Recoverable Robust Knapsacks: Γ-Scenarios*

Christina Büsing, Arie M. C. A. Koster, and Manuel Kutschka

Abstract In this paper, we investigate the recoverable robust knapsack problem, where the uncertainty of the item weights follows the approach of Bertsimas and Sim [3, 4]. In contrast to the robust approach, a limited recovery action is allowed, i.e., upto k items may be removed when the actual weights are known. This problem is motivated by the assignment of traffic nodes to antennas in wireless network planning. Starting from an exponential min-max optimization model, we derive an integer linear programming formulation of quadratic size. In a preliminary computational study, we evaluate the *gain of recovery* using realistic planning data.

1 Introduction

An important problem in the design of wireless networks is the assignment of traffic nodes, e.g., aggregations of users, to antennas. Each antenna has a limited bandwidth capacity to be partitioned among the users in the area covered by the antenna. Users, in general, do not generate a constant traffic rate. Depending on their needs in data traffic or web browsing the requested bitrate fluctuates (e.g., 64 kbps, 384 kbps, 2 Mbps). In the network capacity planning phase usually an average traffic volume is considered. However, during operation, individual users with their actual bitrates need to be admitted to the antenna.

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The actual bitrates are difficult to predict in advance, but using historical data average and peak values can be derived. It is also observed that not all peaks occur simultaneously. Therefore, we may assume that the bitrates of only a limited number Γ of users deviate from their average at the same time. Such behavior is captured by the so-called Γ-scenario set introduced by Bertsimas and Sim [3, 4].

To ensure a good quality of service for all users at any point in time, a robust assignment is appropriate. Such a robust solution is static over time, neglecting the possibility to reassign (a limited number of) users to other antennas, according to the current traffic volume. Focusing on a single antenna, the robust approach reduces to a classical knapsack problem with uncertainty in the weights. Including the possibility to change the assignment during runtime yields a *recoverable robust knapsack problem* (rrKP): Compared to the planning phase, up to k users can be refused a connection at this antenna (and should be reassigned to another one).

In this paper, we study rrKP with Γ-scenarios. In Section 2, we describe the problem in detail and give an overview on previous work. In the next section, we derive an integer linear programming formulation. In Section 4, we present the results of preliminary computational experiments on the gain of recovery using data from wireless network planning. We close with concluding remarks in Section 5.

2 Recoverable Robust Knapsack Problem

Despite its simple structure, the knapsack problem (KP) is weakly **NP**-hard [10] but solvable in pseudo-polynomial time [2]. Alternatively, branch-and-cut algorithms can be used to solve the KP. For a detailed introduction see [11, 13].

Yu [14] defined a robust version of the knapsack problem by introducing uncertainty in the profit values via a discrete set of scenarios. For sets with an unbounded number of scenarios, the decision version of the problem is strongly **NP**-complete and can not be approximated, mentioned by Aissi et al. [1]. On the other hand, if the set contains a constant number of scenarios, the problem is only weakly **NP**-complete, solvable in pseudo-polynomial time [14] and there exists an FPTAS [1]. A recoverable robust knapsack problem with discrete scenarios is investigated by [6]. Here, up to k items can be removed and ℓ items added to a first stage solution.

Recently, Klopfenstein and Nace [12] considered robust knapsacks with uncertainty in the weights based on Γ-scenarios [3, 4]. We will in the following extend this model by a recovery action of deleting up to k items.

Definition 1 (Recoverable Robust Knapsack (rrKP)). Let N be a set of n items with profits p_i, nominal (or default) weight $\underline{w}_i$, and maximum deviation $\hat{w}_i$, $i \in N$. For a given $\Gamma \in \mathbb{N}$, the set $\mathscr{S}_\Gamma$ consists of all scenarios S which define a weight function $w^S : N \to \mathbb{N}$ s.t. $w_i^S \in [\underline{w}_i, \underline{w}_i + \hat{w}_i]$ for all $i \in N$ and $|\{i \in N : w_i^S > \underline{w}_i\}| \leq \Gamma$. For $k \in \mathbb{N}$ and a subset $X \subseteq N$ the *recovery set* $\mathscr{X}_X^k$ consists of all subsets of X with at most $|X| - k$ elements, i.e., $\mathscr{X}_X^k = \{X' \subseteq X : |X \setminus X'| \leq k\}$. Given a knapsack capacity $c \in \mathbb{N}$, the rrKP is to find a set $X \subseteq N$ with maximum profit $p(X) := \sum_{j \in X} p_j$ s.t. for every scenario $S \in \mathscr{S}_\Gamma$ there exists a set $X' \in \mathscr{X}_X^k$ with $w^S(X') \leq c$.

Note that k models the quality of service: for $k = 0$ (the robust case), every user granted connection is connected, whereas for $k = n$ no service guarantee is given. We now focus on a compact formulation of an rrKP instance with Γ-scenarios.

3 A compact ILP Formulation

In this section we present an ILP-formulation for the rrKP. To this end, we define binary variables $x_i \in \{0,1\}$, $i \in N$, denoting the items in the knapsack. Any 0-1 point x satisfying the (exponential many) inequalities

$$\sum_{i\in N} \underline{w}_i x_i + \max_{\substack{\overline{X}\subseteq N\\ |\overline{X}|\leq\Gamma}} \Big(\sum_{i\in\overline{X}} \hat{w}_i x_i - \max_{\substack{Y\subseteq N\\ |Y|\leq k}} \big(\sum_{i\in Y} \underline{w}_i x_i + \sum_{i\in \overline{X}\cap Y} \hat{w}_i \big) \Big) \leq c \tag{1}$$

represents a feasible solution. In the following, we characterize the same polytope by a linear number of constraints. First, we consider a subproblem of finding a scenario $S \in \mathscr{S}_\Gamma$ that imposes the maximum weight on a chosen subset $X \subseteq N$. For given parameters $\Gamma \in \mathbb{N}$ and $k \in \mathbb{N}$ we define the *weight* of a subset $\overline{X} \subseteq X$ as

$$\text{weight}(\overline{X}, X) = \sum_{i\in\overline{X}} \hat{w}_i - \max_{\substack{Y\subseteq X\\ |Y|\leq k}} \Big(\sum_{i\in Y} \underline{w}_i + \sum_{i\in Y\cap\overline{X}} \hat{w}_i \Big).$$

A *maximum weight set* X^k_Γ is a subset of X with $|X^k_\Gamma| \leq \Gamma$ and with maximum weight. The *maximum weight set problem* (MWSP) is to find for a given set X, and parameters Γ and k, a maximum weight set X^k_Γ.

As the following example indicates, there is no inclusion relation between optimal solutions of an MWSP for different Γ values, i.e., in general $X^k_\Gamma \subsetneq X^k_{\Gamma+1}$.

Example 1. Consider the set $X = \{1,\dots,4\}$ with nominal weights $\underline{w} = \{3,3,10,10\}$ and deviations $\hat{w} = \{2,2,5,5\}$ and $k = 1$. For $\Gamma = 1$, the sets $X' = \{1\}$ and $X'' = \{2\}$ are the maximum weight sets for this instance with weight$(X', X) = -8$. But, $\overline{X} = \{3,4\}$ is the maximum weight set with weight$(\overline{X}, X) = -5$ for $\Gamma = 2$, whereas the sets $\{1,3\}$, $\{1,4\}$, $\{2,3\}$ and $\{2,4\}$ have a weight of -8.

Yet, the MWSP can be solved in polynomial time by exploiting linear programming duality. Computing a maximum weight set is formulated by the following ILP

$$\max\Big\{ \sum_{i\in X} \hat{w}_i y_i - \max\big\{ \sum_{i\in X} (\underline{w}_i + \hat{w}_i y_i) z_i \;:\; \sum_{i\in X} z_i \leq k,\; z_i \in \{0,1\} \big\} \;: \\ \sum_{i\in X} y_i \leq \Gamma,\; y_i \in \{0,1\} \Big\} \tag{2}$$

The variables y_i represent the choice, whether an item i is in the maximum weight set X^k_Γ, and z_i, whether the item i is removed due to its high weight.

Given a vector y, (2) can be solved by its linear relaxation, since the matrix is totally unimodular. By duality, we obtain a compact ILP formulation of (2):

$$\begin{aligned}
\max \quad & \sum_{i\in X} \hat{w}_i y_i - k\cdot u - \sum_{i\in X} v_i && && \text{(3a)}\\
s.t. \quad & \sum_{i\in X} y_i && \leq \Gamma && \text{(3b)}\\
& \hat{w}_i \cdot y_i - u - v_i && \leq -\underline{w}_i && \forall i \in X \quad \text{(3c)}\\
& u, v_i && \geq 0 && \forall i \in X \quad \text{(3d)}\\
& y_i && \in \{0,1\} && \forall i \in X \quad \text{(3e)}
\end{aligned}$$

where the dual variable u corresponds to $\sum_{i\in X} z_i \leq k$ and v_i to $z_i \leq 1$ for $i \in X$. Next, we parametrize (3) by the values u can take and denote with $z(u)$ the value of (3).

Lemma 1. *For a fixed parameter u', let $w_i(u') = \min\{\hat{w}_i, -\underline{w}_i + u'\}$ for all $i \in \{1,\dots,n'\}$, $X^-(u') = \{i \in X \mid w_i(u') < 0\}$, and $X(u') \subseteq X \backslash X^-(u')$ maximizing $\sum_{i\in X(u')} w_i(u')$ with $|X(u')| \leq \Gamma$. Then*

$$z(u') = \sum_{i\in X(u')} w_i(u') + \sum_{i\in X^-(u')} w_i(u') - k\cdot u'$$

holds. Furthermore, there always exists an optimal solution (u^*, y^*, v^*) of (3) with $u^* \in U := \{0\} \cup \{\underline{w}_i : i \in X\} \cup \{\underline{w}_i + \hat{w}_i : i \in X\}$.

See [5] for the omitted proof. By Lemma 1, inequality (1) is equivalent to

$$\sum_{i\in N} \underline{w}_i x_i + \max_{u\in U} \Bigg(\sum_{i\in X^-(u)} w_i(u)\cdot x_i - k\cdot u + \max_{\substack{X'\subseteq N\\ |X'|\leq\Gamma}} \sum_{i\in X} w_i(u)\cdot x_i \Bigg) \leq c\,. \tag{4}$$

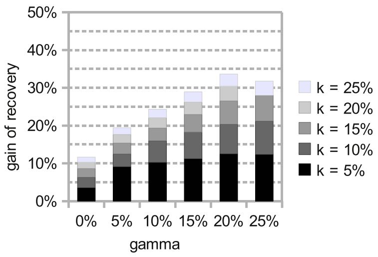
This inequality can be transformed into the following set of constraints

$$\begin{aligned}
\sum_{i\in N} \underline{w}_i x_i + \sum_{i\in X^-(u)} w_i(u)\cdot x_i + \max \sum_{i\in N} w_i(u)\cdot x_i\cdot y_i^u &\leq c + ku && \forall u \in U\\
\sum_{i\in N} y_i^u &\leq \Gamma && \forall u \in U\\
y_i^u &\in \{0,1\} && \forall i \in N, \forall u \in U.
\end{aligned}$$

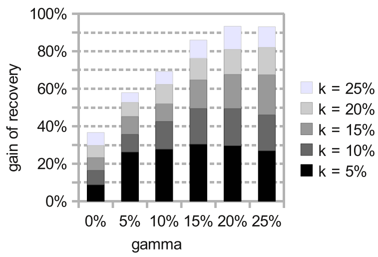
By dualizing the last part, which is totally unimodular, we obtain the following ILP

$$\begin{aligned}
\max & \sum_{i\in N} p_i x_i && && \text{(6a)}\\
s.t. & \sum_{\substack{i\in N:\\ \underline{w}_i<u}} \underline{w}_i x_i + \sum_{\substack{i\in N:\\ \underline{w}_i\geq u}} u x_i + \Gamma\xi^u + \sum_{i\in N} \theta_i^u && \leq c + ku && \forall u \in U \quad \text{(6b)}\\
& \min\{\hat{w}_i, -\underline{w}_i + u\} x_i - \xi^u - \theta_i^u && \leq 0 && \forall i \in N, \forall u \in U \quad \text{(6c)}\\
& x_i \in \{0,1\}, \qquad \xi^u, \theta_i^u && \geq 0 && \forall i \in N, \forall u \in U \quad \text{(6d)}
\end{aligned}$$

with new dual variables ξ^u and θ_i^u. The model contains $\mathcal{O}(n^2)$ variables and $\mathcal{O}(n^2)$ constraints depending on the number of different values of $\underline{w}_i, \hat{w}_i$, $i \in N$.



(a) geometric mean



(b) observed maximum

Fig. 1 Gain of Recovery. For each instance, Γ,and k, the gain of recovery is determined by the objective value normalized to the corresponding case with $k = 0\%$.

4 Computational Experiments

In this section, we present some preliminary results of computational experiments on the gain of recovery for rrKP with Γ-scenarios. As test instances, we consider a wireless network planning problem based on [8]. Given the planning instances, rrKP instances were generated for 51 antennas with 15 to 221 traffic nodes (geometric mean: 87). Uncertain demands are modeled as in [7].

We implemented formulation (6) of the rrKP in C++ using IBM ILOG CPLEX 12.2 [9] as MIP solver. All computations were carried out on a Linux machine with 2.93 GHz Intel Xeon W3540 CPU, 12 GB RAM, and a time limit of one hour per instance. All instances could be solved to optimality.

We investigate the *gain of recovery*, i. e., the (percentual) increase in the objective value by allowing recovery. As values for k and Γ we consider (rounded-up) relative values of $0\,\%, 5\,\%, \dots, 25\,\%$ of the number of traffic nodes.

Comparing all test instances, Figure 1 shows the geometric mean resp. maximum gain of recovery achieved in these instances (normalized to $k = 0$). Further, the added value for each value k is shown.

Fixing k, we observe that in geometric mean a higher gain of recovery is obtained by increasing Γ (e. g., $k = 20\%, \Gamma = 5\%$ yields 18 %, while $k = 20\%, \Gamma = 20\%$ yields 30 %). By evaluating the maximum observed gain of recovery, we estimate the potential added value by recovery. It ranges from 25 % ($k = 5\%$) to 71 % ($k = 25\%$) in geometric mean with an absolute maximum of 93 % ($\Gamma = k = 25\%$).

In summary, the results of our preliminary study show that the recoverable robust approach gives a promising added value to the robust approach for small k already.

5 Concluding Remarks

In this paper, we considered the recoverable robust knapsack problem (rrKP) with Γ-scenarios which is a subproblem in wireless network planning under traffic uncertainties. In detail, we introduced a compact ILP-formulation for this problem which is linear in the input size. Using realistic application-based data, we presented the results of a preliminary computational study evaluating the gain of recovery.

In the future, the polyhedral structure of the rrKP with Γ-scenarios should be studied to improve the overall solving process in a branch-and-cut approach.

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