Lifted Inequalities for 0−1 Mixed-Integer Bilinear Covering Sets*

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Abstract

In this paper, we study 0−1 mixed-integer bilinear covering sets. We derive several families of facet-defining inequalities via sequence-independent lifting techniques. We then show that these sets have polyhedral structures that are similar to those of certain fixed-charge single-node flow sets. As a result, we obtain new facet-defining inequalities for these sets that generalize well-known lifted flow cover inequalities from the integer programming literature.

1 Introduction and motivation

Nonlinear branch-and-bound is a method to solve mixed-integer nonlinear programming (MINLP) problems to global optimality; see [9, 17]. This method has been implemented in commercial solvers such as BARON [27] and LINDO Global [18]. It requires that convex relaxations of the problem be recursively solved over smaller and smaller subsets of the feasible region obtained by branching on variables. Most existing commercial software use a method proposed by McCormick [21] to obtain these convex relaxations for factorable problems. McCormick's relaxation is an instantiation of a more general technique that relaxes (nonconvex) constraints of the form $g(x) \geq r$ into (convex) constraints of the form $\bar{g}(x) \geq r$ where $\bar{g}(x)$ is a concave overestimator of $g(x)$. This technique does not use the right-hand-side of the inequality in the process. As a result, the relaxation obtained is typically not the strongest possible.

Some of the functional forms that appear most frequently in the formulation of nonlinear programs are probably multilinear inequalities and equalities. In particular, bilinear inequalities of the covering type

$$\sum_{j \in N} a_j x_j y_j \geq d, \tag{1}$$

where $a_j > 0$, $x_j \in S \subseteq \mathbb{R}_+$, and $y_j \in S' \subseteq \mathbb{R}_+$ appear in the formulation of various practical problems (including trimloss applications; see Harjunkoski et al. [16] for an example), and are among the simplest nonconvex inequalities that can be studied. Therefore, sets of the form (1) provide an important test bed for the derivation of new, stronger convexification methods that use right-hand-side information. When variables do not have upper bounds, we have derived in [30] closed-form expressions for the convex hull of feasible solutions of (1) over various subsets of the nonnegative orthant. For problems where variables are continuous and have finite upper bounds, we also derived in [29] convex relaxations of (1) that are stronger than McCormick's.

In this paper, we study further the convex hull of feasible solutions to (1) when variables are bounded. In particular, we consider 0−1 mixed-integer bilinear covering sets of the form

$$B = \left\{ (x,y) \in \{0,1\}^n \times [0,1]^n \;\middle|\; \sum_{j=1}^{n} a_j x_j y_j \geq d \right\},$$

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where $n \in \mathbb{Z}_{++}$, $a_j > 0 \; \forall j \in N := \{1, \ldots, n\}$, and $d > 0$. Results similar to those derived in this paper can also be obtained for sets defined through constraints of the form $\sum_{j=1}^{k}(a_j x_j y_j + b_j x_j) + \sum_{j=k+1}^{n} a_j y_j \geq d$. This generalization allows us to extend the applicability of our study to problems where the bounds on y are not 0 and 1 and, in addition, to problems where some of the x variables are fixed. Our proofs extend easily to such a setup because the two sets share strong relationships that are described in Proposition 5.1 and the discussion following it.

In order to guarantee that B is not empty, we impose

Assumption 1. $\sum_{j=1}^{n} a_j \geq d$.

On the theoretical side, we are interested in studying relaxation techniques for B that will take both the right-hand-side d and upper bounds on the variables into account. On the one hand, it follows from the separability of $\sum_{j \in N} a_j x_j y_j$ over j that $\text{conv}\{(x, y, z) \in \{0,1\}^n \times [0,1]^n \times \mathbb{R} \mid z \leq \sum_{j \in N} a_j x_j y_j\}$ is described by the McCormick constraints that overestimate each bilinear term separately [1]. Therefore, the tightest relaxation of the type $\bar{g}(x) \geq d$, where $\bar{g}(x)$ is a concave overestimator of $\sum_{j \in N} a_j x_j y_j$ restricted to $\{0,1\}^n \times [0,1]^n$ over $[0,1]^{2n}$, is the relaxation described above that uses McCormick constraints. On the other hand, if upper bounds on the variables are absent, the convex hull of the bilinear covering set can be obtained explicitly [30]. Yet, as we will see in Proposition 1.1, it is difficult to optimize linear functions over B and therefore the study of *PB* will help us understand better the difficulties that arise from the simultaneous presence of a right-hand-side and upper-bounds on the variables.

On the practical side, we are interested in deriving convex relaxations of B since they directly yield convex relaxations for problems with constraints of the form $\sum_{j=1}^{n} f_j(z) x_j \geq d$, where $z \in \mathbb{R}^p$, by replacing $f_j(z)$ with $a_j y_j + b_j$ where $y_j \in [0,1]$. We are also interested in studying B because of its relations to some important mixed-integer linear sets. In particular, since the set B is a relaxation of the fixed-charge single-node flow set without inflows

$$F = \left\{ (x, y) \in \{0,1\}^n \times [0,1]^n \;\middle|\; \sum_{j=1}^{n} a_j y_j \geq d, \; x_j \geq y_j \; \forall j \in N \right\},$$

see Lemma 4.1 for a proof, valid inequalities for B will also be valid for F. Further, we will show in Section 4 that facets of either F or B can be easily identified if facet-defining inequalities for the other set are known. As a result, the inequalities we derive for B also reveal new families of facet-defining inequalities for the convex hull of F.

We next argue that it is typically difficult to find globally optimal solutions to problems containing B as a constraint by showing that it is NP-hard to optimize a linear function over B. To this end, consider the following optimization problem (Q) that seeks to minimize a linear objective function over the bilinear set B:

$$\text{(Q)} \qquad \min \left\{ \sum_{j=1}^{n} b_j x_j + \sum_{j=1}^{n} c_j y_j \;\middle|\; (x, y) \in B \right\}$$

where $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$.

Proposition 1.1. *Problem (Q) is NP-hard.*

Proof. The proof is by reduction from the $0-1$ knapsack problem, which is proven to be NP-hard in [10]. Consider the following $0-1$ knapsack instance:

$$\text{(K)} \qquad z^K = \min \left\{ \sum_{j=1}^{n} b_j x_j \;\middle|\; \sum_{j=1}^{n} a_j x_j \geq d, \; x_j \in \{0,1\} \; \forall j \in N \right\}.$$

We define a corresponding instance of (Q) by setting $c_j = -1$ for all $j \in N$, i.e.

$$\text{(P)} \qquad z^P = \min \left\{ \sum_{j=1}^{n} b_j x_j - \sum_{j=1}^{n} y_j \;\middle|\; \sum_{j=1}^{n} a_j x_j y_j \geq d, \; x_j \in \{0,1\}, \; y_j \in [0,1] \; \forall j \in N \right\}.$$

The reduction from (K) to (P) is clearly polynomial. Observe further that if x^* is a feasible solution to (K), then $(x^*, \mathbf{1})$ is feasible to (P), therefore showing that $z^P \le z^K - n$. Similarly, if (x^*, y^*) is an optimal solution to (P), then x^* is feasible to (K) as $\sum_{j=1}^n a_j x_j^* \ge \sum_{j=1}^n a_j x_j^* y_j^* \ge d$. Therefore $z^K \le z^P + \mathbf{1}^\intercal y^* \le z^P + n$. We conclude that $z^P = z^K - n$ and that x^* is an optimal solution to (K) if and only if $(x^*, \mathbf{1})$ is an optimal solution to (P).

□

In this paper, we are interested in studying the convex hull of B, conv(B), that we denote by PB. Since B is a finite union of polytopes, PB is polyhedral.

Proposition 1.2. *PB is a polytope.*

It follows that, when studying PB, it is sufficient to consider linear inequalities. Proposition 1.1 suggests that finding a complete closed-form expression for the convex hull of B is difficult. As a result, we will focus our efforts on constructing families of strong cutting planes for optimization problems containing the constraints of B by studying the convex hull of B. To construct these inequalities, we will use lifting. Lifting is a well-known integer programming technique that generates strong inequalities for a given set by transforming an inequality valid for a restricted subset of the feasible region into a globally valid constraint. Early work on lifting in integer programming can be found in Wolsey [32, 33]. A generalization to nonlinear programming is given in Richard and Tawarmalani [24]. In particular, lifting is said to be sequence-independent if the order in which the restrictions are removed does not change the derived inequality. Subadditivity of a certain perturbation function, called the lifting function, is a sufficient condition for lifting to be sequence-independent when the restrictions involve fixing the variables at their bounds; see Proposition 3.2 and [24]. In this paper, we derive new tools to verify that functions are subadditive that we exploit to derive large families of facet-defining inequalities for PB. These results illustrate that lifting can successfully use bounds on variables in the generation of cuts for MINLPs. Further, the results have implications for fixed-charge flow models, a family of problems both theoretically and practically important in mixed-integer linear programming.

The paper is structured as follows. In Section 2, we derive basic polyhedral results about PB. We provide necessary and sufficient conditions for trivial inequalities to be facet-defining. Then, we derive a linear description of PB for the special case where $n = 2$. This result is used to identify the seed inequalities that will be used in lifting procedures. In Section 3, we show that for a general class of multi-dimensional functions, it suffices to check the subadditivity condition at certain points to establish the subadditivity of the function everywhere. Then, using this result, we derive, in closed-form, three families of facet-defining inequalities for PB using sequence-independent lifting techniques. One requires the use of a subadditive approximation of the lifting function. In Section 4, we prove that there are some tight connections between the facets of PB and those of PF. In particular, we show that the lifted inequalities developed for PB generalize certain families of flow cover cuts and yield new facet-defining inequalities for the fixed-charge single-node flow set without inflows F. We summarize the contributions of our work and conclude with directions of future research in Section 5.

2 Basic polyhedral results

In this section, we derive basic results about the polyhedral structure of PB. First, we provide necessary and sufficient conditions for PB to be full-dimensional.

Proposition 2.1. *PB is a full-dimensional polytope if and only if $\sum_{j=1}^n a_j - a_i \ge d$ for all $i \in N$.*

Proof. First, we show that if $\sum_{j=1}^n a_j - a_i \ge d$ for all $i \in N$, then PB is full-dimensional. For all $i \in N$, construct $p^i = (\mathbf{1} - e_i, \mathbf{1})$ and $q^i = (\mathbf{1}, \mathbf{1} - e_i)$. Also define $r = (\mathbf{1}, \mathbf{1})$. The points p^i, q^i, and r belong to B. These points are affinely independent because $r - p^i$ and $r - q^i$ for all $i \in N$ are linearly independent. Since we have described $2n+1$ affinely independent points in PB, we have shown that PB is full-dimensional. Next, we prove that if PB is a full-dimensional polyhedron, then $\sum_{j=1}^n a_j - a_i \ge d$ for all $i \in N$. Assume by contradiction that $\sum_{j=1}^n a_j - a_i < d$ for some $i \in N$. Since $\sum_{j=1}^n a_j \ge d$ from Assumption 1, B is nonempty and so $x_i = 1$ in every feasible solution of B, showing that PB is not full-dimensional. This is the desired contradiction. □

In the remainder of this paper, we will assume that PB is full-dimensional.

Assumption 2. *$\sum_{j=1}^n a_j - a_i \geq d$ for all $i \in N$.*

Observe that Assumption 2 strictly dominates Assumption 1 and implies that $n \geq 2$. We next identify some basic characteristics of the facet-defining inequalities of PB.

Proposition 2.2. *Let*

$$\sum_{j=1}^{n} \alpha_j x_j + \sum_{j=1}^{n} \beta_j y_j \geq \delta \tag{2}$$

be a facet-defining inequality for PB that is not a scalar multiple of $x_i \leq 1$ for $i \in N$ or $y_i \leq 1$ for $i \in N$. Then, (i) $\alpha_i \geq 0$, $\forall i \in N$, (ii) $\beta_i \geq 0$, $\forall i \in N$, and (iii) $\delta \geq 0$.

Proof. Select $i \in N$. Since (2) is a facet-defining inequality for PB that is not a scalar multiple of $x_i \leq 1$, there exists $(x^*, y^*) \in B$ with $x_i^* < 1$ such that

$$\sum_{j=1}^{n} \alpha_j x_j^* + \sum_{j=1}^{n} \beta_j y_j^* = \delta. \tag{3}$$

Consider now $(\bar{x}, \bar{y}) = (x^*, y^*) + (1 - x_i^*)(e_i, 0)$. This point belongs to B and therefore satisfies (2), *i.e.*,

$$\sum_{j=1}^{n} \alpha_j \bar{x}_j + \sum_{j=1}^{n} \beta_j \bar{y}_j \geq \delta. \tag{4}$$

Subtracting (3) from (4), we obtain that $\alpha_i \geq 0$. The proof that $\beta_i \geq 0$ for all $i \in N$ is similar. The fact that $\delta \geq 0$ then follows from (3) after noting that all terms in the left-hand-side are nonnegative. □

The following proposition further studies facet-defining inequalities whose right-hand-sides are zero.

Proposition 2.3. *Let*

$$\sum_{j=1}^{n} \alpha_j x_j + \sum_{j=1}^{n} \beta_j y_j \geq 0 \tag{5}$$

be a facet-defining inequality for PB. Then, (5) is a scalar multiple of $x_j \geq 0$ for $j \in N$ or of $y_j \geq 0$ for $j \in N$.

Proof. Assume for a contradiction that (5) is not a scalar multiple of $x_j \geq 0$ for $j \in N$ or of $y_j \geq 0$ for $j \in N$. Then, for each $i \in N$, there exists $(x^i, y^i) \in B$ such that $x_i^i > 0$ and for which

$$\sum_{j=1}^{n} \alpha_j x_j^i + \sum_{j=1}^{n} \beta_j y_j^i = 0. \tag{6}$$

Since we know from Proposition 2.2 that $\alpha_j \geq 0$ and $\beta_j \geq 0$ for all $j \in N$, we obtain from (6) that

$$0 = \sum_{j=1}^{n} \alpha_j x_j^i + \sum_{j=1}^{n} \beta_j y_j^i \geq \alpha_i x_i^i \geq 0. \tag{7}$$

We conclude that, for each $i \in N$, $\alpha_i = 0$ since $x_i^i > 0$. Similarly, we can establish that $\beta_i = 0 \; \forall i \in N$. This is a contradiction to the fact that (5) is facet-defining for PB. □

We now focus on these inequalities that play a special role in Propositions 2.2 and 2.3 and characterize when they are facet-defining for PB. We refer to these inequalities as *bound inequalities*.

Proposition 2.4. *The upper bound inequalities $x_i \leq 1$, $y_i \leq 1$ are facet-defining for PB for all $i \in N$. Further, for $i \in N$, the lower bound inequalities $x_i \geq 0$, $y_i \geq 0$ are facet-defining for PB if and only if $\sum_{j=1}^n a_j - a_i - a_{l(i)} \geq d$ where $l(i) \in \operatorname{argmax}\{a_j \mid j \in N \setminus \{i\}\}$.*

Proof. The validity of all these inequalities is trivial since they belong to the description of B. Assume for a contradiction that $x_i \le 1$ is not facet-defining for *PB*. Then, it follows from Proposition 2.2 that $(e_i, 0)$ is a recession direction of *PB*, a contradiction to the fact that *PB* is a polytope; see Proposition 1.2. The proof that $y_i \le 1$ is facet-defining for *PB* is similar.

Now, we show that $x_i \ge 0$ is facet-defining if $\sum_{j=1}^n a_j - a_i - a_{l(i)} \ge d$ by describing $2n$ affinely independent points in B that satisfy $x_i = 0$. For $k \in N \setminus \{i\}$, we construct the $2(n-1)$ points, $\bar{p}^k = (\mathbf{1} - e_i - e_k, \mathbf{1} - e_i - e_k)$ and $\bar{q}^k = (\mathbf{1} - e_i - e_k, \mathbf{1} - e_i)$. We also define $r^1 = (\mathbf{1} - e_i, \mathbf{1} - e_i)$ and $r^2 = (\mathbf{1} - e_i, \mathbf{1})$. Clearly, the points r^1, r^2 and $\bar{p}^k$, $\bar{q}^k$ for $k \in N \setminus \{i\}$ satisfy $x_i \ge 0$ at equality and are feasible for B since $\sum_{j=1}^n a_j - a_i \ge \sum_{j=1}^n a_j - a_i - a_k \ge \sum_{j=1}^n a_j - a_i - a_{l(i)} \ge d$. These points are affinely independent since $\bar{p}^k - r^1$, $\bar{q}^k - r^1$, and $r^2 - r^1$ can be easily verified to be linearly independent. To prove the reverse direction, assume now that $x_i \ge 0$ is facet-defining for *PB*. We claim that $\sum_{j=1}^n a_j - a_i - a_{l(i)} \ge d$. Assume for a contradiction that $\sum_{j=1}^n a_j - a_i - a_{l(i)} < d$. This condition implies that every feasible solution (x, y) of *PB* with $x_i = 0$ also must satisfy $x_{l(i)} = 1$. As a result, the dimension of the face defined by $x_i = 0$ is less or equal to $2n - 2$, which is a contradiction. Similarly, it can be proven that $y_i \ge 0$ is facet-defining for *PB* if and only if $\sum_{j=1}^n a_j - a_i - a_{l(i)} \ge d$. □

Observe that the above proofs are also valid when $y_i \in \{0, 1\}$ instead of $y_i \in [0, 1]$ for some subset $J \subseteq N$. We next study another simple facet-defining inequality for *PB*.

Proposition 2.5. *The inequality* $\sum_{j=1}^n a_j y_j \ge d$ *is facet-defining for PB.*

Proof. Validity is easily verified since $\sum_{j=1}^n a_j y_j \ge \sum_{j=1}^n a_j x_j y_j \ge d$. To prove that $\sum_{j=1}^n a_j y_j \ge d$ is facet-defining, we present $2n$ points (x^i, y^i) in B that satisfy $\sum_{j=1}^n a_j y_j^i \ge d$ at equality and such that the system $\alpha x^i + \beta y^i = \delta$ for $i = 1, \ldots, 2n$ only has solutions (α, β, δ) that are scalar multiples of $(\mathbf{0}, a, d)$. Consider the $2n$ points $p^k = (\mathbf{1}, \Delta_k(\mathbf{1} - e_k))$ and $q^k = (\mathbf{1} - e_k, \Delta_k(\mathbf{1} - e_k))$ where $\Delta_k = \frac{d}{\sum_{j=1}^n a_j - a_k}$ for $k \in N$. Note that because of Assumption 2, $0 < \Delta_k \le 1$ for all $k \in N$. Clearly, p^k and q^k belong to B and satisfy $\sum_{j=1}^n a_j y_j \ge d$ at equality. These $2n$ points yield the system:

$$\sum_{j=1}^n \alpha_j + \Delta_k \left(\sum_{j=1}^n \beta_j - \beta_k \right) = \delta \qquad \forall k \in N, \tag{8}$$

$$\sum_{j=1}^n \alpha_j - \alpha_k + \Delta_k \left(\sum_{j=1}^n \beta_j - \beta_k \right) = \delta \quad \forall k \in N. \tag{9}$$

By subtracting (8) from (9), we obtain that $\alpha_k = 0$ for $k \in N$. From (8) and the definition of Δ_k, we then conclude that, for all $k, l \in N$,

$$\sum_{j=1}^n \beta_j - \beta_k = \frac{\delta}{d} \left(\sum_{j=1}^n a_j - a_k \right) \text{ and } \sum_{j=1}^n \beta_j - \beta_l = \frac{\delta}{d} \left(\sum_{j=1}^n a_j - a_l \right).$$

Subtracting these expressions yields $\beta_k - \frac{\delta}{d} a_k = \beta_l - \frac{\delta}{d} a_l$. After defining $\beta_k - \frac{\delta}{d} a_k = \theta$ for $k \in N$ and using these relations in (8), we obtain that $\theta = 0$, which implies $\beta_k = \frac{\delta}{d} a_k$ for $k \in N$. Therefore, we conclude that all solutions (α, β, δ) to the system (8) and (9) are scalar multiples of $(\mathbf{0}, a, d)$. □

In the remainder of this paper, we will often use the term *facet* to refer to a facet-defining inequality. We will also refer to inequalities $x_i \le 1$, $y_i \le 1$, and $\sum_{j=1}^n a_j y_j \ge d$ as *trivial facets* of *PB*. To illustrate the richness of the polyhedral structure of *PB*, we present an example next. The linear inequalities describing the convex hull of this set were obtained using PORTA; see Christof and Löbel [6].

Example 2.6. *Consider the* $0{-}1$ *mixed-integer bilinear covering set*

$$B = \left\{ (x, y) \in \{0, 1\}^4 \times [0, 1]^4 \;\middle|\; 19x_1y_1 + 17x_2y_2 + 15x_3y_3 + 10x_4y_4 \ge 20 \right\}.$$

The linear description of PB has 58 inequalities that are presented in the Appendix. They include:

$$
\begin{array}{rcrcrcrcrcrcrcrcrl}
50x_1 & & & + & 90x_3 & + & 45x_4 & + & 76y_1 & + & 153y_2 & & & & & \geq & 135 & \quad (10)\\
70x_1 & + & 90x_2 & & & + & 27x_4 & + & 38y_1 & & & + & 135y_3 & & & \geq & 117 & \quad (11)\\
19x_1 & + & 17x_2 & & & & & & & & & + & 15y_3 & + & 10y_4 & \geq & 20 & \quad (12)\\
 & & 17x_2 & + & 15x_3 & & & + & 19y_1 & & & & & + & 10y_4 & \geq & 20 & \quad (13)\\
 & & & & & & & & 19y_1 & + & 17y_2 & + & 15y_3 & + & 10y_4 & \geq & 20 & \quad (14)\\
14x_1 & & & + & 10x_3 & + & 5x_4 & & & + & 17y_2 & & & & & \geq & 15 & \quad (15)\\
 & & 12x_2 & + & 10x_3 & + & 5x_4 & + & 19y_1 & & & & & & & \geq & 15 & \quad (16)\\
 & & & & 10x_3 & + & 5x_4 & + & 19y_1 & + & 17y_2 & & & & & \geq & 15 & \quad (17)\\
x_1 & + & x_2 & + & x_3 & & & & & & & & & + & 10y_4 & \geq & 2 & \quad (18)\\
x_1 & + & x_2 & + & x_3 & + & x_4 & & & & & & & & & \geq & 2 & \quad (19)\\
x_1 & & & & & & & & & & & & & & & \geq & 0 & \quad (20)\\
 & & & & & & & & y_1 & & & & & & & \geq & 0 & \quad (21)\\
x_1 & & & & & & & & & & & & & & & \leq & 1 & \quad (22)\\
 & & & & & & & & y_1 & & & & & & & \leq & 1 & \quad (23)
\end{array}
$$

Among the inequalities in Example 2.6, we recognize the upper bound inequalities (22) and (23) that are shown to be facet-defining for *PB* in Proposition 2.4. In this example, the lower bound inequalities (20) and (21) are also facet-defining, as can be established from Proposition 2.4. Further, (14) is the trivial facet-defining inequality studied in Proposition 2.5. Our goal is now to discover families of valid inequalities for *PB* that would explain (10) – (13) and (15) – (19).

To derive these nontrivial facet-defining inequalities, we first study the convex hull of B when $n = 2$ with the goal of identifying seed inequalities for subsequent lifting procedures. We show that the linear description of *PB* has at most three nontrivial inequalities. In this study, Assumption 2 requires that $a_1 \geq d$ and $a_2 \geq d$.

Proposition 2.7. *Let*

$$B^2 = \left\{ (x, y) \in \{0, 1\}^2 \times [0, 1]^2 \;\middle|\; a_1 x_1 y_1 + a_2 x_2 y_2 \geq d \right\},$$

where $a_1 \geq d$, $a_2 \geq d$ and $d > 0$. Then,

$$\text{conv}(B^2) = X := \left\{ (x, y) \in [0, 1]^2 \times [0, 1]^2 \;\middle|\; \begin{array}{rrl} x_1 + & x_2 & \geq 1 \\ dx_1 + & a_2 y_2 & \geq d \\ a_1 y_1 + & dx_2 & \geq d \\ a_1 y_1 + & a_2 y_2 & \geq d \end{array} \right\}.$$

Proof. We prove the result using disjunctive programming techniques; see [5]. We define

$$
\begin{aligned}
X_{10} &:= B^2 \cap \{x_1 = 1, x_2 = 0\} = \{(1, y_1, 0, y_2) \mid \tfrac{d}{a_1} \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1\},\\
X_{01} &:= B^2 \cap \{x_1 = 0, x_2 = 1\} = \{(0, y_1, 1, y_2) \mid 0 \leq y_1 \leq 1,\ \tfrac{d}{a_2} \leq y_2 \leq 1\},\\
X_{11} &:= B^2 \cap \{x_1 = 1, x_2 = 1\} = \{(1, y_1, 1, y_2) \mid a_1 y_1 + a_2 y_2 \geq d,\ 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1\}.
\end{aligned}
$$

It is easily verified that $\text{conv}(B^2) = \text{conv}(X_{10} \cup X_{01} \cup X_{11}) = \text{conv}(X_2 \cup X_{11})$ where $X_2 := \text{conv}(X_{10} \cup X_{01})$. We first use disjunctive programming techniques to obtain a linear description of X_2 and then compute $\text{conv}(B^2)$ as $\text{conv}(X_2 \cup X_{11})$. Using Theorem 2.1 in Balas [5], we write

$$X_2 = \underset{(x,y)}{\text{proj}} \left\{ (x_1, y_1, x_2, y_2, \bar{z}_1, \bar{z}_2, \hat{z}_1, \hat{z}_2, \lambda) \;\middle|\; \begin{array}{l} (x_1, y_1, x_2, y_2) = (\lambda, \bar{z}_1 + \hat{z}_1, 1 - \lambda, \bar{z}_2 + \hat{z}_2), \\ \dfrac{d}{a_1}\lambda \leq \bar{z}_1 \leq \lambda,\ 0 \leq \bar{z}_2 \leq \lambda, \\ 0 \leq \hat{z}_1 \leq 1 - \lambda,\ \dfrac{d}{a_2}(1 - \lambda) \leq \hat{z}_2 \leq 1 - \lambda, \\ 0 \leq \lambda \leq 1 \end{array} \right\}.$$

We then use Fourier-Motzkin elimination [34] to compute the projection. We first eliminate the variables λ, $\hat{z}_1$ and $\hat{z}_2$ using the equations $\lambda = x_1$, $\hat{z}_1 = y_1 - \bar{z}_1$, and $\hat{z}_2 = y_2 - \bar{z}_2$. We obtain

$$x_1 + x_2 = 1,\ 0 \leq x_1 \leq 1,$$

and

$$\begin{aligned} \tfrac{d}{a_1}x_1 \le \quad & \bar{z}_1 \quad \le x_1, \\ x_1 + y_1 - 1 \le \quad & \bar{z}_1 \quad \le y_1, \\ 0 \le \quad & \bar{z}_2 \quad \le 1 - x_2, \\ y_2 - x_2 \le \quad & \bar{z}_2 \quad \le y_2 - \tfrac{d}{a_2}x_2, \end{aligned}$$

from which we project variables $\bar{z}_1$ and $\bar{z}_2$ to obtain

$$X_2 = \operatorname{conv}(X_{10} \cup X_{01}) = \left\{ (x_1, y_1, x_2, y_2) \;\middle|\; \begin{array}{l} x_1 + x_2 = 1,\ x_1 \ge 0,\ x_2 \ge 0, \\ \dfrac{d}{a_1}x_1 \le y_1 \le 1,\ \dfrac{d}{a_2}x_2 \le y_2 \le 1 \end{array} \right\}$$

since $x_1 \le 1$ and $x_2 \le 1$ are implied by $x_1 + x_2 = 1$, $x_1 \ge 0$ and $x_2 \ge 0$. Now, compute $\operatorname{conv}(X_2 \cup X_{11})$ as

$$\operatorname*{proj}_{(x,y)} \left\{ \begin{array}{c} (x_1, y_1, x_2, y_2, \bar{u}_1, \\ \bar{u}_2, \bar{v}_1, \bar{v}_2, \hat{v}_1, \hat{v}_2, \lambda) \end{array} \;\middle|\; \begin{array}{l} (x_1, y_1, x_2, y_2) = (\bar{u}_1 + (1-\lambda), \bar{v}_1 + \hat{v}_1, \bar{u}_2 + (1-\lambda), \bar{v}_2 + \hat{v}_2), \\ \bar{u}_1 + \bar{u}_2 = \lambda,\ \bar{u}_1 \ge 0,\ \bar{u}_2 \ge 0, \\ \dfrac{d}{a_1}\bar{u}_1 \le \bar{v}_1 \le \lambda,\ \dfrac{d}{a_2}\bar{u}_2 \le \bar{v}_2 \le \lambda, \\ a_1\hat{v}_1 + a_2\hat{v}_2 \ge d(1-\lambda), \\ 0 \le \hat{v}_1 \le 1-\lambda,\ 0 \le \hat{v}_2 \le 1-\lambda, \\ 0 \le \lambda \le 1 \end{array} \right\}.$$

We again obtain the projection using Fourier-Motzkin elimination. Using the equations $x_1 = \bar{u}_1 + 1 - \lambda$, $x_2 = \bar{u}_2 + 1 - \lambda$, and $\bar{u}_1 + \bar{u}_2 = \lambda$, we obtain that $\lambda = 2 - (x_1 + x_2)$, $\bar{u}_1 = 1 - x_2$, and $\bar{u}_2 = 1 - x_1$. Using these relations together with $\bar{v}_1 = y_1 - \hat{v}_1$ and $\bar{v}_2 = y_2 - \hat{v}_2$ to eliminate the corresponding variables, we obtain

$$x_1 \le 1,\ x_2 \le 1,\ 1 \le x_1 + x_2 \preccurlyeq 2,$$

and

$$\begin{aligned} y_1 + x_1 + x_2 - 2 \le \quad & \hat{v}_1 \quad \le y_1 - \tfrac{d}{a_1}(1 - x_2), \\ -\tfrac{a_2}{a_1}\hat{v}_2 + \tfrac{d}{a_1}(x_1 + x_2 - 1) \le \quad & \hat{v}_1, \\ 0 \le \quad & \hat{v}_1 \quad \le x_1 + x_2 - 1, \\ y_2 + x_1 + x_2 - 2 \le \quad & \hat{v}_2 \quad \le y_2 - \tfrac{d}{a_2}(1 - x_1), \\ 0 \le \quad & \hat{v}_2 \quad \le x_1 + x_2 - 1, \end{aligned}$$

where inequality $\preccurlyeq$ is clearly redundant. Projecting $\hat{v}_1$, we obtain

$$\begin{gathered} x_1 \le 1,\ x_2 \le 1,\ 1 \le x_1 + x_2,\ y_1 \le 1,\ \tfrac{d}{a_1}(1 - x_2) \le y_1, \\ a_1x_1 + (a_1 - d)x_2 \preccurlyeq 2a_1 - d \end{gathered}$$

and

$$\begin{aligned} \tfrac{d}{a_2}x_1 - \tfrac{a_1}{a_2}y_1 \le \quad & \hat{v}_2, \\ \tfrac{(d-a_1)(x_1+x_2-1)}{a_2} \preccurlyeq \quad & \hat{v}_2, \\ y_2 + x_1 + x_2 - 2 \le \quad & \hat{v}_2 \quad \le y_2 - \tfrac{d}{a_2}(1 - x_1), \\ 0 \le \quad & \hat{v}_2 \quad \le x_1 + x_2 - 1, \end{aligned}$$

where obviously redundant inequalities have been omitted. Again, inequalities $\preccurlyeq$ are redundant since $x_1 \le 1$, $x_2 \le 1$, $x_1 + x_2 \ge 1$, $a_1 \ge d$ and $a_2 \ge d > 0$. Projecting $\hat{v}_2$, we obtain the system

$$x_1 \le 1,\ x_2 \le 1,\ 1 \le x_1 + x_2,\ y_1 \le 1,\ \tfrac{d}{a_1}(1 - x_2) \le y_1,$$

and

$$\begin{aligned} & d \le a_1y_1 + a_2y_2, \\ & a_2 \preccurlyeq (a_2 - d)x_1 + a_1y_1 + a_2x_2, \quad (R) \\ & (a_2 - d)x_1 + a_2x_2 \preccurlyeq 2a_2 - d, \\ & y_2 \le 1, \\ & \tfrac{d}{a_2}(1 - x_1) \le y_2, \\ & 1 \preccurlyeq x_1 + x_2, \end{aligned}$$

where inequalities $\preccurlyeq$ are either repeated or redundant. In particular, (R) is redundant since it can be obtained as a conic combination with weights $(a_2 - d)$ and 1 of valid inequalities $x_1 + x_2 \geq 1$ and $a_1y_1 + dx_2 \geq d$. Therefore, $\operatorname{conv}(X_2 \cup X_{11})$ is defined by bounds and the four inequalities given in the description of X, concluding the proof. □

Next, we give generalizations of the nontrivial facets of $\operatorname{conv}(B^2)$ that we prove are facet-defining for more general instances of $\operatorname{conv}(B)$. In particular, we give a generalization of inequalities $dx_1 + a_2y_2 \geq d$ and $a_1y_1 + dx_2 \geq d$ in Proposition 2.9 and of inequality $x_1 + x_2 \geq 1$ in Proposition 2.11. We will use these generalizations as seed inequalities for lifting procedures in Section 3.

Lemma 2.8. *Inequality*

$$\sum_{j\in N} \min\{dx_j, a_jx_j, a_jy_j\} \geq d \tag{24}$$

is valid for PB.

Proof. We first show that $\sum_{j\in N} \min\{dx_j, a_jx_jy_j\} \geq d$ is valid for B. Consider $(x, y) \in B$. If there exists $j \in N$ such that $dx_j < a_jx_jy_j$ then $x_j = 1$ and, consequently, the inequality is satisfied. Otherwise, the inequality reduces to the defining inequality of B. Since $(x_j, y_j) \in [0, 1]^2$ implies that $x_jy_j \leq \min\{x_j, y_j\}$ and $a_j \geq 0$, it follows that $\min\{dx_j, a_jx_jy_j\} \leq \min\{dx_j, a_jx_j, a_jy_j\}$ and, therefore, (24) is valid for *PB*. □

The set of solutions in $[0, 1]^{2n}$ that satisfy (24) is a subset of the convex relaxations of B discussed in Section 1. In particular, when each bilinear term is outer-approximated using McCormick envelopes, we obtain the inequality $\sum_{j\in N} a_j \min\{x_j, y_j\} \geq d$, which is clearly implied by (24). Further, using orthogonal disjunctions, see [30], it can be shown that

$$O := \operatorname{conv}\Big\{(x, y) \in \mathbb{R}^{2n}_+ \ \Big|\ \sum_{j\in N} a_jx_jy_j \geq d\Big\} = \Big\{(x, y) \in \mathbb{R}^{2n}_+ \ \Big|\ \sum_{j\in N} \sqrt{a_jx_jy_j} \geq \sqrt{d}\Big\}.$$

This convex relaxation is obtained without making use of the bounds or the integrality of the variables x. It follows from the inequality relating elementary means (see Theorem 5 in [15]) that $\sqrt{da_jx_jy_j} \geq \min\{dx_j, a_jy_j\}$. Therefore, the feasible solutions to (24) are contained in O. However, when $(x, y) \in C \subsetneq \mathbb{R}^n_+$, a procedure described in [29] permits strengthening relaxation O by restricting attention to C. When one exploits the fact that $(x, y) \in C = \{0, 1\}^n \times [0, 1]^n$, this construction yields (24).

Proposition 2.9. *Let $L \subseteq N$ be such that $\sum_{j\in N\setminus L} a_j > d$. Define $\bar{a} = \sum_{j\in N\setminus L} a_j - \max_{i\in N\setminus L} a_i$ and assume that $S = \{(x, \bar{y}) \in \{0, 1\}^{|L|} \times [0, 1] \mid \sum_{i\in L} \min\{a_i, d\}x_i + \bar{a}\bar{y} = d\} \neq \emptyset$. Then,*

$$\sum_{j\in L} \min\{a_j, d\}x_j + \sum_{j\in N\setminus L} a_jy_j \geq d \tag{25}$$

is facet-defining for PB. In particular, (25) is facet-defining for PB if (i) $L \cap L^> \neq \emptyset$, or (ii) $L = \emptyset$, or (iii) $\bar{a} \geq \max_{i\in L} \min\{a_i, d\}$, or as a special case (iv) $\bar{a} \geq d$ where $L^> := \{j \in N \mid a_j > d\}$.

Proof. Validity of (25) for *PB* follows from Lemma 2.8. We now prove that (25) is facet-defining for *PB* by providing $2n$ affinely independent points (x^i, y^i) in B that satisfy (25) at equality. Assume without loss of generality that $L = \{1, \ldots, l\}$. Define $n' = |N \setminus L|$ and denote the points as $(x_L, x_{N\setminus L}, y_L, y_{N\setminus L})$.

Let $(x', \bar{y}') \in S$ and define $a' = \sum_{j\in N\setminus L} a_j$. Let $p^0 = (0, \mathbf{1}, 0, \frac{d}{a'}\mathbf{1})$ and $p^j = p^0 + \epsilon(0, 0, 0, \frac{1}{a_j}e_j - \frac{1}{a_{j+1}}e_{j+1})$ for $j = 1, \ldots, n' - 1$. For $i \in L$, define $q^i = (e_i, \mathbf{1}, e_i, \frac{d-\min\{a_i,d\}}{a'-\min\{a_i,d\}}\mathbf{1})$, $r^i = p^0 + (0, 0, e_i, 0)$ if $a_i \leq d$, and $r^i = (e_i, \mathbf{1}, \frac{d}{a_i}e_i, 0)$ if $a_i > d$. For $j \in \{1, \ldots, n'\}$, $s^j = (x'_L, \mathbf{1} - e_j, \mathbf{1}, \bar{y}'\frac{\bar{a}}{\sum_{i\in N\setminus(L\cup\{j\})} a_i}(\mathbf{1} - e_j))$. It can be easily verified that p^0, q^i, s^j and r^i belong to B and that p^j belongs to B when ϵ is sufficiently small.

We now show that the above points are affinely independent. Clearly, for $j \in \{0, \ldots, n' - 2\}$, $p^0, \ldots, p^j$ satisfy $\sum_{i=1}^{j+1} a_i(\frac{d}{a'} - y_{l+i}) = 0$, whereas p^{j+1} does not. Therefore, p^j are affinely independent. Further, for $i \in L$, q^i are affinely independent of p^j since the latter satisfy $(x_i, y_i) = (0, 0)$. For $i \in L$, r^i are independent of p^j and q^j since the latter satisfy $y_i = x_i$. Finally, s^i are affinely independent of p^j, q^j, r^j since the latter satisfy $x_i = 1$. □

The family of inequalities described in Proposition 2.9 is typically exponential in size. In the case of Example 2.6, it contains multiple inequalities including (12-14). More generally, it can be verified that inequalities (10)-(19) in the Appendix are of the form (25).

In the remainder of the paper, we use the following notation extensively. For $N_0, N_1 \subseteq N$ such that $N_0 \cap N_1 = \emptyset$ and $\tilde{N}_0, \tilde{N}_1 \subseteq N$ such that $\tilde{N}_0 \cap \tilde{N}_1 = \emptyset$, we let

$$B(N_0, N_1, \tilde{N}_0, \tilde{N}_1) := \left\{ (x,y) \in B \;\middle|\; \begin{array}{llll} x_j = 0 & \text{for } j \in N_0, & x_j = 1 & \text{for } j \in N_1, \\ y_j = 0 & \text{for } j \in \tilde{N}_0, & y_j = 1 & \text{for } j \in \tilde{N}_1 \end{array} \right\}.$$

We also define $PB(N_0, N_1, \tilde{N}_0, \tilde{N}_1) := \mathrm{conv}(B(N_0, N_1, \tilde{N}_0, \tilde{N}_1))$. In particular, $B(\emptyset, \emptyset, \emptyset, N)$ is equivalent to the classical $0{-}1$ knapsack set

$$\left\{ x \in \{0,1\}^n \;\middle|\; \sum_{j=1}^{n} a_j x_j \geq d \right\},$$

whose polyhedral structure was first studied by Balas [4], Hammer et al. [14] and Wolsey [31]. The following proposition describes, among other things, relations between the bilinear set B and the $0{-}1$ knapsack set $B(\emptyset, \emptyset, \emptyset, N)$.

Proposition 2.10. *Let*

$$\sum_{j \in N} \alpha_j x_j + \sum_{j \in I} \beta_j y_j \geq \delta \tag{26}$$

be an inequality for $PB(\emptyset, \emptyset, \emptyset, N \setminus I)$ that is not a scalar multiple of a bound inequality. Then, (26) is facet-defining for $PB(\emptyset, \emptyset, \emptyset, N \setminus I)$ if and only if (26) is facet-defining for PB.

Proof. We first prove that if (26) is facet-defining for $PB(\emptyset, \emptyset, \emptyset, N \setminus I)$, then (26) is facet-defining for PB. To show that (26) is valid for B, we assume for a contradiction that there exists a point $(x', y') \in B$ with $\sum_{j \in N} \alpha_j x'_j + \sum_{j \in I} \beta_j y'_j < \delta$. Since $(x', y') \in B$, we have that $\sum_{j \in N} a_j x'_j y'_j \geq d$. Next, we define $(\bar{x}, \bar{y})$ as $\bar{x} = x'$, $\bar{y}_j = y'_j$ for $j \in I$, and $\bar{y}_j = 1$ for $j \in N \setminus I$. Observe that $(\bar{x}, \bar{y}) \in B(\emptyset, \emptyset, \emptyset, N \setminus I)$ as $\sum_{j \in I} a_j \bar{x}_j \bar{y}_j + \sum_{j \in N \setminus I} a_j \bar{x}_j \geq \sum_{j \in N} a_j x'_j y'_j \geq d$. Since (26) is valid for $B(\emptyset, \emptyset, \emptyset, N \setminus I)$, $(\bar{x}, \bar{y})$ satisfies $\sum_{j \in N} \alpha_j x'_j + \sum_{j \in I} \beta_j y'_j = \sum_{j \in N} \alpha_j \bar{x}_j + \sum_{j \in I} \beta_j \bar{y}_j \geq \delta$. This is the desired contradiction.

Next, we show that (26) is facet-defining for PB. Since (26) is facet-defining for $PB(\emptyset, \emptyset, \emptyset, N \setminus I)$ and $\delta \neq 0$ as (26) is not a bound, there exist $n + |I|$ linearly independent points in $B(\emptyset, \emptyset, \emptyset, N \setminus I)$, call them (x^k, y^k), that satisfy (26) at equality. Clearly, these points belong to B and satisfy (26) at equality. Now, for each $j \in N \setminus I$, we construct one new point in $B \setminus B(\emptyset, \emptyset, \emptyset, N \setminus I)$ that satisfies (26) at equality. Choose j arbitrarily in $N \setminus I$. Since (26) is not a scalar multiple of $x_j \leq 1$, there exists $k_j \in \{1, \ldots, n + |I|\}$ such that $x_j^{k_j} = 0$. Now define $(\bar{x}^{k_j}, \bar{y}^{k_j})$ such that $\bar{x}_i^{k_j} = x_i^{k_j}\ \forall i \in N$, $\bar{y}_i^{k_j} = y_i^{k_j}\ \forall i \in N \setminus \{j\}$ and $y_j^{k_j} = 0$. Clearly, the point $(\bar{x}^{k_j}, \bar{y}^{k_j})$ belongs to B and satisfies (26) at equality. Further, it is easily seen that the points (x^k, y^k) and $(\bar{x}^{k_j}, \bar{y}^{k_j})$ for $j \in N \backslash I$ are linearly independent and therefore show that (26) is facet-defining for PB.

To prove the reverse implication, we assume that (26) is a facet-defining inequality for PB that is not a scalar multiple of a bound. Validity is trivial since for $B(\emptyset, \emptyset, \emptyset, N \setminus I) \subseteq B$. Now, we show that (26) is facet-defining for $PB(\emptyset, \emptyset, \emptyset, N \setminus I)$. Since $\delta \neq 0$ as (26) is not a bound, the set of $2n$ affinely independent points (x^k, y^k) in B for $k = 1, \ldots, 2n$ that satisfy (26) at equality are also linearly independent. Therefore,

$$\begin{vmatrix} x_1^1 & \ldots & x_n^1 & y_1^1 & \ldots & y_n^1 \\ x_1^2 & \ldots & x_n^2 & y_1^2 & \ldots & y_n^2 \\ & \ldots & & & \ldots & \\ x_1^{2n} & \ldots & x_n^{2n} & y_1^{2n} & \ldots & y_n^{2n} \end{vmatrix} \neq 0.$$

Therefore, there must exist $n + |I|$ rows $i_1, \ldots, i_{n+|I|}$ where $I = \{j_1, \ldots, j_{|I|}\}$ such that

$$\begin{vmatrix} x_1^{i_1} & \ldots & x_n^{i_1} & y_{j_1}^{i_1} & \ldots & y_{j_{|I|}}^{i_1} \\ x_1^{i_2} & \ldots & x_n^{i_2} & y_{j_1}^{i_2} & \ldots & y_{j_{|I|}}^{i_2} \\ & \ldots & & & \ldots & \\ x_1^{i_{n+|I|}} & \ldots & x_n^{i_{n+|I|}} & y_{j_1}^{i_{n+|I|}} & \ldots & y_{j_{|I|}}^{i_{n+|I|}} \end{vmatrix} \neq 0.$$

Hence, we see that the $n+|I|$ points $(x_1^{i_k},\dots,x_n^{i_k};y_{j_1}^{i_k},\dots,y_{j_{|I|}}^{i_k})$ for $k=1,\dots,n+|I|$ are linearly independent. Now, define the points $(\tilde{x}^{i_k},\tilde{y}^{i_k})$ for $k=1,\dots,n+|I|$ such that $\tilde{x}^{i_k}=x^{i_k}$, $\tilde{y}_j^{i_k}=y_j^{i_k}$ for $j\in I$, and $\tilde{y}_j^{i_k}=1$ for $j\in N\setminus I$. The points $(\tilde{x}^{i_k},\tilde{y}^{i_k})$ are feasible to $B(\emptyset,\emptyset,\emptyset,N\setminus I)$ and satisfy (26) at equality. Therefore, we conclude that (26) is facet-defining for $PB(\emptyset,\emptyset,\emptyset,N\setminus I)$. □

Observe that Proposition 2.10 implies that all nontrivial facets of the $0{-}1$ knapsack polytope can be found in B and that it is sufficient to study the facets of B to obtain the facets of the $0{-}1$ knapsack polytope. Next, we use Proposition 2.10 to generalize the inequality $x_1+x_2\ge 1$ of Proposition 2.7 into an inequality that will be used as a seed for lifting procedures in Section 3.4.

Proposition 2.11. *Assume that* $\sum_{j\in N}a_j-a_k-a_m<d$ *for all* $k,m\in N$ *with* $k\neq m$. *The clique inequality*

$$\sum_{j\in N}x_j\ge |N|-1 \tag{27}$$

is facet-defining for PB.

Proof. Because of Proposition 2.10. it is sufficient to prove that (27) is facet-defining for $PB(\emptyset,\emptyset,\emptyset,N)$. To prove validity, assume for a contradiction that there exists $x'\in B(\emptyset,\emptyset,\emptyset,N)$ such that $\sum_{j\in N}a_jx'_j\ge d$ and $\sum_{j\in N}x'_j\le |N|-2$. Since $\sum_{j\in N}x'_j\le|N|-2$, there exist $k,m\in N$ with $k\neq m$ such that $x'_k=0$ and $x'_m=0$. Therefore, $\sum_{j\in N}a_j-a_k-a_m\ge\sum_{j\in N}a_jx'_j\ge d$. This contradicts the assumption that $\sum_{j\in N}a_j-a_k-a_m<d$ for all $k,m\in N$ with $k\neq m$. We next show that (27) is facet-defining for $PB(\emptyset,\emptyset,\emptyset,N)$. It can be easily verified using Assumption 2 that the points $p^k=(\mathbf{1}-e_k,\mathbf{1})$ for $k\in N$ belong to $B(\emptyset,\emptyset,\emptyset,N)$. Since these points are linearly independent and satisfy (27) at equality, we conclude that (27) is facet-defining for $PB(\emptyset,\emptyset,\emptyset,N)$. □

3 Lifted inequalities

In this section, we derive three families of strong valid inequalities for *PB* via lifting. The first two families are obtained using sequence-independent lifting from (25) and are facet-defining for *PB*. In this case, lifting is simple since the lifting function is subadditive. The third inequality is obtained by lifting (27). Although the lifting function associated with this seed inequality is not subadditive, we obtain lifted inequalities using approximate lifting. We then identify conditions under which these inequalities are facet-defining for *PB*.

3.1 Sequence-independent lifting for bilinear covering sets

Sequence-independent lifting is a well-known technique to construct strong valid inequalities for mixed-integer linear programs; see Wolsey [33] and Gu et al. [13]. We next give a brief description of how this technique can be used to derive strong valid inequalities for *PB*. A more general treatment of lifting in nonlinear programming is given in Richard and Tawarmalani [24].

Given $\emptyset\neq S\subsetneq N$, we consider $B(S,\emptyset,S,\emptyset)$, which is the restriction of B obtained when all variables (x_j,y_j) for $j\in S$ are fixed to $(0,0)$. Let $S=\{s,\dots,n\}$ for some $s\ge 2$ and define $S_i=\{i+1,\dots,n\}$ for $i\in S$. Assume that the inequality

$$\sum_{j=1}^{s-1}\alpha_jx_j+\sum_{j=1}^{s-1}\beta_jy_j\ge\delta \tag{28}$$

is facet-defining for $PB(S,\emptyset,S,\emptyset)$. In sequential lifting, we reintroduce the variables (x_j,y_j) for $j\in S$ one at the time in (28). Assuming that variables (x_j,y_j) have already been lifted in the order $j=s,\dots,i-1$, we next review how to lift variables (x_i,y_i) in the inequality

$$\sum_{j=1}^{i-1}\alpha_jx_j+\sum_{j=1}^{i-1}\beta_jy_j\ge\delta, \tag{29}$$

which is assumed to be facet-defining for $PB(S_{i-1}, \emptyset, S_{i-1}, \emptyset)$. To perform this lifting, we first compute the *lifting function*

$$
\begin{aligned}
P^i(w) = \quad \max \quad & \delta - \left\{ \sum_{j=1}^{i-1} \alpha_j x_j + \sum_{j=1}^{i-1} \beta_j y_j \right\} \\
s.t. \quad & \sum_{j=1}^{i-1} a_j x_j y_j \geq d - w \\
& x_j \in \{0,1\},\ y_j \in [0,1] \quad j = 1, \dots, i-1.
\end{aligned}
$$

Once the lifting function $P^i(w)$ is computed, the lifting coefficients (α_i, β_i) are obtained from $P^i(w)$ as follows.

Proposition 3.1 (Richard and Tawarmalani [24]). *Let (29) be a valid inequality for $B(S_{i-1}, \emptyset, S_{i-1}, \emptyset)$. Assume that there exist $(\alpha_i, \beta_i) \in \mathbb{R}^2$ such that*

$$\alpha_i x_i + \beta_i y_i \geq P^i(a_i x_i y_i) \quad \text{for } (x_i, y_i) \in \{0,1\} \times [0,1] \setminus \{0,0\}. \tag{30}$$

Then, the inequality

$$\sum_{j=1}^{i} \alpha_j x_j + \sum_{j=1}^{i} \beta_j y_j \geq \delta \tag{31}$$

is valid for $B(S_i, \emptyset, S_i, \emptyset)$.

The result of Proposition 3.1 can be applied recursively to construct a valid inequality for *PB* from (28). Note that, at each step, the lifting function $P^i(w)$ must be recomputed to account for the changes in the lifted inequality. Further, if $B(S, \emptyset, S, \emptyset)$ is full-dimensional, the seed inequality (28) is facet-defining for $B(S, \emptyset, S, \emptyset)$, and for each $i \in S$, the lifting coefficients (α_i, β_i) of the variables (x_i, y_i) are chosen so that (30) is satisfied at equality by two points (x_i^1, y_i^1) and (x_i^2, y_i^2) such that $(0,0)$, (x_i^1, y_i^1) and (x_i^2, y_i^2) are affinely independent (a feature we refer to as *maximal lifting*), then the final lifted inequality will be facet-defining for *PB*. In this scheme, (re)computing the lifting functions $P^i(w)$ for each $i \in S$ is often the most computationally demanding task. However, this computational work is unnecessary when the lifting function $P^s(w)$ is subadditive. This observation, first made by Wolsey [33], leads to the following result.

Proposition 3.2 (Richard and Tawarmalani [24]). *Assume that (28) is valid for $B(S, \emptyset, S, \emptyset)$. Assume also that (i) $P^s(w)$ is subadditive over $\mathbb{R}_+$, i.e, $P^s(w_1) + P^s(w_2) \geq P^s(w_1 + w_2)\ \forall w_1, w_2 \in \mathbb{R}_+$, and (ii) there exist $(\alpha_i, \beta_i) \in \mathbb{R}^2$ for all $i \in S$ such that*

$$\alpha_i x_i + \beta_i y_i \geq P^s(a_i x_i y_i) \quad \text{for } (x_i, y_i) \in \{0,1\} \times [0,1] \setminus \{0,0\}. \tag{32}$$

Then, the inequality

$$\sum_{j=1}^{n} \alpha_j x_j + \sum_{j=1}^{n} \beta_j y_j \geq \delta \tag{33}$$

is valid for PB. Further, if (i) Inequality (28) is facet-defining for $B(S, \emptyset, S, \emptyset)$, (ii) $B(S, \emptyset, S, \emptyset)$ is full-dimensional and (iii) coefficients (α_i, β_i) are chosen in a way that two linearly independent points satisfy (32) at equality, then (33) is facet-defining for PB.

The fundamental difference between Proposition 3.1 and Proposition 3.2 lies in equations (30) and (32). In the latter, the lifting coefficients of all variables (x_i, y_i) are obtained from the same lifting function $P^s(w)$ while in the former, they are obtained from $P^i(w)$ for $i \in S$. Although this difference might seem minor, it has important practical implications. In particular, the subadditivity of lifting functions typically permits the derivation of closed-form expressions for lifting coefficients that would otherwise be difficult to obtain. Observe also that in Proposition 3.2, the subadditivity of $P^s(w)$ is required only over $\mathbb{R}_+$ since all coefficients a_i in *PB* are assumed to be nonnegative.

Proposition 3.1 describes how to perform lifting when then variables (x_j, y_j) for $j \in S$ are fixed at $(0,0)$. When variables (x_j, y_j) are fixed at $(1,1)$, similar results can be obtained. In this case, condition (30) must be changed to

$$\alpha_i(x_i - 1) + \beta_i(y_i - 1) \geq P^i(a_i x_i y_i - a_i) \quad \text{for } (x_i, y_i) \in \{0,1\} \times [0,1] \setminus \{1,1\}. \tag{34}$$

Similarly, Proposition 3.2 can be adapted to allow sequence-independent lifting for variables (x_j, y_j) fixed at $(1,1)$ by replacing $P^i(w)$ with $P^s(w)$ in (34) and by requiring that the lifting function $P^s(w)$ is subadditive over $\mathbb{R}_-$. Subadditive lifting can also be used to generate facets of PB if $B(\emptyset, S, \emptyset, S)$ is full-dimensional, the seed inequality (28) is facet-defining for $B(\emptyset, S, \emptyset, S)$, and for each $i \in S$, the lifting coefficients (α_i, β_i) of the variables (x_i, y_i) are chosen so that (34) is satisfied at equality by two points (x_i^1, y_i^1) and (x_i^2, y_i^2) such that $(1,1)$, (x_i^1, y_i^1) and (x_i^2, y_i^2) are affinely independent.

We next show in the following proposition that all interesting lifted inequalities that can be obtained by fixing variables (x_i, y_i) at $(0,1)$ or $(1,0)$ can also be obtained by fixing variables (x_i, y_i) at $(0,0)$.

Proposition 3.3. *Assume that (29) defines a nonempty face of $PB(S_{i-1}, \emptyset, S_{i-1}, \emptyset) = PB(S_{i-1}, \emptyset, \emptyset, S_{i-1}) = PB(\emptyset, S_{i-1}, S_{i-1}, \emptyset)$. Then any inequality obtained from maximally lifting (29) in $PB(S_{i-1}, \emptyset, \emptyset, S_{i-1})$ or $PB(\emptyset, S_{i-1}, S_{i-1}, \emptyset)$ could have been obtained by maximally lifting (29) in $PB(S_{i-1}, \emptyset, S_{i-1}, \emptyset)$.*

Proof. First, we consider the case when (x_i, y_i) is fixed at $(1,0)$. In this situation, valid lifting coefficients must satisfy

$$\alpha_i(x_i - 1) + \beta_i y_i \geq P^i(a_i x_i y_i) \quad \text{for } (x_i, y_i) \in \{0,1\} \times [0,1]. \tag{35}$$

We next show that maximal lifting coefficients (α_i, β_i) in (35) must also satisfy

$$\alpha_i x_i + \beta_i y_i \geq P^i(a_i x_i y_i) \quad \text{for } (x_i, y_i) \in \{0,1\} \times [0,1] \tag{36}$$

and be maximal for (36). This is sufficient to prove the result since restricting $(x_i, y_i) = (0,0)$ instead of $(1,0)$ does not change the restricted set and, therefore, the seed inequality is still a face of same dimension. Let $(0, y_i^*)$ satisfy (35) at equality. Such a point exists since lifting is assumed to be maximal. Then,

$$0 \geq \alpha_i = \beta_i y_i^* \geq P^i(a_i y_i^*) \geq 0,$$

where the first inequality follows from (35) by setting $(x_i, y_i) = (0,0)$, the equality holds since $(0, y_i^*)$ satisfies (35) at equality, the second inequality is satisfied from (35) with $(x_i, y_i) = (1, y_i^*)$ and the last inequality is verified since $a_i y_i^* \geq 0$. Therefore, equality holds throughout and, in particular, $\alpha_i = 0$. It follows that $\alpha_i(x_i - 1) + \beta_i y_i = \alpha_i x_i + \beta_i y_i$ and, consequently, (α_i, β_i) is valid and maximal to (36).

Now, we fix (x_i, y_i) at $(0,1)$. Then, we show that any (α_i, β_i) that is valid and maximal to

$$\alpha_i x_i + \beta_i(y_i - 1) \geq P^i(a_i x_i y_i) \tag{37}$$

is also valid and maximal to (36). Let $y_i^* = \min\{y_i \in [0,1] \mid \alpha_i + \beta_i(y_i - 1) = P^i(a_i y_i)\}$, *i.e.*, $(1, y_i^*)$ satisfies (37) at equality. It follows that

$$0 \leq \beta_i(y_i^* - 1) = P^i(a_i y_i^*) - \alpha_i \leq P^i(a_i y_i^*) - P^i(a_i) \leq 0,$$

where the first inequality follows from (37) by substituting $(x_i, y_i) = (0, y_i^*)$, the equality is satisfied since $(1, y_i^*)$ satisfies (37) at equality, the second inequality is verified by substituting $(1,1)$ in (37), and the last inequality holds since $P^i(\cdot)$ is non-decreasing and $a_i y_i^* \leq a_i$. Therefore, the equality holds throughout and, in particular, $\beta_i(y_i^* - 1) = 0$. It follows that either $\beta_i = 0$ or $y_i^* = 1$. We show that $\beta_i = 0$ in the latter case as well. If $y_i^* = 1$, because lifting is assumed to be maximal and because of the definition of y_i^*, there is a $y_i' \in [0,1)$ such that $(0, y_i')$ satisfies (37) at equality. Therefore, $\beta_i(y_i' - 1) = 0$ and so $\beta_i = 0$. It follows that $\alpha_i x_i + \beta_i(y_i - 1) = \alpha_i x_i + \beta_i y_i$ and, consequently, (α_i, β_i) is valid and maximal for (36). ☐

3.2 Subadditivity of lifting functions

In this section, we provide a general result that helps in proving subadditivity of functions. For specific functions, it has been observed (see Proposition 3.10 in [23], Theorem 7 in [11], Proposition 4.2 in [8], and Lemma 21 in [25]) that subadditivity of a function over $\mathbb{R}^n$ can often be established by checking it at a small subset of points. The corresponding proofs are often detailed and are the key step in proving subadditivity. In Theorem 3.4, we identify a fairly large class of functions for which a similar result holds. We use this result to prove subadditivity of functions that arise during lifting of inequalities for mixed-integer $0-1$ bilinear covering set. The scope of applications of Theorem 3.4 is, however, much larger and we provide this general result with the hope that it may be useful in other applications.

Theorem 3.4. *For* $i \in N = \{1, \ldots, n\}$, *let* $b_i \in \mathbb{R}^m$, $f_i \in \mathbb{R}$, *and* $h_i(x) : \mathbb{R}^m \mapsto \mathbb{R}$ *be subadditive functions. Let* $h(x) : \mathbb{R}^m \mapsto \mathbb{R}$ *be a function with* $h(\mathbf{0}) = 0$ *that majorizes* $h_i(x)$ *for all* i. *Let* $B_i = \{x \in \mathbb{R}^m \mid h_i(x - b_i) = h(x - b_i)\}$. *Define* $f(x) : \mathbb{R}^m \mapsto \mathbb{R}$ *as* $f(x) = \min_{i=1}^n \bigl\{f_i + h_i(x - b_i)\bigr\}$. *Then,* $f(x) + h(y - x) \geq f(y)$ *for each* $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m$. *For* $x \in \mathbb{R}^m$, *let* $i(x) \in N$ *be such that* $f(x) = f_{i(x)} + h_{i(x)}(x - b_{i(x)})$. *If* $x \in B_{i(x)}$ *then* $f(b_{i(x)}) - f(b_{i(x)} + y) \leq f(x) - f(x + y)$. *Further, if* $y \in B_{i(y)}$ *as well, then*

$$f(b_{i(x)}) + f(b_{i(y)}) - f(b_{i(x)} + b_{i(y)}) \leq f(x) + f(y) - f(x + y).$$

Proof. Let $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m$. Then,

$$f(x) + h(y - x) \geq f(x) + h_{i(x)}(y - x) = f_{i(x)} + h_{i(x)}(x - b_{i(x)}) + h_{i(x)}(y - x) \geq f_{i(x)} + h_{i(x)}(y - b_{i(x)}) \geq f(y), \quad (38)$$

where the first inequality follows since $h(\cdot) \geq h_{i(x)}(\cdot)$, the second inequality holds since $h_{i(x)}$ is a subadditive function, and the third since the minimum defining $f(y)$ includes a term that equals $f_{i(x)} + h_{i(x)}(y - b_{i(x)})$. Note that $0 \leq h_{i(x)}(\mathbf{0}) \leq h(\mathbf{0}) = 0$ where the first inequality follows from subadditivity of $h_{i(x)}$. Therefore, $h_{i(x)}(0) = 0$. Further, if $x \in B_{i(x)}$:

$$f(x) = f_{i(x)} + h_{i(x)}(x - b_{i(x)}) \geq f(b_{i(x)}) + h_{i(x)}(x - b_{i(x)}) = f(b_{i(x)}) + h(x - b_{i(x)}) \geq f(x),$$

where the first inequality holds since $h_{i(x)}(\mathbf{0}) = 0$ implies $f(b_{i(x)}) \leq f_{i(x)}$ as $f_{i(x)}$ is one of the terms in minimum defining $f(b_{i(x)})$, the second equality since $x \in B_{i(x)}$ and the last inequality by (38). Therefore, equality holds throughout and, in particular, $f_{i(x)} = f(b_{i(x)})$. Now, consider $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m$ with $x \in B_{i(x)}$. Then,

$$f(b_{i(x)}) - f(b_{i(x)} + y) = f(x) - h(x - b_{i(x)}) - f(b_{i(x)} + y) \leq f(x) - f(x + y), \quad (39)$$

where the equality follows from the definition of $i(x)$, $x \in B_{i(x)}$, and $f_{i(x)} = f(b_{i(x)})$, and the inequality follows from (38) since $f(b_{i(x)} + y) + h(x - b_{i(x)}) \geq f(x + y)$. Further, if $y \in B_{i(y)}$,

$$f(b_{i(x)}) + f(b_{i(y)}) - f(b_{i(x)} + b_{i(y)}) \leq f(b_{i(x)}) + f(y) - f(b_{i(x)} + y) \leq f(x) + f(y) - f(x + y),$$

where each of the inequalities follows from (39). □

Any function, say $g(x)$, that is subadditive and satisfies $g(\mathbf{0}) = 0$ can be expressed as $f(x)$, defined in Theorem 3.4, by setting $n = 1$, $b_1 = \mathbf{0}$, $f_1 = 0$, and $h(x) = h_1(x) = g(x)$. Observe that lifting functions derived from seed inequalities that are tight on the restricted set always satisfy the condition $g(\mathbf{0}) = 0$. In other words, Theorem 3.4 can be interpreted as a recursive tool for proving subadditivity of such lifting functions, $f(x)$, that exploits the subadditivity of the constituent simpler functions, $h_i(\cdot)$. For example, Theorem 3.4 shows that it suffices to check the subadditivity of $f(x)$ at a small subset of points if $f(x)$ can be expressed as a minimum of finitely many translates of a subadditive function. Since positively-homogenous convex functions belong to the class of subadditive functions, see Theorem 4.7 in [26], they can be used as building blocks in the application of Theorem 3.4. In Corollaries 3.5 and 3.6, we apply Theorem 3.4 to prove subadditivity of two functions that will be used in Sections 3.3 and 3.4 to derive inequalities for the $0-1$ mixed-integer bilinear covering set. In both cases, functions h_i are univariate positively-homogenous convex functions.

Corollary 3.5. *Let ν and D_i for $i = 0, 1, \ldots, r$ be nonnegative integers that satisfy $\nu > 0$, $D_0 = 0$, and $D_i \geq D_{i-1} + \nu$ for $i = 1, \ldots, r$. Then the function*

$$g(w) := \begin{cases} 0 & \text{if} \quad w < D_0 \\ w - i\nu & \text{if} \quad D_i \leq w < D_{i+1} - \nu, \quad i = 0, \ldots, r-1, \\ D_i - i\nu & \text{if} \quad D_i - \nu \leq w < D_i, \qquad i = 1, \ldots, r-1, \\ D_r - r\nu & \text{if} \quad D_r - \nu \leq w \end{cases}$$

is subadditive over $\mathbb{R}$ if and only if $D_i + D_j \geq D_{i+j}$ for $0 \leq i \leq j \leq r$ with $i + j \leq r$.

Proof. First note that $g(w) = \min_{i=0}^{r}\big\{D_i - i\nu + h_i(w - D_i)\big\}$, where $h_i(w) = \max\{0, w\}$ for $i = 0, \ldots, r-1$ and $h_r(w) = 0$. We observe that $i(x) = 0$ for $x < D_1 - \nu$, $i(x) = i$ for $x \in [D_i - \nu, D_{i+1} - \nu)$ and $i = 1, \ldots, r-1$, and $i(x) = r$ for $x \geq D_r - \nu$. Let $h(w) = \max\{0, w\}$. We also see that $B_0 = B_1 = \ldots = B_{r-1} = \mathbb{R}$ and $B_r = \mathbb{R}_-$.

Assume that $D_i + D_j \geq D_{i+j}$ for $0 \leq i \leq j \leq r$ with $i + j \leq r$. Consider $x, y \in \mathbb{R}$ with $x \leq y$. We argue next that $g(x) + g(y) \geq g(x + y)$. Define $i = i(x)$ and $j = i(y)$. Clearly, $i \leq j$. We consider two cases. Assume first that $j = r$. Then, $g(y) = D_r - r\nu \geq g(x + y)$ and therefore $g(x) + g(y) \geq g(x + y)$ as $g(x) \geq 0$. Assume next that $j \leq r - 1$. Since $x \leq y < D_r - \nu$, it follows that $x \in B_i$ and $y \in B_j$ and, therefore, from Theorem 3.4 that

$$g(D_i) + g(D_j) - g(D_i + D_j) \leq g(x) + g(y) - g(x + y)$$

We next argue that the left-hand-side of the above expression is nonnegative, which proves the result. Let $t = \min\{j, r - i\}$. Then,

$$g(D_i{+}D_j) = g(D_i{+}D_t) = g(D_{i+t}{+}D_i{+}D_t{-}D_{i+t}) \leq g(D_{i+t}){+}D_i{+}D_t{-}D_{i+t} = g(D_i){+}g(D_t) \leq g(D_i){+}g(D_j),$$

where the first equality holds since $t = r - i$ implies $D_i + D_j \geq D_i + D_t \geq D_r$, the first inequality follows from (38) and $D_{i+t} \leq D_i + D_t$, the second equality since $g(D_i) = D_i - i\nu$, and the last inequality from (38) since $D_t \leq D_j$.

We now prove the reverse implication. For $w > 0$,

$$g(D_k - w) \geq g(D_k - \nu) - \max\{0, w - \nu\} = g(D_k) - \max\{0, w - \nu\} > g(D_k) - w, \tag{40}$$

where the first inequality follows from (38) and the last inequality since $\nu > 0$ and $w > 0$. If $i + j \leq r$ and $D_i + D_j < D_{i+j}$ then

$$g(D_i) + g(D_j) - g(D_i + D_j) < g(D_i) + g(D_j) - g(D_{i+j}) - D_i - D_j + D_{i+j} = 0,$$

yields a contradiction to subadditivity of g, where the strict inequality follows from (40) where $k = i + j$ and $w = D_{i+j} - D_i - D_j$ since $D_i + D_j < D_{i+j}$ and the equality holds since $g(D_k) = D_k - k\nu$ for $k \in \{i, j, i+j\}$. ☐

Corollary 3.5 equivalently shows the superadditivity of $w - g(w)$, generalizing prior similar results in the literature. In particular, see Lemmas 6 and 7 in [3] and Definition 4 in [19].

Corollary 3.6. *Let λ and C_i for $i = 0, 1, \ldots, s$ be nonnegative integers that satisfy $\lambda > 0$, $C_0 = 0$ and $C_{i-1} + \lambda \leq C_i$ for $i = 1, \ldots, s$. Then the function*

$$g(w) = \begin{cases} 0 & \text{if} \quad w < C_0 \\ i + \frac{w - C_i}{\lambda} & \text{if} \quad C_i \leq w < C_i + \lambda, \qquad i = 0, \ldots, s, \\ i & \text{if} \quad C_{i-1} + \lambda \leq w < C_i, \quad i = 1, \ldots, s, \\ s + 1 & \text{if} \quad C_s + \lambda \leq w. \end{cases}$$

is subadditive over $\mathbb{R}$ if and only if $C_i + C_j \leq C_{i+j}$ for $0 \leq i \leq j \leq s$ with $i + j \leq s$.

Proof. Let $C_{s+1} = \max\{C_i + C_j \mid i + j = s + 1\}$. Note that $g(w) = \min_{i=0}^{s+1}\big\{i + h_i(w - C_i)\big)\big\}$ where $h_i(w) = \max\{0, \frac{w}{\lambda}\}$ for $i = 0, \ldots, s$ and $h_{s+1}(w) = 0$. Let $h(w) = \max\{0, \frac{w}{\lambda}\}$.

Assume that $C_i + C_j \leq C_{i+j}$ for $0 \leq i \leq j \leq s$ with $i + j \leq s$. Consider $x, y \in \mathbb{R}$ with $x \leq y$. We argue next that $g(x) + g(y) \geq g(x + y)$. Define $i = i(x)$ and $j = i(y)$. Assume first that $j = s + 1$. Then

$g(y) = s+1 \geq g(x+y)$ and therefore, $g(x)+g(y) \geq g(x+y)$ as $g(x) \geq 0$. Next assume that $j \leq s$. Since $x \leq y < C_s + \lambda$, it follows that $x \in B_i$ and $y \in B_j$ and therefore, from Theorem 3.4 that

$$g(C_i) + g(C_j) - g(C_i + C_j) \leq g(x) + g(y) - g(x+y).$$

We next argue that the left-hand-side of the above expression is nonnegative, which proves the result. Let $t = \min\{j, s+1-i\}$. Then,

$$g(C_i + C_j) \leq g(C_{i+t}) = i + t \leq i + j = g(C_i) + g(C_j),$$

where the first inequality follows from (38) and $C_{i+t} \geq C_i + C_j$ when $t = j$ and from $h_{s+1}(w) = 0$ when $t < j$, and the last equality holds because $g(C_k) = k$ for $k \in \{i, j\}$.

We now prove the reverse implication. For $w > 0$ and $i \leq s$,

$$g(C_k + w) \geq g(C_k) + \max\left\{0, \frac{w}{\lambda}\right\} > g(C_k), \tag{41}$$

where the first inequality follows from (38) and the strict inequality from $w > 0$ and $\lambda > 0$. If $i + j \leq s$ and $C_i + C_j > C_{i+j}$ then

$$g(C_i) + g(C_j) - g(C_i + C_j) < g(C_i) + g(C_j) - g(C_{i+j}) = 0,$$

yields a contradiction to subadditivity of $g(\cdot)$ where the strict inequality follows from (41) where $k = i + j$ and $w = C_i + C_j - C_{i+j}$ and the equality holds since $g(C_k) = k$ for $k \in \{i, j, i+j\}$. □

3.3 Lifted inequalities by sequence-independent lifting

In this section, we derive strong inequalities for PB through lifting using (25) as seed inequality. To describe the general form of these inequalities, we use the notion of a cover, which is adapted from the definition of a cover for the $0-1$ knapsack polytope; see Balas [4], Hammer et al. [14], and Wolsey [31].

Definition 3.7. *Let $C \subseteq N$. We say that C is a cover for B if $\sum_{j \in C} a_j > d$. Further, we define the excess of the cover as $\mu = \sum_{j \in C} a_j - d > 0$.*

We create lifted inequalities by first partitioning the set of variables N into $(C', \{l\}, M, T)$ in such a way that:

(A1) $C := C' \cup \{l\}$ is a cover for B with excess μ,

(A2) $a_l \geq a_j$, $\forall j \in C'$,

(A3) $a_l > \mu$,

(A4) $\sum_{j \in C \cup T} a_j > d + a_l$, *i.e.*, $\sum_{j \in T} a_j > a_l - \mu$.

Note that (A1) and (A3) might be reminiscent of conditions that make a cover minimal for the $0-1$ knapsack polytope. We note however that minimal covers require $a_j > \mu$ for all $j \in C$ and not simply $a_l > \mu$. Note also that (A4) implies that $T \neq \emptyset$. To obtain lifted inequalities from $(C', \{l\}, M, T)$, we first fix the variables (x_j, y_j) for $j \in M$ to $(0, 0)$ and the variables (x_j, y_j) for $j \in C'$ to $(1, 1)$. The resulting (full-dimensional) set $B(M, C', M, C')$ is then defined by the inequality

$$a_l x_l y_l + \sum_{j \in T} a_j x_j y_j \geq d - \sum_{j \in C'} a_j = a_l - \mu.$$

Since $a_l > \mu$ and $\sum_{j \in T} a_j > a_l - \mu$ from Conditions (A3) and (A4), we conclude from Proposition 2.9(i) that

$$(a_l - \mu) x_l + \sum_{j \in T} a_j y_j \geq a_l - \mu \tag{42}$$

is facet-defining for $PB(M, C', M, C')$. We will create two different families of lifted inequalities for PB by reintroducing the variables (x_j, y_j) for $j \in M \cup C'$ in different orders. To derive both families, we use the lifting function

$$\begin{aligned} P(w) := \quad \max \quad & (a_l - \mu) - \left\{ (a_l - \mu)x_l + \sum_{j \in T} a_j y_j \right\} \\ s.t. \quad & a_l x_l y_l + \sum_{j \in T} a_j x_j y_j \geq a_l - \mu - w \\ & x_j \in \{0,1\},\ y_j \in [0,1] \quad \forall j \in \{l\} \cup T. \end{aligned} \tag{43}$$

We next derive a closed-form expression for $P(w)$.

Proposition 3.8.

$$P(w) \quad = \quad \begin{cases} -\infty & \text{if} \quad w < -\sum_{j \in T} a_j - \mu, \\ w + \mu & \text{if} \quad -\sum_{j \in T} a_j - \mu \leq w < -\mu, \\ 0 & \text{if} \quad -\mu \leq w < 0, \\ w & \text{if} \quad 0 \leq w < a_l - \mu, \\ a_l - \mu & \text{if} \quad a_l - \mu \leq w. \end{cases}$$

Further, $P(w)$ is subadditive over $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively.

Proof. We first derive a closed-form expression for $P(w)$. Observe that, if (43) is feasible, there exists an optimal solution (x^*, y^*) to (43) for which $x_j^* = 1$ for $j \in T$ and $y_l^* = 1$ since the coefficients of x_j for $j \in T$ and y_l in the objective are equal to 0. Defining $\bar{a} = \sum_{j \in T} a_j$ and $\bar{y} = \frac{\sum_{j \in T} a_j y_j}{\bar{a}}$, we can simplify the formulation of $P(w)$ in (43) as:

$$\begin{aligned} P(w) = \quad \max \quad & (a_l - \mu) - \{(a_l - \mu)x_l + \bar{a}\bar{y}\} \\ s.t. \quad & a_l x_l + \bar{a}\bar{y} \geq a_l - \mu - w \\ & x_l \in \{0,1\},\ \bar{y} \in [0,1]. \end{aligned} \tag{44}$$

When $w < -\bar{a} - \mu$, (44) is infeasible and so $P(w) = -\infty$. When $w \geq a_l - \mu$, the optimal solution is $x_l^* = 0$ and $\bar{y}^* = 0$ with $P(w) = a_l - \mu$. For $-\bar{a} - \mu \leq w < a_l - \mu$, there are two cases. When $-\bar{a} - \mu \leq w < a_l - \bar{a} - \mu$, then every feasible solution $(x_l^*, \bar{y}^*)$ has $x_l^* = 1$. Further, the optimal solution has $\bar{y}^* = \max\{\frac{-\mu - w}{\bar{a}}, 0\}$. It follows that $P(w) = \min\{w + \mu, 0\}$. When $a_l - \bar{a} - \mu \leq w \leq a_l - \mu$, an optimal solution must be found among the solutions $(1, \frac{(-\mu - w)^+}{\bar{a}})$ and $(0, \frac{a_l - \mu - w}{\bar{a}})$. It follows that $P(w) = \max\{(w + \mu)^-, w\}$ from which we obtain the desired expression for $P(w)$ after considering both the cases where $a_l - \bar{a} < 0$ and $a_l - \bar{a} \geq 0$.

Subadditivity of $P(w)$ over $\mathbb{R}_-$ and $\mathbb{R}_+$ follows from Karamata/Hardy-Littlewood-Polya inequality [15], concavity of $P(w)$ over these domains and $P(0) = 0$. □

We note that, although $P(w)$ is subadditive over $\mathbb{R}_+$ and over $\mathbb{R}_-$, $P(w)$ is not subadditive over $\mathbb{R}$ as $P(2a_l - \mu) + P(-a_l) = (a_l - \mu) + (-a_l + \mu) = 0 < a_l - \mu = P(a_l - \mu)$.

3.3.1 Lifted bilinear cover inequalities

To obtain lifted bilinear cover inequalities, we will lift first the variables (x_i, y_i) for $i \in C'$ from $(1,1)$ and then lift the variables (x_i, y_i) for $i \in M$ from $(0,0)$. Since $P(w)$ is subadditive over $\mathbb{R}_-$, we can apply sequence-independent lifting for the variables (x_i, y_i) for $i \in C'$.

Proposition 3.9. *Under Conditions (A1), (A2), (A3) and (A4),*

$$\sum_{j \in C} (a_j - \mu)^+ x_j + \sum_{j \in T} a_j y_j \geq \sum_{j \in C} (a_j - \mu)^+ \tag{45}$$

is facet-defining for $PB(M, \emptyset, M, \emptyset)$.

Proof. The seed inequality (42) is facet-defining for the full-dimensional polytope $PB(M, C', M, C')$. Since $P(w)$ is subadditive over $\mathbb{R}_-$, we obtain from the remark following Proposition 3.2 that the lifting coefficients (α_i, β_i) for (x_i, y_i) for $i \in C'$ are valid if they satisfy

$$\alpha_i(x_i - 1) + \beta_i(y_i - 1) \geq P(a_i x_i y_i - a_i) \quad \text{for } (x_i, y_i) \in \{0,1\} \times [0,1] \setminus \{1,1\}. \tag{46}$$

This condition can be also written as

$$\beta_i \leq \inf_{0 \leq \phi < 1} \frac{-P(a_i\phi - a_i)}{1 - \phi}, \tag{47}$$

$$\alpha_i + \sup_{0 \leq \phi \leq 1} \beta_i(1 - \phi) \leq -P(-a_i). \tag{48}$$

From Conditions (A2) and (A4), we know that $a_i \leq a_l < \sum_{j \in T} a_j + \mu$, $\forall i \in C'$. Therefore, in (47) $a_i\phi - a_i \in (-\sum_{j \in T} a_j - \mu, 0)$ for all $\phi \in [0, 1)$. Since $P(w) \leq 0$ for $w \leq 0$, we conclude that

$$\frac{-P(a_i\phi - a_i)}{1 - \phi} \geq 0, \quad \forall\, 0 \leq \phi < 1,$$

and therefore choosing $\beta_i = 0$ for $i \in C'$ satisfies (47). Further, as $\beta_i = 0$, it is simple to verify that choosing $\alpha_i = -P(-a_i) = (a_i - \mu)^+$ satisfies (48). Finally, observe that (46) is satisfied at equality by the two points $(0, 0)$ and $\left(1, \frac{(a_i - \mu)^+}{a_i}\right)$ that are affinely independent of $(1, 1)$. Therefore, we conclude that (45) is facet-defining for $PB(M, \emptyset, M, \emptyset)$. ☐

Now, we lift the variables (x_j, y_j) for $j \in M$ in (45). The corresponding lifting function is

$$\begin{aligned} P^C(w) := \quad \max \quad & \sum_{j \in C} (a_j - \mu)^+ - \left\{ \sum_{j \in C} (a_j - \mu)^+ x_j + \sum_{j \in T} a_j y_j \right\} \\ s.t. \quad & \sum_{j \in C \cup T} a_j x_j y_j \geq \sum_{j \in C} a_j - \mu - w \\ & x_j \in \{0, 1\},\ y_j \in [0, 1] \quad \forall j \in C \cup T. \end{aligned} \tag{49}$$

We next derive a closed-form expression for $P^C(w)$. To this end, we assume without loss of generality that $C = \{1, \ldots, p\}$ and that $a_1 \geq a_2 \geq \ldots \geq a_p$. We also let $q \in C$ be such that $a_q > \mu \geq a_{q+1}$. We define $A_0 = 0$ and $A_i = \sum_{j=1}^{i} a_j$ for all $i \in \{1, \ldots, q\}$.

Proposition 3.10. *For $w \geq 0$,*

$$P^C(w) = \begin{cases} w - i\mu & \text{if} \quad A_i \leq w < A_{i+1} - \mu, \quad i = 0, \ldots, q-1, \\ A_i - i\mu & \text{if} \quad A_i - \mu \leq w < A_i, \qquad i = 1, \ldots, q-1, \\ A_q - q\mu & \text{if} \quad A_q - \mu \leq w. \end{cases}$$

Proof. First, observe that there exists an optimal solution (x^*, y^*) of (49) in which $x_j^* = 1$ for $j \in T$ and $y_j^* = 1$ for $j \in C$ since the corresponding objective coefficients are zero. Since $a_q > \mu \geq a_{q+1}$, we have $(a_j - \mu)^+ = 0$ for $j = q + 1, \ldots, p$, which similarly implies that we can assume $x_j^* = 1$ for $j = q + 1, \ldots, p$. Defining $\bar{a} = \sum_{j \in T} a_j$ and $\bar{y} = \frac{\sum_{j \in T} a_j y_j}{\bar{a}}$, we simplify the expression of $P^C(w)$ as

$$\begin{aligned} P^C(w) = \quad \max \quad & \sum_{j=1}^{q} (a_j - \mu) - \left\{ \sum_{j=1}^{q} (a_j - \mu) x_j + \bar{a}\bar{y} \right\} \\ s.t. \quad & \sum_{j=1}^{q} a_j x_j + \bar{a}\bar{y} \geq \sum_{j=1}^{q} a_j - \mu - w \\ & x_j \in \{0, 1\}, \quad \forall j = 1 \ldots, q,\ \bar{y} \in [0, 1]. \end{aligned} \tag{50}$$

Next, we solve (50). When $w \geq A_q - \mu$, it is clear that $x_j^* = 0$ for $j = 1, \ldots, q$ and $\bar{y}^* = 0$ is an optimal solution for (50), showing that $P^C(w) = A_q - q\mu$. It is therefore sufficient to consider $w \in [0, A_q - \mu)$. We consider two cases:

1. Assume that $A_i - \mu \le w < A_{i+1} - \mu$ for $i \in \{1 \ldots, q-1\}$. Let $\theta = (A_{i+1} - \mu) - w$. Clearly, $0 < \theta \le a_{i+1}$. Define first the solution $(x^*, \bar{y}^*)$ where $x_j^* = 0$ for $j = 1, \ldots, i+1$, $x_j^* = 1$ for $j = i+2, \ldots, q$, and $\bar{y}^* = \frac{\theta}{\bar{a}}$. When $\theta \le \bar{a}$, $(x^*, \bar{y}^*)$ is a feasible solution to (50) with objective value $z^* = A_{i+1} - (i+1)\mu - \theta = w - i\mu$. Next consider the solution $(x', \bar{y}')$ where $x'_j = 0$ for $j = 1, \ldots, i$, $x'_j = 1$ for $j = i+1, \ldots, q$, and $\bar{y}' = 0$. Solution $(x', \bar{y}')$ is feasible to (50) and has objective value $z' = A_i - i\mu$. It is clear that $z^* \ge z'$ when $\theta \le a_{i+1} - \mu$ and that $z' \ge z^*$ when $a_{i+1} - \mu \le \theta \le a_{i+1}$. Further, solution $(x^*, \bar{y}^*)$ is feasible when $\theta \le a_{i+1} - \mu$ as $a_{i+1} - \mu \le a_1 - \mu \le \bar{a}$ because of Condition (A4). Therefore, we conclude that $P^C(w) \ge w - i\mu$ if $A_i \le w \le A_{i+1} - \mu$ and $P^C(w) \ge A_i - i\mu$ if $A_i - \mu \le w < A_i$.

   We now prove that the proposed solutions are optimal. Pick any feasible solution $(x^\circ, \bar{y}^\circ)$ to (50). Define $N_1 = \{j \in \{1, \ldots, q\} \mid x_j^\circ = 1\}$. Consider first the case where $|N_1| = q - i + k$ for $k \in \{0, \ldots, i\}$. Since $\sum_{j=1}^q a_j x_j^\circ + \bar{a}\bar{y}^\circ \ge \sum_{j=1}^q a_j x_j^\circ \ge A_q - A_{i-k}$, the objective value associated with $(x^\circ, \bar{y}^\circ)$ satisfies $z^\circ = \sum_{j=1}^q (a_j - \mu)(1 - x_j^\circ) - \bar{a}\bar{y}^\circ \le A_{i-k} - (i-k)\mu = A_i - i\mu - \sum_{j=i-k+1}^i (a_j - \mu) \le z'$. Second, consider the case where $|N_1| = q - i - k$ for $k \in \{1, \ldots, q-i\}$. Since $\sum_{j=1}^q a_j x_j^\circ + \bar{a}\bar{y}^\circ \ge A_q - A_{i+1} + \theta$ from feasibility, the corresponding objective value is $z^\circ = \sum_{j=1}^q (a_j - \mu)(1 - x_j^\circ) - \bar{a}\bar{y}^\circ \le A_{i+1} - \theta - (i+k)\mu \le z^*$. Since whenever the solution $(x^*, \bar{y}^*)$ corresponding to z^* is infeasible, $z^* \le z'$, the result is proven.

2. Assume that $0 \le w < A_1 - \mu$. An argument similar to that presented above shows that the feasible solution $x_1^* = 0$, $x_j^* = 1$ for $j = 2, \ldots, q$, and $\bar{y}^* = \frac{A_1 - \mu - w}{\bar{a}}$ is optimal for (50), which implies that $P^C(w) = w$.

□

In the following result, we argue that $P^C(w)$ is subadditive. This result enables us to use Proposition 3.2 to perform sequence-independent lifting for the variables in M.

Corollary 3.11. *The lifting function $P^C(w)$ is subadditive over $\mathbb{R}_+$.*

Proof. In Corollary 3.5, define $\nu = \mu$, $r = q$, and $D_i = A_i$. Since $a_i \ge \mu$ for $i = 1, \ldots, q$, it is clear that $A_i \ge A_{i-1} + \mu$. Further, since A_i is defined as the sum of the largest i coefficients in C, it is clear that $A_i + A_j \ge A_{i+j}$ for $0 \le i, j \le q$ with $i + j \le q$. Therefore, Corollary 3.5 shows that $P^C(w)$ is subadditive over $\mathbb{R}_+$. □

We next illustrate the results of Proposition 3.9, Proposition 3.10, and Corollary 3.11 on an example.

Example 3.12. *Consider the 0−1 mixed-integer bilinear covering set*

$$B = \left\{ (x, y) \in \{0,1\}^5 \times [0,1]^5 \;\middle|\; 21x_1y_1 + 19x_2y_2 + 17x_3y_3 + 15x_4y_4 + 10x_5y_5 \ge 20 \right\}.$$

Let $(C', \{l\}, M, T) = (\{5\}, \{4\}, \{1,2\}, \{3\})$. Clearly, $(C', \{l\}, M, T)$ satisfies Conditions (A1)-(A4) since $C = C' \cup \{l\}$ is a cover with $\mu = 5$, $a_4 \ge a_5$, $a_4 > \mu$ and $\sum_{j \in C \cup T} a_j = 17 + 15 + 10 > 20 + 15 = d + a_l$. By Proposition 3.9, the inequality

$$17y_3 + 10x_4 + 5x_5 \ge 15 \tag{51}$$

is facet-defining for $PB(M, \emptyset, M, \emptyset)$. Using Proposition 3.10, the lifting function $P^C(w)$ is given by

$$P^C(w) = \begin{cases} w & \text{if } \; 0 \le w < 10, \\ 10 & \text{if } \; 10 \le w < 15, \\ w - 5 & \text{if } \; 15 \le w < 20, \\ 15 & \text{if } \; 20 \le w. \end{cases}$$

Function $P^C(w)$ is represented in Figure 1. Corollary 3.11 shows that this function is subadditive over $\mathbb{R}_+$.

We now compute the lifting coefficients of variables (x_i, y_i) for $i \in M$ from $P^C(w)$. It follows from Proposition 3.2 that lifting coefficients (α_i, β_i) for $i \in M$ must be chosen to satisfy

$$\alpha_i x_i + \beta_i y_i \ge P^C(a_i x_i y_i) \quad \text{for } (x_i, y_i) \in \{0,1\} \times [0,1] \setminus \{0,0\}. \tag{52}$$

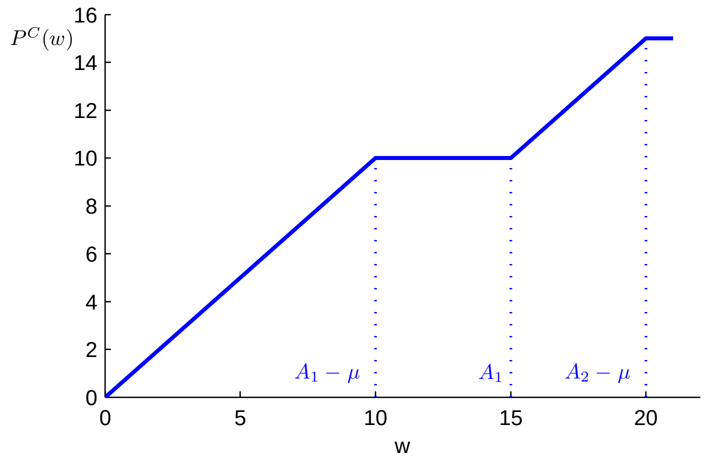


Figure 1: Lifting function $P^C(w)$ of (51)

For the problem described in Example 3.12, $P^C(a_ix_iy_i)$ is represented in Figure 2(a) for $i = 1$. In this figure, we observe that $P^C(a_ix_iy_i)$ is equal to zero when $x_i = 0$ and is equal to $P^C(a_iy_i)$ when $x_i = 1$. Condition (52) requires that the lifting coefficients (α_i, β_i) be chosen in such a way that the plane $\alpha_ix_i+\beta_iy_i$ (passing through the origin $(0, 0)$) overestimates the function $P^C(a_ix_iy_i)$ over $\{0, 1\} \times [0, 1]$. Possible overestimating planes are represented in Figure 2(b). A similar geometric interpretation was used in Richard and Tawarmalani [24] to obtain lifted inequalities for $0-1$ mixed-integer bilinear knapsack sets. It is clear from Figure 2 that good overestimating planes $\alpha_ix_i + \beta_iy_i$ are in direct correspondence with the concave envelope $p(w)$ of $P^C(w)$ over $[0, a_i]$. This observation motivates the following result.

Lemma 3.13. *For $i \in M$, define*

$$q_i := \begin{cases} 0 & \text{if} \quad a_i \le A_1 - \mu, \\ j & \text{if} \quad A_j - \mu < a_i \le A_{j+1} - \mu, \quad j = 1, \dots, q-1, \\ q & \text{if} \quad A_q - \mu < a_i. \end{cases}$$

Let $Q^i_0 = 0$, $Q^i_j = A_j - \mu$ for $j = 1, \dots, q_i$ and $Q^i_{q_i+1} = a_i$. Further, define $\Delta^i_j = Q^i_{j+1} - Q^i_j$ for $j = 0, \dots, q_i$. Define $p^i_j(w) = P^C(Q^i_j) + \frac{P^C(Q^i_{j+1})-P^C(Q^i_j)}{\Delta^i_j}(w - Q^i_j)$ for $j = 0, \dots, q_i$. Then, the function

$$p^i(w) := \min\left\{p^i_j(w) \;\middle|\; j \in \{0, \dots, q_i\}\right\} \tag{53}$$

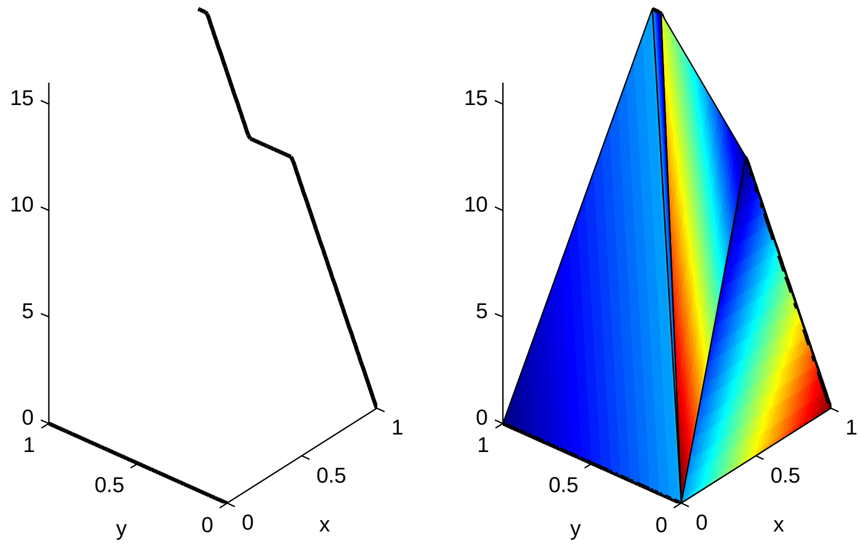
is a concave overestimator of $P^C(w)$ over $[0, a_i]$.

Proof. Clearly, $p^i(w)$ is concave since it is defined as the minimum of affine functions. Observe that, for $j = 0, \dots, q_i$, $p^i_j(w) \ge p^i_{j+1}(w)$ when $w \ge Q^i_{j+1}$, $p^i_j(w) \le p^i_{j+1}(w)$ when $w < Q^i_{j+1}$, and $p^i_j(w) \ge P^C(w)$ when $w \in [Q^i_j, Q^i_{j+1}]$. Now, consider $j \in \{0, \dots, q_i\}$ and $k \ne j$. Then, for $w \in [Q^i_j, Q^i_{j+1}]$, $P^C(w) \le p^i_j(w) \le p^i_k(w)$. □

Observe that the concave overestimator of $P^C(w)$ derived in Lemma 3.13 has q_i+1 linear pieces. Also note that the definition of q_i implies that $\Delta^i_j > 0$ for all $j = 0, \dots, q_i$. Next, we compute maximal lifting coefficients for the variables (x_i, y_i) where $i \in M$ using the sequence-independent lifting result of Proposition 3.2 and Lemma 3.13.

Theorem 3.14. *Under Conditions (A1), (A2), (A3) and (A4), the lifted bilinear cover inequality*

$$\sum_{j\in C}(a_j - \mu)^+x_j + \sum_{j\in T} a_jy_j + \sum_{i\in M} \alpha_ix_i + \sum_{i\in M} \beta_iy_i \ge \sum_{j\in C}(a_j - \mu)^+ \tag{54}$$



(a) (b)

Figure 2: Deriving lifting coefficients for Example 3.15

is facet-defining for PB if

$$(\alpha_i, \beta_i) \in \big(P^C(a_i), 0\big) \bigcup_{j=0}^{q_i} \left(P^C(Q_j^i) - \frac{P^C(Q_{j+1}^i) - P^C(Q_j^i)}{\Delta_j^i} Q_j^i, \frac{P^C(Q_{j+1}^i) - P^C(Q_j^i)}{\Delta_j^i} a_i \right)$$

for $i \in M$ in (54) where Q_j^i, Δ_j^i, and q_i are as defined in Lemma 3.13.

Proof. Because $P^C(w)$ is subadditive over $\mathbb{R}_+$, we know that (54) is valid for *PB* if the lifting coefficients (α_i, β_i) of (x_i, y_i) for $i \in M$ are chosen to satisfy the condition

$$\alpha_i x_i + \beta_i y_i \geq P^C(a_i x_i y_i) \quad \text{for } (x_i, y_i) \in \{0,1\} \times [0,1] \setminus \{0,0\}. \tag{55}$$

Condition (55) can be rewritten as

$$\beta_i \phi \geq P^C(0) \qquad \text{for} \quad 0 < \phi \leq 1, \tag{56}$$
$$\alpha_i + \beta_i \phi \geq P^C(a_i \phi) \quad \text{for} \quad 0 \leq \phi \leq 1. \tag{57}$$

To prove that (54) is facet-defining for *PB*, we also need to show two linearly independent points (x_i, y_i) for which (55) is satisfied at equality. First, consider the case where $(\alpha_i, \beta_i) = (P^C(a_i), 0)$. Condition (56) is satisfied since $\beta_i = 0$ and $P^C(0) = 0$. Condition (57) also holds because $\alpha_i = P^C(a_i)$ and $P^C(w)$ is non-decreasing over $\mathbb{R}_+$. Further, (55) is satisfied at equality at the two points, $(0, 1)$ and $(1, 1)$. Finally, consider

$$(\alpha_i, \beta_i) = \left(P^C(Q_j^i) - \frac{P^C(Q_{j+1}^i) - P^C(Q_j^i)}{\Delta_j^i} Q_j^i, \frac{P^C(Q_{j+1}^i) - P^C(Q_j^i)}{\Delta_j^i} a_i \right)$$

for any $j \in \{0, \ldots, q_i\}$. Clearly, (α_i, β_i) satisfies (56) since $\beta_i \geq 0$ and $P^C(0) = 0$. From Lemma 3.13, we have that

$$\begin{aligned} P^C(a_i\phi) &\leq P^C(Q_j^i) + \tfrac{P^C(Q_{j+1}^i) - P^C(Q_j^i)}{\Delta_j^i} \big(a_i\phi - Q_j^i\big) \\ &= \left(P^C(Q_j^i) - \tfrac{P^C(Q_{j+1}^i) - P^C(Q_j^i)}{\Delta_j^i} Q_j^i \right) + \tfrac{P^C(Q_{j+1}^i) - P^C(Q_j^i)}{\Delta_j^i} a_i\phi \\ &= \alpha_i + \beta_i\phi, \end{aligned}$$

showing that (α_i, β_i) satisfy (57) for $j = 0, \ldots, q_i$. Further, (55) is satisfied at equality at the two points $\left(1, \frac{Q^i_j}{a_i}\right)$ and $\left(1, \frac{Q^i_{j+1}}{a_i}\right)$. Therefore, we conclude that (54) is facet-defining for PB. □

The concave overestimator of Lemma 3.13 is in fact the concave envelope of $P^C(w)$ over $w \in [0, a_i]$. The concave envelope of $P^C(a_i xy)$ over $\{0,1\} \times [0,1]$ implicit in the proof of Theorem 3.14 can also be obtained using the technique for constructing envelopes of functions that satisfy pairwise complementarity described in [28]. We refer to Section 3 of [28] for definitions and, in particular, Proposition 3 therein for relevant constructions. The same construction also yields the concave envelope of $\Psi(a_i xy)$ over $\{0,1\} \times [0,1]$ proved later in Theorem 3.26 using the concave overestimator of $\Psi(w)$ derived in Lemma 3.25.

Recall that Figure 2(a) depicts $P^C(a_1 x_1 y_1)$ for inequality (51). Observe that in Figure 2(b), lifting coefficients $(0, a_1)$ define the plane passing through $(0,0)$ and $(1,0)$ while lifting coefficients $(P^C(a_i), 0)$ define the plane passing through $(0,0)$ and $(0,1)$ (which is identical to the plane obtained when $j = q_1 = 2$). Since there are several choices for the values of each of the pair of lifting coefficients (α_i, β_i), the family of inequalities (54) contains an exponential number of members. Theorem 3.14 therefore provides a new illustration that sequence-independent lifting from a single seed inequality can produce exponentially large families of inequalities, a property that was discussed in a more general setting in Section 2 of [24]. We illustrate this characteristic of lifted bilinear cover inequalities in Example 3.15.

Example 3.15. *In Example 3.12, we established that (51) is facet-defining for PB$(M, \emptyset, M, \emptyset)$ and described the corresponding lifting function $P^C(w)$. We compute that $q_1 = 2$ (with $Q^1_0 = 0$, $Q^1_1 = 10$, $Q^1_2 = 20$, $Q^1_3 = 21$) and $q_2 = 1$ (with $Q^2_0 = 0$, $Q^2_1 = 10$, $Q^2_2 = 19$). Applying Theorem 3.14, we obtain the nine inequalities*

$$\left\{ \begin{array}{ll} & 21y_1 \\ 5x_1 & +\frac{21}{2}y_1 \\ 15x_1 & \end{array} \right\} + \left\{ \begin{array}{ll} & 19y_2 \\ \frac{50}{9}x_2 & +\frac{76}{9}y_2 \\ 14x_2 & \end{array} \right\} + 17y_3 + 10x_4 + 5x_5 \geq 15$$

which are all facet-defining for PB. The three possible choices for the lifting coefficients of (x_1, y_1) are depicted in Figure 2(b). The fact that there are three possible choices for (x_2, y_2) follows similarly with the exception that coefficient a_2 falls in the second interval $(A_1 - \mu, A_2 - \mu]$.

Another look at Figure 2 also suggests that if we had fixed (x_1, y_1) at $(0,1)$ or $(1,0)$, we would only have been able to obtain a single lifted inequality and so fixing variables at $(0,0)$ in this case is crucial in discovering the exponential family of lifted inequalities. This provides a graphical illustration of Proposition 3.3, which states that all interesting lifting coefficients that can be obtained from fixing variables at $(0,1)$ or $(1,0)$ can also be obtained from fixing variables at $(0,0)$.

3.3.2 Lifted reverse bilinear cover inequalities

In Theorem 3.14, we derived lifted bilinear cover inequalities by first lifting the variables (x_j, y_j) for $j \in C'$ and then lifting the variables (x_j, y_j) for $j \in M$. Here, we derive another family of lifted inequalities that we call *lifted reverse bilinear cover inequalities* by changing the lifting order: we start the lifting procedure with the same seed inequality (42), but we now lift the variables (x_j, y_j) for $j \in M$ before the variables (x_j, y_j) for $j \in C'$. In this case, we do not assume that $a_l \geq a_i$ for $i \in C$, *i.e.*, we do not require Condition (A2).

Proposition 3.16. *Under Conditions (A1), (A3), and (A4), the inequality*

$$(a_l - \mu)x_l + \sum_{j \in M} \min\{a_j, a_l - \mu\} x_j + \sum_{j \in T} a_j y_j \geq a_l - \mu \tag{58}$$

is facet-defining for PB$(\emptyset, C', \emptyset, C')$.

Proof. It follows from Proposition 2.9 that

$$(a_l - \mu)x_l + \sum_{j \in T} a_j y_j \geq a_l - \mu$$

is facet-defining for the full-dimensional polytope $PB(M, C', M, C')$. Its lifting function $P(w)$ is derived in Proposition 3.8 where it is also proven to be subadditive over $\mathbb{R}_+$. Therefore, Proposition 3.2 shows that lifting coefficients (α_i, β_i) for (x_i, y_i) for $i \in M$ are valid if they satisfy the condition

$$\alpha_i x_i + \beta_i y_i \geq P(a_i x_i y_i) \quad \text{for } (x_i, y_i) \in \{0,1\} \times [0,1] \setminus \{0,0\}. \tag{59}$$

Condition (59) can be rewritten as

$$\beta_i \phi \geq P(0) \qquad \text{for} \quad 0 < \phi \leq 1, \tag{60}$$
$$\alpha_i + \beta_i \phi \geq P(a_i \phi) \quad \text{for} \quad 0 \leq \phi \leq 1. \tag{61}$$

We now show that $(\alpha_i, \beta_i) = (P(a_i), 0)$ are valid lifting coefficients. Clearly, $\beta_i = 0$ satisfies (60) since $P(0) = 0$. Further, since $P(a_i\phi) = \min\{a_i\phi, a_l - \mu\}$, it is also clear that $\alpha_i = P(a_i) = \min\{a_i, a_l - \mu\} \geq \min\{a_i\phi, a_l - \mu\} = P(a_i\phi)$. To show that (58) is facet-defining for $PB(\emptyset, C', \emptyset, C')$, it suffices to verify that the two points $(0,1)$ and $(1,1)$ satisfy (59) at equality. □

Proposition 3.16 also follows directly from Proposition 2.9. We provided a proof of Proposition 3.16 based on lifting techniques to emphasize that the cover and reverse-cover inequalities are obtained by reversing the order of lifting of M and C'. We remark that the above result does not require Condition (A2). Also, note that lifting coefficients $(\alpha_i, \beta_i) = (0, a_i)$ for $i \in M$ are valid for (59). These coefficients yield facet-defining inequalities for $PB(\emptyset, C', \emptyset, C')$ because (59) is satisfied at equality for $(1,0)$ and $\left(1, \min\left\{1, \frac{a_l - \mu}{a_i}\right\}\right)$. However, these variables could have been treated directly as elements of T in (42) since adding more elements to T will not violate Condition (A4).

To obtain facet-defining inequalities for PB, we lift the remaining variables (x_j, y_j) for $j \in C'$ in (58). To this end, we first compute the function

$$\begin{aligned} P^M(w) := \quad \max \quad & (a_l - \mu) - \left\{ (a_l - \mu)x_l + \sum_{j \in M} \min\{a_j, a_l - \mu\} x_j + \sum_{j \in T} a_j y_j \right\} \\ s.t. \quad & a_l x_l y_l + \sum_{j \in M \cup T} a_j x_j y_j \geq a_l - \mu - w \\ & x_j \in \{0,1\}, \ y_j \in [0,1] \quad \forall j \in \{l\} \cup M \cup T. \end{aligned} \tag{62}$$

Let $M = M_1 \cup M_2$ where $M_1 = \{i \in M \mid a_i > a_l - \mu\}$ and $M_2 = M \setminus M_1$. Assume without loss of generality that $\{l\} \cup M_1 = \{1, \ldots, q\}$ and $a_1 \geq a_2 \geq \ldots \geq a_q$ where $q = |M_1| + 1$. Further, define $A_0 = 0$ and $A_i = \sum_{j=1}^{i} a_j$ for $i = 1, \ldots, q$. Observe that $a_l + \sum_{j \in M \cup T} a_j = A_q + \sum_{j \in M_2} a_j + \sum_{j \in T} a_j$. We derive a closed-form expression for $P^M(w)$ in the following proposition.

Proposition 3.17.

$$P^M(w) = \begin{cases} -\infty & \text{if} \quad w < -\mu - \sum_{j \in M \cup T} a_j, \\ w + A_q - q(a_l - \mu) & \text{if} \quad -\mu - \sum_{j \in M \cup T} a_j \leq w < -A_q + (a_l - \mu), \\ -i(a_l - \mu) & \text{if} \quad -A_{i+1} + (a_l - \mu) \leq w < -A_i, \qquad i = 0, \ldots, q-1, \\ w + A_i - i(a_l - \mu) & \text{if} \quad -A_i \leq w < -A_i + (a_l - \mu), \qquad i = 1, \ldots, q-1, \end{cases}$$

Proof. First, we observe that, if (62) has a feasible solution, then it has an optimal solution (x^*, y^*) that satisfies $x^*_j = 1$ for $j \in T$ and $y^*_j = 1$ for $j \in M \cup \{l\}$ since the objective coefficients corresponding to these variables are zero. Using the notation $\bar{a} = \sum_{j \in T} a_j$ and $\bar{y} = \frac{\sum_{j \in T} a_j y_j}{\bar{a}}$, we simplify the expression of $P^M(w)$ as

$$\begin{aligned} P^M(w) = \quad \max \quad & (a_l - \mu) - \left\{ \sum_{j \in \{l\} \cup M_1} (a_l - \mu)x_j + \sum_{j \in M_2} a_j x_j + \bar{a}\bar{y} \right\} \\ s.t. \quad & \sum_{j \in \{l\} \cup M_1} a_j x_j + \sum_{j \in M_2} a_j x_j + \bar{a}\bar{y} \geq a_l - \mu - w \\ & x_j \in \{0,1\} \quad \forall j \in \{l\} \cup M_1 \cup M_2, \ \bar{y} \in [0,1]. \end{aligned} \tag{63}$$

After introducing $\hat{a} = \sum_{j \in M_2} a_j + \bar{a}$ and $\hat{y} = \frac{\sum_{j \in M_2} a_j x_j + \bar{a}\bar{y}}{\hat{a}}$, we claim that $P^M(w)$ can be written as

$$
\begin{aligned}
P^M(w) = \quad \max \quad & (a_l - \mu) - \left\{ \sum_{j=1}^{q} (a_l - \mu) x_j + \hat{a}\hat{y} \right\} \\
s.t. \quad & \sum_{j=1}^{q} a_j x_j + \hat{a}\hat{y} \geq a_l - \mu - w \qquad (64) \\
& x_j \in \{0,1\} \quad \forall j \in \{1, \ldots, q\}, \; \hat{y} \in [0,1].
\end{aligned}
$$

We next prove that (63) and (64) are equivalent. To do so, we show that (63) has a feasible solution $(x_l^*, x_{M_1}^*, x_{M_2}^*, \bar{y}^*)$ with objective value ζ^* if and only if (64) has a feasible solution $(x_l^*, x_{M_1}^*, \hat{y}^*)$ with objective value ζ^*. On the one hand, given $(x_l^*, x_{M_1}^*, x_{M_2}^*, \bar{y}^*)$, we can obtain $(x_l^*, x_{M_1}^*, \hat{y}^*)$ directly from the definition of $\hat{y}$. The objective values of these two solutions are identical. On the other hand, let $M_2 = \{q+1, \ldots, m\}$. Define $\hat{A}_0 = 0$ and $\hat{A}_i = \sum_{j=q+1}^{q+i} a_j$ for $i = 1, \ldots, m-q$. Then, for a given $(x_l^*, x_{M_1}^*, \hat{y}^*)$, we build $(x_l^*, x_{M_1}^*, x_{M_2}^*, \bar{y}^*)$ as follows. Define $\hat{m} = \max\{i \in \{0, \ldots, m-q\} \mid \hat{A}_i \leq \hat{a}\hat{y}^*\}$ and set $x_{q+j}^* = 1$ for $j \leq \hat{m}$, $x_{q+j}^* = 0$ for $j > \hat{m}$ and $\bar{y}^* = \frac{\hat{a}\hat{y}^* - \hat{A}_{\hat{m}}}{\bar{a}}$. We argue next that this solution is feasible. First observe that $\hat{a}\hat{y}^* - \hat{A}_{\hat{m}} \leq a_{q+\hat{m}+1}$ when $\hat{m} \leq m - q - 1$ and that $\hat{a}\hat{y}^* - \hat{A}_{\hat{m}} \leq \bar{a}$ when $\hat{m} = m - q$. Since $\bar{a} = \sum_{j \in T} a_j > a_l - \mu \geq a_i$ for all $i \in M_2$ because of Condition (A4) and the definition of M_2, we easily conclude that $0 \leq \frac{\hat{a}\hat{y}^* - \hat{A}_{\hat{m}}}{\bar{a}} \leq 1$. Also, $\sum_{j \in \{l\} \cup M_1} a_j x_j^* + \sum_{j \in M_2} a_j x_j^* + \bar{a}\bar{y}^* = \sum_{j \in \{l\} \cup M_1} a_j x_j^* + \hat{A}_{\hat{m}} + \hat{a}\hat{y}^* - \hat{A}_{\hat{m}} = \sum_{j \in \{l\} \cup M_1} a_j x_j^* + \hat{a}\hat{y}^*$. This shows that the proposed solution is feasible for (63) and has the same objective value as $(x_l^*, x_{M_1}^*, \hat{y}^*)$.

Next, we study (64). It is clear that this problem is infeasible if and only if $w < a_l - \mu - A_q - \hat{a} = -\mu - \sum_{j \in M \cup T} a_j$. Therefore assume that $w \geq -\mu - \sum_{j \in M \cup T} a_j$. Consider now any optimal solution $(x^*, \hat{y}^*)$ for which $x_i^* < x_t^*$ and $i < t$ for some $i, t \in \{1 \ldots, q\}$. Then the solution $(\bar{x}, \hat{y}^*)$ where $\bar{x}_k = x_k^*$ if $k \neq i$ and $k \neq t$, $\bar{x}_i = x_t^*$, and $\bar{x}_t = x_i^*$ is also feasible for (64) since $a_i \geq a_t$ and has the same objective value as $(x^*, \hat{y}^*)$. It follows that (64) has an optimal solution that satisfies $x_1^* = \ldots = x_i^* = 1$ and $x_{i+1}^* = \ldots = x_q^* = 0$ for some $i \in \{1, \ldots, q\}$. Consider such a solution further. On the one hand, if $\sum_{j=1}^{i} a_j \geq a_l - \mu - w$, then $\sum_{j=1}^{i-1} a_j < a_l - \mu - w$ and $\hat{y}^* = 0$ otherwise the solution $x_j^\circ = 1$ for $j = 1, \ldots, i-1$, $x_j^\circ = 0$ for $j = i, \ldots, q$ and $\hat{y}^\circ = 0$ would be feasible and would have a better objective value. On the other hand if $\sum_{j=1}^{i} a_j < a_l - \mu - w$ for $i \leq q - 1$ then $\sum_{j=1}^{i+1} a_j \geq a_l - \mu - w$. Otherwise the solution $x_j^\circ = 1$ for $j = 1, \ldots, i+1$, $x_j^\circ = 0$ for $j = i+2, \ldots, q$ and $\hat{y}^\circ = \hat{y}^* - \frac{a_{i+1}}{\hat{a}}$ would be feasible and would have an objective value $a_{i+1} - (a_l - \mu)$ larger than that of (x^*, y^*). This is a contradiction since $a_{i+1} > a_l - \mu$.

We consider two situations. First, assume $-A_q + (a_l - \mu) - \hat{a} \leq w < -A_q + (a_l - \mu)$, it follows from the above discussion that there is an optimal solution $(x^*, \hat{y}^*)$ with $x^* = \mathbf{1}$. Then $\hat{y}^* = \frac{a_l - \mu - w - A_q}{\hat{a}}$. Clearly, $\hat{y}^* \in [0,1]$ and so $P^M(w) = w + A_q - q(a_l - \mu)$. Second, assume $-A_{i+1} + (a_l - \mu) \leq w < -A_i + (a_l - \mu)$ for some $i \in \{0, \ldots, q-1\}$, it follows from the above discussion that one of the following two solutions

$$
\begin{aligned}
& x_1^{\curlywedge} = x_2^{\curlywedge} = \ldots = x_{i+1}^{\curlywedge} = 1, \quad x_{i+2}^{\curlywedge} = \ldots = x_q^{\curlywedge} = 0, \quad \hat{y}^{\curlywedge} = 0, \text{ and} \\
& x_1^{\pm} = x_2^{\pm} = \ldots = x_i^{\pm} = 1, \quad x_{i+1}^{\pm} = \ldots = x_q^{\pm} = 0, \quad \hat{y}^{\pm} = \frac{a_l - \mu - w - A_i}{\hat{a}}
\end{aligned}
$$

with objective values $z^{\curlywedge} = -i(a_l - \mu)$ and $z^{\pm} = -i(a_l - \mu) + (w + A_i)$ is optimal for (64) since $a_l - \mu - w \in (A_i, A_{i+1}]$. Note that the second solution is feasible only when $a_l - \mu - w - A_i \leq \hat{a}$. We now consider two cases. When $w \leq -A_i$ then $z^{\curlywedge} \geq z^{\pm}$ and so $P^M(w) = -i(a_l - \mu)$. When $w > -A_i$, then $z^{\pm} > z^{\curlywedge}$. Further, solution $(x^{\pm}, \hat{y}^{\pm})$ is feasible since $a_l - \mu - w - A_i < a_l - \mu \leq \hat{a}$ because of Condition (A4). It follows that $P^M(w) = -i(a_l - \mu) + (w + A_i)$. □

To perform sequence-independent lifting for the variables (x_j, y_j) for $j \in C'$, we verify that the function $P^M(w)$ is subadditive over $\mathbb{R}^-$.

Proposition 3.18. *The lifting function $P^M(w)$ is subadditive over $\mathbb{R}_-$.*

Proof. First, note that $P^M(w)$ is subadditive over $\mathbb{R}_-$ if it is subadditive over $I = [-\mu - \sum_{j\in M\cup T} a_j, 0]$. Consider Corollary 3.5 and define $D_i = A_i$, $\nu = a_l - \mu$, $r = q$, and notice that $P^M(w) = g(-w) + w$. Clearly, $A_i + A_j \geq A_{i+j}$ for $0 \leq i \leq j \leq q$ with $i + j \leq q$ since A_i is the sum of the largest i coefficients in $M_1 \cup \{l\}$. It then follows from Corollary 3.5 that $P^M(w)$ is subadditive over I, proving the result. □

We next illustrate the results of Propositions 3.16, 3.17, and 3.18 via an example.

Example 3.19. *For the set B of Example 3.12, consider the partition* $(C', \{l\}, M, T) = (\{3\}, \{4\}, \{5\}, \{1, 2\})$. *This partition satisfies Conditions (A1), (A3), and (A4) since* C *is a cover with* $\mu = 12$, $a_4 > \mu$, *and* $\sum_{j\in C\cup T} a_j = 21 + 19 + 17 + 15 > 20 + 15 = d + a_l$. *We obtain from Proposition 3.16 that*

$$3x_4 + 3x_5 + 21y_1 + 19y_2 \geq 3 \tag{65}$$

is facet-defining for PB$(\emptyset, C', \emptyset, C')$. *Further, the lifting function* $P^M(w)$ *over* $\mathbb{R}_-$ *is given by*

$$P^M(w) = \begin{cases} -\infty & \text{if} \quad w < -62 \\ w + 19 & \text{if} \quad -62 \leq w < -22 \\ -3 & \text{if} \quad -22 \leq w < -15 \\ w + 12 & \text{if} \quad -15 \leq w < -12 \\ 0 & \text{if} \quad -12 \leq w \leq 0, \end{cases}$$

as described in Proposition 3.17 since $q = 2$, $A_0 = 0$, $A_1 = 15$ *and* $A_2 = 25$.

Similar to Theorem 3.14, we compute the lifting coefficients for the variables (x_i, y_i) for $i \in C'$ using sequence-independent lifting; refer to the discussion following Proposition 3.2.

Theorem 3.20. *Suppose that Conditions (A1), (A3), and (A4) hold. Then, the lifted reverse bilinear cover inequality*

$$(a_l - \mu)x_l - \sum_{j\in C'} P^M(-a_j)x_j + \sum_{j\in M} \min\{a_j, a_l - \mu\}x_j + \sum_{j\in T} a_j y_j \geq (a_l - \mu) - \sum_{j\in C'} P^M(-a_j) \tag{66}$$

is facet-defining for PB.

Proof. Since $P^M(w)$ is subadditive over $\mathbb{R}_-$, the lifting coefficients (α_i, β_i) of the variables (x_i, y_i) for $i \in C'$ are valid if they are chosen to satisfy

$$\alpha_i(x_i - 1) + \beta_i(y_i - 1) \geq P^M(a_i x_i y_i - a_i) \quad \text{for } (x_i, y_i) \in \{0, 1\} \times [0, 1] \setminus \{1, 1\}. \tag{67}$$

Condition (67) can be rewritten as

$$\beta_i \leq \inf_{0\leq\phi<1} \frac{-P^M(a_i\phi - a_i)}{1 - \phi}, \tag{68}$$

$$\alpha_i + \sup_{0\leq\phi\leq1} \beta_i(1 - \phi) \leq -P^M(-a_i). \tag{69}$$

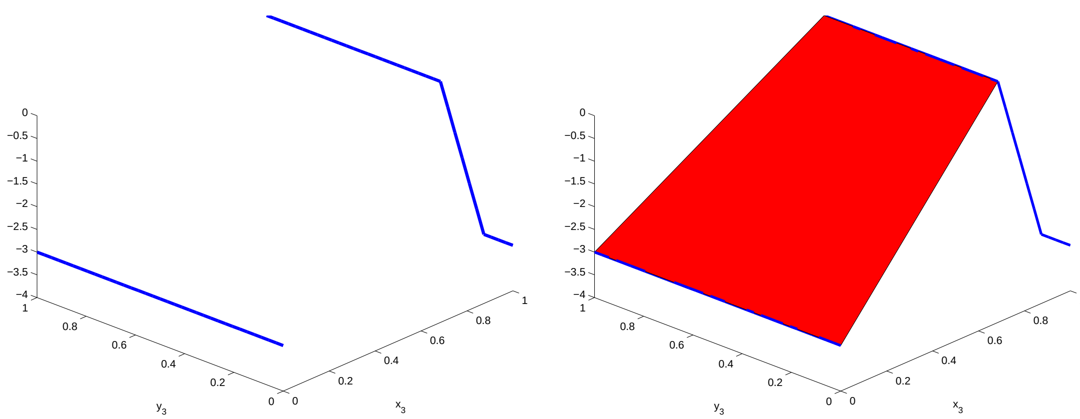
Because of Assumption 2, we know that $a_i \leq \sum_{j\in N} a_j - d = \sum_{j\in C\cup M\cup T} a_j - (\sum_{j\in C} a_j - \mu) = \mu + \sum_{j\in M\cup T} a_j$ for all $i \in C' \subseteq N$ and so $P^M(a_i\phi - a_i) > -\infty$ for all $\phi \in [0, 1)$. Choosing $\beta_i = 0$ satisfies (68) since $P^M(a_i\phi - a_i) \leq 0$ for all $\phi \in [0, 1)$. Moreover, as $\beta_i = 0$, it is easily verified that choosing $\alpha_i = -P^M(-a_i)$ satisfies (69). Finally, note that (67) is tight at the points $(0, 0)$ and $\left(1, \frac{(a_i - A_1 + a_l - \mu)^+}{a_i}\right)$, which proves that (66) is facet-defining for *PB*. □

Note that the lifted reverse bilinear cover inequality (66) we obtained through lifting is unique. This is a significant difference from lifted bilinear cover inequalities (54). We next illustrate in an example the reason that we obtain a single lifted reverse bilinear cover inequality and show that not all lifted reverse bilinear cover inequalities (66) can be derived as lifted bilinear cover inequalities (54).

Example 3.21. *For the partition $(C', \{l\}, M, T) = (\{3\}, \{4\}, \{5\}, \{1, 2\})$, we established in Example 3.19 that (65) is facet-defining for $PB(\emptyset, C', \emptyset, C')$. Applying Theorem 3.20, we obtain the following lifted reverse bilinear cover inequality*

$$3x_3 + 3x_4 + 3x_5 + 21y_1 + 19y_2 \geq 6, \tag{70}$$

which is facet-defining for PB. We represent in Figure 3(a), the function $P^M(a_3x_3y_3 - a_3)$ that was overestimated to construct valid lifting coefficients. We represent in Figure 3(b) the only choice of coefficients that yields an overestimating plane to $P^M(a_3x_3y_3 - a_3)$ over $(x_3, y_3) \in \{0, 1\} \times [0, 1]$ and is tight at $(1, 1)$. Further, Inequality (70) cannot be obtained as a lifted bilinear cover inequality (54). In fact, if (70) was of the form (54), it should be that $C \subseteq \{3, 4, 5\}$. However, none of the four possible covers $C_1 = \{3, 4\}$, $C_2 = \{3, 5\}$, $C_3 = \{4, 5\}$ and $C_4 = \{3, 4, 5\}$ yield (70).



(a) (b)

Figure 3: Deriving lifting coefficients for Example 3.21

3.4 Inequalities through approximate lifting

We now derive another family of lifted inequalities from the seed inequality (27) developed in Proposition 2.11. To this end, we first identify a partition $(K, \overline{M})$ of the set of variables N that satisfies the following conditions

(C1) $\sum_{j \in K} a_j - a_k \geq d$ for all $k \in K$,

(C2) $\sum_{j \in K} a_j - a_k - a_m < d$ for all $k \neq m \in K$, *i.e.*, $a_k + a_m > \mu$ for all $k \neq m \in K$,

where $\mu = \sum_{j \in K} a_j - d$ is the excess of K. Note that Condition (C1) implies that K is a cover. Further, Condition (C1) requires that $K \setminus \{k\}$ is also a cover for all $k \in K$ and so $a_k \leq \mu$ for all $k \in K$. It also follows from Condition (C1) that $|K| \geq 2$. We refer to a set K satisfying Conditions (C1) and (C2) as a *clique*. After fixing the variables (x_i, y_i) for $i \in \overline{M}$ to $(0, 0)$, it follows from Proposition 2.11 that the clique inequality

$$\sum_{j \in K} x_j \geq |K| - 1 \tag{71}$$

is facet-defining for $PB(\overline{M}, \emptyset, \overline{M}, \emptyset)$.

We now lift the remaining variables (x_i, y_i) for $i \in \overline{M}$ in two steps. We assume without loss of generality that $K = \{1, \ldots, r\}$ and that $a_1 \leq a_2 \leq \ldots \leq a_r$. We define $\mu' = a_1 + a_2 - \mu$. We assume that $a_{r+1} \leq \cdots \leq a_n$ and define p such that $\sum_{i=r+1}^{p} a_i < \mu' \leq \sum_{i=r+1}^{p+1} a_i$. (More generally, $\widehat{M}$ can be taken to be any subset of $\overline{M}$ such that $\sum_{i \in \widehat{M}} a_i < \mu'$ without altering the form of the derived inequality.) Let $\widehat{M} = \{a_{r+1}, \ldots, a_p\}$. We show that (71) is facet-defining for $PB(\overline{M} \backslash \widehat{M}, \emptyset, \overline{M} \backslash \widehat{M}, \emptyset)$. First, we show by contradiction that the inequality is valid. Let (x, y) be such that $\sum_{j\in K} x_j < r-1$. Then,

$$\sum_{j=1}^{p} a_j x_j y_j \le \sum_{j=3}^{p} a_j = d - \mu' + \sum_{j=r+1}^{p} a_j < d,$$

where the first inequality holds since $a_1 \le \cdots \le a_r$ and $\sum_{j\in K} x_j < r-1$ and the last inequality follows since $\sum_{j=r+1}^{p} a_j < \mu'$. This inequality implies that $(x, y) \notin B(\overline{M}\backslash\widehat{M}, \emptyset, \overline{M}\backslash\widehat{M}, \emptyset)$, the desired contradiction. By Proposition 2.10, it suffices to show that (71) is facet-defining for $PB(\overline{M}\backslash\widehat{M}, \emptyset, \overline{M}\backslash\widehat{M}, K \cup \widehat{M})$. Define χ such that $\chi_j = 1$ for $j \in K$ and $\chi_j = 0$ for $j \in \widehat{M}$. Then, by (C1), $p^k = \chi - e_k$ for $k \in K$ and $q^k = \chi - e_1 + e_k$ for $k \in \widehat{M}$, are feasible. Since these points are linearly independent, (71) is facet-defining.

We now lift variables (x_i, y_i) for $i \in M = \overline{M}\backslash\widehat{M}$. The lifting function corresponding to (71) is defined as

$$\begin{aligned} \Phi(w) := \quad \max \quad & (|K|-1) - \sum_{j\in K} x_j \\ s.t. \quad & \sum_{j\in K} a_j x_j y_j + \sum_{j\in\widehat{M}} a_j x_j y_j \ge d - w \\ & x_j \in \{0,1\},\ y_j \in [0,1] \quad \forall j \in K. \end{aligned} \tag{72}$$

We define $a' = \sum_{j\in\widehat{M}} a_j$, $\bar{\mu} = \mu' - a'$, $B_0 = 0$, and $B_i = \sum_{j=1}^{i} a_{j+2} - a'$ for $i = 1, \ldots, r-2$. It follows from the definition of $\widehat{M}$ that $\bar{\mu} > 0$. Observe that $B_0 \le B_1$ because $a_3 - a' \ge a_3 - \mu' = a_3 - a_1 - a_2 + \mu \ge -a_2 + \mu \ge 0$, where the last inequality follows from (C1). Also, observe that $B_{r-2} + \bar{\mu} = d - a'$ and, for all $i \in M$, $a_i \ge a_{p+1} \ge \mu' - a' = \bar{\mu}$, where the last inequality follows from the definition of $\widehat{M}$.

Proposition 3.22. *For $w \ge 0$,*

$$\Phi(w) = \begin{cases} 0 & \text{if} \quad 0 \le w < \bar{\mu}, \\ i & \text{if} \quad B_{i-1} + \bar{\mu} \le w < B_i + \bar{\mu}, \quad i = 1, \ldots, r-2, \\ r-1 & \text{if} \quad B_{r-2} + \bar{\mu} \le w. \end{cases}$$

Proof. Problem (72) is feasible for $w \ge 0$ and has an optimal solution (x^*, y^*) that satisfies $(x_j^*, y_j^*) = 1$ for $j \in \widehat{M}$ and $y_j^* = 1$ for $j \in K$ since the objective coefficients of these variables are zero. Hence, $\Phi(w)$ can be rewritten as

$$\begin{aligned} \Phi(w) = \quad \max \quad & (|K|-1) - \sum_{j\in K} x_j \\ s.t. \quad & \sum_{j\in K} a_j x_j \ge d - a' - w \\ & x_j \in \{0,1\} \quad \forall j \in K. \end{aligned} \tag{73}$$

Further, we claim that there exists an optimal solution x^* to (73) in which

$$x_1^* \le x_2^* \le \ldots \le x_r^*. \tag{74}$$

This is because, given any solution x^* to (73) with $x_i^* > x_j^*$ for $i < j$, the solution $\bar{x}$ defined as $\bar{x}_k = x_k^*$ if $k \ne i$ and $k \ne j$, $\bar{x}_i = x_j^*$, and $\bar{x}_j = x_i^*$, is feasible and has the same objective value. It follows from (74) that, given $w \in \big[0, d - a'\big]$, the solution

$$x_j^* = \begin{cases} 0 & \text{if} \quad j = 1, \ldots, t(w) - 1, \\ 1 & \text{if} \quad j = t(w), \ldots, r, \end{cases}$$

where $t(w) = \max\big\{i \;\big|\; \sum_{j=i}^{r} a_j \ge d - a' - w = B_{r-2} + \bar{\mu} - w\big\}$ is optimal for (73) and has an objective value of $t(w) - 2$. When $w \in [0, \bar{\mu})$, $d - a' - w \in (B_{r-2}, B_{r-2} + \bar{\mu}]$ showing that $t(w) = 2$ and $\Phi(w) = 0$. When $w \in [B_{i-1} + \bar{\mu}, B_i + \bar{\mu})$ for $i = 1, \ldots, r-2$, $d - a' - w \in (\sum_{j=i+3}^{r} a_j, \sum_{j=i+2}^{r} a_j]$ showing that $t(w) = i + 2$ and $\Phi(w) = i$. Finally, when $w \ge d - a'$, it is clear that $\Phi(w) = |K| - 1 = r - 1$. □

In Section 3.3.1, all lifting functions were subadditive over appropriate ranges. As a result, strong valid inequalities for *PB* were easily obtained using sequence-independent lifting. The lifting function $\Phi(w)$ derived in Proposition 3.22, however, is not subadditive. To circumvent the difficulties associated with sequence-dependent lifting in such a situation, Gu et al. [13] proposed to use approximate lifting. Following their approach, we say that $\Psi(w)$ is a *valid subadditive approximation* of $\Phi(w)$ if $\Psi(w) \geq \Phi(w)$ for all $w \in \mathbb{R}_+$ and $\Psi(w)$ is subadditive. We say that a valid subadditive approximation $\Psi(w)$ is *nondominated* if there is no other valid subadditive approximation $\Psi'(w)$ of $\Phi(w)$ with $\Psi'(w) \leq \Psi(w)$ for all $w \in \mathbb{R}_+$ and $\Psi'(w') < \Psi(w')$ for some $w' \in \mathbb{R}_+$. We also define the notion of maximal set $E = \{w \in \mathbb{R}_+ \mid \Phi^i(w) = \Phi(w)\ \forall i \in M, \text{for all coefficients } a_i \in \mathbb{R}_+ \text{ and for all lifting orders}\}$. A valid subadditive approximation $\Psi(w)$ of $\Phi(w)$ is called *maximal* if $\Psi(w) = \Phi(w)$ for all $w \in E$. It is clear that a maximal nondominated approximation of Φ leads to strong inequalities that can be obtained efficiently for *PB*. The approximation of $\Phi(w)$ we use has the form presented in Corollary 3.6.

We next describe in Proposition 3.23 a subadditive, nondominated and maximal approximation of $\Phi(w)$ over $\mathbb{R}_+$.

Proposition 3.23. *The function*

$$\Psi(w) := \begin{cases} i + \frac{w-B_i}{\bar{\mu}} & \text{if} \quad B_i \leq w < B_i + \bar{\mu}, \quad i = 0, \ldots, r-2, \\ i & \text{if} \quad B_{i-1} + \bar{\mu} \leq w < B_i, \quad i = 1, \ldots, r-2, \\ r-1 & \text{if} \quad B_{r-2} + \bar{\mu} \leq w, \end{cases}$$

is a valid subadditive approximation of $\Phi(w)$ that is nondominated and maximal over $\mathbb{R}_+$.

Proof. Note that $\Psi(w) = \Phi(w)$ when $w \in [B_{i-1} + \bar{\mu}, B_i]$ for $i \in \{1, \ldots, r-2\}$ and when $w \geq B_{r-2} + \bar{\mu}$. Further,

$$\Psi(w) = \Phi(w) + \frac{w - B_i}{\bar{\mu}} \geq \Phi(w)$$

when $w \in (B_i, B_i + \bar{\mu})$ for $i \in \{0, \ldots, r-2\}$. Next, we show that $\Psi(w)$ is subadditive over $\mathbb{R}_+$. In Corollary 3.6, let $s = r-2$, $C_i = B_i$ and $\lambda = \bar{\mu}$. Since B_i is the sum of the smallest i coefficients in $K \backslash \{1,2\}$, it is clear that $B_i + B_j \leq B_{i+j}$ for $0 \leq i \leq j \leq r-2$ with $i + j \leq r-2$. Therefore, $\Psi(w)$ is subadditive over $\mathbb{R}_+$. We now argue nondominance and maximality over $\mathbb{R}_+$. To this end, we first observe that for all $w' \in \mathbb{R}_+$ there exists $w'' \in \mathbb{R}_+$ such that

$$\Psi(w') + \Psi(w'') = \Phi(w' + w''). \tag{75}$$

In particular, w'' can be chosen to be $B_i + \bar{\mu} - w'$ when $w' \in (B_i, B_i + \bar{\mu})$ and w'' can be chosen to be 0 otherwise. If Ψ' dominates Ψ strictly at w' then $\Psi'(w'+w'') \leq \Psi'(w') + \Psi'(w'') < \Psi(w') + \Psi(w'') = \Phi(w'+w'')$ yielding a contradiction to the assumption that Ψ' is an overestimator of Φ. Similarly, if $\Phi(w') < \Psi(w')$ then (75) implies that $\Phi(w'+w'') - \Phi(w') > \Psi(w'') \geq \Phi(w'')$. Therefore, $\Phi(w'')$ does not yield a valid lifting coefficient for the sequential perturbation of w'' after w'. □

Example 3.24. *For the bilinear set B studied in Example 3.12, consider $K = \{3, 4, 5\}$. Set K satisfies Conditions (C1) and (C2) with $\mu = 22$. It follows from Proposition 2.11 that*

$$x_3 + x_4 + x_5 \geq 2 \tag{76}$$

is facet-defining for $B(\{1,2\}, \emptyset, \{1,2\}, \emptyset)$. Let $\widehat{M} = \emptyset$. The lifting function of (76) obtained using Proposition 3.22 and its valid subadditive approximation $\Psi(w)$ obtained in Proposition 3.23 are given by

$$\Phi(w) = \begin{cases} 0 & \text{if} \quad 0 \leq w < 3, \\ 1 & \text{if} \quad 3 \leq w < 20, \\ 2 & \text{if} \quad 20 \leq w \end{cases} \qquad \textit{and} \qquad \Psi(w) = \begin{cases} \frac{w}{3} & \text{if} \quad 0 \leq w < 3, \\ 1 & \text{if} \quad 3 \leq w < 17, \\ 1 + \frac{w-17}{3} & \text{if} \quad 17 \leq w < 20, \\ 2 & \text{if} \quad 20 \leq w \end{cases}$$

as $r = 3$, $\bar{\mu} = 3$, $B_0 = 0$, and $B_1 = 17$.

In Figure 4, we present the lifting function $\Phi(w)$ of the clique inequality derived in Proposition 3.22 and its valid subadditive approximation $\Psi(w)$ obtained in Proposition 3.23 for the particular case of inequality (76) discussed in Example 3.24. The function $\Phi(w)$ is depicted with a dotted line while $\Psi(w)$ is represented using a solid line. Observe that, for $0 < w \leq \bar{\mu} = 3$, the approximation is exact only when $w = \bar{\mu} = 3$, *i.e.*, $\Psi(\bar{\mu}) = \Phi(\bar{\mu})$. For $w \geq \bar{\mu} = 3$, the approximation is exact when $3 = \bar{\mu} \leq w \leq B_1 = 17$ and $w \geq B_1 + \bar{\mu} = 20$. Next, we obtain a concave overestimator of $\Psi(w)$ in Lemma 3.25 that we will use in Theorem 3.26 to compute lifting coefficients for the variables in M.

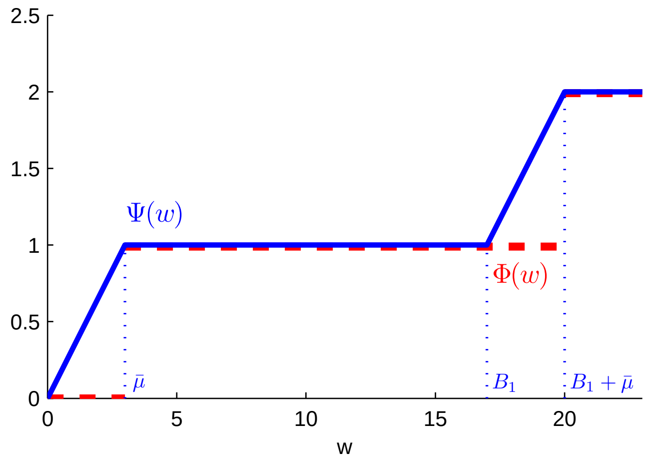


Figure 4: A valid subadditive approximation $\Psi(w)$ of $\Phi(w)$ for Example 3.24.

Lemma 3.25. *For $i \in M$, define*

$$q_i := \begin{cases} 0 & \text{if} \quad a_i \leq \bar{\mu}, \\ j+1 & \text{if} \quad B_j + \bar{\mu} < a_i \leq B_{j+1} + \bar{\mu}, \quad j = 0, \ldots, r-3, \\ r-1 & \text{if} \quad B_{r-2} + \bar{\mu} < a_i. \end{cases}$$

Let $W_0^i = 0$, $W_j^i = B_{j-1} + \bar{\mu}$ for $j = 1, \ldots, q_i$ and $W_{q_i+1}^i = a_i$. Define $\Delta_j^i = W_{j+1}^i - W_j^i$ for $j = 0, \ldots, q_i$. Define also $\psi_j^i(w) = \Psi(W_j^i) + \frac{\Psi(W_{j+1}^i) - \Psi(W_j^i)}{\Delta_j^i}(w - W_j^i)$ for $j = 0, \ldots, q_i$. Then, the function

$$\psi^i(w) := \min \left\{ \psi_j^i(w) \;\middle|\; j \in \{0, \ldots, q_i\} \right\} \tag{77}$$

is a concave overestimator of $\Psi(w)$ over $[0, a_i]$.

Proof. First, $\psi^i(w)$ is concave since it is obtained as the minimum of affine functions. Observe that, for $j = 0, \ldots, q_i$, $\psi_j^i(w) \geq \psi_{j+1}^i(w)$ when $w \geq W_{j+1}^i$, $\psi_j^i(w) \leq \psi_{j+1}^i(w)$ when $w < W_{j+1}^i$, and $\psi_j^i(w) \geq \Psi(w)$ when $w \in [W_j^i, W_{j+1}^i]$. Now, consider $j \in \{0, \ldots, q_i\}$ and $k \neq j$. Then, for $w \in [W_j^i, W_{j+1}^i]$, $\Psi(w) \leq \psi_j^i(w) \leq \psi_k^i(w)$. □

The concave overestimator $\psi^i(w)$ of Lemma 3.25 can be used to obtain lifting coefficients in a manner similar to that of Theorem 3.14. Because of the way the concave overestimator is built, it can be observed that all of its affine pieces (except possibly $\psi_{q_i}^i$) touch the original lifting function Φ at two points and therefore can be used to generate strong lifting coefficients. To describe whether $\psi_{q_i}^i$ touches Φ in two points, we define $I(a_i)$ to be the function that returns 0 if $\Phi(a_i) = \Psi(a_i)$ and returns 1 otherwise, *i.e.*,

$$I(a_i) := \begin{cases} 0 & \text{if} \quad B_{q_i-1} + \bar{\mu} < a_i \leq B_{q_i} \text{ or } a_i > B_{r-2} + \bar{\mu}, \\ 1 & \text{if} \quad B_{q_i} < a_i \leq B_{q_i} + \bar{\mu}. \end{cases}$$

We observe that, when $I(a_i) = 0$, it is possible to derive maximal lifting coefficients (with respect to Φ) from all affine pieces of ψ^i. When $I(a_i) = 1$, however, we can only guarantee the derivation of maximal lifting coefficients (with respect to Φ) from ψ^i for $i = 0, \ldots, q_i - 1$. This intuitive observation is formally proven in the following theorem.

Theorem 3.26. *Under Conditions (C1) and (C2),*

$$\sum_{j\in K} x_j + \sum_{i\in M} \alpha_i^{j_i} x_i + \sum_{i\in M} \beta_i^{j_i} y_i \geq |K| - 1 \tag{78}$$

defines a face of PB of dimension at least $(2n-1) - \sum_{i\in M} I(a_i) \times \mathbf{1}_{\{j_i \geq q_i\}}$ *for all* $j_i \in \{0, \ldots, q_i + 1\}$ *and for all* $i \in M$ *where*

$$(\alpha_i^j, \beta_i^j) = \left(\Psi(W_j^i) - \frac{\Psi(W_{j+1}^i) - \Psi(W_j^i)}{\Delta_j^i} W_j^i, \frac{\Psi(W_{j+1}^i) - \Psi(W_j^i)}{\Delta_j^i} a_i \right) \text{ for } j = 0, \ldots, q_i \tag{79}$$

$$(\alpha_i^{q_i+1}, \beta_i^{q_i+1}) = (\Psi(a_i), 0)$$

and $\bar{\mu}$, W_j^i, Δ_j^i *and* q_i *are as defined in Lemma 3.25. For a given inequality of the form (78), let* $L = \{i \in M \mid j_i \geq q_i, I(a_i) = 1\}$. *In particular, (78) is facet-defining for PB if one of the following conditions holds:*

1. $L = \emptyset$.
2. $\exists \bar{\imath} \in M$ *such that* $j_{\bar{\imath}} = 0$.

Proof. It follows from Proposition 3.23 that $\Psi(w)$ is a valid subadditive approximation of $\Phi(w)$ for $w \geq 0$. Hence, lifting coefficients (α_i, β_i) of (x_i, y_i) for $i \in M$ are valid if they satisfy the condition

$$\alpha_i x_i + \beta_i y_i \geq \Psi(a_i x_i y_i) \quad \text{for } (x_i, y_i) \in \{0,1\} \times [0,1] \setminus \{0,0\}. \tag{80}$$

Condition (80) can be restated as

$$\beta_i \phi \geq \Psi(0) \qquad \text{for} \quad 0 < \phi \leq 1, \tag{81}$$

$$\alpha_i + \beta_i \phi \geq \Psi(a_i \phi) \quad \text{for} \quad 0 \leq \phi \leq 1. \tag{82}$$

To prove that (78) defines a face of PB of dimension at least $(2n-1) - \sum_{i\in M} I(a_i) \times \mathbf{1}_{\{j_i \geq q_i\}}$ when lifting coefficients are chosen according to (79), we will show that, for each $i \in M$,

$$\alpha_i x_i + \beta_i y_i = \Phi(a_i x_i y_i) \tag{83}$$

is satisfied at equality by at least $2 - I(a_i) \times \mathbf{1}_{\{j_i \geq q_i\}}$ independent points.

First, consider the case where $(\alpha_i, \beta_i) = (\Psi(a_i), 0)$. Observe that (81) is satisfied since $\beta_i = 0$ and $\Psi(0) = 0$. Further, (82) holds as $\alpha_i = \alpha_i + \beta_i \phi = \Psi(a_i) \geq \Psi(a_i \phi)$ since Ψ is a nondecreasing function. It is easily verified that (83) is satisfied at equality at the point $(0, 1)$. Further, when $I(a_i) = 0$, then (83) is also satisfied at equality at the point $(1, 1)$.

Second, consider the case where

$$(\alpha_i, \beta_i) = \left(\Psi(W_j^i) - \frac{\Psi(W_{j+1}^i) - \Psi(W_j^i)}{\Delta_j^i} W_j^i, \frac{\Psi(W_{j+1}^i) - \Psi(W_j^i)}{\Delta_j^i} a_i \right).$$

Clearly, (α_i, β_i) satisfies (81) since $\beta_i \geq 0$. From Lemma 3.25, we have that

$$\begin{aligned} \Phi(a_i\phi) \leq \Psi(a_i\phi) \quad & \leq \Psi(W_j^i) + \tfrac{\Psi(W_{j+1}^i) - \Psi(W_j^i)}{\Delta_j^i}(a_i\phi - W_j^i) \\ & = \left(\Psi(W_j^i) - \tfrac{\Psi(W_{j+1}^i) - \Psi(W_j^i)}{\Delta_j^i} W_j^i\right) + \tfrac{\Psi(W_{j+1}^i) - \Psi(W_j^i)}{\Delta_j^i} a_i\phi \\ & = \alpha_i + \beta_i\phi. \end{aligned}$$

We now present points that satisfy (83) at equality. Observe first that, for $j = 0, \ldots, q_i$, the point $(x_i^*, y_i^*) = \left(1, \frac{W_j^i}{a_i}\right)$ satisfies (83) at equality since $\Psi(a_i x_i^* y_i^*) = \Psi(W_j^i) = \Psi(B_{j-1} + \bar{\mu}) = \Phi(B_{j-1} + \bar{\mu})$. Similarly, for

$j = 0, \ldots, q_i - 1$, the point $(x_i^*, y_i^*) = \left(1, \frac{W^i_{j+1}}{a_i}\right)$ satisfies (83) at equality. For $j = q_i$, the point $\left(1, \frac{W^i_{j+1}}{a_i}\right)$ reduces to $(1,1)$ which satisfies (83) at equality when $\Psi(a_i) = \Phi(a_i)$, *i.e.*, when $I(a_i) = 0$. Therefore, we conclude that (78) defines a face of *PB* of dimension at least $(2n-1) - \sum_{i \in M} I(a_i) \times \mathbf{1}_{\{j_i \geq q_i\}}$.

We conclude from the above derivation that when, for all $i \in M$, either $j_i < q_i$ or $I(a_i) = 0$, then the face of *PB* that (78) defines has dimension $2n-1$ showing that (78) is facet-defining for *PB* and proving 1. Now, we show that (78) is also facet-defining if $j_{\bar{\imath}} = 0$ for some $\bar{\imath} \in M$. We first lift $(x_{\bar{\imath}}, y_{\bar{\imath}})$. Since $a_{\bar{\imath}} \geq \bar{\mu}$ (see discussion preceding Proposition 3.22) it follows that $(\alpha^0_{\bar{\imath}}, \beta^0_{\bar{\imath}}) = (0, \frac{a_{\bar{\imath}}}{\bar{\mu}})$. Then, we lift the variables in $M \backslash \{L \cup \{\bar{\imath}\}\}$ and choose any $j_i \leq q_i + 1$ for these variables. The above proof shows that the resulting inequality is facet-defining for $PB(L\backslash\{\bar{\imath}\}, \emptyset, L\backslash\{\bar{\imath}\}, \emptyset)$. Since $PB(L\backslash\{\bar{\imath}\}, \emptyset, L\backslash\{\bar{\imath}\}, \emptyset) \subseteq PB$, all the points tight for (78) are feasible to *PB*. Now, we lift a variable $i' \in L\backslash\{\bar{\imath}\}$. Let

$$F(w,a) = \left\{ (x,y) \in \{0,1\}^n \times [0,1]^n \;\middle|\; \sum_{i \in K} a_i x_i y_i \geq d - a' - w \text{ and } \sum_{i \in K} x_i = |K| - 1 - a \right\}.$$

We show that there exists $p \in F(B_{q_{i'}} + \mu, q_{i'} + 1)$ which is feasible to *PB* and tight on (78). First note that $F(B_{q_{i'}} + \mu, q_{i'} + 1) \neq \emptyset$ because $\Phi(B_{q_{i'}} + \mu) = q_{i'} + 1$. Let $p = (x', y')$. By the definition of $F(w,a)$, we are free to redefine (x'_i, y'_i) for $i \notin K$. Let $x'_i = y'_i = 0$ for $i \in M\backslash\{L \cup \{\bar{\imath}\}\}$ and let $x'_i = y'_i = 1$ for $i \in \widehat{M}$. Let $x'_{\bar{\imath}} = 1$ and $y'_{\bar{\imath}} = \frac{B_{q_{i'}} + \bar{\mu} - a_{i'}}{a_{\bar{\imath}}}$. Since $a_{\bar{\imath}} \geq \bar{\mu}$ and $B_{q_{i'}} < a_{i'} \leq B_{q_{i'}} + \bar{\mu}$ it follows that $0 < y_{\bar{\imath}} \leq 1$. Finally, we set $(x'_{i'}, y'_{i'}) = (1,1)$. Note that $a_{\bar{\imath}} x'_{\bar{\imath}} y'_{\bar{\imath}} + a_{i'} x'_{i'} y'_{i'} = B_{q_{i'}} + \bar{\mu}$ and

$$\alpha^0_{\bar{\imath}} x'_{\bar{\imath}} + \beta^0_{\bar{\imath}} y'_{\bar{\imath}} + \alpha^{j_{i'}}_{i'} x'_{i'} + \beta^{j_{i'}}_{i'} y'_{i'} = \frac{B_{q_{i'}} + \bar{\mu} - a_{i'}}{\bar{\mu}} + q_{i'} + \frac{a_{i'} - B_{q_{i'}}}{\bar{\mu}} = q_{i'} + 1 = \Psi(B_{q_{i'}} + \mu) = \Phi(B_{q_{i'}} + \mu),$$

where the first equality holds since $(\alpha^0_{\bar{\imath}}, \beta^0_{\bar{\imath}}) = (0, \frac{a_{\bar{\imath}}}{\bar{\mu}})$, $\left(\alpha^{j_{i'}}_{i'}, \beta^{j_{i'}}_{i'}\right) = \left(q_{i'} - \theta \frac{B_{q_{i'}-1} + \bar{\mu}}{a_{i'}}, \theta\right)$ when $j_{i'} = q_{i'}$ and $\left(\alpha^{j_{i'}}_{i'}, \beta^{j_{i'}}_{i'}\right) = (\Psi(a_{i'}), 0)$ when $j_{i'} = q_{i'} + 1$ where $\theta = \frac{(\Psi(a_{i'}) - q_{i'}) a_{i'}}{a_{i'} - B_{q_{i'}-1} - \bar{\mu}}$ and $\Psi(a_{i'}) = q_{i'} + \frac{a_{i'} - B_{q_{i'}}}{\bar{\mu}}$. Therefore, $p \in PB$ and is tight for (78). For $j_{i'} = q_{i'}$, we have already demonstrated that there exists a point of *PB* tight for (78) that sets $(x_{i'}, y_{i'}) = \left(1, \frac{W^{i'}_{j_{i'}}}{a_{i'}}\right)$ and for $j_{i'} = q_{i'} + 1$, there is a point of *PB* tight for (78) such that $(x_{i'}, y_{i'}) = (0,1)$. For $j_{i'} = q_{i'}$, affine independence follows since $a_{i'} > W^{i'}_{j_{i'}}$ implies that $(0,0)$, $(1,1)$, and $\left(1, \frac{W^{i'}_{j_{i'}}}{a_{i'}}\right)$ are affinely independent. For $j_{i'} = q_{i'+1}$, affine independence follows from the affine independence of $(0,0)$, $(1,1)$, and $(0,1)$. □

Inequalities (78) can be facet-defining depending on the value of the coefficients a_i and the choice of lifting coefficients (α_i, β_i) for $i \in M$. As mentioned before, $\widehat{M}$ may be chosen to be any subset of $\overline{M}$ that satisfies $\sum_{i \in \widehat{M}} a_i < \mu'$. In this case, (78) will be facet-defining if $\max\{a_i \mid i \in M, j_i = 0\} \geq \bar{\mu}$ but it may not be facet-defining otherwise. The next example illustrates the use of (78) in deriving facets of *PB*.

Example 3.27. *Consider the clique inequality (76) of Example 3.24 and its corresponding approximate lifting function. We have $q_1 = 2$ and $q_2 = 1$ with $W^1_0 = 0$, $W^1_1 = 3$, $W^1_2 = 20$, $W^1_3 = 21$, and $W^2_0 = 0$, $W^2_1 = 3$, $W^2_2 = 19$. Applying Theorem 3.26, we obtain the following nine inequalities*

$$\left\{ \begin{array}{l} \frac{21}{3} y_1 \\ \frac{14}{17} x_1 + \frac{21}{17} y_1 \\ 2 x_1 \end{array} \right\} + \left\{ \begin{array}{l} \frac{19}{3} y_2 \\ \frac{21}{24} x_2 + \frac{19}{24} y_2 \\ \frac{5}{3} x_2 \end{array} \right\} + x_3 + x_4 + x_5 \geq 2,$$

which define faces of PB of dimension at least 8 since $I(a_1) = 0$ and $I(a_2) = 1$. It follows from the first condition of Theorem 3.26 that following three inequalities

$$\left\{ \begin{array}{l} \frac{21}{3} y_1 \\ \frac{14}{17} x_1 + \frac{21}{17} y_1 \\ 2 x_1 \end{array} \right\} + \frac{19}{3} y_2 + x_3 + x_4 + x_5 \geq 2$$

are facet-defining for PB since $j_2 < q_2$. The two inequalities

$$\frac{21}{3} y_1 + \left\{ \begin{array}{l} \frac{21}{24} x_2 + \frac{19}{24} y_2 \\ \frac{5}{3} x_2 \end{array} \right\} + x_3 + x_4 + x_5 \geq 2$$

are also facet-defining for PB since they satisfy the second condition for facet-defining inequalities in Theorem 3.26 as $j_1 = 0$.

4 Relations to fixed-charge single-node flow model without inflows

In Section 3, we derived strong valid inequalities for the bilinear set B using lifting. In this section, we show that many of these lifted inequalities are also facet-defining for the convex hull of the fixed-charge single-node flow model without inflows

$$F = \left\{(x, y) \in \{0,1\}^n \times [0,1]^n \;\middle|\; \sum_{j=1}^{n} a_j y_j \geq d,\ x_j \geq y_j\ \forall j \in N\right\}.$$

In the following lemma, we show that $F \subseteq B$.

Lemma 4.1. *The bilinear covering set B is a relaxation of the flow set F.*

Proof. We prove that $F \subseteq B$. Let $(x, y) \in \{0,1\}^n \times [0,1]^n$ be an arbitrary point of F. It suffices to show that $\sum_{j=1}^{n} a_j x_j y_j \geq d$. Let $N_0 = \{j \in N \mid x_j = 0\}$ and $N_1 = \{j \in N \mid x_j = 1\}$. Since $(x, y) \in F$, $y_j = 0$ for all $j \in N_0$. Then,

$$\sum_{j\in N} a_j x_j y_j = \sum_{j\in N_1} a_j y_j = \sum_{j\in N} a_j y_j \geq d,$$

where the last inequality holds because $(x, y) \in F$. □

Fixed-charge single-node flow sets are important in practice since they can be used as a source of cutting planes for 0−1 mixed-integer programs. Further, they naturally arise in the formulation of fixed-charge network problems; see [2, 12, 19, 20, 22]. The fixed-charge single-node flow set F without inflows was first studied by Padberg et al. [22] under the assumptions that (i) $a_i \leq d$ and (ii) $\sum_{j=1}^{n} a_j > d + a_i$ for all $i \in N$. In the following, we relate the facets of *PF* to those of *PB* without assuming that the sets are full-dimensional.

Lemma 4.2 (Adapted from Proposition 8 in Padberg et al. [22])**.** *Every facet-defining inequality of PF that is not a multiple of $y_i \leq x_i$ can be expressed as $\alpha x + \beta y \geq \delta$, where $\beta \geq 0$.*

Proof. If for some i, $\beta_i < 0$ then the only points tight on this inequality are such that $y_i = x_i$. If F satisfies this equality then we may rewrite the facet-defining inequality as $\alpha x + \beta_i x_i + \beta y - \beta_i y_i \geq \delta$. □

In the following, we refer to the facet-defining inequalities of *PF* that are not a multiple of $y_i \leq x_i$ as *non-trivial* facet-defining inequalities.

Lemma 4.3. $\mathrm{aff}(F) = \mathrm{aff}(B)$*.*

Proof. Clearly, $\mathrm{aff}(F) \subseteq \mathrm{aff}(B)$ since $F \subseteq B$ by Lemma 4.1. It therefore remains to prove that $\mathrm{aff}(B) \subseteq \mathrm{aff}(F)$. Consider any point $(x, y) \in B$. If $(x, y) \in F$, then clearly $(x, y) \in \mathrm{aff}(F)$. We may therefore assume that $(x, y) \in B \backslash F$. Define $p = (x', y')$ where $(x'_i, y'_i) = (x_i, x_i y_i)$ for $i \in N$. It is easy to see that $\sum_{i\in N} a_i y'_i = \sum_{i\in N} a_i x_i y_i \geq d$ and $y'_i \leq x'_i$ for $i \in N$ and so $p \in F$. Let $I' = \{i \in N \mid y_i > x_i\}$. We show next that for each $i \in I'$, $p^i = p + (0, e_i) \in \mathrm{aff}(F)$. To this end, observe that $x'_i = 0$ for each $i \in I'$. It follows easily that $q^i = p + (e_i, 0)$ and $r^i = p + (e_i, e_i)$ belong to F. Therefore, $p^i = p + (r^i - q^i) \in \mathrm{aff}(F)$. Now, observe that $(x, y) = p + \sum_{i\in I'} y_i(p^i - p) \in \mathrm{aff}(F)$. It follows that $B \subseteq \mathrm{aff}(F)$ and therefore $\mathrm{aff}(B) \subseteq \mathrm{aff}(F)$. □

Proposition 4.4. *Assume that*

$$\alpha x + \beta y \geq \delta \tag{84}$$

is valid for PF and, for each $i \in N$, either $\alpha_i \leq 0$ or $\beta_i \geq 0$. Then, (84) is valid for PB. Further, for every non-trivial facet (84) of PF with $\beta \geq 0$, (84) is facet-defining inequality for PB.

Proof. We first show that (84) is valid for B. Consider $(x,y) \in B$. Let $I = \{i \in N \mid \alpha_i \le 0\}$. Define (x', y') such that $(x'_i, y'_i) = (1, y_i)$ for $i \in I$ and $(x'_i, y'_i) = (x_i, x_i y_i)$ for $i \in N \setminus I$. Then,

$$\sum_{i\in N} a_i y'_i = \sum_{i \in I} a_i y_i + \sum_{i \in N\setminus I} a_i x_i y_i \ge \sum_{i\in N} a_i x_i y_i \ge d,$$

where the last inequality holds because $(x,y) \in B$. Further, since $y'_i \le x'_i$, it follows that $(x', y') \in F$. Then,

$$\delta \le \alpha x' + \beta y' \le \alpha x + \beta y,$$

where the first inequality holds because $(x', y') \in F$ and the second inequality is satisfied since, by construction, $\alpha(x' - x) + \beta(y' - y) \le 0$. It follows that (84) is valid for *PB*.

Consider a non-trivial facet-defining inequality $\alpha' x + \beta' y \ge \delta'$ of *PF* with $\beta' \ge 0$. Then, by the first part of this result, it follows that $\alpha' x + \beta' y \ge \delta'$ is valid for *PB*. Since, by Lemmas 4.1 and 4.3 respectively, $B \supseteq F$ and $\dim(B) = \dim(F)$, it follows that $\alpha' x + \beta' y \ge \delta$ defines a facet of B. □

In Proposition 4.4, the assumption that $\beta \ge 0$ for a facet-defining inequality is without loss of generality because of Lemma 4.2. As an immediate consequence of Proposition 4.4, it can be shown that lifting functions associated with inequalities $\alpha x + \beta y \ge \delta$, such that for each i either $\alpha_i \le 0$ or $\beta_i \ge 0$ are identical when they are computed over B or over F. Since the inequalities derived in Section 3 as well as the seed inequalities satisfy these assumptions, many of our results in Section 3 extend to the study of F. We record this observation in the following corollary.

Corollary 4.5. *Let $(\alpha, \beta) \in \mathbb{R}^{2n}$ and, for each $i \in N$, assume that either $\alpha_i \le 0$ or $\beta_i \ge 0$. Let $B(w) = \big\{(x,y) \in \{0,1\}^n \times [0,1]^n \bigm| \sum_{i\in N} a_i x_i y_i \ge d - w\big\}$, and $F(w) = \big\{(x,y) \in \{0,1\}^n \times [0,1]^n \bigm| \sum_{i \in N} a_i y_i \ge d - w$ and $y_i \le x_i$ for all $i \in N\big\}$, where $a_i \ge 0$ for all $i \in N$. Let $z_B(w) = \min\big\{\alpha x + \beta y \bigm| (x,y) \in B(w)\big\}$ and $z_F(w) = \min\big\{\alpha x + \beta y \bigm| (x,y) \in F(w)\big\}$. Then, $z_B(w) = z_F(w)$.*

Proof. By Lemma 4.1, $B(w) \supseteq F(w)$. It follows that $z_B(w) \le z_F(w)$. We now argue that $z_B(w) \ge z_F(w)$. By the definition of $z_F(w)$, $\alpha x + \beta y \ge z_F(w)$ is valid for $F(w)$, which is a flow-set. Let $(x', y') \in \operatorname{argmin}\{\alpha x + \beta y \mid (x,y) \in B(w)\}$. Then, $z_B(w) = \alpha x' + \beta y' \ge z_F(w)$, where the inequality follows from Proposition 4.4. We conclude that $z_B(w) = z_F(w)$. □

Now, we illustrate Proposition 4.4 via an example.

Example 4.6. *Consider the fixed-charge single-node flow set without inflows*

$$F = \left\{(x,y) \in \{0,1\}^4 \times [0,1]^4 \;\middle|\; 19y_1 + 17y_2 + 15y_3 + 10y_4 \ge 20,\ x_j \ge y_j,\ \forall j = 1,\ldots,4\right\},$$

corresponding to the bilinear covering set B discussed in Example 2.6. We obtained a complete linear description of PF using PORTA; see Christof and Löbel [6]. This linear description is given in the Appendix. We observe that inequalities (10), (11), (17), and (18) are facets for both PB and PF. However, it can be verified that inequalities (12), (13), (15), and (16) are facet-defining for PB but not for PF. We mention that the inequalities of PF described in the Appendix have been numbered according to their counterparts in PB.

Proposition 4.4 is surprising in light of Lemma 4.1 because on the one hand $F \subsetneq B$ and on the other hand the nontrivial facets of *PF* are facets of *PB*. In other words, a polyhedral description of *PF* can be derived from that of *PB* by adding the trivial facets of *PF*. The converse, however, is not true. As an illustration, inequality (20) in the Appendix is a non-trivial facet-defining inequality of *PB* that is not facet-defining for *PF*. Surprisingly, a partial converse to Proposition 4.4 does hold.

We will show that an inequality description of *PB* can be obtained given the facet-defining inequalities for *PF*. The key to this construction is the result of Lemma 4.7 which shows that F and B can be viewed as projections of the same set onto different subspaces. Let

$$S = \left\{(x,y,z) \in \{0,1\}^n \times [0,1]^n \times \mathbb{R}^n \;\middle|\; \sum_{j=1}^n a_j z_j \ge d, z_j = x_j y_j, \forall j \in N\right\}.$$

Lemma 4.7. *The projection of S onto the (x,z) space is F while the projection of S onto (x,y) space is B. Consequently,* $\operatorname{proj}_{(x,z)} \operatorname{conv}(S) = PF$ *and* $\operatorname{proj}_{(x,y)} \operatorname{conv}(S) = PB$.

Proof. First, we show that $\operatorname{proj}_{(x,z)} S = F$. If $(x,y,z) \in S$, it is clear that $(x,z) \in F$ since $0 \le z_j \le x_j$ and $\sum_{j=1}^n a_j z_j \ge d$. If $(x,z) \in F$, then $0 \le z_j \le x_j$ and $x_j \in \{0,1\}$ imply that $z_j = x_j z_j$. Therefore, $(x,z,z) \in S$. Second, we show that $\operatorname{proj}_{(x,y)} S = B$. This follows by substituting $x_j y_j$ for z_j in $\sum_{j=1}^n a_j z_j \ge d$. The last statement follows since $\operatorname{conv}(AS) = A\operatorname{conv}(S)$ for any linear transformation A. □

Surprisingly, $\operatorname{conv}(S)$ can be described using facet-defining inequalities for PF. We write that $(\alpha,\beta,\gamma) \in \mathcal{F}(PF)$ if $\alpha x + \beta y \ge \gamma$ is a facet-defining inequality of PF that is not a multiple of $y_j \le x_j$. Define

$$G = \{(x,y,z) \mid \alpha x + \beta z \ge \gamma \, \forall (\alpha,\beta,\gamma) \in \mathcal{F}(PF), z \le \min\{x,y\}, y \le z + \mathbf{1} - x\}.$$

Theorem 4.8. $G = \operatorname{conv}(S)$.

Proof. ($\supseteq$) To show that $\operatorname{conv}(S) \subseteq G$, it suffices to show that $S \subseteq G$ because G is convex. Consider $(x,y,z) \in S$. Then, by Lemma 4.7, $(x,z) \in F$ and, therefore, $\alpha x + \beta z \ge \gamma$ for all (α,β,γ) in $\mathcal{F}(PF)$. Further, by McCormick inequalities, $x_j + y_j - 1 \le x_j y_j \le \min\{x_j, y_j\}$. Therefore, (x,y,z) satisfies the defining inequalities of G.

($\subseteq$) Now, we show that $G \subseteq \operatorname{conv}(S)$. If $(x,y,z) \in G$, then $(x,z) \in PF$ and $z \le y \le z + \mathbf{1} - x$. Therefore, there exists I such that $(x,z) = \sum_{i\in I} \lambda_i (x^i, z^i)$ where $(x^i, z^i) \in F$, $\lambda_i \ge 0$ for $i \in I$, and $\sum_{i \in I} \lambda_i = 1$. We define $f_j = \frac{y_j - z_j}{1 - x_j}$ if $x_j < 1$ and 0 otherwise. Let $I_j^1 = \{i \in I \mid x_j^i = 1\}$. Now, consider (x^i, y^i, z^i) where $y_j^i = z_j^i$ if $i \in I_j^1$ and $y_j^i = f_j$ if $i \in I \setminus I_j^1$. Then, $z_j^i \le x_j^i$ and $x_j^i \in \{0,1\}$ imply that $z_j^i = x_j^i y_j^i$. Further,

$$\sum_{i \in I} \lambda_i y_j^i = \sum_{i \in I_j^1} \lambda_i z_j^i + \sum_{i \in I \setminus I_j^1} \lambda_i f_j = z_j + (1 - x_j) f_j = y_j,$$

where the second equality follows since $z_j = \sum_{i \in I} \lambda_i z_j^i = \sum_{i \in I_j^1} \lambda_i z_j^i$, $\sum_{i \in I_j^1} \lambda_i = x_j$, and $\sum_{i\in I} \lambda_i = 1$, and the last equality since $x_j = 1$ implies that $z_j = y_j$. Therefore, $(x,y,z) = \sum_{i \in I} \lambda_i (x^i, y^i, z^i) \in \operatorname{conv}(S)$. □

Finally, we show that the projections of G to (x,z) space and (x,y) space are not altered even if G is relaxed in a certain way. Let

$$R = \{(x,y,z) \mid \alpha x + \beta z \ge \gamma \, \forall (\alpha,\beta,\gamma) \in \mathcal{F}(PF), z \le \min\{x,y\}, y \le \mathbf{1}\}.$$

Corollary 4.9. $PF = \operatorname{proj}_{(x,z)} R$ *and* $PB = \operatorname{proj}_{(x,y)} R$.

Proof. We will show that $\operatorname{proj}_{(x,z)} R = \operatorname{proj}_{(x,z)} G$ and $\operatorname{proj}_{(x,y)} R = \operatorname{proj}_{(x,y)} G$. Then, the result follows from Lemma 4.7 and Theorem 4.8. Since $z + \mathbf{1} - x \le \mathbf{1}$, it follows that $R \supseteq G$. First, we show that $\operatorname{proj}_{(x,z)} R \subseteq \operatorname{proj}_{(x,z)} G$. Assume that $(x,y,z) \in R$. Then, define $y' = z + \mathbf{1} - x$. Since $z + \mathbf{1} - x \ge z$ it follows that $(x, y', z) \in G$. Second, we show that $\operatorname{proj}_{(x,y)} R \subseteq \operatorname{proj}_{(x,y)} G$. Assume that $(x,y,z) \in R$. Then, let $z' = \max\{z, x + y - \mathbf{1}\}$. By Lemma 4.2, for all $(\alpha,\beta,\gamma) \in \mathcal{F}(PF)$, $\beta \ge 0$. Therefore, $\alpha x + \beta z' \ge \alpha x + \beta z \ge \gamma$. Further, $z' = \max\{z, x + y - \mathbf{1}\} \le \min\{x,y\}$ since $z \le \min\{x,y\}$ and $x, y \in [0,1]^2$. Finally, by construction, $y \le z' + \mathbf{1} - x$. Therefore, $(x,y,z') \in G$. □

Corollary 4.9 implies every non-trivial facet of PB arises as a conic combination of a single non-trivial facet of PF and (possibly multiple) trivial facet-defining inequalities $y_j \le x_j$.

Corollary 4.10. *Let $\alpha x + \beta y \ge \gamma$ be a facet-defining inequality for PB where $\beta \ge 0$. Then, $\alpha x + \beta y \ge \gamma$ defines a non-empty face of F. Further, there exists (α', β') and $\lambda \ge 0$ such that $(\alpha, \beta) = (\alpha' + \lambda, \beta' - \lambda)$, where $\alpha' x + \beta' y \ge \gamma$ is facet-defining for PF and for $j = 1, \ldots, n$, $\lambda_j \beta_j = 0$.*

Proof. Let $\delta = \min\{\alpha x + \beta y \mid (x,y) \in PF\}$. Since, by Lemma 4.1, $F \subseteq B$, it follows that $\delta \ge \gamma$. By Proposition 4.4, $\alpha x + \beta y \ge \delta$ is valid for PB. Therefore, $\delta \le \gamma$. In other words, $\delta = \gamma$ and $\alpha x + \beta y \ge \gamma$ defines a non-empty face of F. By Corollary 4.9 and Fourier-Motzkin elimination of z from R it follows that,

$$PB = \{(x,y) \mid \alpha' x + {\beta'}^J x + {\beta'}^{N \setminus J} y \ge \gamma' \forall (\alpha', \beta', \gamma') \in \mathcal{F}(PF) \text{ and } J \subseteq N, y \le 1\},$$

where ${\beta'}^J_j = \beta'_j$ if $j \in J$ and ${\beta'}^J_j = 0$ otherwise. Since (α,β,γ) is not a multiple of $y_j \le 1$, it follows that there exists $J \subseteq N$ and $(\alpha', \beta', \gamma') \in \mathcal{F}(PF)$ such that $(\alpha, \beta) = (\alpha' + {\beta'}^J, \beta' - {\beta'}^J)$. □

Example 4.11. *Consider the inequality* $126x_1+90x_3+45x_4+153y_2 \geq 135$ *that is facet-defining for the bilinear covering set of Example 2.6 but not facet-defining for the corresponding flow set presented in Example 4.6; see Appendix for a complete description of facet-defining inequalities of PB where this inequality is numbered (20). Then, as Corollary 4.10 proves, this inequality can be expressed as a sum of* $50x_1+90x_3+45x_4+76y_1+153y_2 \geq 135$ *and* $76x_1 - 76y_1 \geq 0$*, which are the facet-defining inequalities of the corresponding flow-set numbered (1) and (f1) in the Appendix.*

Proposition 4.4 and Corollary 4.9 show that a polyhedral description of either *PF* or *PB* can be derived explicitly given the facet-defining inequalities of the other. In fact, Proposition 4.4 also shows that an affine function over either B or F can be optimized if we have an oracle for optimizing an affine function over the other set. We discuss the reduction below. Let $l(x,y) = \alpha x + \beta y - \gamma$ and define $I = \{i \in N \mid \alpha_i > 0, \beta_i < 0\}$. Let $z_B(l) = \min\{l(x,y) \mid (x,y) \in B\}$ and $z_F(l) = \min\{l(x,y) \mid (x,y) \in F\}$. Define $l'(x,y) = \alpha x + \sum_{i \in N \setminus I} \beta_i y_i + \sum_{i \in I} \beta_i - \gamma$. While minimizing $l(x,y)$ over B, y_i can be set to 1 whenever $\beta_i \leq 0$. Therefore, it follows that $z_B(l) = z_B(l')$. However, by Proposition 4.4, $z_F(l') = z_B(l')$. Therefore, $z_B(l) = z_F(l')$. If (x,y) is the optimal solution to $z_F(l')$, the optimal solution to $z_B(l)$ is (x, y') where $y'_i = 1$ if $i \in I$ and $y'_i = y_i$ otherwise. Now, we consider the converse. Define $l''(x,y) = \alpha x + \sum_{i \in I} \beta_i x_i + \sum_{i \in N \setminus I} \beta_i y_i - \gamma$. While minimizing $l(x,y)$ over F, y_i can be set to x_i whenever $\beta_i \leq 0$. Therefore, $z_F(l) = z_F(l'')$. But, then by Proposition 4.4, $z_F(l'') = z_B(l'')$. Therefore, $z_F(l) = z_B(l'')$. The optimal solution can be obtained as in the proof of Proposition 4.4.

Given the relationships between the polyhedra *PB* and *PF*, it is reasonable to expect that the inequalities we developed in Section 3 reveal facets of *PF*. We now provide a detailed discussion of which inequalities are facet-defining for *PF*. For the remainder of this section, we assume, as we did for *PB*, that

Assumption 3. $\sum_{j=1}^{n} a_j \geq d + a_i$ *for all* $i \in N$.

Under Assumption 3, it follows from Lemma 4.3 that *PF* is a full-dimensional polytope.

Theorem 4.12. *A lifted bilinear cover inequality (54) is facet-defining for PF if and only if*

$$(\alpha_i, \beta_i) \in \bigcup_{j=0}^{q_i} \left(P^C(Q^i_j) - \frac{P^C(Q^i_{j+1}) - P^C(Q^i_j)}{\Delta^i_j} Q^i_j, \frac{P^C(Q^i_{j+1}) - P^C(Q^i_j)}{\Delta^i_j} a_i \right) \tag{85}$$

for all $i \in M$.

Proof. The proof of Proposition 2.9 already shows that (42) is facet-defining for $PF(M, C', M, C')$ since all the points considered are feasible to the flow set.

Now, it suffices to show that sufficiently many of the tight points added when lifting variables (x_i, y_i) for $i \in M \cup C'$ also belong to *PF*. When we lifted variables (x_i, y_i) for $i \in C'$ in the proof of Proposition 3.9, we added the two affinely independent points $(0,0)$ and $\left(1, \frac{(a_i-\mu)^+}{a_i}\right)$ that both correspond to feasible solutions of F; see (42) and Corollary 4.5. When lifting the variables (x_i, y_i) for $i \in M$ in Theorem 3.14, we added the two points $\left(1, \frac{Q^i_j}{a_i}\right)$ and $\left(1, \frac{Q^i_{j+1}}{a_i}\right)$ that both correspond to feasible solutions of F; see (45) and Corollary 4.5.

Next, we show that if (54) is facet-defining for *PF*, then (α_i, β_i) must be chosen as in (85). It suffices to show that if $(\alpha_i, \beta_i) = (P^C(a_i), 0)$ for some $i \in M$ and if at least one of the coefficients pair $(P^C(a_i), 0)$ does not reduce to coefficients studied before (which happens when $P^C(a_v) \neq P^C(Q^v_{q_v})$ for some v), then (54) is not facet-defining for *PF*. We will do so by showing that in such a case, (54) can be obtained by combining a different (facet-defining) inequality of the form (54) for *PF* with trivial facets $y_i \leq x_i$ of *PF*. Let $V \subseteq M$ be the set of lifting coefficients (α_v, β_v) chosen to be $(P^C(a_v), 0)$. Inequality (54) then reduces to

$$\sum_{v \in V} P^C(a_v) x_v + \sum_{i \in C} (a_i - \mu)^+ x_i + \sum_{j \in T} a_j y_j + \sum_{i \in M \setminus V} \alpha_i x_i + \sum_{i \in M \setminus V} \beta_i y_i \geq \sum_{i \in C} (a_i - \mu)^+. \tag{86}$$

Using the first part of this proof, we know that choosing lifting coefficients

$$\left(\left(P^C(Q^v_{q_v}) - \frac{P^C(Q^v_{q_v+1}) - P^C(Q^v_{q_v})}{\Delta^v_{q_v}} Q^v_{q_v} \right), \left(\frac{P^C(Q^v_{q_v+1}) - P^C(Q^v_{q_v})}{\Delta^v_{q_v}} a_v \right) \right)$$

for $v \in V$ yields the following facet-defining inequality

$$\sum_{v \in V} \left(P^C(Q^v_{q_v}) - \frac{P^C(Q^v_{q_{v+1}}) - P^C(Q^v_{q_v})}{\Delta^v_{q_v}} Q^v_{q_v} \right) x_v + \left(\frac{P^C(Q^v_{q_v+1}) - P^C(Q^v_{q_v})}{\Delta^v_{q_v}} a_v \right) y_v \tag{87}$$

$$+ \sum_{i \in C} (a_i - \mu)^+ x_i + \sum_{j \in T} a_j y_j + \sum_{i \in M \setminus V} \alpha_i x_i + \sum_{i \in M \setminus V} \beta_i y_i \geq \sum_{i \in C} (a_i - \mu)^+$$

for *PF*. Summing (87) with

$$\left(\frac{P^C(Q^v_{q_v+1}) - P^C(Q^v_{q_v})}{\Delta^v_{q_v}} a_v \right) (x_v - y_v) \geq 0, \forall v \in V \tag{88}$$

we obtain (86) since $Q^v_{q_v+1} = a_v$ and $\Delta^v_{q_v} = a_v - Q^v_{q_v}$. Since we assumed that $P^C(a_v) - P^C(Q^v_{q_v}) > 0$ for some $v \in V$ and because it is easy to see that (88) does not define the same face of *PF* that (87) defines, we conclude that (86) is not facet-defining for *PF*. □

We remark that in the proof of Theorem 4.12, we proved that a few inequalities of the type (54) are facet-defining for *PB* but not for *PF*. This was shown by expressing these inequalities using another non-trivial facet of *PF* and the inequalities $y_j \leq x_j$. We have already shown in Corollary 4.10 that this construction can be used to describe all facet-defining inequalities of *PB* that are not facet-defining for *PF*. We will use similar constructions later in the section. As a consequence of Theorem 4.12, we obtain the following result initially obtained by Padberg et al. [22].

Corollary 4.13. *(Adapted from Proposition 12 in Padberg et al. [22]) Assume that (i) $\mathcal{C}$ is a cover with excess $\bar{\mu} = \sum_{j \in \mathcal{C}} a_j - d$ such that $\bar{a} = \max_{j \in \mathcal{C}} a_j > \bar{\mu}$ and (ii) $\mathcal{L} \subseteq N \setminus \mathcal{C}$ is chosen so that $0 < \bar{a} - \bar{\mu} < a_k \leq \bar{a}$ for all $k \in \mathcal{L}$ and $\sum_{j \in N \setminus \mathcal{L}} a_j > d + \bar{a}$. Then*

$$\sum_{j \in \mathcal{C}} (a_j - \bar{\mu})^+ x_j + \sum_{j \in \mathcal{L}} (\bar{a} - \bar{\mu}) x_j + \sum_{j \in N \setminus (\mathcal{C} \cup \mathcal{L})} a_j y_j \geq \sum_{j \in \mathcal{C}} (a_j - \bar{\mu})^+ \tag{89}$$

is facet-defining for PF.

Proof. Let $\mathcal{C}$ and $\mathcal{L} \subseteq N \setminus \mathcal{C}$ be given that satisfy conditions (i) and (ii) of Corollary 4.13. Select $l \in \operatorname{argmax}\{a_j \mid j \in \mathcal{C}\}$. Define $C' = \mathcal{C} \setminus \{l\}$, $M = \mathcal{L}$, and $T = N \setminus (\mathcal{C} \cup \mathcal{L})$. Clearly, $\mu = \bar{\mu}$. Observe further that $a_l = \bar{a} > \mu$ and that $\sum_{j \in T} a_j > a_l - \bar{\mu}$ since $\sum_{j \in N \setminus \mathcal{L}} a_j > d + \bar{a}$. It follows that $(C', \{l\}, M, T)$ is a partition of N that satisfies Conditions (A1), (A2), (A3), and (A4) of Theorem 3.14. We obtain from Assumption (ii) that $A_1 - \mu < a_i \leq A_1 < A_2 - \mu$ for $i \in M$, which implies that $q_i = 1$ for all $i \in M$ in Lemma 3.13. Further, since $Q^i_1 = A_1 - \mu$ and $Q^i_2 = a_i$ for $i \in M$, we can select (α_i, β_i) as $(A_1 - \mu, 0)$ in (54), yielding

$$\sum_{j \in C} (a_j - \mu)^+ x_j + \sum_{j \in T} a_j y_j + \sum_{j \in M} (A_1 - \mu) x_j \geq \sum_{j \in C} (a_j - \mu)^+,$$

which is exactly (89) after performing the substitutions $C = \mathcal{C}$, $T = N \setminus (\mathcal{C} \cup \mathcal{L})$, $M = \mathcal{L}$, $A_1 = \bar{a}$ and $\mu = \bar{\mu}$. □

Observe that in (89), for each $j \in N$, either the coefficient of x_j or that of y_j is zero, whereas this is not the case for (54). Therefore, the facet-defining inequalities obtained via (89) are strictly contained in the facet-defining inequalities obtained via (54). In Padberg et al. [22], the authors did not explicitly impose the condition $\sum_{j \in N \setminus \mathcal{L}} a_j > d + \bar{a}$. However, in its absence, the inequalities are not necessarily facet-defining as we show in Example 4.14. The authors' proof implicitly made use of this assumption during an induction step. The next example illustrates that without this assumption (89) may not define a facet of the flow set.

Example 4.14. *Consider the flow set defined by*

$$F = \left\{ (x, y) \in \{0,1\}^4 \times [0,1]^4 \;\middle|\; 7y_1 + 6y_2 + 5y_3 + 4y_4 \geq 10, x_j \geq y_j \, \forall j = 1, \ldots, 4 \right\}.$$

Define $\mathcal{C} = \{1, 3\}$ and $\mathcal{L} = \{2\}$ where $\bar{a} = 7$ and $\bar{\mu} = 2$. Clearly $\bar{a} - \bar{\mu} < a_2 \leq \bar{a}$. However, the assumption that $\sum_{j \in N \setminus \mathcal{L}} a_j > d + \bar{a}$ does not hold. Inequality (89) takes the form

$$5x_1 + 5x_2 + 3x_3 + 4y_4 \geq 8. \tag{90}$$

Observe that whenever (90) is satisfied at equality by a point of F, the inequality $x_1 + x_2 \geq 1$ is also tight. Since $x_1 + x_2 \geq 1$ is clearly valid for F, it follows that (90) is not facet-defining for PF.

We next show that the family of lifted bilinear cover inequalities that are proven to be facet-defining for *PF* in Theorem 4.12 is larger than the family given by (89).

Example 4.15. *As established in Example 4.6, (10) and (11) are facet-defining lifted bilinear cover inequalities (54) for both PB and PF. They are obtained by choosing $(C', l, M, T) = (\{4\}, \{3\}, \{1\}, \{2\})$ and $(C', l, M, T) = (\{4\}, \{2\}, \{1\}, \{3\})$ respectively in Theorem 3.14. However, as mentioned above, (10) and (11) cannot be obtained using Corollary 4.13.*

Next, we show that the lifted reverse bilinear cover inequalities (66) that were shown to define facets of *PB* in Section 3 also define facets of *PF*.

Theorem 4.16. *Lifted bilinear reverse cover inequalities (66) are facet-defining for PF if and only if $a_i > a_l - \mu$ for all $i \in M$.*

Proof. Assume first that $a_i > a_l - \mu$ for all $i \in M$. It is clear that (66) is valid for F since $F \subseteq B$. Recall that (66) is obtained in Section 3 by lifting the seed inequality (42) which is facet-defining for $PB(M, C', M, C')$. We have shown in the proof of Theorem 4.12 that (42) is facet-defining for $PF(M, C', M, C')$. Now, we show that the tight points added when lifting variables (x_i, y_i) for $i \in M \cup C'$ also belong to *PF*. When we lifted the variables (x_i, y_i) for $i \in M$ in the proof of Proposition 3.16, we added the two linearly independent points $(0, 1)$ and $(1, 1)$. The first of these points is not feasible for F and cannot be used for the present derivation. However, when $a_i > a_l - \mu$, the solution $(1, 1 - \epsilon)$ for ϵ sufficiently small is feasible for F and satisfies (59) at equality. Therefore, $(1, 1)$ and $(1, 1 - \epsilon)$ provide the desired two tight independent feasible solutions of F; see (42) and Corollary 4.5. When lifting the variables (x_i, y_i) for $i \in C'$ in Theorem 3.20, we added the two points $(0, 0)$ and $\left(1, \frac{(a_i - A_1 + a_l - \mu)^+}{a_i}\right)$ which are affinely independent of $(1, 1)$ and correspond to feasible solutions of F; see (58) and Corollary 4.5. This proves that (66) is facet-defining for *PF*.

Assume now that $a_i \leq a_l - \mu$ for some $i \in M$. Define $M_2 = \{i \in M \mid a_i \leq a_l - \mu\} \neq \emptyset$ and $M_1 = M \setminus M_2$. Inequality (66) can be written as

$$(a_l - \mu)x_l - \sum_{j \in C'} P^M(-a_j)x_j + \sum_{j \in M_1} (a_l - \mu)x_j + \sum_{j \in M_2} a_j x_j + \sum_{j \in T} a_j y_j \geq (a_l - \mu) - \sum_{j \in C'} P^M(-a_j). \tag{91}$$

Observe next that partition $(C', \{l\}, M_1, T \cup M_2)$ satisfies Conditions (A1), (A2) and (A4) since $(C', \{l\}, M, T)$ does. Further, since $a_i > a_l - \mu$ for $i \in M_1$, it follows from the first part of this proof that the lifted reverse bilinear cover inequality

$$(a_l - \mu)x_l - \sum_{j \in C'} P^{M_1}(-a_j)x_j + \sum_{j \in M_1} (a_l - \mu)x_j + \sum_{j \in T \cup M_2} a_j y_j \geq (a_l - \mu) - \sum_{j \in C'} P^{M_1}(-a_j) \tag{92}$$

is facet-defining for *PF*, where it is easy to verify that $P^{M_1}(w) = P^M(w)$ for $w \leq 0$. Now, observe that (91) can be obtained by summing (92) and inequalities

$$a_j(x_j - y_j) \geq 0 \tag{93}$$

for $j \in M_2$. Since (92) and (93) define different facets of the full-dimensional polyhedron *PF*, we conclude that (91) is not facet-defining for *PF*. □

The inequalities of Theorem 4.16 are known to be valid for *PF*, as first shown in Gu et al. [12].

Corollary 4.17 (Adapted from Theorem 12 in Gu et al. [12])**.** *Assume that (i) $\mathcal{C} \subseteq N$ is a generalized cover for F such that $\sum_{j\in\mathcal{C}} a_j = d - \lambda$ with $\lambda > 0$ and (ii) $\mathcal{L} \neq \emptyset$ and $\sum_{j\in N\setminus\mathcal{L}} a_j > d$ where $\mathcal{L} = \{j \in N\setminus\mathcal{C} \mid a_j > \lambda\}$. Assume also that $\mathcal{L} = \{j_1, j_2, \ldots, j_r\}$ with $a_{j_i} \geq a_{j_{i+1}}$ for $i = 1, \ldots, r-1$. Let $r = |\mathcal{L}|$, $A_0 = 0$, and $A_i = \sum_{k=1}^{i} a_{j_k}$ for $i = 1, \ldots, r$. Further, let $d' = \sum_{j\in N\setminus\mathcal{C}} a_j - \lambda$. Define*

$$f(z) = \begin{cases} i\lambda & \text{if} \quad A_i \leq z \leq A_{i+1} - \lambda, \quad i = 0, \ldots, r-1, \\ z - A_i + i\lambda & \text{if} \quad A_i - \lambda \leq z \leq A_i, \qquad i = 1, \ldots, r-1, \\ z - A_r + r\lambda & \text{if} \quad A_r - \lambda \leq z \leq d'. \end{cases} \tag{94}$$

Then, the lifted simple generalized flow cover inequality (LSGFCI)

$$\sum_{j\in\mathcal{L}} \lambda x_j + \sum_{j\in\mathcal{C}} f(a_j) x_j + \sum_{j\in N\setminus(\mathcal{C}\cup\mathcal{L})} a_j y_j \geq \lambda + \sum_{j\in\mathcal{C}} f(a_j) \tag{95}$$

is facet-defining for PF.

Proof. For a given *generalized cover* $\mathcal{C}$ of F, we define $C = \mathcal{C} \cup \{l\}$ where $l \in \mathcal{L} \neq \emptyset$. Set C is a cover since $a_j > \lambda$ for all $j \in \mathcal{L}$. Further, $\sum_{j\in C} a_j = d + a_l - \lambda > d$ and so $\mu = a_l - \lambda > 0$, i.e. C satisfies Conditions (A1) and (A3) in Theorem 3.20. Now set $M = \mathcal{L} \setminus \{l\}$ in (66). Condition (A4) in Theorem 3.20 also holds since $\sum_{j\in N\setminus(\mathcal{L}\setminus\{l\})} a_j - d = \sum_{j\in N\setminus\mathcal{L}} a_j + a_l - d > 0$. Next, we observe that $C \cup M = \mathcal{C} \cup \mathcal{L}$ and that $\min\{a_i, a_l - \mu\} = \min\{a_i, \lambda\} = \lambda = a_l - \mu$ for all $i \in M$. Substituting $a_l - \mu = \lambda$ in Proposition 3.17, we obtain that $f(w) = -P^M(-w)$ since $M \cup \{l\} = \mathcal{L}$. Therefore, we conclude that (95) is a lifted reverse bilinear cover inequality (66). ☐

Because in Gu et al. [12] the fixed-charge single-node flow set studied is more general than F, the authors focused mainly on the derivation of valid inequalities and discussed only indirectly whether the resulting inequalities are facet-defining. The result of Corollary 4.17 is therefore different from that of Theorem 12 in Gu et al. [12] in two ways. First we added the condition $\sum_{j\in N\setminus\mathcal{L}} a_j > d$. This condition guarantees that the simple generalized flow cover inequality (SGFCI) that is used as seed inequality for lifting procedures in Gu et al. [12] is facet-defining for the problem restriction. Second, we replaced the statement that inequality (95) is valid for *PF* with the stronger statement that it is facet-defining for *PF*.

We conclude this section by presenting conditions under which the lifted clique inequalities (78) are facet-defining for the flow set *PF*.

Theorem 4.18. *A lifted clique inequality (78) is facet-defining for PF if (i) $\sum_{j\in K} a_j - a_k > d$ for all $k \in K$ and (ii) lifting coefficients are chosen according to (79) and (iii) one of the following conditions holds:*

1. *$L = \emptyset$.*
2. *$\exists \bar\imath \in M$ such that $j_{\bar\imath} = 0$ and, for all $i \in L\setminus\{\bar\imath\}$, $j_i = q_i$.*

Proof. Using a proof technique similar to that used in Theorems 4.12 and 4.16, we show that seed inequality (71) is facet-defining for $PF(\overline{M}\setminus\widehat{M}, \emptyset, \overline{M}\setminus\widehat{M}, \emptyset)$ and that lifting (x_i, y_i) for $i \in M$ adds two tight independent points in (78) that belong to F. Let $K = \{1, \ldots, l\}$ and $\widehat{M} = \{l+1, \ldots, h\}$. Define χ such that $\chi_j = 1$ for $j \leq l$ and 0 for $l+1 \leq j \leq k$. Consider $p^i = (\chi - e_i, \chi - e_i)$ for $i = 1, \ldots, l$, $q^i = (\chi - e_i, \chi - e_i - \epsilon e_{i+1})$ for $i = 1, \ldots, l-1$, $q^l = (\chi - e_l, \chi - e_l - \epsilon e_1)$ where ϵ is positive, and, for $j = l+1, \ldots, h$, $r^j = \big(\chi - e_1 + e_j, \chi - e_1\big)$ and $s^j = (\chi - e_1 + e_j, \chi - e_1 + e_j)$. These points satisfy (71) at equality, are affinely independent and, because of Assumption (i), belong to F when ϵ is sufficiently small. This shows that (71) is facet-defining for $PF(M, \emptyset, M, \emptyset)$. Assume first that $L = \emptyset$ and consider now the lifting of variables (x_i, y_i) for $i \in M$ in the proof of Theorem 3.26. For $j_i \in \{0, \ldots, q_i\}$, lifting adds the two independent points $\Big(1, \frac{W^i_j}{a_i}\Big)$ and $\Big(1, \frac{W^i_{j+1}}{a_i}\Big)$ that both correspond to feasible solutions of F because of (71) and Corollary 4.5, proving the result. Then it follows from the first part of this proof that the inequality obtained after lifting the variables in $M\setminus L$ is facet-defining for $PF(L\setminus\{\bar\imath\}, \emptyset, L\setminus\{\bar\imath\}, \emptyset)$. Consider now the lifting of variables (x_i, y_i) for $i \in L\setminus\{\bar\imath\}$. When $j_i = q_i$, we derived in the proof of Theorem 3.26 that lifting adds the two independent points $(1, 1)$ and $(1, \frac{W^i_{j_i}}{a_i})$ that both correspond to feasible solutions of F because the first point sets $(x_{\bar\imath}, y_{\bar\imath}) = \big(1, \frac{B_{q_i} + \bar\mu - a_i}{a_{\bar\imath}}\big)$, and the structure of (71) satisfies the assumptions of Corollary 4.5. ☐

To the best of our knowledge, Theorem 4.18 presents a new family of facet-defining inequalities for fixed-charge single-node flow models without inflows.

5 Discussion and Conclusion

Many of the results presented in this paper extend to $0-1$ mixed integer sets defined by constraints of the form $\sum_{i=1}^{k}(a_i x_i y_i + b_i x_i) + \sum_{i=k+1}^{n} a_i y_i \geq d$, *i.e.* bilinear covering sets where a linear term has been added to the left-hand-side. The primary reason the inequalities extend without significant changes is summarized in the next simple observation.

Proposition 5.1. *Consider an inequality $\alpha x + \beta y \geq \gamma$ such that for each i, $\alpha_i \beta_i = 0$. Let $H^+ = \{(x,y) \in \mathbb{R}^{2n} \mid \alpha x + \beta y \geq \gamma\}$. Let $I = \{i \in N \mid \alpha_i = 0\}$ and $I^c = N \setminus I$. Let $a_i \in \mathbb{R}$ and $(b_i, c_i) \in \mathbb{R}^2_+$ be such that, for each $i \in N$, $a_i + \min\{b_i, c_i\} \geq 0$. Consider the sets $A(w) = \{(x,y) \in \{0,1\}^n \times [0,1]^n \mid \sum_{i=1}^{n}(a_i x_i y_i + b_i x_i + c_i y_i) \geq d - w\}$ and $B(w) = \{(x,y) \in \{0,1\}^n \times [0,1]^n \mid \sum_{i \in I}(a_i + c_i)x_i y_i + \sum_{i \in I^c}(a_i + b_i) x_i y_i \geq d - \sum_{i \in I} b_i - \sum_{i \in I^c} c_i - w\}$. Then, $\min\{\alpha x + \beta y \mid (x,y) \in A(w)\} = \min\{\alpha x + \beta y \mid (x,y) \in B(w)\}$. Further, $H^+ \supseteq A(w)$ if and only if $H^+ \supseteq B(w)$.*

Proof. Consider a point (x', y') with $x'_i = 1$ for $i \in I$ and $y'_i = 1$ for $i \in I^c$. Then, for $i \in I$, $a_i x'_i y'_i + b_i x'_i + c_i y'_i = (a_i + c_i) x'_i y'_i + b_i$. Similarly, for $i \in I^c$, $a_i x'_i y'_i + b_i x'_i + c_i y'_i = (a_i + b_i) x'_i y'_i + c_i$. In other words, $\sum_{i \in I}(a_i + c_i) x'_i y'_i + \sum_{i \in I^c}(a_i + b_i) x'_i y'_i - \sum_{i \in I} b_i - \sum_{i \in I^c} c_i = \sum_{i=1}^{n}(a_i x'_i y'_i + b_i x'_i + c_i y'_i)$. Therefore, $(x', y') \in A(w)$ if and only if $(x', y') \in B(w)$. Now, it easy to see that

$$\begin{aligned} z_{A(w)} &= \min\{\alpha x + \beta y \mid (x,y) \in A(w)\} = \min\{\alpha x + \beta y \mid (x,y) \in A(w), x_i = 1 \forall i \in I, y_i = 1 \forall i \in I^c\} \\ &= \min\{\alpha x + \beta y \mid (x,y) \in B(w), x_i = 1 \forall i \in I, y_i = 1 \forall i \in I^c\} = \min\{\alpha x + \beta y \mid (x,y) \in B(w)\} = z_{B(w)}, \end{aligned}$$

where the second and the second last equality follow from the assumptions which imply that $a_i x_i y_i + b_i x_i + c_i y_i \leq \min\{a_i x_i + b_i x_i + c_i, a_i y_i + b_i + c_i y_i\}$, $(a_i + c_i) x_i y_i \leq (a_i + c_i) y_i$, and $(a_i + b_i) x_i y_i \leq (a_i + b_i) x_i$. Since $z_{A(w)} = z_{B(w)}$ and $H^+ \supseteq A(w)$ (resp. $H^+ \supseteq B(w)$) if and only if $z_{A(w)} \geq \gamma$ (resp. $z_{B(w)} \geq \gamma$), it follows that $H^+ \supseteq A(w)$ if and only if $H^+ \supseteq B(w)$. □

Note that the seed inequalities and the intermediate inequalities we derive during lifting satisfy the condition $\alpha_i \beta_i = 0$ for all i. Then, Proposition 5.1 essentially shows that the lifting functions derived for the problem with only bilinear terms on the left-hand-side also carry over to problems containing a linear term. For detailed derivations of facet-defining inequalities for bilinear covering sets with linear terms, we refer the reader to [7].

In this paper, we study the polyhedral structure of the $0-1$ mixed-integer bilinear covering set. We give a complete linear description of its convex hull when $n = 2$. We also show that, for a fairly large class of functions, it is sufficient to check that subadditivity holds on a subset of points of the domain to show that the function is subadditive over $\mathbb{R}^n$. This result enables short subadditivity proofs for many practically useful functions. In particular, we use this result to derive three families of strong inequalities for *PB* that can be obtained using sequence-independent lifting. Among them, two families have an exponential number of members. We study relations between $0-1$ mixed-integer bilinear covering sets and fixed-charge single-node flow sets without inflows. We show that valid inequalities for bilinear sets are also valid for flow sets and prove that all nontrivial facets of *PF* can be obtained through the study of facets of *PB*. We then show that the inequalities we derive generalize two classical families of lifted flow cover inequalities for *PF* and provide a new family for *PF*. Future research will focus on evaluating the computational benefits of using these lifted cuts in branch-and-bound frameworks for both linear and nonlinear mixed integer programming.

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Appendix

Linear descriptions of $\text{conv}(B)$ and $\text{conv}(F)$

The linear description of the convex hulls of the bilinear set B and the flow set F are obtained by PORTA as the following:

$$B = \left\{ (x, y) \in \{0,1\}^4 \times [0,1]^4 \;\middle|\; 19x_1y_1 + 17x_2y_2 + 15x_3y_3 + 10x_4y_4 \geq 20 \right\}$$

(1)	$50x_1$		$+90x_3$	$+45x_4$	$+76y_1$	$+153y_2$			≥ 135
(2)	$70x_1$	$+90x_2$		$+27x_4$	$+38y_1$		$+135y_3$		≥ 117
(3)	$25x_1$		$+65x_3$	$+45x_4$	$+76y_1$	$+153y_2$			≥ 110
(4)		$+50x_2$	$+70x_3$	$+35x_4$	$+133y_1$	$+34y_2$			≥ 105
(5)		$+25x_2$	$+45x_3$	$+35x_4$	$+133y_1$	$+34y_2$			≥ 80
(6)	$21x_1$	$+41x_2$		$+27x_4$	$+38y_1$		$+135y_3$		≥ 68
(7)	$30x_1$	$+35x_2$	$+21x_3$		$+19y_1$			$+70y_4$	≥ 56
(8)	$18x_1$	$+23x_2$	$+21x_3$		$+19y_1$			$+70y_4$	≥ 44
(9)	$19x_1$	$+17x_2$					$+15y_3$	$+10y_4$	≥ 20
(10)	$19x_1$		$+15x_3$			$+17y_2$		$+10y_4$	≥ 20
(11)	$19x_1$			$+10x_4$		$+17y_2$	$+15y_3$		≥ 20
(12)	$19x_1$					$+17y_2$	$+15y_3$	$+10y_4$	≥ 20
(13)		$+17x_2$	$+15x_3$		$+19y_1$			$+10y_4$	≥ 20
(14)		$+17x_2$		$+10x_4$	$+19y_1$		$+15y_3$		≥ 20
(15)		$+17x_2$			$+19y_1$		$+15y_3$	$+10y_4$	≥ 20
(16)			$+15x_3$	$+10x_4$	$+19y_1$	$+17y_2$			≥ 20
(17)			$+15x_3$		$+19y_1$	$+17y_2$		$+10y_4$	≥ 20
(18)				$+10x_4$	$+19y_1$	$+17y_2$	$+15y_3$		≥ 20
(19)					$+19y_1$	$+17y_2$	$+15y_3$	$+10y_4$	≥ 20
(20)	$14x_1$		$+10x_3$	$+5x_4$		$+17y_2$			≥ 15

(21)		$+12x_2$	$+10x_3$	$+5x_4$	$+19y_1$				≥ 15
(22)			$+10x_3$	$+5x_4$	$+19y_1$	$+17y_2$			≥ 15
(23)	$12x_1$	$+10x_2$		$+3x_4$			$+15y_3$		≥ 13
(24)		$+10x_2$	$+10x_3$	$+3x_4$	$+19y_1$				≥ 13
(25)		$+10x_2$		$+3x_4$	$+19y_1$		$+15y_3$		≥ 13
(26)	$10x_1$	$+10x_2$		$+x_4$			$+15y_3$		≥ 11
(27)	$10x_1$		$+10x_3$	$+x_4$		$+17y_2$			≥ 11
(28)	$10x_1$			$+x_4$		$+17y_2$	$+15y_3$		≥ 11
(29)	$7x_1$	$+5x_2$	$+3x_3$					$+10y_4$	≥ 8
(30)		$+5x_2$	$+3x_3$		$+19y_1$			$+10y_4$	≥ 8
(31)		$+5x_2$	$+3x_3$	$+5x_4$	$+19y_1$				≥ 8
(32)		$+3x_2$	$+3x_3$	$+3x_4$	$+19y_1$				≥ 6
(33)	$5x_1$		$+x_3$			$+17y_2$		$+10y_4$	≥ 6
(34)	$5x_1$	$+5x_2$	$+x_3$					$+10y_4$	≥ 6
(35)	$5x_1$		$+x_3$	$+5x_4$		$+17y_2$			≥ 6
(36)	$3x_1$	$+x_2$					$+15y_3$	$+10y_4$	≥ 4
(37)	$3x_1$	$+x_2$	$+3x_3$					$+10y_4$	≥ 4
(38)	$3x_1$	$+x_2$		$+3x_4$			$+15y_3$		≥ 4
(39)	x_1	$+x_2$	$+x_3$					$+10y_4$	≥ 2
(40)	x_1	$+x_2$		$+x_4$			$+15y_3$		≥ 2
(41)	x_1		$+x_3$	$+x_4$		$+17y_2$			≥ 2
(42)	x_1	$+x_2$	$+x_3$	$+x_4$					≥ 2
(43)	x_1								≥ 0
(44)		x_2							≥ 0
(45)			x_3						≥ 0
(46)				x_4					≥ 0
(47)					y_1				≥ 0
(48)						y_2			≥ 0
(49)							y_3		≥ 0
(50)								y_4	≥ 0
(51)								y_4	≤ 1
(52)							y_3		≤ 1
(53)						y_2			≤ 1
(54)					y_1				≤ 1
(55)				x_4					≤ 1
(56)			x_3						≤ 1
(57)		x_2							≤ 1
(58)	x_1								≤ 1

$$F = \left\{ (x, y) \in \{0,1\}^4 \times [0,1]^4 \;\middle|\; 19y_1 + 17y_2 + 15y_3 + 10y_4 \geq 20, x_j \geq y_j \; \forall j = 1, \dots, 4 \right\}$$

(1)	$50x_1$		$+90x_3$	$+45x_4$	$+76y_1$	$+153y_2$			≥ 135
(2)	$70x_1$	$+90x_2$		$+27x_4$	$+38y_1$		$+135y_3$		≥ 117
(3)	$25x_1$		$+65x_3$	$+45x_4$	$+76y_1$	$+153y_2$			≥ 110
(4)		$+50x_2$	$+70x_3$	$+35x_4$	$+133y_1$	$+34y_2$			≥ 105
(5)		$+25x_2$	$+45x_3$	$+35x_4$	$+133y_1$	$+34y_2$			≥ 80
(6)	$21x_1$	$+41x_2$		$+27x_4$	$+38y_1$		$+135y_3$		≥ 68
(7)	$30x_1$	$+35x_2$	$+21x_3$		$+19y_1$			$+70y_4$	≥ 56
(8)	$18x_1$	$+23x_2$	$+21x_3$		$+19y_1$			$+70y_4$	≥ 44
(19)					$+19y_1$	$+17y_2$	$+15y_3$	$+10y_4$	≥ 20
(22)			$+10x_3$	$+5x_4$	$+19y_1$	$+17y_2$			≥ 15
(24)		$+10x_2$	$+10x_3$	$+3x_4$	$+19y_1$				≥ 13

(25)		$+10x_2$		$+3x_4$	$+19y_1$		$+15y_3$		≥ 13
(26)	$10x_1$	$+10x_2$		$+x_4$			$+15y_3$		≥ 11
(27)	$10x_1$		$+10x_3$	$+x_4$		$+17y_2$			≥ 11
(28)	$10x_1$			$+x_4$		$+17y_2$	$+15y_3$		≥ 11
(30)		$+5x_2$	$+3x_3$		$+19y_1$			$+10y_4$	≥ 8
(31)		$+5x_2$	$+3x_3$	$+5x_4$	$+19y_1$				≥ 8
(32)		$+3x_2$	$+3x_3$	$+3x_4$	$+19y_1$				≥ 6
(33)	$5x_1$		$+x_3$			$+17y_2$		$+10y_4$	≥ 6
(34)	$5x_1$	$+5x_2$	$+x_3$					$+10y_4$	≥ 6
(35)	$5x_1$		$+x_3$	$+5x_4$		$+17y_2$			≥ 6
(36)	$3x_1$	$+x_2$					$+15y_3$	$+10y_4$	≥ 4
(37)	$3x_1$	$+x_2$	$+3x_3$					$+10y_4$	≥ 4
(38)	$3x_1$	$+x_2$		$+3x_4$			$+15y_3$		≥ 4
(39)	x_1	$+x_2$	$+x_3$					$+10y_4$	≥ 2
(40)	x_1	$+x_2$		$+x_4$			$+15y_3$		≥ 2
(41)	x_1		$+x_3$	$+x_4$		$+17y_2$			≥ 2
(42)	x_1	$+x_2$	$+x_3$	$+x_4$					≥ 2
(47)					y_1				≥ 0
(48)						y_2			≥ 0
(49)							y_3		≥ 0
(50)								y_4	≥ 0
(55)				x_4					≤ 1
(56)			x_3						≤ 1
(57)		x_2							≤ 1
(58)	x_1								≤ 1
$(f1)$	x_1				$-y_1$				≥ 0
$(f2)$		x_2				$-y_2$			≥ 0
$(f3)$			x_3				$-y_3$		≥ 0
$(f4)$				x_4				$-y_4$	≥ 0