Interior-Point Algorithms for a Generalization of Linear Programming and Weighted Centering

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Abstract

We consider an extension of ordinary linear programming (LP) that adds weighted logarithmic barrier terms for some variables. The resulting problem generalizes both LP and the problem of finding the weighted analytic center of a polytope. We show that the problem has a dual of the same form and give complexity results for several different interior-point algorithms. We obtain an improved complexity result for certain cases by utilyzing a combination of the volumetric and logarithmic barriers. As an application we consider the complexity of solving the Eisenberg-Gale formulation of a Fisher equilibrium problem with linear utility functions.

Keywords: Linear programming, Weighted analytic center, Interior-point algorithm, Fisher equilibrium, Volumetric barrier.

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1 Introduction

In this paper we consider an optimization problem of the form

$$\begin{aligned} \text{(LPWC)} \quad \min \quad & g(x) := c^T x - \sum_{i=1}^{n} w_i \ln(x_i) \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

where A is an $m \times n$ matrix with independent rows and $w \geq 0$. The name of the problem is chosen to suggest that LPWC is a natural generalization of two other well-known optimization problems. The first, when $w = 0$, is ordinary linear programming (LP). The second, when $c = 0$ and $w > 0$, is the problem of computing the weighted analytic center of the polyhedron $\{x \geq 0 \,|\, Ax = b\}$. Complexity results for the latter problem were given by Atkinson and Vaidya [3] and Freund [5], and are described in more detail below.

In [11, 12], Ye describes an interesting instance of LPWC corresponding to the Eisenberg-Gale formulation of a Fisher equilibrium problem with linear utility functions. In this instance $c = 0$ but some components of w are zero, so the results of [3] and [5], which both assume that $w > 0$, do not apply. Ye shows that a modification of the well-known primal-dual path following algorithm for LP can be used to solve LPWC with essentially the same complexity as in the LP case.

In the next section we show that LPWC has a natural dual problem, DPWC, with a very similar form. In section 3, we describe how the complexity theory based on self-concordant functions and barriers from Nesterov and Nemirovskii [7] can be applied in several different ways to solve the problems LPWC/DPWC. The complexity results obtained are consistent with those of [5] and [12]. We then consider the situation where both m and the number of logarithmic barrier terms $|\{i \,|\, w_i > 0\}|$ are much smaller than n, which is the case in the application considered in [11, 12]. For this case we obtain a lower iteration complexity for DPWC by utilizing a combination of the volumetric and logarithmic barriers. In section 4 we apply this result to the Fisher equilibrium problem.

Notation For vectors $x \in \Re^n$, $s \in \Re^n$, $x \circ s$ denotes the Hadamard product $(x \circ s)_i = x_i s_i$, $i = 1, \ldots, n$, and $\text{Diag}(s)$ is the diagonal matrix with $\text{Diag}(s)_{ii} = s_i$, $i = 1, \ldots, n$. We use e to denote a vector with each component equal to one. For a square matrix X, $\text{ldet}\, X = \ln \det X$.

2 The dual problem

We begin by defining a dual problem for LPWC,

$$\begin{aligned}
\text{(DPWC)} \quad \max \quad & h(y) := \gamma(w) + b^T y + \sum_{i=1}^{n} w_i \ln(c_i - a_i^T y) \\
\text{s.t.} \quad & A^T y \le c,
\end{aligned}$$

where a_i is the ith column of A and $\gamma(w) := e^T w - \sum_{i=1}^{n} w_i \ln(w_i)$. We use DP to refer to an instance of DPWC with $w = 0$. The relationship between LPWC and DPWC is summarized in the following theorem. Let $\mathcal{P} = \{i \,|\, w_i > 0\}$. We call x a *strictly feasible* solution for LPWC if x is feasible for the constraints of LPWC and $x_i > 0$, $i \in \mathcal{P}$; similarly we call y strictly feasible for DPWC if $s = c - A^T y \ge 0$ and $s_i > 0$, $i \in \mathcal{P}$.

Theorem 1 *Assume that x is strictly feasible for LPWC, and y is strictly feasible for DPWC. Then*

1. *(weak duality) $h(y) \le g(x)$;*
2. *(optimality conditions) x and y are optimal solutions of LPWC and DPWC, respectively, if and only if $x \circ s = w$ where $s = c - A^T y$;*
3. *(strong duality) LPWC and DPWC have optimal solutions $\bar{x}$ and $\bar{y}$ with $g(\bar{x}) = h(\bar{y})$ and $c^T \bar{x} - b^T \bar{y} = e^T w$.*

Proof: We have $c^T x - b^T y = c^T x - y^T A x = (c^T - y^T A)x = s^T x$, so

$$\begin{aligned}
g(x) - h(y) &= x^T s - \gamma(w) - \sum_{i=1}^{n} w_i \ln(x_i s_i) \\
&= -\gamma(w) + \sum_{i=1}^{n} (x_i s_i - w_i \ln(x_i s_i)).
\end{aligned}$$

Let $v_i := x_i s_i \ge 0$. If $w_i = 0$, the minimum possible value of $v_i - w_i \ln(v_i)$ is obviously zero, and this value occurs uniquely at $v_i = w_i = 0$. If $w_i > 0$, it is easy to compute that the minimum value of $v_i - w_i \ln(v_i)$ also occurs uniquely at $v_i = w_i$, and therefore $v_i - w_i \ln(v_i) \ge w_i - w_i \ln(w_i)$. It follows that

$$g(x) - h(y) \ge -\gamma(w) + \sum_{i=1}^{n} w_i - w_i \ln(w_i) = 0,$$

proving 1). This argument also shows that $g(x) - h(y) = 0$ if and only if $v = x \circ s = w$, in which case we also have $c^T x - b^T y = e^T w$. It is then immediate that if $x \circ s = w$, then x and y are optimal for LPWC and DPWC respectively, which is half of part 2). To prove the other half of 2), assume that x is optimal in LPWC. Since $g(\cdot)$ is convex and the constraints of LPWC are linear, the KKT conditions must hold. The KKT conditions state that there exists a vector $\bar{y}$ so that $\bar{s} = c - A^T \bar{y}$ satisfies $\bar{s}_i - w_i/x_i = 0$ if $w_i > 0$, and $\bar{s}_i \geq 0$, $\bar{s}_i x_i = 0$ if $w_i = 0$. These conditions are obviously equivalent to $\bar{s} = c - A^T \bar{y} \geq 0$, $x \circ \bar{s} = w$, so $\bar{y}$ is optimal in DPWC and $g(x) - h(\bar{y}) = 0$. It follows that if y is any other optimal solution in DPWC then it must be that $x \circ s = w$, as required to complete 2). To complete the proof of 3) we must only show that an optimal solution of LPWC exists. Since LPWC is strictly feasible, and the objective value is bounded from below, this could only fail if the optimal value was not attained. To argue that this is impossible, it suffices to bound the region in which a feasible solution x with $g(x)$ below some value must occur. It is easy to see that the existence of a strictly feasible dual solution y bounds the components x_i, $i \in \mathcal{P}$. The feasible region of LPWC with these bounds added can then be written as the sum of a bounded polyhedron and the cone $\{x \geq 0 \,|\, Ax = 0, x_i = 0, i \in \mathcal{P}\}$, and clearly $c^T x \geq 0$ for any x in this cone since DPWC is feasible. Discarding the cone, we obtain a bounded set which must contain an optimal solution $\bar{x}$ of LPWC. □

Parts (2) and (3) of Theorem 1 generalize the well-known characterization for points x and $s = c - A^T y$ on the "central path" for a linear programming problem and its dual, respectively, corresponding to LPWC/DPWC with $w = \mu e$, where μ is a positive scalar. It is also worth noting that LPWC is obviously equivalent to the same problem with (c, w) replaced by $(\theta c, \theta w)$ for any $\theta > 0$. It is then easy to compute that if $w \neq 0$, $\gamma(\theta w) = 0 \iff \ln(\theta) = \gamma(w)/e^T w$. It follows that the objective in any instance of LPWC with $w \neq 0$ can be rescaled so that the constant term that appears in the objective of DPWC disappears.

3 Complexity results

To describe the complexity of algorithms for solving LPWC/DPWC we will utilize the theory of self-concordant functions developed by Nesterov and Nemirovskii [7]. We begin by defining necessary terminology.

Definition 1 *Let $\mathcal{F}$ be a closed convex subset of $\Re^n$, and let $f(\cdot)$ be a C^3, convex mapping from* $\text{Int}(\mathcal{F})$ *to $\Re$, where* $\text{Int}(\cdot)$ *denotes interior.*

1. *$f(\cdot)$ is called a–self–concordant on $\mathcal{F}$ if $|D^3 f(x)[\xi,\xi,\xi]| \le 2a^{-1/2} \left(D^2 f(x)[\xi,\xi]\right)^{3/2}$ for every $x \in \text{Int}(\mathcal{F})$ and $\xi \in \Re^n$. If in addition $f(\cdot)$ tends to infinity for any sequence approaching a boundary point of $\mathcal{F}$ then $f(\cdot)$ is called strongly a–self–concordant on $\mathcal{F}$.*

2. *$F(\cdot)$ is called a ϑ–self–concordant barrier for $\mathcal{F}$ if $F(\cdot)$ is strongly 1–self–concordant on $\mathcal{F}$, and in addition $\nabla F(x)\nabla^2 F(x)^{-1}\nabla F(x)^T \le \vartheta$ for every $x \in \text{Int}(\mathcal{F})$.*

3. *Let $F(\cdot)$ be a ϑ–self–concordant barrier for $\mathcal{F}$. Then $f(\cdot)$ is called β–compatible with $F(\cdot)$ if $|D^3 f(x)[\xi,\xi,\xi]| \le \beta \left(3D^2 f(x)[\xi,\xi]\right)\left(3D^2 F(x)[\xi,\xi]\right)^{1/2}$ for every $x \in \text{Int}(\mathcal{F})$ and $\xi \in \Re^n$.*

The results in the following proposition are easy to prove using basic properties of the logarithmic barrier and results from [7].

Proposition 1 *Let $\mathcal{F} = \Re^n_+$, and for $x > 0$ let $f(x) = -\sum_{i=1}^n w_i \ln(x_i)$, $F(x) = -\sum_{i=1}^n \ln(x_i)$. Let $w_{\min} := \min\{w_i \,|\, w_i > 0\}$. Then*

1. *$f(\cdot)$ is $O(1)$-compatible with $F(\cdot)$;*

2. *If $w > 0$ then $f(\cdot)$ is $w_{\min}$-self-concordant on $\mathcal{F}$;*

3. *If $w > 0$ then $(1/w_{\min})f(\cdot)$ is a ϑ–self–concordant barrier for $\mathcal{F}$, for $\vartheta = e^T w/w_{\min}$.*

It follows from Proposition 1 and [7, Section 2.2.3] that if $w > 0$ and one attempts to solve LPWC by directly minimizing $g(\cdot)$ using a damped Newton method, then the number of steps required to obtain an $O(1)$-optimal solution will be proportional to $\Delta_g^0/w_{\min}$, where Δ_g^0 is the gap between an initial objective value $g(x^0)$ and the solution value for LPWC. This is exactly the result obtained by Freund [4, Remark 7.2][1] who considered the weighted centering problem corresponding to LPWC with $c = 0$, $w > 0$, $e^T w = 1$.

A better complexity result was obtained by Atkinson and Vaidya [3], who considered DPWC with $b = 0$, $w_{\min} = 1$. Atkinson and Vaidya develop a specialized algorithm based

[1]The bound in [4, Remark 7.2] does not explicitly appear in the published version [5], but is a simple consequence of [5, Lemma 6.2].

on following a path of approximate maximizers y^k corresponding to a sequence of weight vectors w^k, $k = 0, \ldots, K$ where $w^0 = e$ and $w^K = w$. The main result of [3] is that by appropriately rescaling w^i to obtain w^{i+1}, an $O(1)$-optimal solution y^K for DPWC is obtained for $K = O(\ln w_{\max})$, where $w_{\max} := \max\{w_i\}$, and the work on each iteration k is dominated by the computations required to execute $O(\sqrt{n})$ Newton steps. Hence the algorithm requires a total of $O(\sqrt{n} \ln w_{\max})$ Newton steps to obtain an $O(1)$-optimal solution of the problem, starting with an approximation of the ordinary analytic center.

The algorithms of [4, 5] and [3] both assume that $w > 0$, and the linear term in either LPWC or DPWC is zero. However, part 1 of Proposition 1 immediately implies that general results on the complexity of barrier algorithms from [7] can be applied to these problems. Consider for example DPWC, and let

$$F_t(y) := -th(y) - \sum_{i=1}^{n} \ln(c_i - a_i^T y), \qquad t \geq 0.$$

It follows from [7, Section 3.2.2] that if y^0 is the minimizer of $F_0(\cdot)$ (that is, the ordinary analytic center), and $t_0 > 0$ satisfies

$$\lambda(F_{t_0}, y^0) := \left[\nabla F_{t_0}(y^0)(\nabla^2 F_{t_0}(y^0))^{-1} \nabla F_{t_0}(y^0)^T\right]^{\frac{1}{2}} < \delta, \tag{1}$$

where $\delta > 0$ is a suitable constant, then the number of Newton iterations required to obtain an ϵ-optimal solution y^K of DPWC is

$$K = O(\sqrt{n} |\ln(\epsilon t_0)|). \tag{2}$$

To obtain an overall complexity result, it remains only to provide a lower bound on $t_0 > 0$ satisfying (1). General results of this type are provided in [7], but below we give a bound tailored to DPWC.

Theorem 2 *Suppose that y^0 is the minimizer of $F_0(\cdot)$. Then (1) is satisfied for $1/t_0 = O(\Delta_b^0 + \|w\|)$, where $\Delta_b^0 := \max\{b^T y \mid c - A^T y \geq 0\} - b^T y^0$.*

Proof: It is straightforward to compute that if $s = c - A^T y > 0$, then

$$\begin{aligned} \nabla F_t(y)^T &= -tb + AS^{-1}(e + tw) \\ \nabla^2 F_t(y) &= AS^{-2}(I + tW)A^T, \end{aligned}$$

where $W = \mathrm{Diag}(w)$ and $S = \mathrm{Diag}(s)$. Moreover for $s^0 = c - A^T y^0$, $S_0 = \mathrm{Diag}(s^0)$ we have $\nabla F(y^0)^T = AS_0^{-1}e = 0$. It follows that

$$\begin{aligned} \lambda(F_t, y^0) &\leq t\left[b^T(AS_0^{-2}(I+tW)A^T)^{-1}b\right]^{\frac{1}{2}} + t\left[w^T S_0^{-1}A^T(AS_0^{-2}(I+tW)A^T)^{-1}AS_0^{-1}w\right]^{\frac{1}{2}} \\ &\leq t\left[b^T(AS_0^{-2}A^T)^{-1}b\right]^{\frac{1}{2}} + t\left[w^T S_0^{-1}A^T(AS_0^{-2}A^T)^{-1}AS_0^{-1}w\right]^{\frac{1}{2}} \\ &\leq t(\Delta_b^0 + \|w\|), \end{aligned}$$

where the final inequality uses the fact that the Dikin ellipsoid

$$\{y \,|\, (y - y^0)^T \nabla^2 F_0(y^0)(y - y^0) \leq 1\}$$

is contained in the feasible region of DPWC, and $b^T y^0 + [b^T(AS_0^{-2}A^T)^{-1}b]^{\frac{1}{2}}$ is the maximal value of $b^T y$ for y in this ellipsoid. $\square$

Corollary 1 *Let y^0 be the minimizer of $F_0(\cdot)$. Then the barrier algorithm for DPWC can be initialized at y^0, $t_0 > 0$, so that an ϵ-optimal solution of DPWC is obtained after $O(\sqrt{n}\ln((\Delta_b^0 + \|w\|)/\epsilon))$ Newton iterations, each requiring $O(m^2 n)$ operations.*

Note that if $b = 0$, Corollary 1 implies that starting at y^0, an ϵ-optimal solution of DPWC can be obtained in $O(\sqrt{n}\ln(nw_{\max}/\epsilon))$ Newton iterations. This is exactly the complexity obtained by Ye [12, Theorem 1], who generalized the primal-dual path following algorithm for LP/LD to apply to LPWC with $c = 0$. In addition, it is easy to show that Theorem 2 and Corollary 1 hold exactly as stated if the assumption that y^0 is the minimizer of $F_0(\cdot)$ is replaced with the assumption that $\lambda(F_0, y^0) < \delta' < \delta$ for $\lambda(\cdot,\cdot)$ given in (1), where $\delta - \delta' = O(1)$. Finally, when $w = 0$, the result of Corollary 1 is exactly the well-known complexity of the barrier algorithm applied to the dual linear programming problem DP.

It is well known that if the "partial updating" techinique is used [6, 8], then the worst-case average number of operations per iteration of the barrier algorithm can be reduced to $O(\sqrt{n}m^2 + nm)$. However, there are some cases where partial updating actually cannot improve the overall complexity of the algorithm. For example, consider the case when $n = \Theta(m^2)$ but each column of A has only $O(1)$ nonzero entries. In this case the Hessian $\nabla^2 F_t(y)$ can be formed in only $O(n)$ operations, and inverted or factorized in $O(m^3)$ operations. The remaining work per iteration is $O(n + m^2)$, so the total work per iteration is $O(n + m^3) = O(m^3)$. If partial updating is used to update the inverse, or a factorization, of the Hessian, then the average updating work per iteration is $O(\sqrt{n}m^2) = O(m^3)$, which is already equal to the original complexity per iteration. If partial updating is applied to the Hessian itself then the updating work averages only $\sqrt{n}$ operations per iteration, but the inversion or factorization of the Hessian again requires $O(m^3)$ operations.

An alternative to using partial updating, which attempts to reduce the work per iteration, is to reduce the number of Newton iterations required to obtain an ϵ-optimal solution. This requires a self-concordant barrier for $\mathcal{F}_D = \{y \mid A^T y \leq c\}$ with a parameter $\vartheta = o(n)$. Such a barrier is known in the case where $m = o(n)$. The *volumetric barrier*[2] for $\mathcal{F}_D$ is the function

$$V(y) = \frac{1}{2} \operatorname{ldet} \nabla^2 F(s(y)) = \frac{1}{2} \operatorname{ldet} AS(y)^{-2} A^T,$$

where $s(y) = c - A^T y$, $S(y) = \operatorname{Diag}(s(y))$ and $F(\cdot)$ is the logarithmic barrier for $\Re^n_+$. The volumetric barrier was introduced by Vaidya [9] (see also [1]) in the construction of a cutting-plane algorithm for convex feasibility problems whose complexity is lower than that of the ellipsoid algorithm. The hybrid volumetric-logarithmic barrier

$$V_\rho(y) = V(y) + \rho F(s(y)),$$

with $\rho = O(m/n)$ was subsequently used by Vaidya and Atkinson [10] to construct a path-following algorithm that required only $O((mn)^{1/4} \ln(\Delta_b^0/\epsilon))$ Newton iterations to obtain an ϵ-optimal solution of DP, compared to $O(n^{1/2} \ln(\Delta_b^0/\epsilon))$ iterations when using the usual logarithmic barrier $F(\cdot)$. It was shown in [2] that the work per iteration for such an algorithm can be held to $O(m^2 n)$ iterations, resulting in an overall complexity reduction when $m = o(n)$. Self-concordancy results for the barriers $V(\cdot)$ and $V_\rho(\cdot)$ are obtained in [7, Section 5.5] and [2, Section 5]; the latter shows that $361\sqrt{n}V(\cdot)$ is an $O(m\sqrt{n})$-self-concordant barrier for $\mathcal{F}_D$, and for $\rho = (m-1)/(n-1)$, $361\sqrt{n/m}V_\rho(\cdot)$ is an $O(\sqrt{mn})$-self-concordant barrier for $\mathcal{F}_D$.

Our interest here is to apply the barrier $V_\rho(\cdot)$ to DPWC, but unfortunately the weighted logarithmic barrier $f(s(y))$ that appears in the objective of DPWC is not $O(1)$-compatible with $O(\sqrt{n/m})V_\rho(\cdot)$. However, a complexity reduction can still be obtained in the case

[2] In much of the literature on the volumetric barrier the variables m and n are interchanged, and the constraint matrix A is replaced by A^T.

where $|\{i \mid w_i > 0\}| = O(m)$. To do this, we will utilize the barrier

$$G(y) = 361\sqrt{n/m}V_\rho(y) - \sum_{i=1}^{r} \ln(s_i(y)),$$

where $\rho = (m-1)/(n-1)$ and we assume that $w_i = 0$, $i > r$.

Proposition 2 *Assume that $w_i = 0$, $i > r$. Then*

1. *$G(\cdot)$ is an $O(\sqrt{mn} + r)$-self-concordant barrier for $\mathcal{F}_D$,*

2. *$-\sum_{i=1}^{r} w_i \ln(s_i(y))$ is $O(1)$-compatible with $G(\cdot)$.*

Using Proposition 2 we can obtain a complexity result for the barrier algorithm applied to DPWC using $G_t(\cdot)$ in place of $F_t(\cdot)$. To accomplish this we require a result similar to that given for $F_t(\cdot)$ in Theorem 2. For simplicity we give this result assuming that $b = 0$, which holds for our application in the next section.

Theorem 3 *Assume that $b = 0$, $w_i = 0$ for $i > r$ and the first r columns of A have rank m. Let y^0 be the minimizer of $G_0(\cdot)$. Then $\lambda(G_{t_0}, y^0) < \delta = O(1)$ is satisfied for $1/t_0 = O(\|w\|)$.*

Proof: Using $b = 0$ and the fact that $\nabla G_0(y^0) = 0$, we have

$$\begin{aligned} \lambda(G_t, y^0) &= t\left[w^T S_0^{-1} A^T (\nabla^2 G(y^0) + tAS^{-2}WA^T)^{-1} AS_0^{-1} w\right]^{\frac{1}{2}} \\ &\le t\left[w^T S_0^{-1} A^T \left(\sum_{i=1}^{r} a_i a_i^T/(s_i^0)^2\right)^{-1} AS_0^{-1} w\right]^{\frac{1}{2}} \\ &\le t\|w\|, \end{aligned}$$

where the final inequality uses the fact that $w_i = 0$, $i > r$. □

Corollary 2 *Assume that $w_i = 0$, $i > r$, and the first $r = O(m)$ columns of A have rank m. Let y^0 be the minimizer of $G_0(\cdot)$. Then if $b = 0$, the barrier algorithm using $G_t(\cdot)$ can be initialized at y^0, $t_0 > 0$ so that an ϵ-optimal solution of DPWC is obtained in $O((mn)^{1/4} \ln(\|w\|/\epsilon))$ Newton-like iterations, each requiring $O(m^2 n)$ operations.*

The statement of Corollary 2 refers to "Newton-like" as opposed to "Newton" iterations because some care must be taken when working with $V(\cdot)$ to insure that the work per iteration remains $O(m^2 n)$; see [2] for details. When $m = o(n)$, the complexity result of Corollary 2 is better than the result of Corollary 1 with $b = 0$. In addition, in some situations it may be possible to reduce the upper bound of $O(m^2 n)$ operations per iteration required in both cases. We consider such an example in the next section.

4 The Fisher equilibrium problem

The Eisenberg-Gale formulation of the Fisher equilibrium problem, with linear utility functions, is [11, 12]

$$
\begin{aligned}
\text{(EGF)} \quad \min \quad & -\sum_{i=1}^{p} w_i \ln u_i \\
\text{s.t.} \quad & \sum_{i=1}^{p} x_{ij} = 1, \quad j = 1, \dots, q \\
& u_i - \sum_{j=1}^{q} u_{ij} x_{ij} = 0, \quad i = 1, \dots, p \\
& u_i \geq 0, \;\; x_{ij} \geq 0, \forall i, j,
\end{aligned}
$$

where w_i is the initial endowment of consumer i, and u_{ij} is the marginal utility for consumer i's consumption of good j. We assume that $w > 0$, $U \geq 0$, $Ue > 0$, $e^T U > 0$. Clearly EGF is an instance of LPWC with $n = pq + p = p(q+1)$, $m = p + q$, $c = 0$. It is straightforward to compute that the dual of EGF is the problem

$$
\begin{aligned}
\max \quad & \gamma(w) - e^T y + \sum_{i=1}^{p} w_i \ln z_i \\
\text{s.t.} \quad & y_j - u_{ij} z_i \geq 0, \quad i = 1, \dots, p, \;\; j = 1, \dots, q \\
& z \geq 0.
\end{aligned}
$$

Note that the feasible region of this problem is a convex cone; if (y, z) is feasible, then $(\theta y, \theta z)$ is also feasible for any $\theta \geq 0$. Using this fact it is easy to show that the optimal solution of the problem must have $e^T y = e^T w$, and therefore the dual of EGF may be rewritten in the form

$$
\begin{aligned}
\text{(DEGF)} \quad \max \quad & \gamma'(w) + \sum_{i=1}^{p} w_i \ln z_i \\
\text{s.t.} \quad & y_j - u_{ij} z_i \geq 0, \quad i = 1, \dots, p, \;\; j = 1, \dots, q \\
& e^T y \leq e^T w, \\
& z \geq 0,
\end{aligned}
$$

where $\gamma'(w) = \gamma(w) - e^T w = -\sum_{i=1}^{p} w_i \ln(w_i)$. Thus DEGF corresponds to an instance of DPWC with a bounded feasible region and $b = 0$. For $i = 1, \dots, p$, let $\tilde{U}_i$ be the $p \times q$ matrix whose ith row is the ith row of U, and all of whose other entries are zero. The constraint matrix A for the problem DEGF then has the form

$$A = \begin{pmatrix} 0 & -I & -I & \cdots & -I & e \\ -I & \tilde{U}_1 & \tilde{U}_2 & \cdots & \tilde{U}_p & 0 \end{pmatrix},$$

where the first block of q rows of A corresponds y, and the remaining block of p rows corresponds to z. Note that each column of A except for the last has at most 2 nonzero entries.

To simplify the statement of complexity results for DEGF, we assume henceforth that $q = O(p)$. Applying Corollary 1, the barrier algorithm based on the usual logarithmic barrier $F(y, z)$, suitably initialized, requires $O(\sqrt{n} \ln(\|w\|/\epsilon)) = O(p \ln(\|w\|/\epsilon))$ Newton iterations to obtain an ϵ-optimal solution, and it is easy to show that due to the structure of A the work per iteration is only $O(m^3) = O(p^3)$ operations. However, DEGF satisfies all of the assumptions of Corollary 2, with $r = p + q = m$. It follows that if the barrier $G(y, z)$ is used in place of $F(y, z)$ then the barrier algorithm, suitably initialized, requires only $O((mn)^{1/4} \ln(\|w\|/\epsilon)) = O(p^{3/4} \ln(\|w\|/\epsilon))$ Newton-like iterations to obtain an ϵ-optimal solution, and it is easy to show that the work per iteration remains $O(p^3)$.

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