A polynomial relaxation-type algorithm for linear programming

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Abstract

The paper proposes a polynomial algorithm for solving systems of linear inequalities. The algorithm uses a polynomial relaxation-type procedure which either finds a solution for $Ax = b, \mathbf{0} \leq x \leq \mathbf{1}$, or decides that the system has no integer solutions.

1 Polynomial algorithm

The polynomial algorithm described in [2] either solves a linear system $Ax = b, \mathbf{0} \leq x \leq \mathbf{1}$, or decides that it has no 0,1-solutions. Here, $\mathbf{0}$ and $\mathbf{1}$ denote all-zero and all-one vectors, respectively. The core of the algorithm is a divide-and-conquer procedure which may be considered as a generalization of the relaxation method (see [1] and [5]).

This short paper describes a way of converting the mentioned polynomial algorithm, which is further called the LP ALGORITHM, into a polynomial algorithm for solving linear systems. This approach substantially differs from well known polynomial algorithms like the ellipsoid method by Khachiyan [4] and the projective algorithm by Karmarkar [3].

Consider a linear system

$$Ax = b, \mathbf{0} \leq x \leq u, \tag{1.1}$$

where all coefficients are integer. Without loss of generality we assume that $u_j > 0$ for all $j = 1, \dots, n$.

Let us introduce additional variables ξ and $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)^T$ and consider the system

$$\begin{aligned} & Ax - b\xi = b, \\ & \bar{x} + x - u\xi = u, \\ & \mathbf{0} \leq \bar{x} \leq (1 + T^n)u, \\ & \mathbf{0} \leq x \leq (1 + T^n)u, \\ & 0 \leq \xi \leq T^n, \end{aligned} \tag{1.2}$$

where T is $n!$ times absolute value of the product of all components of A, b, and u. The size of T is polynomially bounded.

Lemma 1.1 *The system (1.2) has an integer solution if and only if (1.1) is feasible.*

Proof. Indeed, if $(x^*, \bar{x}^*, \xi^*)$ is an integer solution of (1.2), then $\frac{1}{1+\xi^*}x^*$ is feasible for (1.1). Now let us assume that the system (1.1) is feasible. Then (1.1) defines a bounded polyhedron. Consider a vertex $x^{\#}$ of this polyhedron. Since all coefficients of the system are integer, Cramer's theorem implies that $x^{\#}$ is representable as $x^{\#} = \left(\frac{x'_1}{x''_1}, \dots, \frac{x'_n}{x''_n}\right)^T$ where $|x'_j|$ and $|x''_j|$ are integers bounded by T. Let $\xi^* = |\prod_{l=1}^n x''_l| - 1$. The ξ^* is a nonnegative integer bounded by T^n. Let us set

$$x^* := (1 + \xi^*)x^{\#}$$

and

$$\bar{x}^* := (1 + \xi^*)u - x^*.$$

We have $Ax^* = (1 + \xi^*)b$ because $Ax^{\#} = b$. The vector x^* is nonnegative because $\xi^* \geq 0$ and $x^{\#} \geq \mathbf{0}$. Moreover, it is integer because

$$(1 + \xi^*)|x_j^{\#}| = |\prod_{l=1}^{n} x''_l| \cdot \frac{|x'_j|}{|x''_j|} = |\prod_{l \neq j} x''_l| \cdot |x'_j|.$$

We also have $x^* \leq (1+T^n)u$ because $x^\# \leq u$ and $\xi^* \leq T^n$. The vector $\bar{x}^*$ is integer because x^* and u are integer vectors and ξ^* is integer. The vector $\bar{x}^*$ satisfies $\bar{x}^* = (1+\xi^*)(u-x^\#) \geq \mathbf{0}$, because $x^\# \leq u$, and $\bar{x}^* \leq (1+T^n)u$ because $x^* \geq \mathbf{0}$. It follows that $(x^*, \bar{x}^*, \xi^*)$ is an integer solution of (1.2). ■

Let (1.2) be written in the form

$$A'x' = b', \mathbf{0} \leq x' \leq u'. \tag{1.3}$$

Recalling that

$$2^s = \sum_{l=0}^{s-1} 2^l + 1,$$

and using the binary representation of u'_j, we derive

$$u'_j = \sum_{s=0}^{\lceil \log u'_j \rceil} \sigma_{js} 2^s = \sum_{s=0}^{\lceil \log u'_j \rceil} \sigma_{js} \left(\sum_{l=0}^{s-1} 2^l + 1 \right).$$

Here, $\sigma_{js} \in \{0,1\}$ are the digits in the binary representation of u'_j. (We assume that $\sum_{l=0}^{-1}$ equals zero.)

Let us denote the elements of A' by a'_{ij} and consider another system whose size is polynomial in the size of (1.3):

$$\begin{array}{c} \sum_{j=1}^{n} a'_{ij} \left(\sum_{s=0}^{\lceil \log u'_j \rceil} \sigma_{js} \left(\sum_{l=0}^{s-1} 2^l z_{jsl} + z_{jss} \right) \right) = b'_i, \ i = 1, \ldots, m, \\ \mathbf{0} \leq z \leq \mathbf{1}. \end{array} \tag{1.4}$$

Lemma 1.2 *The system (1.4) has an integer solution if and only if (1.3) has an integer solution.*

Proof. If (1.4) has an integer solution z^*, we may set

$$x^*_j = \sum_{s=0}^{\lceil \log u'_j \rceil} \sigma_{js} \left(\sum_{l=0}^{s-1} 2^l z^*_{jsl} + z^*_{jss} \right) \tag{1.5}$$

for every $j = 1, \ldots, n$. Using the fact that the components of z^* are nonnegative and not greater than one, we get

$$0 \leq x^*_j \leq \sum_{s=0}^{\lceil \log u'_j \rceil} \sigma_{js} \left(\sum_{l=0}^{s-1} 2^l + 1 \right) = u'_j.$$

At the same time, $A'x^* = b'$ due to (1.4). It follows that the obtained solution x^* is feasible for (1.3).

Let x^* be an integer solution of (1.3). Since $x_j^* \leq u_j'$, we may represent x_j^* in the form

$$x_j^* = \sum_{s=0}^{\lceil \log u_j' \rceil} \sigma_{js}^* 2^s$$

where $\sigma_{js}^* \in \{0,1\}$ are the digits in the binary representation of x_j^*. If $x_j = u_j'$, we put $z_{jsl}^* = 1$ for all s and l. Otherwise, since $x_j^* < u_j'$, there exists s_j^* such that $\sigma_{js_j^*}^* = 0$ and $\sigma_{js_j^*} = 1$, and the digits σ_{js}^* with $s > s_j^*$ in the binary representation of x_j^* are equal to the corresponding digits σ_{js} in the binary representation of u_j'. Then

$$x_j^* = \sum_{s=s_j^*+1}^{\lceil \log u_j' \rceil} \sigma_{js}^* 2^s + \sum_{l=0}^{s_j^*-1} \sigma_{jl}^* 2^l = \sum_{s=s_j^*+1}^{\lceil \log u_j' \rceil} \sigma_{js} 2^s + 1 \cdot \sum_{l=0}^{s_j^*-1} (2^l \cdot \sigma_{jl}^* + 1 \cdot 0)$$

$$= \sum_{s=s_j^*+1}^{\lceil \log u_j' \rceil} \sigma_{js} \left(\sum_{l=0}^{s-1} 2^l \cdot 1 + 1 \cdot 1 \right) + \sigma_{js_j^*} \sum_{l=0}^{s_j^*-1} (2^l \cdot \sigma_{jl}^* + 1 \cdot 0).$$

According to the first sum in the latter expression, we set $z_{jss}^* := 1$ and $z_{jsl}^* := 1$ for $s > s_j^*$ and $l = 0, \ldots, s-1$. According to the second sum we set $z_{js_j^*l}^* := \sigma_{jl}^*$ for all $l = 0, \ldots, s_j^* - 1$. For the remaining components we set $z_{jsl}^* := 0$.

The $\{0,1\}$-solution z^* obtained in the above way satisfies (1.5) for all j. Since $A'x^* = b'$, it follows that z^* is feasible for (1.4). ■

Thus, we get the following procedure: Given the system (1.1), transform it to the corresponding system (1.2). Write this system as (1.3). Then transform (1.3) to the corresponding system (1.4). Let (1.4) be written as

$$A''x'' = b'', \mathbf{0} \leq x'' \leq \mathbf{1}. \tag{1.6}$$

Apply the LP ALGORITHM to (1.6). If the algorithm decides that (1.6) has no integer solution, then (1.3) also has no integer solution and we decide that the original system (1.1) is infeasible. Otherwise, let x'' be a solution of (1.6). Of course, it must not necessarily be integer. Then we translate x'' to a solution x of (1.1).

Theorem 1.1 *The above procedure gives a polynomial algorithm for solving systems of inequalities of the form $Ax = b, \mathbf{0} \leq x \leq u$, where all coefficients are integer.*

Proof. By Lemma 1.2, the system (1.6) has an integer solution if and only if (1.3) has an integer solution. By Lemma 1.1, the system (1.3) has an integer solution if and only if (1.1) is feasible. If the LP ALGORITHM delivers a solution x'' for (1.6), then it can be translated to a solution x' of (1.3) in polynomial time, by the procedure which obtains x^* from z^* by the formula (1.5) in the proof of Lemma 1.2, because the sizes of the systems are polynomially bounded in the size of the original system (1.1). It is proved in [2] that the LP ALGORITHM runs in polynomial time. The solution x' can in turn be translated to a solution of (1.1) in polynomial time. To see this, it is sufficient to note that if $x' = (x^*, \bar{x}^*, \xi^*)$, where $(x^*, \bar{x}^*, \xi^*)$ satisfies (1.2), then $\frac{1}{1+\xi^*} x^*$ is feasible for (1.1). If the LP ALGORITHM decides that (1.6) has no integer solution, then (1.1) is infeasible by Lemmas 1.2 and 1.1. ∎

References

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