

$$\text{find } u \in C \quad \text{such that} \quad -G(u) \in N_C(u)$$

where $N_C(u)$ is the normal cone of C at u. Such a problem can be reformulated (Proposition 2.2 as finding the maxinf-point of the following Ky Fan bifunction, namely,

$$K(u, v) := \langle G(u), v - u \rangle \quad \text{with} \quad \operatorname{dom} K = C \times C.$$

Clearly, this bifunction is convex (linear) in v, it's usc in u on C when G is continuous and moreover, $K(u, u) = 0$. Ky Fan's inequality [20, Lemma 2] then guarantees the existence of maxinf-point, say $\bar{u}$, that is a solution of this variational inequality with $K(\bar{u}, \cdot) \geq 0$.

From the convergence results in §3, Corollary 3.4 and Theorem 3.2, it follows that a sequence of Ky Fan bifunctions $\{K^\nu\}_{\nu \in N}$ lop-converges ancillary tightly to the bifunction K will yield a maxinf-point of the limit bifunction K if the sequence of maxinf-points of the bifunctions K^ν admits a cluster point. The implication for variational inequalities is the following:

5.1 Proposition (convergence of variational inequalities). *The variational inequality: find $u \in C$ such that $-G(u) \in N_C(u)$ where $G : C \to I\!R^n$ and $\emptyset \neq C \subset I\!R^n$ is convex will have a solution as long as we can find a sequence of sets $\{C^\nu\}_{\nu \in N}$ convex, compact, converging to C, and a sequence of lsc functions $\{G^\nu : C^\nu \to I\!R^n\}_{\nu \in N}$ converging continuously to G with respect to the sequence $C^\nu \to C$.*

Proof. The functions $G^\nu : I\!R^n \to I\!R^n$ converge continuously to G (on the sequence $C^\nu \to C$), i.e., for all $x^\nu \in C^\nu \to x \in C$, $G^\nu(x^\nu) \to G(x)$. Define

$$K^\nu(u, x) = \langle G^\nu(u), v - u \rangle \quad \text{on} \quad \operatorname{dom} K^\nu = C^\nu \times C^\nu.$$

Then, K^ν are Ky Fan bifunctions that converge lopsided ancillary tightly to K, i.e., any cluster point of the solutions of the approximating variational inequalities is a solution of the limit one. □

Requiring continuous convergence might appear to be bit too stringent, but possibly only in some very specific instances, it's unavoidable. To support this assertion, let's go through a completely different argument that expresses a variational inequality as a generalized equation, or equivalently, as a set-valued inclusion[§].

We work with the same collection of variational inequalities with $C^\nu \to C$ and G^ν converging appropriately to the continuous function G. It's this 'appropriately' that we want to investigate. Let's rewrite the systems to be solved as follows,

$$u \in C \quad \text{such that} \quad A(u) \ni 0 \quad \text{where} \quad A(u) = N_C(u) + G(u)$$

[§]Let's note that Robinson [23], see also [17], working to obtain error bounds, under significantly more restrictive conditions such as differentiability conditions, is also led to require continuous convergence of the mappings G^ν to G.

and for $\nu = 1, \ldots,$

$$u \in C^\nu \ \text{ such that } \ A^\nu(u) \ni 0 \ \text{ where } \ A^\nu(u) = N_{C^\nu}(u) + G^\nu(u)$$

It really suffices to concern ourselves with conditions under which the solutions of $A^\nu(u) \ni 0$ will converge to those of $A(u) \ni 0$. In view of [25, Theorem 5.37], the question essentially boils down to asking if $A^\nu \xrightarrow{g} A$. Now, if $C^\nu \to C$, by Attouch's Theorem [25, Theorem 12.35] $N_{C^\nu} \xrightarrow{g} N_C$, in fact because these are convex cones $N_{C^\nu} \xrightarrow{t} N_C$, i.e. *totally converge to* [25, Theorem 4.25(b)]. Since by continuity $\operatorname{rge} G$, the range of G, is bounded on C and $\operatorname{gph} G$ is clearly connected, it follows that for ν sufficiently large, $\operatorname{gph} G^\nu$ are uniformly bounded if $G^\nu \xrightarrow{g} G$ [25, Corollary 4.12]. Moreover, in this situation, when $\operatorname{gph} G^\nu \to \operatorname{gph} G$, they actually totally converge [25, Theorem 4.25(d)]. There now remains only to appeal [25, Exercise 4.29(c)] to conclude that $A^\nu \xrightarrow{g} A$ when $G^\nu \xrightarrow{g} G$ and $N_C \cap (\operatorname{gph} -G)^\infty = \{0\}$. This last condition is innocuous since G is bounded on C. We now turn to [25, Corollary 5.45] to conclude that since G is single-valued and bounded, then for all practical purposes $G^\nu \xrightarrow{g} G$ is equivalent to G^ν converges continuously to G.

One might still hope that one could escape 'continuous convergence' in the nicest of all situations: the functions G^ν and G are monotone and with domain $I\!R^n$. They are, then, the gradients of convex functions, say g^ν and g and the solutions of the variational inequalities are the optimal solutions of the convex optimization problems

$$\min g^\nu + \iota_{C^\nu} \quad \text{and} \quad \min g + \iota_C$$

where ι_D is the indicator function of the set D. The question about the convergence of the solutions then comes down to the epi-convergence of these functions. Since $C^\nu \to C$, $\iota_{C^\nu} \xrightarrow{e} \iota_C$, the sums will converge when $\operatorname{dom} g$ and C cannot be separated, certainly satisfied when $\operatorname{dom} g$ is $I\!R^n$, and the functions g^ν epi-converge to g [25, Exercise 7.47(b)]. But, now again, via Attouch's Theorem [25, Theorem 12.35], $g^\nu \xrightarrow{e} g$ cannot occur unless their gradients $G^\nu = \nabla g^\nu$ converge graphically to $G = \nabla g$ which brings us to continuous convergence via [25, Theorem 12.35] already cited earlier.

Let's conclude this section by summarizing our results as follows:

5.2 Theorem (convergence of the solution sets of V.I.). *Suppose $\{C^\nu \subset I\!R^n,\, \nu \in I\!N\}$ is a sequence of convex, compact sets converging to $C \neq \emptyset$, necessarily convex but not not necessarily compact, and the functions $\{G^\nu : I\!R^n \to I\!R^n,\, \nu \in I\!N\}$ converge continuously to a (continuous) bounded function G. Then, the solution set*

$$D = \{u \mid G(u) + N_C(u) \ni 0\}$$

of the limiting variational inequality contains the outer-limit of the solutions sets of the variational inequalities, i.e., $\operatorname{Limsup}_\nu D^\nu \subset D$ *where*

$$D^\nu = \{u \mid G^\nu(u) + N_{C^\nu}(u) \ni 0\}, \qquad \nu \in I\!N.$$

Moreover, if $\bar{u} \in D$, then there exists approximate solutions $u^\nu \in C^\nu$ of the variational inequalities $-G^\nu(u) \in N_{C^\nu}(u) \ni 0$ such that $u^\nu \to \bar{u}$.

Proof. Let's begin by observing that in these circumstances the sets D^ν are nonempty as observed at the beginning of this section and $(u^\nu \in D^\nu) \to u$ yields $u \in D$ by Proposition 5.1.

On the other hand, if $\bar u \in D$, then $\bar u \in \operatorname{argmax}_{u\in C}(\inf_{v\in C}\langle G(u), v-u\rangle)$ which in view of Theorem 3.2 means that there exists $\{\varepsilon_\nu \searrow 0, u^\nu \in \varepsilon_\nu\text{-}\operatorname{argmax}_{u\in C^\nu}[\inf_{v\in C^\nu}\langle G^\nu(u), v-u\rangle]\}$, or equivalently

$$\exists\, \varepsilon_\nu \searrow 0 \text{ and } (u^\nu \in C^\nu) \to \bar u \text{ such that } \langle -G^\nu(u^\nu), v-u^\nu\rangle \le \varepsilon_\nu,\ \ \forall\, v \in C^\nu,$$

which yields the sequence of approximating solutions. □

In [18, §3], Gürkan and Pang make a thorough analysis of the convergence of the variational inequalities associated with finding Nash equilibrium points of non-cooperative games. Their results are a bit more specific since they deal with a class of non-cooperative games where the strategies sets are 'fixed', i.e., don't change with the convergence parameter, here ν. Interestingly enough, they are also led to impose continuous convergence on the gradients of the reward functions that in this framework correspond to the functions the G^ν.

6 Convergence of fixed points

Brouwer's Fixed Point Theorem, and its classical generalizations, can be derived from Ky Fan's inequality. Indeed, with C a non-empty, compact, convex subset of $I\!R^n$ and $G: C\times C$ a continuous function, let's define the bifunction

$$F(x,y) = \langle x - G(x), y - x\rangle, \quad \text{i.e.,} \quad F: C\times C \to I\!R.$$

F is clearly a Ky Fan bifunction defined on a product of compact, convex sets and for all $x \in I\!R^n$, $F(x,x) = \langle x - G(x), x-x\rangle \ge 0$. Hence, from Lemma 3.3 follows the existence of a maxinf-point $\bar x$ of F such that

$$\forall\, y \in I\!B: \quad F(\bar x, y) = \langle \bar x - G(\bar x), y - \bar x\rangle \ge 0.$$

Since $G(\bar x) \in C$, recall $G: C \to C$, and since $F(\bar x, \cdot) \ge 0$, again by Lemma 3.3, choosing $y = G(\bar x)$, one has

$$\langle \bar x - G(\bar x), G(\bar x) - \bar x\rangle = -|\bar x - G(\bar x)|^2 \ge 0$$

and this can only occur when $G(\bar x) = \bar x$. We encapsulate this conclusion in the well-known theorem[¶]:

6.1 Theorem (existence of a fixed point). *Let $G: C \to C$ be continuous where $C \subset I\!R^n$ is (nonempty), compact and convex. Then, G admits a fixed point, i.e., for some $\bar x \in C$, $G(\bar x) = \bar x$.*

Let's now turn to approximation issues and consider the following situation, the compact, convex sets $C^\nu \to C$ which is nonempty, convex and compact. In particular, this means [25, Corollary 4.11] that for ν sufficiently large, all sets C^ν are contained in $C + \eta I\!B$ for some $\eta > 0$. The continuous functions $G^\nu: C^\nu \to C^\nu$ are converging continuously to $G: C \to C$ with respect to $C^\nu \to C$: $\forall\, x^\nu \in C^\nu \to x \in C$, $G^\nu(x^\nu) \to G(x)$. It's then straightforward to verify that the bifunctions $K^\nu(x,y) = \langle x - G^\nu(x), y - x\rangle$ defined on $C^\nu \times C^\nu$ *lop-converge* tightly to $K: \langle x - G(x), y-x\rangle$ defined on $C\times C$ and, via 3.2, this implies:

[¶]in [5] a coercivity condition had been imposed on G: $\langle x, G(x)\rangle \le |x|^2$ to obtain this result via the Ky Fan inequality.

6.2 Theorem (convergence of fixed points). *In the situation described here above,*

1. *for all ν, the functions G^ν have a least one fixed point in C^ν as is also the case for $G : C \to C$,*
2. *for $C^\#$ the set of cluster points of the fixed points of the functions G^ν, $\emptyset \neq C^\# \subset C$,*
3. *if $\bar{x} \in C^\#$, then $\bar{x}$ is a fixed point of G on C,*
4. *if $\bar{x} \in C$ is a fixed point of G, there exist approximate fixed points of G^ν, $x^\nu \in C^\nu$ converging to $\bar{x}$.*

Proof. All the assertions are covered by the paragraph preceding the theorem provided it's understood that by "approximate fixed point," say x_ε, one means that it belongs to the ε- argmax - inf of the corresponding bifunction. □

7 Stability and existence of Nash equilibrium

We next turn to non-cooperative games as introduced in §2.3 and deal with the existence and the stability of Nash equilibrium points. We consider a game $\mathcal{G} = \{(C_a, r_a) \,|\, a \in \mathcal{A}\}$ and an approximating sequence $\mathcal{G}^\nu = \{(C_a^\nu, r_a^\nu) \,|\, a \in \mathcal{A}\}$ with a finite number $|\mathcal{A}|$ of players. For each player $a \in \mathcal{A}$, the sets C_a or C_a^ν, subsets of $I\!R^n$, determine the available strategies x_a and the associated reward is $r_a(x_a, x_{-a})$, or $r_a^\nu(x_a, x_{-a})$, where x_{-a} is the vector of the strategies selected by the remaining players. *Nash equilibrium points* are strategies that satisfy

$$\forall\, a \in \mathcal{A}: \quad x_a^* \in \text{argmax}_{x_a \in C_a} r_a(x_a, x_{-a}^*) \text{ or } x_a^{\nu,*} \in \text{argmax}_{x_a \in C_a^\nu} r_a(x_a, x_{-a}^{\nu,*}).$$

Existence and continuity results for Nash equilibrium points will be derived via the properties of the maxinf-points of the bivariate *Nikaido-Isoda bifunctions*, $N : C \times C \to I\!R$ where $C = \prod_{a \in \mathcal{A}} C_a$,

$$N(x, y) = \sum_{a \in \mathcal{A}} \Big(r_a(x_a, x_{-a}) - r_a(y_a, x_{-a}) \Big),$$

and $N^\nu : C^\nu \times C^\nu \to I\!R$ where $C^\nu = \prod_{a \in \mathcal{A}} C_a^\nu$,

$$N^\nu(x, y) = \sum_{a \in \mathcal{A}} \Big(r_a^\nu(x_a, x_{-a}) - r_a^\nu(y_a, x_{-a}) \Big).$$

Existence, a well-known result (cf. [7, Theorem 4.2], for example), is obtained here as a direct consequence of the Ky Fan Inequality Lemma 3.3 and its extension Corollary 3.4, refer to the last few paragraphs of §3 and the fact that the Nikaido-Isoda bifunction N is finite-valued on $C \times C$.

7.1 Theorem (existence of Nash equilibrium points). *If for all $a \in \mathcal{A}$ the sets C_a are convex and compact and the Nikaido-Isoda bifunction $N : I\!R^{n \times |\mathcal{A}|} \times I\!R^{n \times |\mathcal{A}|} \to \overline{I\!R}$ satisfy:*
(a) *for all $y \in I\!R^{n \times |\mathcal{A}|}$, $x \mapsto N(x, y)$ is usc,*
(b) *for all $x \in I\!R^{n \times |\mathcal{A}|}$, $y \mapsto N(x, y)$ is convex,*
then $\mathcal{G}$ has a Nash equilibrium point.

Proof. One simply appeals to Ky Fan's Inequality 3.3 after observing that $N(x, x) \geq 0$. □

It's immediate from the definition of the Nikaido-Isoda bifunction that it will be a Ky Fan bifunction under the following conditions:

7.2 Proposition (Nikaido-Isoda as a Ky Fan bifunction). *A sufficient condition for N to be a Ky Fan bifunction is the following: for all $a \in \mathcal{A}$, the sets C_a are convex and,*

a) *r_a is usc and for all $x_a \in I\!R^n$, $r_a(x_a, \cdot)$ is lsc;*

b) *for all $x \in I\!R^{n\times|\mathcal{A}|}$, $r_a(\cdot, x_{-a})$ is concave.*

Let's now turn to stability issues related to perturbations of both the strategy sets and the payoffs. For this purpose we introduce the following convergence notion for a sequence of (approximating) games: $\mathcal{G}^\nu = \{(C_a^\nu, r_a^\nu), a \in \mathcal{A}\}$ for $\nu \in I\!N$.

7.3 Definition (convergence of non-cooperative games). *Convergence of a sequence of games $\{\mathcal{G}^\nu, \nu \in I\!N\}$ to a game $\mathcal{G}$ is defined in the following terms: for all $a \in \mathcal{A}$,*

a) *the nonempty compact convex sets C_a^ν converge to the nonempty compact set C_a;*

b) *the sequence of Nikaido-Isoda bifunctions N^ν associated with the games $\mathcal{G}^\nu$ lop-converge ancillary tightly to the Nikaido-Isoda bifunction N associated with game $\mathcal{G}$.*

When the collection of sets C_a^ν, nonempty, compact, convex converge to C_a, this limit set is also convex [25, Proposition 4.15]. Also, if $C_a^\nu \to C_a$ for all $a \in \mathcal{A}$, then $\prod_{a\in\mathcal{A}} C_a^\nu \to \prod_{a\in\mathcal{A}} C_a$ [25, Exercise 4.29].

An extension of continuous convergence usually defined relative to a fixed set, say C, cf. [25, §7.C] turns out to be a sufficient condition for lopsided convergence of the Nikaido-Isoda bifunction sequence N^ν. Indeed, a sequence of bifunctions $\{f^\nu : I\!R^n \to \overline{I\!R}, \nu \in I\!N\}$ is said to *converge continuously to* $f : I\!R^n \to \overline{I\!R}$ *relative to a sequence of sets* $C^\nu \to C$ if for all $x^\nu \to x$ such that for all $\nu \in N$, $x^\nu \in C^\nu$: $f^\nu(x^\nu) \to f(x)$. Thus, it's easy to see that condition a) in the definition of convergence of games jointly with the continuous convergence of r_a^ν relative to C_a^ν for all $a \in \mathcal{A}$ imply the ancillary tight lopsided convergence of Nikaido-Isoda bifunctions associated with the games $\mathcal{G}^\nu$.

Combining the previous results and observations, we can formulate our first stability result for Nash equilibrium points.

7.4 Theorem (convergence of Nash equilibrium points). *Suppose the games $\{\mathcal{G}^\nu, \nu \in I\!N\}$ converge to a game $\mathcal{G}$ and that the Nikaido-Isoda bifunctions N^ν associated to them are Ky Fan, then, there exist strategies $\{\bar{x}^\nu = (\bar{x}_a^\nu, a \in \mathcal{A}), \nu \in I\!N\}$ such that, for all $\nu \in I\!N$, are Nash equilibrium points of $\mathcal{G}^\nu$, and any cluster point, say $\{\bar{x} = (\bar{x}_a, a \in \mathcal{A})\}$ of such a sequence is itself a Nash equilibrium point of the game $\mathcal{G}$.*

Proof. Since the games $\{\mathcal{G}^\nu, \nu \in I\!N\}$ converge to a game $\mathcal{G}$ and the bifunctions N^ν are finite on $\prod_{a\in\mathcal{A}} C_a^\nu$, we obtain from Theorem 7.1 and the properties of lopsided convergence (Theorem 3.2) the assertions of the theorem. □

The next corollary translates the stability result in terms of the original formulation of the games.

7.5 Corollary (sufficient conditions for convergence of Nash points). *Suppose the games $\{\mathcal{G}^\nu, \nu \in I\!N\}$ are such that for all $a \in \mathcal{A}$ the sets C_a are convex and compact and that the payoff functions satisfy:*

a) *r_a^ν is usc and for all $x_a \in I\!R^n$, $r_a^\nu(x_a, \cdot)$ is lsc;*

b) *for all* $x \in I\!R^{n\times|\mathcal{A}|}$, $r_a^\nu(\cdot, x_{-a})$ *is concave.*

Suppose also that for all $a \in \mathcal{A}$, r_a^ν *continuously converges relatively to* C_a^ν. *Then, there exist a Nash equilibrium* $\{\bar{x}^\nu = (\bar{x}_a^\nu,\, a \in \mathcal{A}),\, \nu \in I\!N\}$ *of* $\mathcal{G}^\nu$ *for all* $\nu \in I\!N$, *and any cluster point of this sequence is a Nash equilibrium of the game* $\mathcal{G}$.

Independently, and at about the same time we started to circulate some of the results in this article, Gürkan and Pang [18] developed an alternative approach to the convergence of Nash equilibrium points based on *multi-epi convergence*, more precisely, multi-hypo convergence since we are dealing rewards-maximization problems. The sequence of games $\mathcal{G}^\nu$, they consider, is somewhat less general than those being analyzed here. In their framework, the strategies sets C_a are constant, i.e., don't depend on the approximation parameter ν. We allow for this dependence in the more general definition below. Let

$$C_{-a} = \prod\nolimits_{a' \in \mathcal{A}\setminus\{a} C_{a'} \quad \text{and similarly} \quad C_{-a}^\nu = \prod\nolimits_{a' \in \mathcal{A}\setminus\{a} C_{a'}^\nu.$$

7.6 Definition (multi-hypo convergence á la Gürkan-Pang). *For all* $a \in \mathcal{A}$, *suppose that for all* $\{x_{-a}^\nu \in C_{-a}^\nu,\, \nu \in I\!N\}$ *converging to* x_{-a}, *the functions* $\{r_a^\nu(\cdot, x_{-a}^\nu),\, \nu \in I\!N\}$ *hypo-converge to* $r_a(\cdot, x_{-a})$; *for the definition of hypo-convergence for finite-valued functions defined on subsets of* $I\!R^n$, *cf. [20, §2].*

The following theorem allows us to relate the Gürkan-Pang approach to the one based on lopsided convergence.

7.7 Theorem (hypo-convergence of the reward functions). *Suppose the games* $\{\mathcal{G}^\nu,\, \nu \in I\!N\}$ *are such that their reward functions multi-hypo converge to the reward functions of* $\mathcal{G} = \{(C_a, r_a), a \in \mathcal{A}\}$. *Then the corresponding sequence of Nikaido-Isoda bifunctions* $\{N^\nu,\, \nu \in I\!N\}$ *lop-converges to* N, *the Nikaido-Isoda bifunction associated vith* $\mathcal{G}$.

Proof. For each player $a \in \mathcal{A}$, hypo-convergence of the functions $\{r_a^\nu(\cdot, x_{-a}^\nu),\, \nu \in I\!N\}$ to $r_a(\cdot, x_{-a})$ means [20, Proposition 2]

a)-a$^\infty$) $\forall x^\nu \in C_a^\nu \to x \in C_a, \limsup_\nu r_a^\nu(x^\nu) \geq r_a(x)$ and $r_a^\nu(x^\nu) \searrow -\infty$ when $x \notin C_a$,

b) $\forall x_a \in C_a, \exists\, x_a^\nu \to x_a$ such that $\liminf_\nu r_a^\nu(x^\nu) \geq r_a(x)$.

Let's first verify that condition 3.1(a) is satisfied, namely for all $y \in C$ and $x^\nu \in C^\nu \to x$, $\exists\,(y^\nu \in C^\nu) \to y$ such that $\limsup_\nu N^\nu(x^\nu, y^\nu) \leq N(x, y)$ when $x \in C$ and $\limsup_\nu N^\nu(x^\nu, y^\nu) \searrow -\infty$ when $x \notin C$, or equivalently that given any $y \in C$ and any $x^\nu \in C^\nu \to x \in C$ one can find $y^\nu \to y$ such that

$$\text{limsup}_\nu \sum_{a \in \mathcal{A}} (r_a(x^\nu) - r_a(y_a^\nu, x_{-a}a^\nu)) \leq \sum_{a \in \mathcal{A}} (r_a(x) - r_a(y_a, x_{-a}))$$

when $x \in C$ and otherwise

$$\text{limsup}_\nu \sum_{a \in \mathcal{A}} (r_a(x^\nu) - r_a(y_a^\nu, x_{-a}a^\nu)) \searrow -\infty.$$

Condition b) guarantees that for all $y_a \in C$ and $x_{-a}^\nu \in C_{-a}^\nu \to x_{-a} \in C_{-a}$, one can find $y_a^\nu \in C_a^\nu \to y_a \in C_a$ such that $\liminf_\nu r_a^\nu(y_a^\nu, x_{-a}^\nu) \geq r_a(y_a, x_{-a})$. On the other hand, given the sequence $x_{-a}^\nu \in C_{-a}^\nu \to x_{-a} \in C_{-a}$ and any sequence $x_a^\nu \in C_a^\nu$ either $\limsup_\nu r_a(x^\nu) \leq r_a(x = (x_a, x_{-a}))$ or

$r_a(x^\nu)\searrow -\infty$. This means that for any sequence $x^\nu \in C^\nu \to x$ and $y \in C$ with, for all $a \in \mathcal{A}$, the appropriate choice of $y_a^\nu \to y_a$ one has

$$\limsup_\nu [\, r_a^\nu(x^\nu) - r_a^\nu(y_a^\nu, x_{-a}^\nu)\,] \le \limsup_\nu r_a^\nu(x^\nu) - \liminf_\nu r_a^\nu(y_a^\nu, x_{-a}^\nu)$$
$$\le \begin{cases} r_a(x) - r_a(y_a, x_{-a}) & \text{if } x_a \in C_a, \\ -\infty & \text{if } x_a \notin C_a. \end{cases}$$

Clearly, this remains valid once we take the sum with respect to $a \in \mathcal{A}$ and thus the first condition for the lopsided convergence of the sequence Nikaido-Isoda N^ν to N is satisfied.

Let's now turn to verifying 3.1(b). Given $x \in C$, let's choose our sequence $(x_a^\nu, x_{-a}^\nu) \in C^\nu \to (x_a, x_{-a})$ in such a way that for each $a \in \mathcal{A}$, $\liminf_\nu r_a^\nu(x^\nu) \ge r_a(x)$ as predicated by the hypo-convergence of the reward functions, more specifically, by assumption b). Given any sequence $\{y^\nu \in C^\nu,\, \nu \in I\!N\}$ converging to y, from a)-a$^\infty$) one knowns that $\limsup_\nu r_a^\nu(y_a^\nu, x_{-a}^\nu) \le r_a(y_a, x_{-a})$ when $y_a \in C_a$ or $r_a^\nu(y_a^\nu, x_{-a}^\nu)\searrow -\infty$ otherwise, or equivalently, $\liminf_\nu -r_a^\nu(y_a^\nu, x_{-a}^\nu) \ge -r_a(y_a, x_{-a})$ when $y_a \in C_a$ or $r_a^\nu(y_a^\nu, x_{-a}^\nu)\nearrow\infty$ otherwise. This immediately carries over to the Nikaido-Isoda bifunctions by taking sums. □

In [18], Gürkan and Pang make a thorough analysis of the convergence of the variational inequality associated with finding Nash equilibrium points for non-cooperative games. Their results are a bit more specific since they deal with a class of non-cooperative games where the strategy sets are 'fixed', i.e., don't change with the convergence parameter ν. Interestingly, they are also led to impose continuous convergence on the gradients of the reward functions that in this case correspond to the functions G^ν; refer to the comments about continuous convergence in §5.

8 Existence and stability of Walras equilibria

One can rewrite the Walras barter problem as a non-cooperative game and then apply the results of the two preceding sections, in particular §7. Let's begin by following this path but in the end, as we shall see, a more direct approach via the Walrasian bifunction turns out to be more expedient.

Returning to the Walras model introduced in §2.4, our collection of players will consist of the individual agents $a \in \mathcal{A}$ in addition to a so-called *Walrasian auctioneer*, a panoptic player whose main function is to choose a market price system aimed at securing a market equilibrium. The reward functions are

$$\forall a \in \mathcal{A}: \qquad r_a(x, x_{-a}, p) = u_a(x) \quad \text{when} \quad \langle p, e_a - x\rangle \ge 0,\ x \in C_a$$

and for the Walrasian auctioneer,

$$r_W(p, x_{\mathcal{A}}) = \inf_{q \ge 0} \langle q, \sum\nolimits_{a\in\mathcal{A}} (e_a - x_a)\rangle \quad \text{if } 0 \le p \ne 0$$

where $x_{\mathcal{A}}$ consists of the strategies of all the individual agents. In this model,

- r_a doesn't depend on x_{-a}, it's included in the arguments simply for consistency,
- when $0 \le p \ne 0$, $r_W(p, x_{\mathcal{A}}) = -\infty$ unless $\sum_{a\in\mathcal{A}}(e_a - x_a) \ge 0$ in which case $r_W = 0$, its upper bound.

Now, suppose that $(\bar{x}_{\mathcal{A}}, \bar{p})$, with $\bar{p} \neq 0$, is an equilibrium point of the Walras barter model which implies that the excess supply $\sum_{a \in \mathcal{A}} (e_a - \bar{x}_a) \geq 0$. Then, clearly, for all $a \in \mathcal{A}$, $\bar{x}_a \in \operatorname{argmax}_x r_a(x, \bar{x}_{-a}, \bar{p})$ and $\bar{p} \in \operatorname{argmax}_{0 \leq p \neq 0} r_W(\cdot, \bar{x}_{\mathcal{A}})$. The reward function of the Walrasian auctioneer doesn't depend explicitly on p but actually does so indirectly; in fact, once a price system has be chosen so that $\sum_{a \in \mathcal{A}} (e_a - x_a) \geq 0$ given that for all $a \in \mathcal{A}$, $x_a \in \operatorname{argmax}_x r_a(x, x_{-a}, p)$, in theory any other non-negative price system would "maximize" r_W but that wouldn't guarantee that each $\bar{x}_a$ would then maximize $r_a(\cdot, \bar{x}_{-a}, \tilde{p})$. Hence, $(\bar{x}_{\mathcal{A}}, \bar{p})$ is an Nash equilibrium point for the non-cooperative game defined by the rewards functions $\big((r_a, a \in \mathcal{A}), r_W\big)$.

On the other hand, if $(\bar{x}_{\mathcal{A}}, \bar{p})$ is a Nash equilibrium point of a game with reward functions $\big((r_a,\, a \in \mathcal{A}), r_W\big)$, then necessarily for all $a \in \mathcal{A}$, $\bar{x}_a \in \operatorname{argmax}_x \{u_a(x) \,\big|\, \langle \bar{p}, e_a - x \geq 0\}$ and $\sum_{a \in \mathcal{A}} (e_a - \bar{x}_a) \geq 0$ since otherwise $\operatorname{argmax} r_W$ would be empty. This means that $0 \leq \bar{p} \neq 0$ with $\bar{x}_{\mathcal{A}}$ is an equilibrium point of the Walras barter model.

One can then proceed to writing down the corresponding Nikaido-Isoda bifunction,

$$N\big(((x, p), (y, q)\big) = \sum\nolimits_{a \in \mathcal{A}} \big(r_a(x_a, x_{-a}, p) - r_a(y_a, x_{-a}, p)\big) + \big(r_W(p, x_{\mathcal{A}}) - r_W(q, x_{\mathcal{A}})\big)$$

but an explicit expression, in terms of the given utility functions and the associated budgetary constraints, gets a little unwieldy that makes the analysis and, in particular, the study of the convergence properties, more involved than they should be.

So, let's proceed as in §2.4 and work with the Walrasian bifunctions and deduce both existence and stability from the general results for bifunctions. Recall that $\mathcal{E} = \big\{(u_a, C_a, e_a),\, a \in \mathcal{A}\big\}$ provides the description of the economy, $W(p, q) = \langle s(p), q\rangle$ on $\Delta \times \Delta$ is the Walrasian with Δ the unit simplex in $\mathbb{R}^n$ and $s(\cdot)$ is the excess supply function.

8.1 Theorem *Suppose that $p \mapsto s(p)$ is usc on Δ, then $\mathcal{E}$ has at least one Walras equilibrium point, say $\bar{p}$. Moreover, $W(\bar{p}, \cdot) \geq 0$.*

Proof. When s is usc, so is $W(\cdot, q)$ for all $q \in \Delta$ and hence W is then a Ky Fan bifunction, finite-valued on the compact convex set $\Delta \times \Delta$. Since for any $p \in \Delta$, $W(p, p) \geq 0$, Ky Fan's inequality 3.3 immediately yields the existence of an maxinf-point. Moreover, $W(\bar{p}, \cdot) \geq 0$ when $\bar{p}$ is a maxinf-point, again by Lemma 3.3. □

Conditions on the original data of the economy $\mathcal{E}$ under which the excess supply function is usc are provided in the next proposition; basically, the same conditions as those used by Arrow and Debreu [3] to derive their existence result.

8.2 Proposition *If for all $a \in \mathcal{A}$ the utility functions u_a are usc and concave on C_a, and the initial endowments $e_a \in \operatorname{int} C_a$, then $W(., q)$ is usc on Δ, for all q, i.e., W is a Ky Fan bifunction.*

Proof. This is an immediate consequence of the classical results for the sup-projection of a bivariate function, here $s_a(p) = e_a - \sup \big\{u(x) \,\big|\, \langle p, x - e_a\rangle \leq 0, x \in C_a\big\}$, cf. for example, [25, Chapter 7]. □

Next, we consider a sequence of economies $\mathcal{E}^\nu = \big\{(u_a^\nu, C_a^\nu, e_a^\nu) \,\big|\, a \in \mathcal{A}\big\}$, that can be interpreted as perturbation of the utility functions, survival sets and initial endowments $e_a^\nu \operatorname{int} C_a^\nu$.

8.3 Definition *A sequence of economies $\{\mathcal{E}^\nu, \nu \in I\!N\}$ is said to be converging to an economy $\mathcal{E}$ if the Walrasians $W^\nu(p,q)$ associated with the economy $\mathcal{E}^\nu$ lop-converge ancillary tightly to the Walrassian W associated with $\mathcal{E}$; this means that convergence of the economies is defined in term of the convergence of their associated Walrasians.*

8.4 Theorem *Suppose the economies $\{\mathcal{E}^\nu, \nu \in I\!N\}$ converge to $\mathcal{E}$ and that the Walrasians W^ν associated with them are Ky Fan, then for every $\mathcal{E}^\nu$ comes with Walras equilibrium prices $\{\bar{p}^\nu, \nu \in I\!N\}$. Moreover, any sequence of such equilibrium points has at least one cluster point and any such cluster point is a Walras equilibrium for $\mathcal{E}$.*

Proof. As the sequence $\mathcal{E}^\nu$ converges to $\mathcal{E}$ and the functions W^ν are finite on $\Delta \times \Delta$, Theorem 8.1 allows us to infer the existence of equilibrium points for each one of the economies $\mathcal{E}^\nu$. The conclusions follow from Theorem 3.2 about the convergence of maxinf-points and the fact that Δ is compact. □

8.5 Remark Sufficient conditions to guarantee the lop-convergence ancillary tightly of the Walrasian bifunctions W^ν were given in an earlier paper by the same authors [21]. Indeed, if for all $a \in \mathcal{A}$ the utility functions u_a^ν are usc and concave with the same domain, the initial endowments satisfy $e_a \in \text{int}\, I\!R^n_+$ and the u_a^ν continuously converge to u_a then W^ν lop-converge ancillary tightly to W.

9 Convergence of solutions to generalized equations

There is an extensive literature dealing with the local behavior of the solutions to generalized equations, or inclusions, under perturbations of some specific parameters, cf. the recent monograph of Dontchev and Rockafellar [12] and references therein; see also the work of Ait Mansour, in particular [1, 2]. A more global approach was already followed in [8] but with a boundedness condition on the coderivative of the mappings. Here, the conditions imposed are on the properties of the mappings rather than their codrivatives. Moreover, we limit ourselves to a simplified situation; a full analysis deserve an independent treatment, a special case was dealt with in §5 as an alternative approach to the convergence of variational inequalities, see the discussion that follows Proposition 5.1. Our immediate aim is to illustrate how lop-convergence can be exploited to get us on the way to getting a convergence result for the solutions $S^\nu(x) \ni d^\nu$ to the solutions of $S(x) \ni 0$ when the mappings S^ν approximate S and $d^\nu \to 0$.

Let $S^\nu, S : I\!R^n \rightrightarrows I\!R^m$ be convex-valued, osc mappings and consider the inclusions $S^\nu(x) \ni d^\nu$, $S(x) \ni 0$ with $d^\nu \to 0$. For $x \in \text{dom}\, S^\nu = \{x \mid S^\nu(x) \neq \emptyset\}$, let's designate by $\sigma^\nu(x, \cdot)$ the *support function* of $S^\nu(x)$ and define $\sigma(x, \cdot)$ as the support function of S. Of course, for all x, these functions are convex.

The need for the upper semicontinuity of the support functions leads us to impose some further conditions on the collection of mappings $\{S, S^\nu, \nu \in I\!N\}$. Although it's far from a minimal condition, let's carry on with the assumption that our mapping are also *locally bounded*. When that's the case, the outer semicontinuity of the mappings is enough to guarantee that the mappings $\sigma^\nu(\cdot, v)$, $\sigma(\cdot, v)$ are usc. The argument is provided for the function σ, it's similar for all σ^ν. Indeed, first note that then the mapping S is compact-valued and for all $x \in \text{dom}\, S$, $\sup\{\langle v, u\rangle \mid u \in S(x)\}$ is attained at some $u \in S^\nu(x)$. If $x^k \to x$ in $\text{dom}\, S$, the corresponding points $u^k \in \text{argmax}\{\langle v, u\rangle \mid u \in S(x^k)\}$ for any subsequence have a cluster point $u \in S(x)$ as follows from local boundedness and outer semicontinuity of S. Hence,

$$\text{limsup}_k\, \sigma(x^k, v) = \text{limsup}_k \langle v, u^k \rangle \le \sigma(x, v).$$

So, let's define $K^\nu = \sigma^\nu - \langle d^\nu, \cdot \rangle$ and show that they lop-converge to $K = \sigma$ when the mappings S^ν graphically converge to S, i.e., $\text{gph}\, S^\nu \to \text{gph}\, S$.

9.1 Lemma (range of a compact-valued osc mapping). *Compact-valued osc mappings defined on a compact domain have compact range and, hence, are locally bounded.*

Proof. Let $S : D \rightrightarrows I\!R^m$ be such a mapping and let's show that $\text{rge}\, S$ is bounded. To the contrary, suppose $\text{rge}\, S$ was unbounded which means there exists $u^k \in S(x^k)$ with $|u^k| \nearrow \infty$ and $x^k \in D$. Since D is compact, passing to a subsequence, if necessary, $x^k \to \bar{x} \in D$ and since S is compact-valued and osc at $\bar{x}$ it follows that for any $\varepsilon > 0$ arbitrarily small and k sufficiently large, $S(x^k) \subset S(\xi, \bar{x}) + \varepsilon I\!B$ which negates the possibility of having such a sequence $\{u^k\}$ and consequently, $\text{rge}\, S$ must be bounded. That $\text{rge}\, S$ is also closed follows from a similar argument: when $u^k \in S(x^k) \to \bar{u}$, again passing to a subsequence if necessary, implies that $x^k \to \bar{x} \in D$ and since S is osc, $\bar{u} \in S(\bar{x}) \subset \text{rge}\, S$. □

9.2 Proposition (inclusions: lop-convergence of support functions). *Suppose $d^\nu \to 0$, the osc mappings $\{S^\nu : D^\nu \subset I\!R^n \rightrightarrows I\!R^m,\ \nu \in I\!N\}$ are uniformly locally bounded and converge graphically to $S : D \subset I\!R^n \rightrightarrows I\!R^m$. Then, the bifunctions*

$$\{K^\nu = \sigma^\nu - \langle d^\nu, \cdot \rangle : D^\nu \times I\!R^m \to I\!R\} \text{ lop-converge to } K = \sigma : D \times I\!R^m \to I\!R.$$

Proof. It really suffices to prove that the support functions σ^ν lop-converge to σ. The graphical convergence of the mappings S^ν to S implies that $\text{gph}\, S$ is closed and thus, S is also osc. Moreover, since the graphical limit at some point $\bar{x} \in D$ can be expressed as

$$\bigcup_{\{x^\nu \to \bar{x}\}} \text{Limsup}_{\nu \to \infty}\, S^\nu(x^\nu) \ \subset\ S(\bar{x}) \subset \bigcup_{\{x^\nu \to \bar{x}\}} \text{Liminf}_{\nu \to \infty}\, S^\nu(x^\nu),$$

see [25, Proposition 5.33], it immediately follows that S is locally bounded since the mappings S^ν are uniformly locally bounded.

Let's now turn to condition 3.1(a). So, given $\bar{v} \in D$ and $x^\nu \in D^\nu \to \bar{x}$, we have to exhibit $v^\nu \to v^\nu$ such that $\text{limsup}_\nu\, \sigma^\nu(x^\nu, v^\nu) \le \sigma(\bar{x}, \bar{v})$. Let's simply choose $v^\nu \equiv \bar{v}$ and let $u^\nu \in \text{argmax}\,\{\langle \bar{v}, u \rangle\} \,\big|\, u \in S^\nu(x^\nu)\}$. Now recall that the mappings S^ν and S are uniformly locally bounded and, consequently, every subsequence of $\{u^\nu,\ \nu \in I\!N\}$ comes with a further converging subsequence. For any cluster point $\bar{u}$ of $\{u^\nu,\ \nu \in I\!N\}$, restricting our attention to the subsequence converging to $u^\nu \underset{N}{\to} \bar{u}, N \subset I\!N$, one has

$$\lim_{\nu \in N} \sigma^\nu(x^\nu, \bar{v}) = \lim_{\nu \in N} \langle u^\nu, \bar{v} \rangle = \langle \bar{u}, \bar{v} \rangle \le \sigma(\bar{x}, \bar{v})$$

from which one immediately concludes that $\text{limsup}\, \sigma^\nu(x^\nu, \bar{v}) \le \sigma(\bar{x}, \bar{v})$.

Condition 3.1(b) is verified as follows. For $\bar{x} \in D$ and $v^\nu \to \bar{v}$, let $\bar{u} \in \text{argmax}\,\{\langle \bar{v}, u \rangle \,\big|\, u \in S(\bar{x})\}$. Graphical convergence, in particular $\text{Liminf}_\nu\, \text{gph}\, S^\nu \supset \text{gph}\, S$, implies that there exists a sequence $(x^\nu, u^\nu) \in \text{gph}\, S^\nu \to (\bar{x}, \bar{u})$. Taking this into account, with

(i) $\sigma(x^\nu, v^\nu) \geq \langle v^\nu, u^\nu \rangle$, and

(ii) $\langle v^\nu, u^\nu \rangle \to \langle \bar{v}, \bar{u} \rangle$,

one obtains, $\liminf_\nu \sigma(x^\nu, v^\nu) \geq \sigma(\bar{x}, \bar{v})$. □

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