SOME REGULARITY RESULTS FOR THE PSEUDOSPECTRAL ABSCISSA AND PSEUDOSPECTRAL RADIUS OF A MATRIX

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Abstract. The ε-pseudospectral abscissa α_ε and radius ρ_ε of an $n \times n$ matrix are respectively the maximal real part and the maximal modulus of points in its ε-pseudospectrum, defined using the spectral norm. It was proved in [A.S. Lewis and C.H.J. Pang. Variational analysis of pseudospectra. SIAM Journal on Optimization, 19:1048-1072, 2008] that for fixed $\varepsilon > 0$, α_ε and ρ_ε are Lipschitz continuous at a matrix A except when α_ε and ρ_ε are attained at a critical point of the norm of the resolvent (in the nonsmooth sense), and it was conjectured that the points where α_ε and ρ_ε are attained are not resolvent-critical. We prove this conjecture, which leads to the new result that α_ε and ρ_ε are Lipschitz continuous, and also establishes the Aubin property with respect to both ε and A of the ε-pseudospectrum for the points $z \in \mathbb{C}$ where α_ε and ρ_ε are attained. Finally, we give a proof showing that the pseudospectrum can never be "pointed outwards".

Keywords. Pseudospectrum, pseudospectral abscissa, pseudospectral radius, eigenvalue perturbation, Lipschitz multifunction, Aubin property

1. Introduction. Let $\|\cdot\|$ denote the vector or matrix 2-norm (spectral norm). For real $\varepsilon \geq 0$, the ε-*pseudospectrum* of a matrix $A \in \mathbb{C}^{n\times n}$ [TE05] is the union of the spectra of nearby matrices,

$$\Lambda_\varepsilon(A) = \{z \in \mathbf{C} : z \in \Lambda(A+E) \text{ for some } E \in \mathbb{C}^{n\times n} \text{ with } \|E\| \leq \varepsilon\} \tag{1.1}$$

where $\Lambda(A)$ denotes the spectrum (set of eigenvalues) of A. Equivalently, Λ_ε is the upper level set of the norm of the resolvent of $A - zI$,

$$\Lambda_\varepsilon(A) = \{z : \|(A - zI)^{-1}\| \geq \frac{1}{\varepsilon}\} \tag{1.2}$$

and the lower level set of the smallest singular value of $A - zI$,

$$\Lambda_\varepsilon(A) = \{z \in \mathbf{C} : \sigma_n(A - zI) \leq \varepsilon\}. \tag{1.3}$$

The ε-*pseudospectral abscissa* of A is the largest of the real parts of the elements of the pseudospectrum, i.e.,

$$\alpha_\varepsilon(A) = \max\{\text{Re } z : z \in \Lambda_\varepsilon(A)\}. \tag{1.4}$$

The case $\varepsilon = 0$ reduces to the spectral abscissa $\alpha(A)$, which is negative if and only if the continuous time system $\dot{x} = Ax$ has solutions that decay to zero for all initial states. The pseudospectral abscissa α_ε ranges, as ε is varied from 0 to ∞, from measuring *asymptotic* growth/decay to *initial* growth/decay, and also measures robust stability with respect to perturbations bounded in norm by ε.

The analogous robust measure for discrete-time systems $x_{k+1} = Ax_k$ is the ε-*pseudospectral radius*

$$\rho_\varepsilon(A) = \max\{|z| : z \in \Lambda_\varepsilon(A)\}.$$

The case $\varepsilon = 0$ reduces to $\rho(A)$, the spectral radius of A, which is less than one if and only if solutions decay to zero for all initial states.

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Below, we will refer to points where α_ε or ρ_ε is attained. By this we mean the points $z \in \Lambda_\varepsilon$ where the real part or the modulus respectively is maximized.

For fixed ε, $\Lambda_\varepsilon : A \to \Lambda_\varepsilon(A)$ is continuous [Kar03, Theorem 2.3.7]. Since Λ_ε is set-valued, continuity is to be understood in the Hausdorff metric. Recently, Lewis and Pang [LP08] proved that Λ_ε has further regularity properties. Specifically, they showed that Λ_ε has a local Lipschitz property known as the Aubin property everywhere except at *resolvent-critical* points (to be defined in the next section). It was also proved that for fixed $\varepsilon > 0$, α_ε (respectively ρ_ε) is Lipschitz continuous at a matrix A if the points where α_ε (respectively ρ_ε) are attained are not resolvent-critical (a consequence of [LP08, Corollary 7.2] and [LP08, Theorem 5.2]). The fact that for a fixed matrix A the number of resolvent-critical points is finite leads to the property that Λ_ε, α_ε and ρ_ε are Lipschitz around a given matrix A for all but finitely many $\varepsilon > 0$ [LP08, Corollary 8.5]. It was conjectured that the points where α_ε is attained are not resolvent-critical [LP08, Conjecture 8.9]. We prove this conjecture, which implies that for fixed $\varepsilon > 0$, α_ε is locally Lipschitz continuous on $\mathbb{C}^{n\times n}$. Our proof also applies to ρ_ε, proving that it is also locally Lipschitz. We also prove the Aubin property of the ε-pseudospectrum with respect to both ε and A for the points $z \in \mathbb{C}$ where α_ε and ρ_ε are attained. Finally, we give a proof showing that Λ_ε can never be "pointed outwards".

2. Previous Results and Notation. Before stating the conjecture, we need the following known results and definitions from [LP08]. We write MSV : $M^n \rightrightarrows C^n \times C^n$, with

$$\mathrm{MSV}(A) := \{(u, v) \mid u, v \text{ minimal left and right singular vectors of } A\}.$$

In this definition, u, v are minimal left and right singular vectors of A if they are unit vectors satisfying

$$\sigma_n(A)u = Av \quad \text{and} \quad \sigma_n(A)v = A^*u,$$

where A^* is the Hermitian transpose of A. The set

$$Y(A) := \{v^*u \mid (u, v) \in MSV(A)\}$$

will be a key tool in our analysis since, for a fixed A and for $z \notin \Lambda(A)$, we have [LP08, Proposition 4.5]

$$Y(A - zI) = \partial(-\sigma_n(A - zI)), \tag{2.1}$$

where ∂ is the subdifferential in the sense of [RW98, Definition 8.3]. This leads to:

Definition 2.1. *A point $z \in C$ is resolvent-critical for $A \in \mathbb{C}^{n\times n}$ if either $z \in \Lambda(A)$ or $0 \in Y(A - zI)$.*

Thus, a resolvent-critical point is either an eigenvalue of A or a critical point of the norm of the resolvent in the nonsmooth sense (see [LP08, Proposition 4.7 and Definition 4.8]).

Now we are ready to state Lewis and Pang's conjecture:

Conjecture 2.1. *[LP08, Conjecture 8.9] The points $z \in \Lambda_\varepsilon(A)$ where the pseudospectral abscissa $\alpha_\varepsilon(A)$ is attained are not resolvent-critical.*

In the following, we will also need Aubin property, a local Lipschitz property for set-valued mappings.

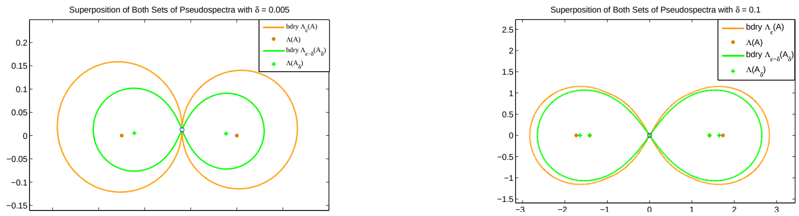

FIG. 3.1. *Figure illustrates the inclusion* $\Lambda_{\varepsilon-\delta}(A_\delta) \subset \Lambda_\varepsilon(A)$. *On the left,* A *is the* 4×4 *matrix with tangential coalescence given in [ABBO10, right panel of Figure 1] with* $\varepsilon = 0.0136$ *and* $\delta = 0.005$. *On the right, Gracia's example,* A *is the reverse diagonal matrix with entries 1,1,3 and 2,* $\varepsilon = 1$ *and* $\delta = 0.1$. *The smallest singular value of* $A - zI$ *has multiplicity 2 in both cases, and* $m = 2$ *in both cases. The plots are obtained with the software package EigTool [Wri02].*

DEFINITION 2.2. *(see [RW98, Definition 9.36])A mapping* $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *has the Aubin property at* $\bar{x}$ *for* $\bar{u}$, *where* $\bar{x} \in \mathbb{R}^n$ *and* $\bar{u} \in S(\bar{x})$, *if gph* S *is locally closed at* $(\bar{x}, \bar{u})$ *and there are neighborhoods* V *of* $\bar{x}$ *and* W *of* $\bar{u}$, *and a constant* $\kappa \in R_+$ *such that*

$$S(x') \cap W \subset S(x) + \kappa |x' - x| \mathbb{B} \text{ for all } x, x' \in V.$$

3. New Results. We now state our main result, which is based on a result in [ABBO10]:

THEOREM 3.1. *If* $z \in bd\ \Lambda_\varepsilon(A)$ *is resolvent-critical for some* $\varepsilon > 0$ *and a matrix* A, *then there exists an integer* $m \geq 2$, $\tilde{\theta}$ *real and* $\tilde{\rho}$ *positive real such that for all* $\omega < \pi/m$, Λ_ε *contains* m *equally spaced circular sectors of angle at least* ω *centered at* z, *that is*

$$\Lambda_\varepsilon(A) \supset \{z + \rho e^{i\theta} \mid \theta \in [\tilde{\theta} + 2\pi k/m - \omega/2, \tilde{\theta} + 2\pi k/m + \omega/2], \rho \leq \tilde{\rho}\}$$

for all $k = 0, 1, 2, \ldots, m-1$.

Proof. Assume that $z \in \text{bd}\ \Lambda_\varepsilon(A)$ is resolvent-critical. Since $\varepsilon > 0$, $z \notin \Lambda(A)$, so there exists a pair of singular vectors $(\tilde{u}, \tilde{v}) \in \text{MSV}(A - zI)$ such that $\tilde{v}^* \tilde{u} = 0$. If the smallest singular value of $A - zI$ is simple, then it follows from [ABBO10, Theorem 9 and subsequent discussion] that there exists an integer $m \geq 2$ such that for all $\omega < \pi/m$, Λ_ε contains m circular sectors of angle at least ω centered at z as stated, and so the result is proved. If the smallest singular value of $A - zI$ is not simple, consider a perturbed matrix $A_\delta = A - \delta \tilde{u} \tilde{v}^*$ for $\delta \in (0, \varepsilon)$. Then, the smallest singular value of $A_\delta - zI$ is simple with value $\varepsilon - \delta$ and corresponding singular vectors $\tilde{u}, \tilde{v}$ with $\tilde{u}^* \tilde{v} = 0$. Thus, we can apply [ABBO10, Theorem 9] to A_δ, finding that for all $\omega < \pi/m$, $\Lambda_{\varepsilon-\delta}(A_\delta)$ contains $m \geq 2$ circular sectors of angle at least ω centered at z. But immediately from the definition, using the triangle inequality for the norm, we have (see Figure 3.1)

$$\Lambda_\varepsilon(A) \supset \Lambda_{\varepsilon-\delta}(A_\delta),$$

proving the result. □

We conjecture that the only possible value for m in Theorem 3.1 is 2. See [ABBO10, Figure 3].

Clearly, at a point where the pseudospectral abscissa or pseudospectral radius is attained, the pseudospectrum cannot contain $m \geq 2$ circular sectors as defined above. As a consequence, we have:

Corollary 3.2. *For any $\varepsilon > 0$, the points where the pseudospectral abscissa α_ε or pseudospectral radius ρ_ε are attained are not resolvent-critical.*

Thus, Conjecture 2.1 is proved. Furthermore, Theorem 3.1 implies the following regularity results about $\alpha_\varepsilon, \rho_\varepsilon$ and Λ_ε:

Corollary 3.3. *Let $\varepsilon > 0$ be given, and $z_* \in \text{bd}\, \Lambda_\varepsilon(A_*)$ be a point where the pseudospectral abscissa $\alpha_\varepsilon(A_*)$ or pseudospectral radius $\rho_\varepsilon(A_*)$ is attained for some matrix A_*. Then, the map $A \to \Lambda_\varepsilon(A)$ has the Aubin property at A_* for z_*.*

Proof. By Corollary 3.2, z_* is not resolvent-critical. The result follows from [LP08, Theorem 5.2]. □

Corollary 3.4. *For any fixed $\varepsilon > 0$, α_ε and ρ_ε are Lipschitz continuous at any matrix A.*

Proof. Let $A \in \mathbb{C}^{n\times n}$ be given. By Corollary 3.3, Λ_ε has the Aubin property at A at all the points where the pseudospectral abscissa or pseudospectral radius is attained. An application of [LP08, Corollary 7.2(a)] with $F = \Lambda_\varepsilon$, $g(x) = \text{Re}(-x)$ proves the Lipschitz continuity of α_ε while using $F = \Lambda_\varepsilon$, $g(x) = -|x|$ proves the Lipschitz continuity of ρ_ε. □

Corollary 3.5. *Let $z_* \in \mathbb{C}$ be a point where the pseudospectral abscissa $\alpha_{\varepsilon_*}(A)$ or pseudospectral radius $\rho_{\varepsilon_*}(A)$ is attained for some $\varepsilon_* > 0$ and $A \in C^{n\times n}$. Then the map $\varepsilon \to \Lambda_\varepsilon(A)$ has the Aubin property at ε_* for z_*.*

Proof. From (2.1) and Corollary 3.2, we have $0 \notin Y(A - z_*I) = \partial(-\sigma_n(A - z_*I))$. The result then follows from [LP08, Proposition 5.3], using, as is done there, the inclusion $-\partial(\sigma_n(A - zI)) \subset \partial(-\sigma_n(A - zI))$. □

In the terminology of [BLO03, Definition 4.5 and its corrigendum], a point z is called nondegenerate with respect to $\Lambda_\varepsilon(A)$ if $Y(A - zI) \neq \{0\}$. Thus, Corollary 3.2 implies that a point where the pseudospectral abscissa or pseudospectral radius is attained is nondegenerate. This leads to the following generalization of [BLO03, Proposition 4.8]:

Proposition 3.6. *Let z_* be a point where $\alpha_\varepsilon(A)$ or $\rho_\varepsilon(A)$ is attained for some $\varepsilon > 0$ and a matrix A. Then the boundary of $\Lambda_\varepsilon(A)$ is differentiable at z_*.*

Proof. This follows from [BLO03, Proposition 4.8] and the fact that z_* is nondegenerate. □

It was proved in [BLO03, Proposition 4.8] that the pseudospectrum cannot be "pointed outwards" at nondegenerate points. By this, one means that around a nondegenerate point z_*, the pseudospectrum can never be contained in a circular sector of angle strictly less than π centered at z_*. It was further stated that a more detailed analysis due to Trefethen shows that the pseudospectrum is never pointed outwards. Since the latter result, based on eigenvalue perturbation theory, was never published, we give a new proof here.

Proposition 3.7. *Let z_* be on the boundary of the pseudospectrum, i.e. $z_* \in \text{bd}\, \Lambda_\varepsilon(A)$ for some $\varepsilon > 0$ and a matrix A. The pseudospectrum cannot be contained in a circular sector of angle $< \pi$ centered at z_*, that is, for all $\omega \in [0, \pi)$, $\tilde\theta \in [0, 2\pi)$ and $\tilde\rho$ positive real, there exists a point $y \in \Lambda_\varepsilon(A)$ such that y is not contained in the circular sector*

$$z + \{\rho e^{i\theta} \mid [\tilde\theta - \omega/2, \tilde\theta + \omega/2], \rho \leq \tilde\rho\}.$$

Proof. If z_* is nondegenerate, then the result follows from [BLO03, Proposition 4.7]. Otherwise, z_* is resolvent-critical and the result follows from Theorem 3.1. □

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